MULTIPLICATIVE STRUCTURES AND THE TWISTED BAUM-CONNES ASSEMBLY MAP

NOÉ BÁRCENAS, PAULO CARRILLO ROUSE, AND MARIO VELÁSQUEZ

Abstract

Using a combination of Atiyah-Segal ideas in one side and of Connes and Baum-Connes ideas in the other, we show that the Total Twisted geometric K-homology group of a Lie groupoid (includes Lie groups and discrete groups) admits a ring structure (or module structure for the odd group). This group is the left hand side of the twisted geometric Baum-Connes assembly map recently constructed in [5]. For the case of proper groupoids, for which the assembly map is an isomorphism, our multiplicative structures coincide with the ones defined by Tu and Xu in [15].

1. Introduction

In recent years twisted K-theory and twisted index theory have benefited of a great deal of interest from several groups of mathematicians and theoretical physicists. Besides its relations with string theory and theoretical physics in general, one of the main mathematical motivations was the series of works by Freed, Hopkins and Teleman in which they describe a ring structure on an equivariant twisted K-theory of a group (compact connected Lie group) and in which they give a ring isomorphism with the Verlinde algebra of the group.

For discrete or non compact Lie groups it is not clear how these multiplicative structures should be defined directly or even if they exist at all. In this paper we give a step to try to understand these issues. Our approach is a mixture of Atiyah-Segal ideas in one side and of Connes and Baum-Connes ideas in the other. Indeed, if the group in question acts properly in a nice space then one can use a homotopy theoretical model for the twisted K-theory groups and use Atiyah-Segal ideas for defining a product in this setting. On the other hand, following Baum-Connes ideas one might expect that the analytically defined twisted equivariant K-theory can be approached (or assembled to be precise) by groups defined by using only proper actions (the so called left hand side). The main result of this paper is to define a ring structure on the left hand side of a twisted Baum-Connes assembly map associated to every Lie groupoid, proper or not. We explain this below with more details but before let us mention why we abruptly changed our terminology from groups to groupoids. We have at least two big reasons for this, first, the category of Lie groupoids encodes much more that groups and group actions, many singular situations can be handled using appropriate groupoids; second, our constructions and proofs are largely simplified by the use Connes deformation groupoids techniques (see explanation below).

We pass now to the explicit content of the paper. For proper groupoids one can define the twisted K-theory groups by a generalization of Atiyah-Jänich Fredholm model for classical topological K-theory. More precisely, if G is a proper Lie groupoid with connected units M and P is a G-equivariant $PU(H)-$principal bundle over M, the *Twisted G-equivariant K-theory* groups of M twisted by P are defined as the homotopy groups of the G-equivariant sections

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$$K_G^{-p}(M,P) := \pi_p\left(\Gamma(M;\mathrm{Fred}^{(0)}(\widehat{P}))^G, s\right)$$

where $\mathrm{Fred}^{(0)}(\widehat{P}) \to M$ is a certain bundle constructed from P with fibers an space of Fredholm operators, see definition 3.6 for more details. Using this suitable choice of Fredholm bundles we follow Atiyah-Segal for defining an associative product

$$K_G^{-p}(M,P) \times K_G^{-q}(M,P') \to K_G^{-(p+q)}(M, P \otimes P').$$

These twisted K-theory groups for proper groupoids are isomorphic to the K-theory of some C^*-algebras associated to the twisting, theorem 3.14 in [16]. In [15] the authors constructed a ring structure on Twisted K-theory for proper groupoids (and more generally for crossed modules) using the C^*-algebraic model and Kasparov's KK-theory techniques. We check in section 3 below that under the mentioned isomorphism the multiplicative structures coincide.

For non necessarily proper groupoids one does not dispose of a Fredholm model for defining the multiplicative structure as above and even if there is a C^*-algebraic model for twisted K-theory it is not clear how to define this product directly. Following Baum-Connes ideas one might expect that the K-theory of the twisted algebra can be approached by K-theory groups using only proper actions.

Given a Lie groupoid G (not necessarily proper) together with G-equivariant $PU(H)$-principal bundle P over the units of G, the authors in [5] generalize to the twisted case, Connes construction of the geometric K-homology group, denoted by $K_*^{geo}(G,P)$, and the construction of the geometric Baum-Connes assembly map. The main theorem in order to prove that this group and the assembly map are well defined is the wrong way functoriality of the pushforward construction associated to oriented smooth G-maps (theorem 4.2 in [5]). Given two isomorphic G-equivariant $PU(H)$-bundles their associated twisted K-theory groups and their associated twisted geometric K-homology groups are isomorphic as well, also the twisted Baum-Connes map mentioned above is compatible with these isomorphisms (theorem 6.4 in [5] gives a vast generalization of this fact). Denote by $H^1(G,PU(H))$ the set of isomorphism classes of G-equivariant $PU(H)$-principal bundles[1]. Consider the Total twisted geometric K-homology group

$$K_{TW,*}^{geo}(G) := \bigoplus_{\alpha \in H^1(G,PU(H))} K_*^{geo}(G, P_\alpha)$$

where P_α is a G-equivariant $PU(H)$-principal bundle in the class of α. The group $K_{TW,*}^{geo}(G)$ is well defined up to isomorphism, there is no canonical choice for a representative in a given isomorphism class.

We want to briefly describe a product on this Total twisted geometric K-homology group. By definition each $K_*^{geo}(G,P)$ is generated by cycles of the form (X,x) where X is a $G-$proper co-compact manifold and $x \in K_G^{-p}(X,P_X)$ (where P_X is the twisting over X induced by P and where $K_G^{-p}(X,P_X)$ denotes the equivariant twisted K-theory group associated to the action groupoid $X \rtimes G$, see definition 3.6 for more details) and with main relation given by the pushforward maps (see definition 6.1 for more precisions).

Let P and Q two twistings on G. Let (X,x) with $x \in K_G^{-p}(X,P_X)$ and (Y,y) with $y \in K_G^{-q}(Y,Q_Y)$, the product looks like follows

$$(X,x)\cdot(Y,y) := (X \times_{G_0} Y, \pi_X^* x \cdot \pi_Y^* y) \in K_G^{-p-q}(X \times_{G_0} Y, P_{X\times_{G_0}Y} \otimes Q_{X\times_{G_0}Y}) \tag{1.1}$$

[1] $H^1(G,PU(H))$ is of course the 1st Cech cohomology group of G with values in $PU(H)$ but for the purpose of this paper we do not need to use it in these terms. There are canonical maps $H^1(G,PU(H)) \to H^2(G,S^1) \to H^3(G,\mathbb{Z})$ which in general are surjective and isomorphisms for the case of proper groupoids, see for instance [16] for a detailed discution on this subject.

where π_X, π_Y stand for the respective projections from $X \times_{G_0} Y$ to X and Y and where the pullback is defined in section 6 below, it is easy to see that the product described above is compatible with isomorphism classes of $PU(H)-$principal bundles. To prove that the definition above induces a product on the Total twisted K-homology group one needs to prove two main properties:

(i) The compatibility of the product with respect to the pushforward maps, proposition 5.18 below.

(ii) The compatibilty of the pushforward and the pullback constructions, proposition 6.7 below.

For the two properties above the use of the groupoid language becomes very useful. First of all the construction of the pushforward maps can be completely realized in the Fredholm picture by using Connes deformation groupoids, and hence adapting to this model the main results and constructions from [5] for the case of proper groupoids, we explain this in section 5. Second, the proofs become conceptually very simple, for example to prove the first property above amounts to check that the morphisms induced by restriction are compatible with the product. So even if one is only interested in the group case (Lie or discrete for instance) the use of deformation groupoids gives a unified way to construct the pushforwrd maps, to prove their functoriality and to prove their compatibility with the product.

Our main theorem is the following one:

Theorem 1.2. *For any Lie groupoid, the product described above induces*

- *a ring structure on the even Total twisted geometric K-homology group* $K^{geo}_{TW,0}(G)$, *and*

- *a* $K^{geo}_{TW,0}(G)$*-module structure on the odd Total twisted geometric K-homology group* $K^{geo}_{TW,1}(G)$.

Consider, for any Lie groupoid G with an equivariant $PU(H)$-prinicipal bundle P, the twisted K-theory group $K_*(C^*_r(G,P))$, for $* = 0, 1$. We can consider as well, the Total Twisted K-theory group $K^*_{TW}(G) := \bigoplus_{\alpha \in H^1(G,PU(H))} K_*(C^*_r(G,P_\alpha))$ and the associated Baum-Connes assembly map

$$K^{geo}_{TW,*}(G) \xrightarrow{\mu_{TW}} K^*_{TW}(G) \tag{1.3}$$

given in each component by the Baum-Connes assembly map

$$K^{geo}_*(G,P) \xrightarrow{\mu_{TW}} K_*(C^*_r(G,P)) \tag{1.4}$$

constructed in [5].

By the theorem above one could expect to transpose the ring structure (resp. module structure for the odd case) to the even (resp. odd) Total Twisted K-theory group via the assembly map. This is of course the case when this twisted Baum-Connes map is an isomorphism. After the discussion in the last section of [5] one might expect that this occurs whenever the untwisted Baum-Connes conjecture is verified. Another interesting question would be if it is possible to construct directly these multiplicative structures on the Total twisted K-theory groups such that the assembly map is a ring/module isomorphism. These questions will be discussed elsewhere.

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2. Preliminaries on groupoids

In this section, we review the notion of twistings on Lie groupoids and discuss some examples which appear in this paper. Let us recall what a groupoid is:

Definition 2.1. A *groupoid* consists of the following data: two sets G and M, and maps

(1) $s, r : G \to M$ called the source map and target map respectively,
(2) $m : G^{(2)} \to G$ called the product map (where $G^{(2)} = \{(\gamma, \eta) \in G \times G : s(\gamma) = r(\eta)\}$),

together with two additional maps, $u : M \to G$ (the unit map) and $i : G \to G$ (the inverse map), such that, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, we have

(i) $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.
(ii) $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$, $\forall \gamma, \eta, \delta \in G$ whenever this makes sense.
(iii) $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$, $\forall \gamma, \eta \in G$ with $s(\gamma) = x$ and $r(\eta) = x$.
(iv) $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$, $\forall \gamma \in G$.

For simplicity, we denote a groupoid by $G \rightrightarrows M$.

In this paper we will only deal with Lie groupoids, that is, a groupoid in which G and M are smooth manifolds, and s, r, m, u are smooth maps (with s and r submersions).

2.1. The Hilsum-Skandalis category.

Lie groupoids form a category with strict morphisms of groupoids. It is now a well-established fact in Lie groupoid's theory that the right category to consider is the one in which Morita equivalences correspond precisely to isomorphisms. We review some basic definitions and properties of generalized morphisms between Lie groupoids, see [16] section 2.1, or [8, 13, 11] for more detailed discussions.

Definition 2.2 (Generalized homomorphisms)**.** Let $G \rightrightarrows M$ and $H \rightrightarrows M'$ be two Lie groupoids. A generalized groupoid morphism, also called a Hilsum-Skandalis morphism, from H to G is given by the isomorphism class of a principal G-bundle over H, that is, a right principal G-bundle over M' which is also a left H-bundle over M such that the the right G-action and the left H-action commute, formally denoted by

$$f : \ H - - \to G$$

or by

$$\begin{array}{ccccc} H & & P_f & & G \\ \downdownarrows & \swarrow & & \searrow & \downdownarrows \\ M' & & & & M. \end{array}$$

if we want to emphasize the bi-bundle P_f involved.

As the name suggests, generalized morphism generalizes the notion of strict morphisms and can be composed. Indeed, if P and P' give generalized morphisms from H to G and from G to L respectively, then

$$P \times_G P' := P \times_M P'/(p, p') \sim (p \cdot \gamma, \gamma^{-1} \cdot p')$$

gives a generalized morphism from H to L. Consider the category $Grpd_{HS}$ with objects Lie groupoids and morphisms given by generalized morphisms. There is a functor

$$Grpd \longrightarrow Grpd_{HS} \tag{2.3}$$

where $Grpd$ is the strict category of groupoids.

Definition 2.4 (Morita equivalent groupoids)**.** Two groupoids are called Morita equivalent if they are isomorphic in $Grpd_{HS}$.

We list here a few examples of Morita equivalence groupoids which will be used in this paper.

Example 2.5 (Pullback groupoid)**.** Let $G \rightrightarrows M$ be a Lie groupoid and let $\phi : M \to M$ be a map such that $t \circ pr_2 : M \times_M G \to M$ is a submersion (for instance if ϕ is a submersion), then the pullback groupoid $\phi^* G := M \times_M G \times_M M \rightrightarrows M$ is Morita equivalent to G, the strict morphism $\phi^* G \to G$ being a generalized isomorphism. For more details on this example the reader can see [11] examples 5.10(4).

Example 2.6 (Discrete groups)**.** Let Γ be a discret group. Let M be a manifold together with a generalized morphism

$$M - -- > \Gamma$$

(in this case this is equivalent a continuous map $M \to B\Gamma$) given by a Γ-principal bundle $\widetilde{M} \to M$ over M (*i.e.*, a Γ-covering). Consider the (Connes-Moscovici) groupoid

$$\widetilde{M} \times_\Gamma \widetilde{M} \rightrightarrows M$$

where $\widetilde{M} \times_\Gamma \widetilde{M} := \widetilde{M} \times \widetilde{M} / \triangle\Gamma$ and with structural maps $s(\tilde{x}, \tilde{y}) = y$, $t(\tilde{x}, \tilde{y}) = x$ and product

$$(\tilde{x}, \tilde{y}) \cdot (\tilde{y}, \tilde{z}) := (\tilde{x}, \tilde{z}).$$

The groupoids $\widetilde{M} \times_\Gamma \widetilde{M} \rightrightarrows M$ and $\Gamma \rightrightarrows \{e\}$ are Morita equivalent.

2.2. **Twistings on Lie groupoids.** In this paper, we are only going to consider $PU(H)$-twistings on Lie groupoids where H is an infinite dimensional, complex and separable Hilbert space, and $PU(H)$ is the projective unitary group $PU(H)$ with the topology induced by the norm topology on the unitary group $U(H)$.

Definition 2.7. A twisting α on a Lie groupoid $G \rightrightarrows M$ is given by a generalized morphism

$$\alpha : G - - \to PU(H).$$

Here $PU(H)$ is viewed as a Lie groupoid with the unit space $\{e\}$.

So a twisting on a Lie groupoid G is given by a locally trivial right principal $PU(H)$-bundle P_α over G.

Remark 2.8. The definition of generalized morphisms given in the last subsection was for two Lie groupoids. The group $PU(H)$ it is not a finte dimensional Lie group but it makes perfectly sense to speak of generalized morphisms from Lie groupoids to this infinite dimensional groupoid following exactly the same definition.

Example 2.9. For a list of various twistings on some standard groupoids see example 1.8 in [6]. Here we will only a few basic examples.

(i) (Twisting on manifolds) Let X be a C^∞-manifold. We can consider the Lie groupoid $X \rightrightarrows X$ where every morphism is the identity over X. A twisting on X is given by a locally trivial principal $PU(H)$-bundle over X. In particular, the restriction of a twisting α on a Lie groupoid $G \rightrightarrows M$ to its unit M defines a twisting α_0 on the manifold M.

(ii) (Orientation twisting) Let X be a manifold with an oriented real vector bundle E. The bundle $E \to X$ defines a natural generalized morphism

$$X \dashrightarrow SO(n).$$

Note that the fundamental unitary representation of $Spin^c(n)$ gives rise to a commutative diagram of Lie group homomorphisms

$$\begin{tikzcd} Spin^c(n) \arrow[r] \arrow[d] & U(\mathbb{C}^{2^n}) \arrow[d] \\ SO(n) \arrow[r] & PU(\mathbb{C}^{2^n}). \end{tikzcd}$$

With a choice of inclusion $\mathbb{C}^{2^n}$ into a Hilbert space H, we have a canonical twisting, called the orientation twisting, denoted by

$$\beta_E : X \dashrightarrow PU(H).$$

(iii) (Pull-back twisting) Given a twisting α on G and for any generalized homomorphism $\phi : H \to G$, there is a pull-back twisting

$$\phi^*\alpha : H \dashrightarrow PU(H)$$

defined by the composition of ϕ and α. In particular, for a continuous map $\phi : X \to Y$, a twisting α on Y gives a pull-back twisting $\phi^*\alpha$ on X. The principal $PU(H)$-bundle over X defines by $\phi^*\alpha$ is the pull-back of the principal $PU(H)$-bundle on Y associated to α.

(iv) (Twisting on fiber product groupoid) Let $N \overset{p}{\to} M$ be a submersion. We consider the fiber product $N \times_M N := \{(n, n') \in N \times N : p(n) = p(n')\}$,which is a manifold because p is a submersion. We can then take the groupoid

$$N \times_M N \rightrightarrows N$$

which is a subgroupoid of the pair groupoid $N \times N \rightrightarrows N$. Note that this groupoid is in fact Morita equivalent to the groupoid $M \rightrightarrows M$. A twisting on $N \times_M N \rightrightarrows N$ is given by a pull-back twisting from a twisting on M.

(v) (Twisting on a Lie group) By definition a twisting on a Lie group G is a projective representation

$$G \xrightarrow{\alpha} PU(H).$$

2.3. **Deformation groupoids.** One of our main tools will be the use of deformation groupoids. In this section, we review the notion of Connes' deformation groupoids from the deformation to the normal cone point of view.

Deformation to the normal cone

Let M be a C^∞-manifold and $X \subset M$ be a C^∞-submanifold. We denote by $\mathcal{N}_X^M$ the normal bundle to X in M. We define the following set

$$\mathcal{D}_X^M := \left(\mathcal{N}_X^M \times 0\right) \bigsqcup (M \times \mathbb{R}^*)\,. \tag{2.10}$$

The purpose of this section is to recall how to define a C^∞-structure in $\mathcal{D}_X^M$. This is more or less classical, for example it was extensively used in [8].

Let us first consider the case where $M = \mathbb{R}^p \times \mathbb{R}^q$ and $X = \mathbb{R}^p \times \{0\}$ (here we identify X canonically with $\mathbb{R}^p$). We denote by $q = n - p$ and by $\mathcal{D}_p^n$ for $\mathcal{D}_{\mathbb{R}^p}^{\mathbb{R}^n}$ as above. In this case we have that $\mathcal{D}_p^n = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ (as a set). Consider the bijection $\psi : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \to \mathcal{D}_p^n$ given by

$$\psi(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, t\xi, t) & \text{if } t \neq 0 \end{cases} \tag{2.11}$$

whose inverse is given explicitly by

$$\psi^{-1}(x,\xi,t) = \begin{cases} (x,\xi,0) & \text{if } t=0 \\ (x,\frac{1}{t}\xi,t) & \text{if } t \neq 0 \end{cases}$$

We can consider the C^∞-structure on $\mathcal{D}_p^n$ induced by this bijection.

We pass now to the general case. A local chart $(\mathcal{U},\phi)$ of M at x is said to be a X-slice if

1) $\mathcal{U}$ is an open neighbourhood of x in M and $\phi : \mathcal{U} \to U \subset \mathbb{R}^p \times \mathbb{R}^q$ is a diffeomorphsim such that $\phi(x) = (0,0)$.
2) Setting $V = U \cap (\mathbb{R}^p \times \{0\})$, then $\phi^{-1}(V) = \mathcal{U} \cap X$, denoted by $\mathcal{V}$.

With these notations understood, we have $\mathcal{D}_V^U \subset \mathcal{D}_p^n$ as an open subset. For $x \in \mathcal{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x),0)$, then

$$\phi_1 : \mathcal{V} \to V \subset \mathbb{R}^p$$

is a diffeomorphism. Define a function

$$\tilde{\phi} : \mathcal{D}_{\mathcal{V}}^{\mathcal{U}} \to \mathcal{D}_V^U \tag{2.12}$$

by setting $\tilde{\phi}(v,\xi,0) = (\phi_1(v), d_N\phi_v(\xi),0)$ and $\tilde{\phi}(u,t) = (\phi(u),t)$ for $t \neq 0$. Here $d_N\phi_v : N_v \to \mathbb{R}^q$ is the normal component of the derivative $d\phi_v$ for $v \in \mathcal{V}$. It is clear that $\tilde{\phi}$ is also a bijection. In particular, it induces a C^∞ structure on $\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}$. Now, let us consider an atlas $\{(\mathcal{U}_\alpha,\phi_\alpha)\}_{\alpha\in\Delta}$ of M consisting of $X-$slices. Then the collection $\{(\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha},\tilde{\phi}_\alpha)\}_{\alpha\in\Delta}$ is a C^∞-atlas of $\mathcal{D}_X^M$ (Proposition 3.1 in [4]).

Definition 2.13 (Deformation to the normal cone)**.** Let $X \subset M$ be as above. The set $\mathcal{D}_X^M$ equipped with the C^∞ structure induced by the atlas of X-slices is called the deformation to the normal cone associated to the embedding $X \subset M$.

One important feature about the deformation to the normal cone is the functoriality. More explicitly, let $f : (M,X) \to (M',X')$ be a C^∞-map $f : M \to M'$ with $f(X) \subset X'$. Define $\mathcal{D}(f) : \mathcal{D}_X^M \to \mathcal{D}_{X'}^{M'}$ by the following formulas:

1) $\mathcal{D}(f)(m,t) = (f(m),t)$ for $t \neq 0$,
2) $\mathcal{D}(f)(x,\xi,0) = (f(x), d_Nf_x(\xi),0)$, where d_Nf_x is by definition the map
$$(\mathcal{N}_X^M)_x \xrightarrow{d_Nf_x} (\mathcal{N}_{X'}^{M'})_{f(x)}$$
induced by $T_xM \xrightarrow{df_x} T_{f(x)}M'$.

Then $\mathcal{D}(f) : \mathcal{D}_X^M \to \mathcal{D}_{X'}^{M'}$ is a C^∞-map (Proposition 3.4 in [4]). In the language of categories, the deformation to the normal cone construction defines a functor

$$\mathcal{D} : \mathcal{C}_2^\infty \longrightarrow \mathcal{C}^\infty, \tag{2.14}$$

where $\mathcal{C}^\infty$ is the category of C^∞-manifolds and $\mathcal{C}_2^\infty$ is the category of pairs of C^∞-manifolds.

Given an immersion of Lie groupoids $G_1 \overset{\varphi}{\to} G_2$, let $G_1^N = \mathcal{N}_{G_1}^{G_2}$ be the total space of the normal bundle to φ, and $(G_1^{(0)})^N$ be the total space of the normal bundle to $\varphi_0 : G_1^{(0)} \to G_2^{(0)}$. Consider $G_1^N \rightrightarrows (G_1^{(0)})^N$ with the following structure maps: The source map is the derivation in the normal direction $d_Ns : G_1^N \to (G_1^{(0)})^N$ of the source map (seen as a pair of maps) $s : (G_2,G_1) \to (G_2^{(0)},G_1^{(0)})$ and similarly for the target map.

The groupoid G_1^N may fail to inherit a Lie groupoid structure (see counterexample just before section IV in [8]). A sufficient condition is when $(G_1^{(0)})^N$ is a G_1^N-vector bundle over $G_1^{(0)}$. This is the case when $G_1^x \to G_2^{\varphi(x)}$ is étale for every $x \in G_1^{(0)}$ (in particular if the groupoids are étale) or when one considers a manifold with two foliations $F_1 \subset F_2$ and the induced immersion (again 3.1, 3.19 in [8]).

The deformation to the normal bundle construction allows us to consider a C^∞ structure on

$$G_\varphi := \left(G_1^N \times \{0\}\right) \bigsqcup \left(G_2 \times \mathbb{R}^*\right),$$

such that $G_1^N \times \{0\}$ is a closed saturated submanifold and so $G_2 \times \mathbb{R}^*$ is an open submanifold. The following results are an immediate consequence of the functoriality of the deformation to the normal cone construction.

Proposition 2.15 (Hilsum-Skandalis, 3.1, 3.19 [8])**.** *Consider an immersion $G_1 \overset{\varphi}{\to} G_2$ as above for which $(G_1)^N$ inherits a Lie groupoid structure. Let $G_{\varphi_0} := \left((G_1^{(0)})^N \times \{0\}\right) \bigsqcup \left(G_2^{(0)} \times \mathbb{R}^*\right)$ be the deformation to the normal cone of the pair $(G_2^{(0)}, G_1^{(0)})$. The groupoid*

$$G_\varphi \rightrightarrows G_{\varphi_0} \tag{2.16}$$

with structure maps compatible with the ones of the groupoids $G_2 \rightrightarrows G_2^{(0)}$ and $G_1^N \rightrightarrows (G_1^{(0)})^N$, is a Lie groupoid with C^∞-structures coming from the deformation to the normal cone.

One of the interest of these kind of groupoids is to be able to define family indices. First we recall the following elementary result.

Proposition 2.17. *Given an immersion of Lie groupoids $G_1 \overset{\varphi}{\to} G_2$ as above and a twisting α on G_2. There is a canonical twisting α_φ on the Lie groupoid $G_\varphi \rightrightarrows G_{\varphi_0}$, extending the pull-back twisting on $G_2 \times \mathbb{R}^*$ from α.*

Proof. The proof is a simple application of the functoriality of the deformation to the normal cone construction. Indeed, the twisting α on G_2 induces by pullback (or composition of cocycles) a twisting $\alpha \circ \varphi$ on G_1. The twisting α on G_2 is given by a $PU(H)$-principal bundle P_α with a compatible left action of G_2, and by definition the twisting $\alpha \circ \varphi$ on G_1 is given by the pullback of P_α by $\varphi_0 : G_1^{(0)} \to G_2^{(0)}$. In particular, $P_{\alpha\circ\varphi} = G_1^{(0)} \times_{G_2^{(0)}} P_\alpha$ Hence the action map $G_2 \times_{G_2^{(0)}} P_\alpha \to P_\alpha$ can be considered as an application in the category of pairs:

$$(G_2 \times_{G_2^{(0)}} P_\alpha, G_1 \times_{G_1^{(0)}} P_{\alpha\circ\varphi}) \longrightarrow (G_2^{(0)} \times_{G_2^{(0)}} P_\alpha, G_1^{(0)} \times_{G_1^{(0)}} P_{\alpha\circ\varphi}).$$

We can then apply the deformation to the normal cone functor to obtain the desire $PU(H)$-principal bundle with a compatible G_φ-action, which gives the desired twisting. □

3. Twisted equivariant K-theory

The crucial diference to [3] is the use of graded Fredholm bundles, which are needed for the definition of the multiplicative structure.

Let $\mathcal{H}$ be a separable Hilbert space and

$$\mathcal{U}(\mathcal{H}) := \{U : \mathcal{H} \to \mathcal{H} \mid U \circ U^* = U^* \circ U = \mathrm{Id}\}$$

the group of unitary operators acting on $\mathcal{H}$. Let $\mathrm{End}(\mathcal{H})$ denote the space of endomorphisms of the Hilbert space and endow $\mathrm{End}(\mathcal{H})_{c.o.}$ with the compact open topology. Consider the inclusion

$$\begin{aligned} \mathcal{U}(\mathcal{H}) &\to \mathrm{End}(\mathcal{H})_{c.o.} \times \mathrm{End}(\mathcal{H})_{c.o.} \\ U &\mapsto (U, U^{-1}) \end{aligned}$$

and induce on $\mathcal{U}(\mathcal{H})$ the subspace topology. Denote the space of unitary operators with this induced topology by $\mathcal{U}(\mathcal{H})_{c.o.}$ and note that this is different from the usual compact open topology on $\mathcal{U}(\mathcal{H})$. Let $\mathcal{U}(\mathcal{H})_{c.g}$ be the compactly generated topology associated to the compact open topology, and topologize the group $P\mathcal{U}(\mathcal{H})$ from the exact sequence

$$1 \to S^1 \to \mathcal{U}(\mathcal{H})_{c.g.} \to P\mathcal{U}(\mathcal{H}) \to 1.$$

Definition 3.1. Let $\mathcal{H}$ be a separable Hilbert space. The space $\mathrm{Fred}'(\mathcal{H})$ consist of pairs (A, B) of bounded operators on $\mathcal{H}$ such that $AB - 1$ and $BA - 1$ are compact operators. Endow $\mathrm{Fred}'(\mathcal{H})$ with the topology induced by the embedding

$$\begin{array}{rcl} \mathrm{Fred}'(\mathcal{H}) & \to & \mathsf{B}(\mathcal{H}) \times \mathsf{B}(\mathcal{H}) \times \mathsf{K}(\mathcal{H}) \times \mathsf{K}(\mathcal{H}) \\ (A, B) & \mapsto & (A, B, AB - 1, BA - 1) \end{array}$$

where $\mathsf{B}(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$ with the compact open topology and $\mathsf{K}(\mathcal{H})$ denotes the compact operators with the norm topology.

We denote by $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ to a $\mathbb{Z}_2$-graded, infinite dimensional Hilbert space.

Definition 3.2. Let $U(\widehat{\mathcal{H}})_{c.g.}$ be the group of even, unitary operators on the Hilbert space $\widehat{\mathcal{H}}$ which are of the form

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_i denotes a unitary operator in the compactly generated topology defined as before.

We denote by $P\mathcal{U}(\widehat{\mathcal{H}})$ the group $U(\widehat{\mathcal{H}})_{c.g.}/S^1$ and recall the central extension

$$1 \to S^1 \to \mathcal{U}(\widehat{\mathcal{H}}) \to P\mathcal{U}(\widehat{\mathcal{H}}) \to 1$$

Definition 3.3. The space $\mathrm{Fred}''(\widehat{\mathcal{H}})$ is the space of pairs $(\widehat{A}, \widehat{B})$ of self-adjoint, bounded operators of degree 1 defined on $\widehat{\mathcal{H}}$ such that $\widehat{A}\widehat{B} - I$ and $\widehat{B}\widehat{A} - I$ are compact.

Given a $\mathbb{Z}/2$-graded Hilbert space $\widehat{\mathcal{H}}$, the space $\mathrm{Fred}''(\widehat{\mathcal{H}})$ is homeomorphic to $\mathrm{Fred}'(H)$.

Definition 3.4. We denote by $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ the space of self-adjoint degree 1 Fredholm operators A in $\widehat{\mathcal{H}}$ such that A^2 differs from the identity by a compact operator, with the topology coming from the embedding $A \mapsto (A, A^2 - I)$ in $\mathcal{B}(\mathcal{H}) \times \mathcal{K}(\mathcal{H})$.

The following result was proved in [1], Proposition 3.1 :

Proposition 3.5. *The space* $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ *is a deformation retract of* $\mathrm{Fred}''(\widehat{\mathcal{H}})$.

The above discussion can be concluded telling that $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ is a representing space for K-theory. The group $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}$ of degree 0 unitary operators on $\widehat{\mathcal{H}}$ with the compactly generated topology acts continuously by conjugation on $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$, therefore the group $P\mathcal{U}(\widehat{\mathcal{H}})$ acts continuously on $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ by conjugation. In [3] twisted K-theory for proper actions of discrete groups was defined using the representing space $\mathrm{Fred}'(\mathcal{H})$, but in order to have multiplicative structure we proceed using $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Let us choose the operator

$$\widehat{I} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

as the base point in $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Choosing the identity as a base point on the space $\mathrm{Fred}^{'}(\mathcal{H})$, gives a diagram of pointed maps

$$\begin{tikzcd} \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) \arrow[r, "i"] & \mathrm{Fred}''(\widehat{\mathcal{H}}) \arrow[r, "f"] \arrow[d, "r"] & \mathrm{Fred}'(\mathcal{H}) , \\ & \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) & \end{tikzcd}$$

where i denotes the inclusion, r is a strong deformation retract and f is a homeomorphism. Moreover, the maps are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}, \mathcal{U}(\mathcal{H})_{c.g.}$ and the map $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.} \to \mathcal{U}(\mathcal{H})_{c.g.}$.

Let X be a proper G-space and let $P \to X$ be a projective unitary G-equivariant bundle over X. Denote by $\widehat{P}$ the projective unitary bundle obtained by performing the tensor product with the trivial bundle $\mathbb{P}(\widehat{\mathcal{H}})$, $\widehat{P} = P \otimes \mathbb{P}(\widehat{\mathcal{H}})$.

The space of Fredholm operators is endowed with a continuous right action of the group $P\mathcal{U}(\widehat{\mathcal{H}})$ by conjugation, therefore we can take the associated bundle over X

$$\mathrm{Fred}^{(0)}(\widehat{P}) := \widehat{P} \times_{P\mathcal{U}(\widehat{\mathcal{H}})} \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}),$$

and with the induced G action given by

$$g \cdot [(\lambda, A))] := [(g\lambda, A)]$$

for g in G, λ in $\widehat{P}$ and A in $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Denote by

$$\Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))$$

the space of sections of the bundle $\mathrm{Fred}^{(0)}(\widehat{P}) \to X$ and choose as base point in this space the section which chooses the base point $\widehat{I}$ on the fibers. This section exists because the $P\mathcal{U}(\widehat{\mathcal{H}})$ action on $\widehat{I}$ is trivial, and therefore

$$X \cong \widehat{P}/P\mathcal{U}(\widehat{\mathcal{H}}) \cong \widehat{P} \times_{P\mathcal{U}(\widehat{\mathcal{H}})} \{\widehat{I}\} \subset \mathrm{Fred}^{(0)}(\widehat{P});$$

let us denote this section by s.

Definition 3.6. Let X be a connected proper G-space and P a projective unitary G-equivariant bundle over X. The *Twisted G-equivariant K-theory* groups of X twisted by P are defined as the homotopy groups of the G-equivariant sections

$$K_G^{-p}(X; P) := \pi_p \left(\Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))^{X \rtimes G}, s \right)$$

where the base point $s = \widehat{I}$ is the section previously constructed.

3.1. **Additive structure.** There exists a natural map

$$\Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))^{X \rtimes G} \times \Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))^{X \rtimes G} \to \Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))^{X \rtimes G},$$

inducing an abelian group structure on the twisted equivariant K- theory groups, which we will define below. Consider for this the following commutative diagram.

$$\begin{tikzcd} \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) \times \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) \arrow[r, "f \circ i"] \arrow[d, dashed] & \mathrm{Fred}'(\widehat{\mathcal{H}}) \times \mathrm{Fred}'(\widehat{\mathcal{H}}) \arrow[d] \arrow[dl, phantom, "\circ"] \\ \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) & \mathrm{Fred}'(\widehat{\mathcal{H}}) \arrow[l, "f^{-1} \circ r"'] \end{tikzcd}$$

where the vertical map denotes composition. As the maps involved in the diagram are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g}$, respectively $\mathcal{U}(\mathcal{H})_{c.g}$ and G, for any projective unitary G-equivariant bundle P, this induces a pointed map

$$\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^{X\rtimes G},s)\times(\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^{X\rtimes G},s)\to(\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^{X\rtimes G},s).$$

Which defines an additive structure in $K_G^{-p}(X;P)$.

3.2. **Multiplicative structure.** We define an associative product on twisted K-theory.

$$K_G^{-p}(X;P)\times K_G^{-q}(X;P')\to K_G^{-(p+q)}(X;P\otimes P')$$

Induced by the map

$$(A,A')\mapsto A\widehat{\otimes}I+I\widehat{\otimes}A'$$

defined in $\mathrm{Fred}^0(\widehat{\mathcal{H}})$, and $\widehat{\otimes}$ denotes the graded tensor product, see [2] in pages 24-25 for more details. We denote this product by $\bullet$.

Definition 3.7 (Equivariant Total Twisted K-theory)**.** For every $p\in\mathbb{N}$, the degree p, G-equivariant Total Twisted K-theory group is given by

$$K_{TW,G}^{-p}(X):=\bigoplus_{\alpha\in H^1(G,PU(H))}K_G^{-p}(X,P_\alpha). \tag{3.8}$$

By the discussion above

$$K_{TW,G}^{ev}(X):=\bigoplus_{p\in\mathbb{N},\,even}K_{TW,G}^{-p}(X) \tag{3.9}$$

has a ring structure and

$$K_{TW,G}^{odd}(X):=\bigoplus_{p\in\mathbb{N},\,odd}K_{TW,G}^{-p}(X) \tag{3.10}$$

is a $K_{TW,G}^{ev}(X)$-module. The last groups above are called the even, respectively odd, $G-$equivariant Total Twisted K-theory groups of X.

3.3. **Topologies on Fredholm Operators.** In [16] a Fredholm picture of twisted K-theory is introduced. Denote by $\mathrm{Fred}'(\mathcal{H})_{s*}$ the space whose elements are the same as $\mathrm{Fred}'(\mathcal{H})$ but with the strong *-topology on $B(\mathcal{H})$.

Definition 3.11. [16, Thm. 3.15] Let X be a connected G-proper space and P a projective unitary G-equivariant bundle over X. The *Twisted G-equivariant K-theory* groups of X (in the sense of Tu-Xu-Laurent) twisted by P are defined as the homotopy groups of the G-equivariant strong*-continuous sections

$$\mathbb{K}_G^{-p}(X;P):=\pi_p\left(\Gamma(X;\mathrm{Fred}'(P)_{s^*})^G,s\right).$$

The bundle $\mathrm{Fred}'(P)_{s^*}$ is defined in a similar way as $\mathrm{Fred}'(P)$.

We will prove that the functors $K_G^*(-,P)$ and $\mathbb{K}_G^*(-,P)$ are naturally equivalent.

Lemma 3.12. *The spaces* $\mathrm{Fred}'(\mathcal{H})$ *and* $\mathrm{Fred}'(\mathcal{H})_{s^*}$ *are $PU(\mathcal{H})$-weakly homotopy equivalent.*

Proof. The strategy is to prove that $\mathrm{Fred}'(\mathcal{H})_{s^*}$ is a representing of equivariant K-theory. The same proof for $\mathrm{Fred}'(\mathcal{H})$ in [1, Prop. A.22] applies. In particular $GL(\mathcal{H})_{s^*}$ is G-contractible because the homotopy h_t constructed in [1, Prop. A.21] is continuous in the strong*-topology and then the proof applies. □

Using the above lemma one can prove that the identity map defines an equivalence between (twisted) cohomology theories $K_G^*(-,P)$ and $\mathbb{K}_G^*(-,P)$. Then we have that the both definitions of twisted K-theory are equivalents. Summarizing

Proposition 3.13. *For every proper G-manifold X and every projective unitary G-equivariant bundle over X. We have an isomorphism*

$$K_G^{-p}(X;P) \cong \mathbb{K}_G^{-p}(X;P).$$

Remark 3.14. In order to simplify the notation from now on we denote by $\mathcal{H}$ to a $\mathbb{Z}_2$-graded separable Hilbert space and we denote by $\mathrm{Fred}^{(0)}(P)$ to the bundle $\mathrm{Fred}^{(0)}(\widehat{P})$.

3.4. **Relation with the Kasparov external product.** In [16] twisted K-theory for Lie groupoids is defined and in Prop. 6.11 of that work this group is described as a KK-group for the case of proper groupoids.

Proposition 3.15. [16, Prop. 6.11] *If $G \rightrightarrows M$ is a proper Lie groupoid and M/G is compact, then for $i = 0, 1$, there is a natural isomorphism $\chi : KK_G^i(C_0(M), B_P) \to K_G^i(M, P)$, where B_P is certain C^*-algebra associated to the twisting P.*

Using the external Kasparov product they define a product

$$K_G^i(M,P) \otimes K_G^j(M,P') \xrightarrow{\quad \bullet_{TXL} \quad} K_G^{i+j}(M, P \otimes P').$$

Using the functoriality of both products $\bullet$ and $\bullet_{TXL}$ one can prove that they are the same.

Definition 3.16. (i) If Φ is a $KK_G(A,B)$-cycle we denote by Φ_* to the homomorphism

$$\begin{aligned} \Phi_* : KK_G(C_0(M), B_0) &\to KK_G(C_0(M), B_P) \\ x &\mapsto x \bullet_{TXL} \Phi. \end{aligned}$$

(ii) If $s \in \Gamma^G(\mathrm{Fred}^{(0)}(P))$ we denote by s_* the homomorphism

$$\begin{aligned} s_* : K_G^i(X) &\to K_G^i(X,P) \\ [f] &\mapsto [s \bullet f]. \end{aligned}$$

Proposition 3.17. *If $\Phi \in KK_G^i(C_0(M), B_P)$ and $\Psi \in KK_G^i(C_0(M), B_{P'})$, then $\chi(\Phi \bullet_{tuxustacks} \Psi) = \chi(\Phi) \bullet \chi(\Psi)$.*

Proof.

$$\begin{aligned} \chi(\Phi \bullet_{TXL} \Psi) &= \chi(\Phi_*(1_{C_0(M)}) \bullet_{TXL} \Psi_*(1_{C_0(M)})) \\ &= \chi(\Phi_*(\Psi_*(1_{C_0(M)}))) \\ &= (\chi(\Phi))_*((\chi(\Psi))_*(\chi(1_{C_0(M)})) \\ &= (\chi(\Phi))_*(1_M) \bullet (\chi(\Psi))_*(1_M) \\ &= (\chi(\Phi)) \bullet (\chi(\Psi)). \end{aligned}$$

□

The above result implies that both products are the same module the equivalence χ.

4. Thom isomorphism

Let $G \rightrightarrows G_0$ be a Lie groupoid and P a twisting. Consider a G-oriented vector bundle $E \longrightarrow X$. In particular since we will assume that G acts properly on P and on E, we can assume E admits a G-invariant metric, see for instance [14] proposition 3.14 and [7] theorem 4.3.4.. As explained in [5] appendix A (specially proposition A.3), in this situation there is a natural isomorphism

$$Th : \mathbf{K_G^*(X,P)} \to \mathbf{K_G^{*-rank(E)}}(\mathbf{E}, \pi^*(\mathbf{P} \otimes \beta_{\mathbf{E}}))$$

where $\mathbf{K}^*_{\mathbf{G}}(\mathbf{X},\mathbf{P})$ stands for the K-theory of the twisted groupoid C^*-algebra $C^*_r(X \rtimes G, P)$ and where β_E is the orientation G-twisting over E defined in example ((ii)) in 2.9 above. The fact that it is indeed the Thom isomorphism comes from the functoriality and the naturality with respect to the Kasparov products of the Le Gall's descent construction [10] theorem 7.2. This is explained in details in the appendix cited above or in [12] in the context of real groupoids (the same arguments apply in the complex case).

Now, in [16] theorem 3.14 the authors prove that for proper Lie groupoids the groups $\mathbf{K}^*_{\mathbf{G}}(\mathbf{X},\mathbf{P})$ and $\mathbb{K}^{-p}_G(X;P)$ are naturally isomorphic. We thus obtain, by proposition 3.13, the Thom isomorphism

$$Th : K^*_G(X,P) \to K_G^{*-rank(E)}(E, \pi^*(P \otimes \beta_E)).$$

It is possible however to construct the Thom isomorphism directly in the Fredholm picture of the twisted K-theory (whenever the respectives action groupoids are proper), we will perform this construction for the benefit of the reader.

The spin representation and twisted K-Theory. Let n be an even natural number.

Let $\mathbb{R}^n$ denote the euclidean, n-dimensional vector space denoted with the euclidean scalar product.

The Clifford algebra $\mathrm{Cliff}(\mathbb{R}^n)$ is defined as the complexification of the quotient of the tensor algebra $T\mathbb{R}^n = \bigotimes_{j=0}^{\infty} \mathbb{R}^n$ by the two-sided ideal defined by elents of the form $x \otimes x - \langle x, x\rangle$, where $\langle \quad \rangle$ denotes the euclidean scalar product.

It is generated as $\mathbb{C}$-algebra by the elements of a an orthogonal basis e_i of $\mathbb{R}^n$ with the relations $e_i \cdot e_j = -2\delta_{i,j}$.

The algebra $\mathrm{Cliff}(\mathbb{R}^n)$ is isomorphic as a vector space to the exterior algebra $\Lambda^*(\mathbb{R}^n) = \bigoplus_{j=0}^{n} \Lambda^j \mathbb{R}^n$ [9], Proposition 1.3 in page 10, in particular, it has complex dimension 2^n .

The map given by Clifford multiplication with the element $e_1 \cdot, \dots, \cdot e_n$ defines a linear operator on $\mathrm{Cliff}(\mathbb{R}^n)$. The Clifford algebra then decomposes as a vector space $\mathrm{Cliff}(\mathbb{R}^n) = S^+ \oplus S^-$, where S^+ is the eigenspace associated to $+1$ and S^- is the one associated with -1. An element in S^+ is called even, an element in S^- is said to be odd.

The group $\mathrm{Spin}(\mathbb{R}^n)$ consists of the multiplicative group of even units in the Clifford algebra, in symbols $\mathrm{Spin}(\mathbb{R}^n) = \mathrm{Cliff}(\mathbb{R}^n)^* \cap S^+$.

The group $\mathrm{Spin}(\mathbb{R}^n)$ is the universal covering of the special orthogonal group $\mathrm{SO}(n)$. The map

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(\mathbb{R}^n) \to SO(n) \to 1$$

is a model for the universal central extension of $SO(n)$.

This extension is classified by the nontrivial class $\tau \in H^2(SO(n), S^1) \cong \mathbb{Z}_2$.

The group $\mathrm{Spin}(\mathbb{R}^n)$ has a complex linear representation $\rho : \mathrm{Spin}(\mathbb{R}^n) \to U(2^n)$, given by the identification of $\mathbb{C}\mathrm{liff}(\mathbb{R}^n) = \mathrm{Cliff}(\mathbb{R}^n) \otimes \mathbb{C}$ with the complex vector space of dimension 2^n as an algebra, and the linear operator given by $\rho(x) : v \mapsto x^{-1}vx$.

The representation ρ gives rise to a continuous group homomorphism β as in the following diagram:

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] \arrow[d] & \mathrm{Spin}^c(\mathbb{R}^n) \arrow[r] \arrow[d, "\rho"] & SO(n) \arrow[r] \arrow[d, "\beta"] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] & U(2^n) \arrow[r] & PU(2^n) \arrow[r] & 1 \end{tikzcd}$$

Definition 4.1. The spin representation is the homomorphism $\beta : \mathrm{SO}(n) \to P\mathcal{U}(\mathcal{H})$

Remark 4.2. Let n be a even positive integer. Consider a proper oriented G-vector bundle $E \xrightarrow{\pi} X$ over a proper G-manifold X. We can suppose that the chart data is given by a generalized morphism

$$X \rtimes G \overset{O_E}{- - \to} \mathrm{SO}(n)\ .$$

Composing the generalized morphism O_E with the spin representation β we obtain a twisting $\beta_E : X \rtimes G - - \to P\mathcal{U}(\mathcal{H})$, called the *orientation twisting.*

We will construct now the Thom class in the Fredholm picture. If X is a proper G-manifold, by Theorem 2.3 in [17] for every $x \in X$ there is a open neighbourhood U of x contractible to the orbit of x in $X \rtimes G$ with action of the isotropy group G_x such that there is a Lie groupoid isomorphism

$$(X \rtimes G) \mid U \cong U \rtimes G_x.$$

We have an isomorphism

$$K^{-n}_{G_x}(U, \beta_E \mid_U) \cong R_{S^1}(\widetilde{G_x}), \tag{4.3}$$

where $\widetilde{G_x}$ is the S^1-central extension of G_x associated to the twisting $\beta_E \mid_U$. On the other hand, $E \mid_{\{x\}}$ is a real representation of G_x, since it can be viewed as a homomorphism $\eta_x : G_x \to SO(n)$. The composition $\beta \circ \eta_x : G \to P\mathcal{U}(\mathcal{H})$ is a projective representation and its isomorphism class determines an element of $R_{S^1}(\widetilde{G_x})$. Using the identification 4.3, it can be viewed as an element of $K^{-n}_{G_x}(U, \beta_E \mid_U)$. We denote this element by λ^U_{-1}.

Taking a covering of $X \rtimes G$ one can see that these local elements are the same on intersections. The local trivializations define a global element

$$[\lambda^E_{-1}] \in K^{-n}_G(X, \beta_E),$$

we call it the *Thom class.*

Given $s \in \Gamma^G(P \times_{P\mathcal{U}(\mathcal{H})} \mathrm{Fred}^{(0)}(\mathcal{H}))$, where $P \to X$ is a twisting, we define the Thom isomorphism

$$\begin{aligned} Th : K^*_G(X, P) \to & K^{*-n}_G(E, \pi^*(P \otimes \beta_E)) \\ [s] \mapsto & [e \mapsto s(\pi(e)) \bullet \lambda^E_{-1}(\pi(e))]. \end{aligned}$$

When the vector bundle E is odd dimensional, using the classic suspension isomorphism and the previous Thom isomorphism for $E \oplus \mathbb{R}$, one gets as well a Thom isomorphism as above.

Since the Thom isomorphism is natural with respect to the Kasparov product we can resume the discussion above in the following statement.

Theorem 4.4. *[Thom isomorphism] With notations as above, there is a natural isomorphism*

$$Th : K^*_G(X, P) \to K^{*-rank(E)}_G(E, \pi^*(P \otimes \beta_E))$$

which gives the Thom isomorphism.

(i) If E is a $Spin^c$ $G-$ vector bundle of even ranck, then β_E is trivial and one obtains a ring isomorphism

$$Th : K^{ev}_{TW,G}(X) \to K^{ev}_{TW,G}(E)$$

and an isomorphism of $K^{ev}_{TW,G}(X)$-modules

$$Th : K^{odd}_{TW,G}(X) \to K^{odd}_{TW,G}(E).$$

(ii) If E is a $Spin^c$ $G-$ vector bundle of odd rank, then β_E is trivial and one obtains a $K^{ev}_{TW,G}(X)$-module isomorphism

$$Th : K^{ev}_{TW,G}(X) \to K^{odd}_{TW,G}(E)$$

and a $K^{ev}_{TW,G}(X)$-module isomorphism

$$Th : K^{odd}_{TW,G}(X) \to K^{ev}_{TW,G}(E).$$

5. Pushforward Map

In this section we will recall how to define the pushforward morphism associated to any smooth G-map $f : X \to Y$ between to G- manifolds, definition 4.1 in [5]. For the purpose of this paper we will perform the construction in the case of K-oriented maps. By this we mean that the bundle $T^*X \oplus f^*(TY)$ admits a $Spin^c-$structure.

The difference in the present construction with respect to ref.cit. is that we will not make reference to C^*-algebras and we will perform the construction using the Fredholm picture of the twisted K-theory, in particular the construction below works only for G-proper manifolds.

We will need to state some general statements about groupoids that will simplify the particular constructions we are interested in.

Lemma 5.1. *Let $G \rightrightarrows G_0$ be a proper Lie groupoid together with a twisting P. Let $H \rightrightarrows H_0$ be a proper Lie saturated closed subgroupoid.*

(i) There is a canonical restriction morphism

$$K^{-p}_G(G_0, P) \to K^{-p}_H(H_0, P|_{H_0}) \tag{5.2}$$

(ii) Suppose G decomposes as the union of two saturated proper subgroupoids $G = H \sqcup H' \rightrightarrows H_0 \sqcup H'_0$ with H closed subgroupoid. There is a long exact sequence

$$\longrightarrow K^{-p}_{H'}(H'_0, P|_{H'_0}) \longrightarrow K^{-p}_G(G_0, P) \longrightarrow K^{-p}_H(H_0, P|_{H_0}) \longrightarrow K^{-p-1}_{H'}(H'_0, P|_{H'_0}) \longrightarrow \tag{5.3}$$

Lemma 5.4. *Let $G \rightrightarrows G_0$ be a proper Lie groupoid together with a twisting P, consider the product groupoid $G \times (0, 1] \rightrightarrows G_0 \times (0, 1]$ with the pullback twisting $P_{(0,1]}$. For every $p \in \mathbb{Z}$*

$$K^{-p}_{G\times(0,1]}(G_0 \times (0, 1], P_{(0,1]}) = 0.$$

The two previous lemmas are classic in the C^*-algebraic context, *i.e.,* once we use that the isomorphism between the twisted K-theory with the C^*-picture and the twisted K-theory with the Fredholm picture (theorem 3.14 [16]).

The following result is an immediate consequence of lemmas 5.1 and 5.4 above.

Proposition 5.5. *Given an immersion of proper Lie groupoids $G_1 \overset{\varphi}{\to} G_2$ and a twisting α on G_2, consider the twisted deformation groupoid (G_φ, P_α) of section 2.3 (propositions 2.15 and 2.17). The morphism in K-theory induced by the restriction at zero,*

$$K^{-p}_{G_\varphi}(G^{(0)}_\varphi, P_\varphi) \xrightarrow{\quad e_0 \quad} K^{-p}_{G_2}(G^{(0)}_2, P_2) \tag{5.6}$$

is an isomorphism.

Definition 5.7 (Index associated to a groupoid immersion)**.** Given an immersion of proper Lie groupoids $G_1 \overset{\varphi}{\to} G_2$ as above and a twisting α on G_2, we let

$$Ind_\varphi : K^{-p}_{G_1^N}((G_1^{(0)})^N, P_1^N) \to K^{-p}_{G_2}(G_2^{(0)}, P_2) \tag{5.8}$$

to be the morphism in K-theory given by $Ind_\varphi := e_1 \circ e_0^{-1}$.

We are ready to define the shriek map. Let $G \rightrightarrows G_0$ be a Lie groupoid together with a twisting P. Let X, Y be two G-proper manifolds and let $f : X \to Y$ be a smooth G-map with $T^*X \oplus f^*TY$ a G-$Spin^c$ vector bundle that we will assume in a first time to have even rank. We will also assume the moment maps $X \to G_0$ and $Y \to G_0$ to be submersions, then $T^*X \oplus f^*TY$ being $Spin^c$ is equivalent to $V_f := T_v^*X \oplus f^*T_vY$ being $Spin^c$. The shriek morphism

$$f! : K^{-p}_G(X, P_X) \xrightarrow{f_!} K^{-p-d_f}_G(Y, P_Y), \tag{5.9}$$

where $d_f := rank\, V_f$, will be given as the composition of the following three morphism

I. The twisted G-equivariant Thom isomorphism

$$K^{-p}_G(X, P_X) \xrightarrow[\cong]{T} K^{-p-d_f}_G(T_v^*X \bigoplus f^*T_vY, P_{V_f}). \tag{5.10}$$

II. We consider now the index morphism

$$K^{-p-d_f}_{(T_vX \bigoplus f^*T_vY)\rtimes G}(f^*T_vY, P) \xrightarrow{Ind} K^{-p-d_f}_{f^*T_vY \rtimes (T_vX \rtimes G)}(f^*T_vY, P) \tag{5.11}$$

associated to the immersion

$$f^*T_vY \rtimes G \longrightarrow f^*T_vY \rtimes (T_vX \rtimes G)$$

given by the product of the identity in G and the inclusion of the units f^*T_vY in the groupoid $f^*T_vY \rtimes T_vX$.

III. Consider the groupoid immersion

$$X \rtimes G \xrightarrow{\tilde{f}} (Y \times_{G_0} (X \times_{G_0} X)) \rtimes G, \tag{5.12}$$

where $\tilde{f} := (f \times \triangle) \times Id_G$. Then the induced deformation groupoid is

$$G_f \rtimes G$$

where

$$G_f \rightrightarrows G_f^{(0)}$$

is the groupoid given by

$$G_f := f^*(T_vY) \rtimes T_vX \times \{0\} \bigsqcup Y \times_{G_0} (X \times_{G_0} X) \times (0,1] \text{ and} \tag{5.13}$$

$$G_f^{(0)} = f^*T_vY \times \{0\} \bigsqcup Y \times_{G_0} X \times (0,1] \tag{5.14}$$

Notice that $Y \times_{G_0} (X \times_{G_0} X)$ and Y are Morita equivalent groupoids with Morita equivalence the canonical projection.

Let α_f the twisting on $G_f \rtimes G$ given by proposition 2.17. It is immediate to check that $\alpha_f|_{(f^*(T_vY)\rtimes T_vX)\rtimes G} = \pi^*_{f^*T_vY \rtimes T_vX}\alpha$.

We can hence consider the twisted deformation index morphism associated to $(G_f \rtimes G, \alpha_f)$:

$$
\begin{tikzcd}
K^{-p-d_f}_{f^*T_vY\rtimes(T_vX\rtimes G)}(f^*T_vY,P) \arrow[r, "Ind_f"] & K^{-p-d_f}_{(Y\times_{G_0}(X\times_{G_0}X))\rtimes G}(Y\times_{G_0}X,P) \arrow[d, "\mu", "\cong"'] \\
& K^{-p-d_f}_G(Y,P)
\end{tikzcd}
\tag{5.15}
$$

For composing 5.10 with 5.11 remember that by the Fourier isomorphism proved in proposition 2.12 in [6] and by theorem 3.14 in [16] we have an isomorphism

$$K^*_G(T^*_vX\bigoplus f^*T_vY,P_{V_f})\approx K^*_{(T_vX\bigoplus f^*T_vY)\rtimes G}(f^*T_vY,P).$$

We can now give the following definition:

Definition 5.16 (Pushforward morphism for twisted G-manifolds)**.** Let X,Y be two manifolds and $f:X\longrightarrow Y$ a smooth map. Under the presence of a twisting P on G we let

$$K^{-p}_G(X,P_X)\xrightarrow{f_!}K^{-p-d_f}_G(Y,P_Y) \tag{5.17}$$

to be the morphism given by the composition of the three morphisms described above, 5.10 followed by 5.11 followed by 5.15.

One of the main results is that the pushforward maps induce a ring morphisms between the total twisted K-theory rings:

Proposition 5.18. *Let $G\rightrightarrows G_0$ be a Lie groupoid. Let X,Y be two G-proper manifolds and let $f:X\to Y$ be a G-smooth K-oriented map. We have that*

(i) If $T^*_vX\oplus f^*T_vY$ has even rank, the pushforward map

$$f!:K^{ev}_{TW,G}(X)\to K^{ev}_{TW,G}(Y)$$

is a ring morphism and the pushforward map

$$f!:K^{odd}_{TW,G}(X)\to K^{odd}_{TW,G}(Y)$$

is a $K^{ev}_{TW,G}(X)$-module morphism.

(ii) If $T^*_vX\oplus f^*T_vY$ has odd rank, the pushforward map

$$f!:K^{ev}_{TW,G}(X)\to K^{odd}_{TW,G}(Y)$$

is $K^{ev}_{TW,G}(X)$-module morphism and the pushforward map

$$f!:K^{odd}_{TW,G}(X)\to K^{odd}_{TW,G}(Y)$$

is a $K^{ev}_{TW,G}(X)$-module morphism.

Proof. By definition $f!$ is constructed by means of a Thom isomorphism and of two deformation indices. These indices are at their turn constructed by restriction (or evaluation) morphisms. To conclude the proof one has only to observe that restrictions induce ring/modules morphisms together with the fact that Thom is a ring/module isomorphism, see 4.4. □

The main theorem in [5] (theorem 4.2) can be now written as follows

Theorem 5.19. *The above push-forward morphism is functorial, that means, if we have a composition of smooth K-oriented G-maps between $G-$ proper manifolds with $T^*_vX\oplus f^*T_vY$ and $T^*_vY\oplus f^*T_vZ$ of even rank:*

$$X\xrightarrow{f}Y\xrightarrow{g}Z, \tag{5.20}$$

Then the following diagram of ring morphisms commutes

$$\begin{tikzcd} K^{ev}_{TW,G}(X) \arrow[rr, "(g\circ f)_!"] \arrow[dr, "f_!"'] & & K^{ev}_{TW,G}(Z) \\ & K^{ev}_{TW,G}(Y) \arrow[ur, "g_!"'] & \end{tikzcd}$$

In the case the bundles $T_v^*X \oplus f^*T_vY$ and $T_v^*Y \oplus f^*T_vZ$ are not both simultaneously of even rank a similar conclusion is obtained for modules morphisms.

6. Connes Approach to twisted K Homology for Lie groupoids

The pushforward functoriality theorem (thm. 4.2 in [5]) allows us to give the following definition:

Definition 6.1 (Twisted geometric K-homology)**.** Let $G \rightrightarrows M$ be a Lie groupoid with a twisting P. By the "Twisted geometric K-homology group" associated to (G, P) we mean the abelian group denoted by $K^{geo}_*(G, P)$ with generators the cycles (X, x) where

(1) X is a smooth co-compact G-proper manifold,
(2) $\pi_X : X \to M$ is the smooth momentum map which supposed to be a K-oriented submersion and
(3) $x \in K^{-p}_G(X, P_X)$ for some $p \in \mathbb{N}$,

and relations given by

$$(X, x) \sim (X', g_!(x)) \tag{6.2}$$

where $g : X \to X'$ is a smooth G-equivariant map.

The group defined above admits a $\mathbb{Z}_2$-gradation

$$K^{geo}_*(G, P) = K^{geo}_0(G, P) \bigoplus K^{geo}_1(G, P).$$

where $K^{geo}_j(G, P)$ is the subgroup generated by cycles (X, x) for which T_vX has rank congruent to j modulo 2.

We will now describe a product between two cycles with possibly different twistings by using the product structure defined in previous sections. For that we will need first to recall the a basic operation:

The Pullback: Let $A \xrightarrow{h} B$ be a smooth G-equivariant map (A, B G-proper manifolds). Suppose we have a twisting P on G. We are going to consider, for every $q \in \mathbb{N}$, the pullback

$$h^* : K^{-q}_G(B, P_B) \longrightarrow K^{-q}_G(A, P_A) \tag{6.3}$$

given as follows: If $\gamma : S^q \to \Gamma(B, Fred(\hat{P_B}))^G$ is a continuous map with $\gamma(*) = s$ one let

$$h^*\gamma : S^q \to \Gamma(A, Fred(\hat{P_A}))^G$$

to be given by

$$(h^*\gamma)(z)(a) := \gamma(z)(h(a)),$$

it is then classic to show that it induces a map between the homopoty classes.

More generally we will need a pullback map associated to a G-equivariant Hilsum-Skandalis map. We explain next what do we mean by this.

Consider a Lie groupoid $H_A \rightrightarrows A$, we say that it is a G-groupoid if G acts on H_A, on A and the source and target maps of H_A are $G-$equivariant. Under this situation we might form the semi-direct product groupoid

$$H_A \rtimes G \rightrightarrows A.$$

Suppose now that we have two G-proper (all the actions are required to be proper) Lie groupoids H_A and H_B together with a generalized morphism $h : H_A --- > H_B$ between them, that is, suppose we are given a H_B-principal bundle P_h over H_A, putting this in a diagram:

$$\begin{tikzcd} H_A \arrow[d, shift left] \arrow[d, shift right] & & P_h \arrow[dll, "t_h"'] \arrow[drr, "s_h"] & & H_B \arrow[d, shift left] \arrow[d, shift right] \\ A & & & & B \end{tikzcd}$$

We are going to consider, for every $q \in \mathbb{N}$, the pullback

$$h^* : K^{-q}_{H_B \rtimes G}(B, P_B) \longrightarrow K^{-q}_{H_A \rtimes G}(A, P_A) \tag{6.4}$$

given as follows: If $\gamma : S^q \to \Gamma(B, Fred(\hat{P_B}))^{H_B \rtimes G}$ is a continuous map with $\gamma(*) = s$ one let

$$h^*\gamma : S^q \to \Gamma(A, Fred(\hat{P_A}))^{H_A \rtimes G}$$

to be given by

$$(h^*\gamma)(z)(a) := \gamma(z)(b)$$

where $b = s_h(v)$ for some $v \in t_h^{-1}(a)$. One proves using the invariance of γ together with the identification $P_h \times_{H_B \rtimes G} Fred(\hat{P_B}) = Fred(\hat{P_A})$ that the definition of $h^*\gamma$ does not depend on the choice of v.

Lemma 6.5. *The pullback is natural. The following properties hold:*

(i) $Id^* = Id$
(ii) $(h_2 \circ h_1)^* = h_1^* \circ h_2^*$

Remark 6.6 (On Le Gall's descent functors)**.** The definition of the pullback above recalls Le Gall's pullback construction on the untwisted case which generalizes Kasparov descent morphisms. The simplicity of our construction is due to the fact that we are only dealing with the proper action case. In the general case is certainly possible to adapt Le Gall's to S^1-central extensions and then to apply it to the Twisted K-theory case. We do not need to do it in this generality in this paper.

The main property is the naturality of the pushforward maps with respect to pullbacks, this is the new and the main key technical result in this paper.

Proposition 6.7. *Let $G \rightrightarrows G_0$ be a Lie groupoid together with a twisting P. Suppose we have a commutative diagram of G-smooth K-oriented maps between G-proper manifolds*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "p"'] & A' \arrow[d, "q"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Then we have the following equality between K-theory morphisms

$$g! \circ p^* = q^* \circ f!$$

Proof. We have to show that the following diagram is commutative

$$\begin{tikzcd} K^*_G(A, P_A) \arrow[r, "g!"] & K^*_G(A', P_{A'}) \\ K^*_G(B, P_B) \arrow[u, "p^*"] \arrow[r, "f!"'] & K^*_G(B', P_{B'}) \arrow[u, "q^*"'] \end{tikzcd} \tag{6.8}$$

We will split the above diagram in four commutative diagrams:

Diagram I. Consider the following commutative diagram of groupoid morphisms which are equivariant with respect to the G-action:

$$\begin{tikzcd}
A' \times_{G_0} (A \times_{G_0} A) \arrow[r, "Id_{A'} \times Pr_{G_0}"] \arrow[d, "q\times\triangle(p)"'] & A' \times_{G_0} G_0 \arrow[d, "q\times Id_{G_0}"] \\
B' \times_{G_0} (B \times_{G_0} B) \arrow[r, "Id_{B'} \times Pr_{G_0}"] & B' \times_{G_0} G_0
\end{tikzcd}$$

Once identifying $A' \times_{G_0} G_0$ with A' (and respectively for B') we have that $Id_{A'} \times Pr_{G_0}$ induces the Morita equivalence of groupoids between $A' \times_{G_0} (A \times_{G_0} A)$ and A' with inverse a Hilsum-Skandalis isomorphism that induces the isomorphism μ in K-theory. Hence the diagram above induces the following commutative diagram in K-theory:

$$\begin{tikzcd}
K^*_{G\ltimes(A'\times_{G_0}(A\times_{G_0}A))}(A' \times_{G_0} A, P_{A'\times_{G_0}A}) \arrow[r, "\mu", "\approx"'] \arrow[dr, phantom, "\mathbf{I}"] & K^*_G(A', P_{A'}) \\
K^*_{G\ltimes(B'\times_{G_0}(B\times_{G_0}B))}(B' \times_{G_0} B, P_{B'\times_{G_0}B}) \arrow[r, "\approx", "\mu"'] \arrow[u, "(q\times_{G_0}\triangle(p))^*"] & K^*_G(B', P_{B'}) \arrow[u, "q^*"']
\end{tikzcd} \tag{6.9}$$

Diagram II. Remember the G-groupoid immersions

$$A \overset{g\times\triangle}{\longrightarrow} A' \times_{G_0} (A \times_{G_0} A)$$

and

$$B \overset{f\times\triangle}{\longrightarrow} B' \times_{G_0} (B \times_{G_0} B)$$

used above to construct the deformation indices (see (5.12) and 5.15). They fit in the following commutative diagram of G-morphisms:

$$\begin{tikzcd}
A' \times_{G_0} (A \times_{G_0} A) \arrow[r, "q\times\triangle(p)"] & B' \times_{G_0} (B \times_{G_0} B) \\
A \arrow[r, "p"'] \arrow[u, "g\times\triangle"] & B \arrow[u, "f\times\triangle"']
\end{tikzcd} \tag{6.10}$$

By the functoriality of the deformation to the normal cone we have a morphism of G-groupoids (see (5.13) for notations)

$$\begin{tikzcd}
G_g \arrow[r, "\widetilde{q\times\triangle(p)}"] \arrow[d, shift left] \arrow[d, shift right] & G_f \arrow[d, shift left] \arrow[d, shift right] \\
G_g^{(0)} \arrow[r, "(\widetilde{q\times\triangle(p)})_0"'] & G_f^{(0)}
\end{tikzcd}$$

whose restriction at $t = 1$ gives $q \times \triangle(p)$ and whose restriction at $t = 0$ gives $d^v p \ltimes d^v q : T_v A \ltimes g^* T_v A' \to T_v B \ltimes f^* T_v B'$ as a morphism of G-groupoids where $d^v p$ (resp. $d^v q$) stands for the derivative in the tangent vertical direction. Since pullbacks obviuosly commutes with restrictions we have the following commutative

diagram
(6.11)

$$\begin{tikzcd}
K^*_{G\ltimes(T_vA\ltimes g^*T_vA')}(g^*T_vA',P_{g^*T_vA'}) \arrow[r, "Ind_{\tilde{g}}"] \arrow[dr, phantom, "\mathbf{II}"] & K^*_{G\ltimes(A'\times_{G_0}(A\times_{G_0}A))}(A'\times_{G_0}A,P_{A'\times_{G_0}A}) \\
K^*_{G\ltimes(T_vB\ltimes f^*T_vB')}(f^*T_vB',P_{f^*T_vB'}) \arrow[r, "Ind_{\tilde{f}}"'] \arrow[u, "(d^vp\ltimes d^vq)^*"] & K^*_{G\ltimes(B'\times_{G_0}(B\times_{G_0}B))}(B'\times_{G_0}B,P_{B'\times_{G_0}B}) \arrow[u, "(q\times_{G_0}\triangle(p))^*"']
\end{tikzcd}$$

Diagram III. The groupoid morphism (equivariant w.r. to G)

$$d^vp\ltimes d^vq : T_vA\ltimes g^*T_vA'\to T_vB\ltimes f^*T_vB'$$

induces (again by functoriality of the deformation to the normal cone) a G-groupoid morphism between the respective tangent groupoids

$$(d^vp\ltimes d^vq)^{tan} : (T_vA\ltimes g^*T_vA')^{tan}\to (T_vB\ltimes f^*T_vB')^{tan}$$

whose restriction at $t=1$ gives $d^vp\ltimes d^vq$ and whose restriction at zero gives $d^vp\oplus d^vq : T_vA\oplus g^*T_vA'\to T_vB\oplus f^*T_vB'$. For the same reason as diagram II we have the following commutative diagram in K-theory:

(6.12)

$$\begin{tikzcd}
K^*_{G\ltimes(T_vA\oplus g^*T_vA')}(g^*T_vA',P_{g^*T_vA'}) \arrow[r, "Ind"] \arrow[dr, phantom, "\mathbf{III}"] & K^*_{G\ltimes(T_vA\ltimes g^*T_vA')}(g^*T_vA',P_{g^*T_vA'}) \\
K^*_{G\ltimes(T_vB\oplus f^*T_vB')}(f^*T_vB',P_{f^*T_vB'}) \arrow[r, "Ind"'] \arrow[u, "(d^vp\oplus d^vq)^*"] & K^*_{G\ltimes(T_vB\ltimes f^*T_vB')}(f^*T_vB',P_{f^*T_vB'}) \arrow[u, "(d^vp\ltimes d^vq)^*"']
\end{tikzcd}$$

Diagram IV. The commutativity of the following diagram follows from the naturality of Thom isomorphism:

(6.13)
$$\begin{tikzcd}
K^*_G(A,P_A) \arrow[r, "Thom", "\approx"'] & K^*_{G\ltimes(T_vA\oplus g^*T_vA')}(g^*T_vA',P_{g^*T_vA'}) \\
K^*_G(B,P_B) \arrow[r, "\approx", "Thom"'] \arrow[u, "p^*"] & K^*_{G\ltimes(T_vB\oplus f^*T_vB')}(f^*T_vB',P_{f^*T_vB'}) \arrow[u, "(d^vp\oplus d^vq)^*"']
\end{tikzcd}$$

By definition, diagram (6.8) decomposes, with the previous diagrams, in the following form:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[dr, phantom, "\mathbf{IV}"] & \cdot \arrow[r] \arrow[dr, phantom, "\mathbf{III}"] & \cdot \arrow[r] \arrow[dr, phantom, "\mathbf{II}"] & \cdot \arrow[r] \arrow[dr, phantom, "\mathbf{I}"] & \cdot \\
\cdot \arrow[r] \arrow[u] & \cdot \arrow[r] \arrow[u] & \cdot \arrow[r] \arrow[u] & \cdot \arrow[r] \arrow[u] & \cdot \arrow[u]
\end{tikzcd}$$

and hence it is commutative. □

The product of two cycles: Let P and Q two twistings on G. Let (X,x) with $x\in K_G^{-p}(X,P_X)$ and (Y,y) with $y\in K_G^{-q}(Y,Q_Y)$ we put
(6.14)

$$(X,x)\cdot(Y,y) := (X\times_{G_0}Y,\pi_X^*x\cdot\pi_Y^*y)\in K_G^{-p-q}(X\times_{G_0}Y,P_{X\times_{G_0}Y}\otimes Q_{X\times_{G_0}Y})$$

where π_X,π_Y stand for the respective projections from $X\times_{G_0}Y$ to X and Y.

The following is the main result of this paper:

Theorem 6.15. *For any Lie groupoid, the Total twisted geometric K-homology group*

$$K^{geo}_{TW,0}(G) := \bigoplus_{\alpha\in H^1(G,PU(H))} K_0^{geo}(G,P_\alpha)$$

has a ring structure with the product described above and

$$K^{geo}_{TW,1}(G) := \bigoplus_{\alpha \in H^1(G,PU(H))} K^{geo}_1(G, P_\alpha)$$

has a $K^{geo}_{TW,0}(G)$*-module structure.*

Proof. We only have to prove that the product described above is well defined in the K-homology group. Let P and Q two twistings on G. Let (X, x) with $x \in K^{-p}_G(X, P_X)$ and (Y, y) with $y \in K^{-q}_G(Y, Q_Y)$. Suppose we have smooth maps $X \xrightarrow{g} X'$ and $Y \xrightarrow{f} Y'$. We would finish if we can show that

$$(X \times_{G_0} Y, \pi^*_X x \cdot \pi^*_Y y) \sim (X' \times_{G_0} Y', \pi^*_{X'} g!x \cdot \pi^*_{Y'} f!y).$$

In fact we can consider the smooth map

$$X \times_{G_0} Y \xrightarrow{g\times f} X' \times_{G_0} Y'$$

which fits the following commutative diagrams

$$\begin{tikzcd} X \times_{G_0} Y \arrow[r, "g\times f"] \arrow[d, "\pi_X"'] & X' \times_{G_0} Y' \arrow[d, "\pi_{X'}"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

and

$$\begin{tikzcd} X \times_{G_0} Y \arrow[r, "g\times f"] \arrow[d, "\pi_Y"'] & X' \times_{G_0} Y' \arrow[d, "\pi_{Y'}"] \\ Y \arrow[r, "f"'] & Y' \end{tikzcd}$$

The result now follows from proposition 6.7 and proposition 5.18 since they imply

$$(g \times f)!(\pi^*_X x \cdot \pi^*_Y y) = (g \times f)!(\pi^*_X x) \cdot (g \times f)!(\pi^*_Y y) = \pi^*_{X'} g!x \cdot \pi^*_{Y'} f!y$$

and hence

$$(X \times_{G_0} Y, \pi^*_X x \cdot \pi^*_Y y) \sim (X' \times_{G_0} Y', \pi^*_{X'} g!x \cdot \pi^*_{Y'} f!y).$$

□

7. The Baum-Connes map

Recall that in [5] the Baum-Connes assembly map

$$K^{geo}_*(G, P_\alpha) \xrightarrow{\mu_\alpha} K^{-*}(G, P_\alpha) \tag{7.1}$$

was constructed for every twisting α on G where $K^*(G, P_\alpha) := K_{-*}(C^*_r(G, P_\alpha))$ stands for the $K-$theory of the reduced C^*-algebra associated to the twisted groupoid (G, P_α), (there is also the assembly map taking values on the maximal C^*-algebra). The definition of the Baum-Connes map is given by

$$\mu_\alpha(X, x) := \pi_X!(x) \in K^*(G, P_\alpha)$$

where $\pi_X!$ is the pushforward map defined in [5]. Consider the Total Twisted K-theory group

$$K^*_{TW}(G) := \bigoplus_{\alpha \in H^1(G,PU(H))} K^*(G, P_\alpha).$$

By the theorem above we have a ring (module for the odd case) structure on the image of the Total twisted Baum-Connes assembly map

$$K^{geo}_{TW,0}(G) \xrightarrow{\mu_{TW}} K^0_{TW}(G) \tag{7.2}$$

where $\mu_{TW} := \oplus\mu_\alpha$ whenever μ_{TW} is injective. In particular if μ_{TW} is an isomorphism then $K^0_{TW}(G)$ has a ring (module for the odd case) structure such that μ_{TW} is a ring isomorphism.

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Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán MÉXICO 58089

E-mail address: barcenas@matmor.unam.mx

URL: http://www.matmor.unam.mx̃barcenas

Institut de Mathématiques de Toulouse., 118, route de Narbonne F-31062 Toulouse, FRANCE

E-mail address: paulo.carrillo@math.univ-toulouse.fr

URL: http://www.math.univ-toulouse.fr/~carrillo

Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán MÉXICO 58089

Departamento de Matemáticas., Pontificia Universidad Javeriana, Cra. 7 No. 43-82 - Edificio Carlos Ortíz 5to piso, Bogotá D.C, Colombia

E-mail address: mavelasquezm@gmail.com

URL: https://sites.google.com/site/mavelasquezm/