BREDON COHOMOLOGY, K-THEORY AND K-HOMOLOGY OF PULLBACKS OF GROUPS.

NOÉ BÁRCENAS, DANIEL JUAN-PINEDA, AND MARIO VELÁSQUEZ

ABSTRACT. We develop a spectral sequence of Eilenberg-Moore type to compute Bredon Cohomology of spaces with an action of a group given as a pullback.

Using several other spectral sequences, and positive results on the Baum-Connes Conjecture, we are able to compute Equivariant K-Theory and K-homology of the reduced C^*-algebra of a 6-dimensional crystallographic group Γ introduced by Vafa and Witten. We also use positive results on the Farrell-Jones conjecture to give a vanishing result for the algebraic K-theory of the group ring of the group Γ in negative degrees.

1. INTRODUCTION

We develop a method to compute Bredon Cohomology and equivariant (co)-homology theories of spaces with an action of a discrete group Γ obtained as a pullback of discrete groups over a finite group.

Condition 1.1. Let Γ be a group which is obtained as a pullback diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{p_2} & H \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ G & \xrightarrow{\pi_1} & K \end{array}$$

where K is a finite group.

Given such a pullback diagram, the group Γ can be viewed as a subgroup of $G \times H$, namely $\Gamma = \{(g,h) \in G \times H \mid \pi_1(g) = \pi_2(h)\}$. When the maps in the pullback are clear from the context, we denote this pullback by $G \times_K H$. We develop a spectral sequence in Theorem 2.19 converging to Bredon cohomology groups of spaces with a Γ-action when Γ is defined as a pullback as in condition 1.1.

Theorem. 2.19 [Eilenberg-Moore spectral sequence for Bredon cohomology] Let Γ be a group as in 1.1. Assume that X is a G-CW complex with isotropy in a family $\mathcal{F}$ of finite subgroups of G and Y is an H-CW complex with isotropy in a family $\mathcal{F}'$ of finite subgroups of H. Let M be a Bredon coefficient system, which we suppose endowed with a Green functor structure. Assume that M satisfies the projectivity condition 2.9. Then, there is a spectral sequence with E_2 term given by

$$\operatorname{Tor}_{M(K/K)}^{p,q}(H_{\mathcal{F}}^*(X, M), H_{\mathcal{F}'}^*(Y, M))$$

which converges to

$$\operatorname{Tor}_{M(K/K)}^{p,q}(\underline{\mathrm{C}}_G^*(X, M), \underline{\mathrm{C}}_H^*(Y, M)).$$

The groups $H_{\mathcal{F}'}^*(Y, M)$ denote Bredon co-homology with coefficients in the contravariant functor M, defined on a family $\mathcal{F}'$ containing the isotropy groups of Y. The groups $\underline{\mathrm{C}}_G^*(X, M)$ denote Bredon cochain complexes with a differential graded structure explained in detail in Section 2. The groups $\operatorname{Tor}_{M(K/K)}^{p,q}(\ ,\ )$ are derived functors of differential graded algebras and modules.

The spectral sequence gives a method to compute Γ-equivariant Bredon cohomology groups out of the (potentially easier to calculate) G- respectively, H-equivariant cohomology groups of X and Y, together with knowledge about their structure as modules over the ring $M(K/K)$.

The pullback structure in 1.1 appears in the computations of Bredon cohomology of crystallographic groups Γ with a given (finite) point group K. These groups are given as an extension

$$(1.2) \qquad\qquad 1 \to \mathbb{Z}^n \to \Gamma \to K \to 1$$

where K is finite and the conjugation action on $\mathbb{Z}^n$ is given by a representation $\rho : K \to Gl_n(\mathbb{Z})$. In this situation, the space $\mathbb{R}^n$ with the induced action is a model for $\underline{E}\Gamma$. This is a consequence of Proposition 1.12, page 30 in [CK90].

Splitting the representation $\rho : K \to Gl_n(\mathbb{Z})$ gives a pullback structure on Γ. More precisely, assume that $\mathbb{Z}^n$ with the action given by ρ has a K-invariant decomposition $\mathbb{Z}^n = A \oplus B$.

Denote by G the semidirect product $A \rtimes K$ and by H the semidirect product $B \rtimes K$. Then, the group Γ is isomorphic to the pullback $G \times_K H$. See Remark 2.24 for details on this.

The main application of Theorem 2.19 will be a method for the computation of equivariant (co)-homology theories evaluated on classifying spaces for families of subgroups of Γ as in Theorem 3.18.

The interest in these computations comes from the fact that the assembly maps in the Baum-Connes conjecture [BCH94]

$$(1.3) \qquad K_*^{\Gamma}(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma))$$

and in the Farrell-Jones Conjecture [DL98]

$$(1.4) \qquad \mathbb{H}_*^{\Gamma}(E_{\mathcal{VC}}(\Gamma), \mathbb{K}^{-\infty}(R)) \to K_*(R\Gamma)$$

involve equivariant homology theories evaluated on these spaces. Computations of Bredon (co)-homology groups associated to these (co)-homology theories give inputs to a spectral sequence of Atiyah-Hirzebruch type [DL98], abutting to the relevant equivariant (co)-homology groups.

Until now, the methods developed for the computation of K-theory and K-homology groups of extensions Γ as in 1.2 include assumptions on the maximality of finite, respectively virtually cyclic subgroups relevant to the computation, as well as strong hypotheses on their normalizers. This concerns particularly conditions M and NM in [LS00], [DL13], or explicit computations related to the Weyl groups of them, as in [LL12]. All of them restrict the class of extensions to those arising from conjugation actions which are free outside of the origin.

In another direction, extensive knowledge of models for both spaces, using the classification of crystallographic groups in a given dimension also gives information about the homology groups relevant to the Farrell-Jones Conjecture, as it is done in [FO12].

The methods derived from the spectral sequence in Theorem 2.19 rely neither on the dimension, as the use of specific models in [FO12], nor on freeness of the conjugation action as in [LL12], [LS00], [DL13] and the existence of a unique maximal finite subgroup.

To illustrate our method, we concentrate in a group extension

$$1 \to \mathbb{Z}^6 \to \Gamma \to \mathbb{Z}/4\mathbb{Z} \to 1,$$

which gained interest in theoretical physics [VW95].

Example 1.5. [The 6- dimensional Vafa-Witten toroidal orbifold quotient] Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{Z}^{\oplus 6}$ induced from the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{C}^3$, given by

$$k(z_1, z_2, z_3) = (-z_1, iz_2, iz_3).$$

The associated semidirect product

$$(1.6) \qquad 1 \to \mathbb{Z}^6 \to \Gamma \to \mathbb{Z}/4\mathbb{Z} \to 1$$

will be called the Vafa-Witten group, this splits as a multiple pullback

$$(\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}).$$

Where the first two semidirect products are taken with respect to the representation $-1 : \mathbb{Z}/4 \to Gl_1(\mathbb{Z})$ given by scalar multiplication by -1, and the two last ones are given by the representation $i : \mathbb{Z}/2 \to \mathbb{Z}^2$ given by complex multiplication by i. Notice that the action determined by Γ in $\underline{E}\Gamma$, and more specifically, the one coming from the block given by the action $\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ is not free outside of the origin. Condition M of [Lüc05] is not satisfied in this case, because if we denote by 1 the generator of $\mathbb{Z}$ and by t a generator of $\mathbb{Z}/4\mathbb{Z}$, we have that the group $\langle (0, t^2) \rangle \cong \mathbb{Z}/2\mathbb{Z}$ is contained in two maximal finite subgroups namely $\langle (0, t) \rangle$ and $\langle (1, t) \rangle$, although our methods readily apply to this situation.

For the group described in Example 1.5, we show that the spectral sequence from Theorem 2.19 collapses at the E_2-term for the specific choice of the complex representation ring as a Bredon coefficient system. With the use of a Universal Coefficient Theorem for Bredon Cohomology, Theorem 1.13 in [BV14], completely determines the equivariant K-Homology of the classifying space for proper actions.

Theorem. 3.18 [Topological K-Theory] Let Γ be the group $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$ acting on $\mathbb{R}^6$ as in 1.5. The topological K-theory of the reduced C^*-algebra of Γ is as follows:

- $K_0(C_r^*(\Gamma)) \cong K_0^\Gamma(\underline{E}\Gamma) \cong \mathbb{Z}^{\oplus 47}$ and
- $K_1(C_r^*(\Gamma)) \cong K_1^\Gamma(\underline{E}\Gamma) = 0$.

The ideas developed in Theorem 2.19 and subsequent computations are particularly well-suited to families of subgroups which are well-behaved under products. The example for such a family is, notably, the family of finite subgroups. Although the Eilenberg-Moore method 2.19 does not transfer directly to the family of virtually cyclic subgroups due to its bad behaviour under products, we are able to deduce using positive results on the Farrell-Jones Conjecture [Tsa95], [JP03], [FJ] and computations of lower algebraic K-Theory [Cara], [Carb], the following result, computing the negative algebraic K-theory of the group ring RG.

Theorem. 3.22 [Negative algebraic K-Theory] Let Γ be the group determined by the extension 1.6. Let R be a ring of algebraic integers. Then,

$$K_i(R\Gamma) = 0, \text{ for all } i < 0.$$

This paper is organized as follows:

Contents

2. The Eilenberg-Moore spectral sequence and Bredon Cohomology

Definition 2.1. Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1 \leq n} X_n$ with $X_{-1} = \emptyset$, $X_0 = A$ and every space is inductively obtained from the previous one by attaching cells with pushout diagrams

$$
\begin{array}{ccc}
\coprod_\lambda S^{n-1} \times G/H_\lambda & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_\lambda D^n \times G/H_\lambda & \longrightarrow & X_n
\end{array}
$$

Definition 2.2. Let $\mathcal{F}$ be a family of subgroups which is closed under subgroups and conjugation. A model for the classifying space for the family $\mathcal{F}$ is a G-CW complex X satisfying

- All isotropy groups of X lie in $\mathcal{F}$.
- For any G-CW complex Y with isotropy in $\mathcal{F}$, there exists up to G-homotopy a unique G-equivariant map $f : Y \to X$.

A model for the classifying space of the family $\mathcal{F}$ will be usually denoted by $E_\mathcal{F}(G)$.

Particularly relevant is the classifying space for proper actions, the classifying space for the family $\mathcal{FIN}$ of finite subgroups, denoted by $\underline{E}G$ and the space $E_{\mathcal{VC}}(G)$ for the family $\mathcal{VC}$ of virtually cyclic subgroups.

Let X be a G-CW-complex. The Bredon chain complex is defined as the contravariant functor to the category of chain complexes $C_*^G(X) : \mathsf{Or}_{\mathcal{F}}(G) \to \mathbb{Z} - \mathsf{CHCOM}$ which assigns to every object G/H the cellular $\mathbb{Z}$-chain complex of the H-fixed point complex $C_*(X^H) \cong C_*(\mathrm{Map}_G(G/H, X))$ with respect to the cellular boundary maps $\underline{\partial}_*$. The n-chains of the Bredon chain complex evaluated on an object G/K of the orbit category, consist of elements of free abelian groups $\bigoplus_\lambda \mathbb{Z}[e_\lambda]$, where e_λ denotes the cell orbits of type $D^n \times G/K$ in the cell decomposition above and $\mathbb{Z}[e_\lambda]$ denotes the free abelian group generated by e_λ.

Let G be a discrete group, let $\mathsf{Or}(G)$ be the orbit category of G, where objects are homogeneous sets G/H and morphisms are G-equivariant maps.

Convention 2.3. Let R be a ring and $\mathcal{F}$ be a family of groups. Recall that a contravariant Bredon functor M with values on R-modules is a contravariant functor defined on the full subcategory $\mathsf{Or}(G, \mathcal{F})$ consisting of homogeneneous spaces G/H, where $H \in \mathcal{F}$. In this paper we suppose that Bredon functors are endowed with a representable Green functor structure. For a definition of Green functors see [Bou97].

Definition 2.4 (Bredon cochain complex)**.** Given a contravariant Bredon functor M, the Bredon cochain complex $C_G^*(X; M)$ is defined as the abelian group of natural transformations of functors defined on the orbit category $C_G^*(X) \to M$. In symbols,

$$C_G^n(X; M) = \mathrm{Hom}_{\mathsf{Or}_{\mathcal{F}(G)}}(C_n^G(X), M),$$

where $\mathcal{F}(G)$ is a family containing the isotropy groups of X.

Given a set $\{e_\lambda\}$ of representatives of the orbits of n-cells of the G-CW complex X, and isotropy subgroups P_λ of the cells e_λ, the abelian groups $C_G^n(X, M)$ satisfy:

$$C_G^n(X, M) = \prod_\lambda Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/P_\lambda))$$

with one summand for each orbit representative e_λ. They afford a differential $\delta^n : C_G^n(X, M) \to C_G^{n+1}(X, M)$ determined by $\underline{\partial}_*$ and maps $M(\phi) : M(G/P_\xi) \to M(G/P_\lambda)$ for morphisms $\phi : G/P_\lambda \to G/P_\xi$.

Given a Bredon functor M, the Bredon cochain complex has cup products

$$\cup : C_G^m(X, M) \otimes C_G^n(X, M) \to C_G^{n+m}(X, M).$$

See [Bre67], Chapter I.8 in pages 19-20.

We will list now some algebraic definitions.

Definition 2.5 (Differential Graded Algebra)**.** Let A be a graded algebra. A is said to be a differential graded algebra (DGA) if there exists a group homomorphism $d : A \to A$ of degree $+1$ satisfying

(i) $d^2 = 0$
(ii) $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ where $\mid a \mid$ is the degree of the element $a \in A$.

Definition 2.6 (Differential Modules over a Differential Graded Algebra)**.** Let (A, d_A) be a differential graded algebra. A differential graded module over A is a graded A-module $\mathcal{M}$ together with differentials $d_{\mathcal{M}}$ satisfying $d_{\mathcal{M}}(am) = d_A(a)m + (-1)^{|m|}ad_{\mathcal{M}}(m)$

Remark 2.7 (DGA Associated to a Bredon Module)**.** Let M be a contravariant Bredon functor. The differential graded algebra $C_G^*(M)$ is defined as the inverse limit

$$C_G^0(M) = \lim_{G/P \in \mathcal{F}} M(G/P)$$

where the limit is taken in the category of commutative rings with 1, $C_G^i(M) = 0$ for $i \neq 0$, and $d_i = 0$ for all i. Note that if the group G is finite, and $\mathcal{F}$ is the family of finite subgroups, $C_G^0(M) = M(G/G)$.

The full Bredon cochain complex $\bigoplus_n C_G^n(X, M)$ together with the differential graded $C_G^*(M)$-module structure will be denoted by $C_G^*(X, M)$.

Definition 2.8 (Bredon cohomology)**.** The Bredon cohomology groups with coefficients in M, denoted by $H_G^*(X, M)$ are the cohomology groups of the cochain complex $(C_{\mathcal{F}}^*(X, M), \delta^*)$.

We will now assume the following condition, which simplifies the differential graded structure involved in the cochain complexes.

Condition 2.9 (Condition P)**.** We will assume that Γ fits in a pullback diagram as in condition 1.1.

Consider a contravariant Bredon functor M defined on the orbit categories $\mathsf{Or}(\Gamma, \mathcal{F})$, $\mathsf{Or}(G, p_1(\mathcal{F}))$ and $\mathsf{Or}(H, p_2(\mathcal{F}))$. Given elements G/P and H/Q in $\mathsf{Or}(G, p_1(\mathcal{F}))$, respectively $\mathsf{Or}(H, p_2(\mathcal{F}))$,

- The maps $\pi_1^* : M(K/\pi_1(P)) \to M(G/P)$, $\pi_2^* : M(K/\pi_2(P)) \to M(H/Q)$ furnish $M(G/P)$ and $M(H/Q)$ with structure of projective modules over the ring $M(K/\pi_1(P)) = M(K/\pi_2(P))$.

The following fact is crucial for our computations related to the family of finite subgroups.

Lemma 2.10 (Structure Lemma for families of finite subgroups). *Let Γ be a group given as a pullback as in 1.1.*

- *The structure maps p_1 and p_2 give a bijective correspondence between the elements of the family $\mathcal{FIN}(\Gamma)$ and the family $\mathcal{FIN}(G) \times_K \mathcal{FIN}(H) = \{(P \times_{\pi_1(P)} Q) \mid P \in \mathcal{FIN}(G), Q \in \mathcal{FIN}(H)\}$.*
- *Let X and Y be proper G-, respectively H-CW-complexes. Then, the isotropy groups of the action of $G \times_K H$ in $X \times Y$ are contained in the family $\mathcal{FIN}(G) \times_K \mathcal{FIN}(H)$.*

Proposition 2.11. *Given a pullback diagram as in condition 1.1, and a restriction of the pullback to finite subgroups*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{p_2} & Q \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ P & \xrightarrow{\pi_1} & \pi_1(P) \end{array}$$

there is a natural isomorphism of $M(K/\pi_1(P))$-modules

$$M(\Gamma/\Gamma_1) \cong M(G/P) \otimes_{M(K/\pi_1(P))} M(H/Q).$$

Where the $M(K/\pi_1(P))$-module structure in both sides is given by the pullback diagram.

Proof. As the functor M is representable it preserve colimits, then it sends a pullback diagram 1.1 in a pushout diagram

$$\begin{array}{ccc} M(\Gamma/\Gamma_1) & \xleftarrow{p_2^*} & M(H/Q) \\ {\scriptstyle p_1^*}\uparrow & & \uparrow{\scriptstyle \pi_2^*} \\ M(G/P) & \xleftarrow{\pi_1^*} & M(K/\pi_1(P)). \end{array}$$

On the category of algebras, pushouts corresponds to tensor products, then one have $M(\Gamma/\Gamma_1) \cong M(G/P) \otimes_{M(K/\pi_1(P))} M(H/Q)$.   □

Definition 2.12. Let Γ be a group given as a pullback as in Condition 1.1. Let M be a contravariant Bredon functor defined on the orbit categories $\mathsf{Or}(\Gamma, \mathcal{F})$, $\mathsf{Or}(G, p_1(\mathcal{F}))$ and $\mathsf{Or}(H, p_2(\mathcal{F}))$. We will denote by $M^G = M \mid \mathsf{Or}(G, p_1(\mathcal{F}))$, respectively $M^H = M \mid \mathsf{Or}(H, p_2(\mathcal{F}))$.

Consider the category $\mathsf{Or}(G, \mathcal{FIN}) \times \mathsf{Or}(H, \mathcal{FIN})$ and consider the functor defined on its objects $G/R \times H/Q$ as $M(G/R) \otimes_{\mathbb{Z}} M(H/Q)$ and its morphism in a similar way. We will denote the restriction of this

functor to $\mathrm{Or}(G \times_K H, \mathcal{FIN})$ by $M^G \otimes_{\mathbb{Z}} M^H$. This functor is defined on each object $G \times_K H/P \times_{\pi_1(P)} Q$ as,

$$M^G \otimes_{\mathbb{Z}} M^H : \mathrm{Or}_{\mathcal{F} \times_K \mathcal{F}'}(G \times_K H) \quad \to \mathrm{RINGS}$$
$$(G \times_K H)/(P \times_{\pi_1(P)} Q) \mapsto M(G/P) \otimes_{\mathbb{Z}} M(H/Q).$$

Convention 2.13. We can define a further equivalence relation over the restriction of this tensor product, we say $\alpha \cdot \rho_1 \otimes \rho_2 \sim \rho_1 \otimes \alpha \cdot \rho_2$, where $\alpha \in M(G/\pi_1(P))$ and the products in both sides are defined via the maps π_1^* and π_2^*. Denote the quotient by $(M^G \otimes_{M^K} M^H)$.

Lemma 2.14. *The isomorphism in Proposition 2.11 can be promoted to a natural equivalence between the functors M^Γ and $(M^G \otimes_{M^K} M^H)$.*

Proof. Given a G-map

$$G \times_K H/P \times_{\pi_1(P)} Q \to G \times_K H/P' \times_{\pi_1(P')} Q',$$

this map is characterized by an element in $G \times_K H$ that conjugates $P \times_{\pi_1(P)} Q$ to a subgroup of $P' \times_{\pi_1(P')} Q'$. Now, taking the restriction to subconjugate subgroups commutes with taking the pullbacks with respect to p_1 and p_2 due to the Structure Lemma 2.10. $\square$

Taking the associated differential algebra structure, one obtains:

Proposition 2.15. *Under the assumptions of lemma 2.10, there is a natural isomorphism of differential graded algebras*

$$C_G^*(M) \otimes_{C_K^*(M)} C_H^*(M) \to C_{G \times_K H}^*(M).$$

Proof. As the differential map is zero on $C_{G \times_K H}^*(M)$ it is a consequence of Lemma 2.14. $\square$

Note that if X is a proper G-CW-complex and Y is a proper H-CW-complex, the product $X \times Y$ has a natural structure of $(G \times H)$-CW-complex (the cells correspond to product of cells of X and Y). From this structure we can construct a $(G \times_K H)$-CW-complex structure in $X \times Y$. Given a $(G \times H)$-equivariant cell $e_\lambda = D^n \times (G \times H/P \times Q)$, set

$$e_{\lambda,t} = D^n \times (G \times_K H/P \times_{\pi_1(P)} Q)$$

for $t \in (G \times H/G \times_K H)/(P \times Q/P \times_H Q)$. Notice that $C_*^{G \times_K H}(X \times Y)$ can be obtained as the composition

$$\mathrm{Or}_{\mathcal{F} \times_K \mathcal{F}'}(G \times_K H) \xrightarrow{i_\sharp} \mathrm{Or}_{\mathcal{F} \times \mathcal{F}'}(G \times H) \xrightarrow{C_*^{G \times H}(X \times Y)} \mathbb{Z} - \mathrm{CHCOM}$$

where $i_\sharp$ is the map induced by the inclusion

$$i : G \times_K H \to G \times H.$$

Proposition 2.16. *There is an isomorphism of* $\mathsf{Or}_{\mathcal{F} \times_K \mathcal{F}'}(G \times_K H)$-*chain complexes*

$$C_*^{G \times_K H}(X \times Y) \cong i_\sharp(C_*^G(X) \otimes C_*^H(Y)).$$

Moreover, the isomorphism is compatible with the Differential Graded Algebra structure.

Proof. The identification of the isotropy groups of the second part of Lemma 2.10, and the usual Eilenberg-Zilber argument identify up to chain homotopy eqivalence the chain complexes over the orbit category

$$C_*^{G \times H}(X \times Y) \cong C_*^G(X) \otimes C_*^H(Y)$$

as $\mathsf{Or}_{\mathcal{F} \times \mathcal{F}'}(G \times H)$-chain complexes.

The differential graded structure is preserved since the differential graded algebra $C_K^*(M)$ is concentrated in degree zero and the cup product agrees with the module structure over the commutative ring $M(K/K)$.

$\square$

We can refine Proposition 2.11 to an isomorphism of differential graded algebras:

Proposition 2.17. *There is an isomorphism of differential graded algebras*

$$\mathrm{Hom}(i_\sharp(C_*(X) \otimes C_*(Y)), C_G^*(M) \otimes_{C_K^*(M)} C_H^*(M)) \xrightarrow{\cong}$$
$$C_G^*(X, M) \otimes_{C_K^*(M)} C_H^*(Y, M).$$

Proof. Notice that in degree n the left hand side cochain complex is

$$\bigoplus_{\lambda,\mu} \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}[e_\lambda] \otimes_\mathbb{Z} \mathbb{Z}[f_\mu], M(G/P_\lambda) \otimes_{M(K/\pi_1(P_\lambda))} M(H/Q_\mu)),$$

where e_λ denotes a cell in X with isotropy group P_λ and f_μ denotes a cell in Y with isotropy group Q_μ, and the sum is taken over the pairs λ and μ such that $\dim(e_\lambda) + \dim(f_\mu) = n$. Note that $\mathbb{Z}[e_\lambda] \otimes_\mathbb{Z} \mathbb{Z}[f_\mu]$ is isomorphic as abelian group to $\mathbb{Z}$. Then, each summand in the direct sum is isomorphic to

$$M(G/P_\lambda) \otimes_{M(K/\pi_1(P_\lambda))} M(H/Q_\mu),$$

and the left hand side cochain complex in degree n is isomorphic to

$$\bigoplus_{\lambda,\mu} M(G/P_\lambda) \otimes_{M(K/\pi_1(P_\lambda))} M(H/Q_\mu).$$

The right hand side cochain complex in degree n is

$$\bigoplus_\lambda \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/P_\lambda)) \otimes_{M(K/\pi_1(P_\lambda))} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[f_\mu], M(H/Q_\mu)).$$

Using 2.16, this term is isomorphic to $M(G/P_\lambda) \otimes_{M(K/\pi_1(P_\lambda))} M(H/Q_\mu)$ and a straightforward calculation shows that the isomorphism are compatible with the coboundary maps. $\qquad\square$

Recall the construction of the Eilenberg-Moore spectral sequence, page 241 in Chapter 7 of [McC01].

Theorem 2.18 (First Eilenberg-Moore Theorem). *Let A be a differential graded algebra over the ring R, let M and N be differential graded A-modules. Assume A and the graded R-Module given by the homology of A (denoted by $H(A)$), are flat modules over R. Then, there is a second quadrant spectral sequence with*

$$E_2^{p,q} = \mathrm{Tor}_{H(A)}^{p,q}(H(M), H(N))$$

converging to $\mathrm{Tor}_A^{p,q}(M, N)$.

Specializing to the Bredon cochain complex and the differential graded module structure, we have

Theorem 2.19. *Let $\Gamma = G \times_K H$ be a group satisfying condition 1.1. Let M be a contravariant Bredon Functor taking values on the category of commutative rings. Assume that X is a proper G-CW complex and Y is a proper H-CW complex. Then, there is a spectral sequence with E_2 term given by*

$$\mathrm{Tor}_{H(C_K^*(M))}^{p,q}(H_{\mathcal{F}}^*(X, M), H_{\mathcal{F}'}^*(Y, M))$$

which converges to

$$\mathrm{Tor}_{C_K^*(M)}^{p,q}(\underline{C}_G^*(X, M), \underline{C}_H^*(Y, M)).$$

Notice that, as the differential graded algebra $C_K^*(M)$ is concentrated in degree 0 and has no differentials, the E_2 term can be identified with

$$\mathrm{Tor}_{C_K^*(M)}^{p,q}(H_{\mathcal{F}}^*(X, M), H_{\mathcal{F}'}^*(Y, M)).$$

Proposition 2.20. *Denote by $\mathcal{R}$ the Bredon functor given by the representation ring. Then, $C_G^*(X, \mathcal{R})$ is a $C_G^*(\mathcal{R})$-projective module.*

Proof. The cochain complex $C_G^n(X, \mathcal{R})$ in degree n is isomorphic to a module of the form

$$Hom(\bigoplus_\lambda \mathbb{Z}[e_\lambda], \mathcal{R}),$$

where e_λ is an orbit cell of the type $G/H_\lambda \times D^n$. Using the Yoneda lemma, this is isomorphic to $\mathrm{colim}_{H_\lambda} \mathcal{R}(G/H_\lambda)$, where the limit is taken with respect to G-maps $G/H \to G/K$. Since the representation ring is semisimple due to the Schur-Artin-Wedderburn theorem, this is a projective module over $C_G^*(\mathcal{R})$. $\square$

If conditions 2.9 are satisfied, $C_G^*(X, M)$ is a projective $C_K^*(M)$-module. Moreover if $H_G^*(X, M)$ is a projective $C_K^*(M)$-module, then by Lemma 7.3 in [McC01] the spectral sequence of Theorem 2.19 collapses at level 2 with

$$\mathrm{Tor}_{C_K^*(M)}^{p,q}(H_{\mathcal{F}}^*(X, M), H_{\mathcal{F}'}^*(Y, M)) \cong H_{\mathcal{F}}^p(X, M) \otimes_{C_K^*(M)} H_{\mathcal{F}'}^q(Y, M)),$$

and

$$\mathrm{Tor}_{C_K^*(M)}^{p,q}(C_G^*(X, M), C_H^*(Y, M)) \cong H_p(C_G^*(X, M)) \otimes_{C_K^*(M)} C_H^q(Y, M)).$$

Proposition 2.15, 2.16, and 2.17 yield

Theorem 2.21 (Bredon cohomology of pullbacks). *If conditions 2.9 are satisfied and $H_G^*(X, M)$ is a projective $C_K^*(M)$-module, there is an isomorphism of $C_K^*(M)$-modules*

$$H_{\mathcal{F} \times_K \mathcal{F}'}^*(X \times Y, M) \cong H_{\mathcal{F}}^*(X, M) \otimes_{C_K^*(M)} H_{\mathcal{F}'}^*(Y, M).$$

When we take rational coefficients, the spectral sequence constructed above collapses and we obtain a Künneth formula. Let $M^{\mathbb{Q}}(?)$ be the functor M with rational coefficients i.e. $M^{\mathbb{Q}}(G/H) = M(G/H) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Corollary 2.22 (Rationalized Bredon cohomology of pullbacks). *There is an isomorphism of $M^{\mathbb{Q}}$-modules*

$$H_{\mathcal{F} \times_K \mathcal{F}'}^*(X \times Y, M^{\mathbb{Q}}) \cong H_{\mathcal{F}}^*(X, M^{\mathbb{Q}}) \otimes_{M^{\mathbb{Q}}(K/K)} H_{\mathcal{F}'}^*(Y, M^{\mathbb{Q}}).$$

In order to apply Theorem 2.21 in the following section, we will need the following elementary lemma.

Lemma 2.23. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of projective R-modules and I be an ideal in R, then, the sequence

$$0 \to A/I \to B/I \to C/I \to 0$$

is exact.

Bredon cohomology of Crystallographic groups of arbitrary dimension with a given point group.

Remark 2.24. Let Γ be a group extension

$$1 \to \mathbb{Z}^n \to \Gamma \to K \to 1$$

given by the conjugation action of a representation $\rho : K \to Gl_n(\mathbb{Z})$ of a finite group K. Then,

- The space $\mathbb{R}^n$ with the induced action is a model for $\underline{E}\Gamma$. This is a consequence of Proposition 1.12, page 30 in [CK90].
- Let $1 \to \mathbb{Z}^n \to \Gamma \to K \to 1$ be a group extension coming from a representation of a finite group $\rho : K \to Gl_n(\mathbb{Z})$. Assume that $\mathbb{Z}^n$ with the action given by ρ has a K-invariant decomposition $\mathbb{Z}^n = A \oplus B$. Denote by G the semidirect product $A \rtimes K$ and by H the semidirect product $B \rtimes K$. Then, the group Γ is isomorphic to the pullback $G \times_K H$, as it can be readily seen from the following diagram,

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & G \\
\downarrow & & \downarrow \\
H & \longrightarrow & K
\end{array}
$$

  Here, the maps $\Gamma \to H$ and $\Gamma \to G$ are determined by the projections onto the invariant K-submodules $\mathbb{Z}^n \to A$ and $\mathbb{Z}^n[\rho] \to B$, which in turn induce group homomorphisms $\Gamma = \mathbb{Z}^n \rtimes K \to G = A \rtimes K$, $\Gamma = \mathbb{Z}^n \rtimes K \to G = B \rtimes K$ giving the relevant group homomorphisms out of Γ.

The spectral sequence constructed in Theorem 2.19 suggests a method to compute the Bredon cohomology groups $H_\Gamma^*(\underline{E}\Gamma, M)$:

- Decompose the representation ρ as direct sum $\rho = \oplus n_i \rho_i$ of indecomposable representations $\rho_i : K \to Gl_{n_i}(\mathbb{Z})$.
- Consider the group extensions

$$1 \to \mathbb{Z}^{n_i} \to \Gamma_i \to K \to 1$$

- Compute the (potentially easier) Bredon cohomology groups $H_{\Gamma_i}^*(\underline{E}\Gamma_i, M)$
- Feed the spectral sequence 2.19 with the cohomology groups.
- Establish the relevant differential graded module structures and obtain information about $H_\Gamma^*(\underline{E}\Gamma, M)$.

For finite groups K for which any prime p, the p-Sylow subgroup is of order less than p^3, there is a finite number of irreducible such representations ρ_i [HR62].

We will specialize in crystallographic groups with point group $\mathbb{Z}/4\mathbb{Z}$ for an specific example and carry out this program obtainting complete integral information in the next section.

3. Computations for the Vafa-Witten group Γ.

Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{Z}^{\oplus 6}$ induced from the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{C}^3$, given by

$$k(z_1, z_2, z_3) = (-z_1, iz_2, iz_3).$$

The associated semidirect product

$$1 \to \mathbb{Z}^6 \to \Gamma \to \mathbb{Z}/4\mathbb{Z} \to 1$$

will be called the Vafa-Witten Group and splits as a multiple pullback

(3.1) $(\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}).$

The first two semidirect products are taken with respect to the of $\mathbb{Z}/4$ on $\mathbb{Z}$ given by scalar multiplication with -1, and the two last ones are given by the action on $\mathbb{Z}^2$ given by complex multiplication by i.

First we will apply the Spectral sequence constructed in previous sections to compute the equivariant K-Theory and K-homology of the classifying space $\underline{E}\Gamma$. Using the universal coefficient Theorem 1.13 in [BV14] and the Atiyah-Hirzebruch spectral sequence, we obtain the equivariant K-homology groups relevant to the Baum-Connes conjecture.

Finally, we classify the virtually cyclic subgroups appearing in Γ, and using results on the algebraic K-theory in degrees lower than -1, we wil conclude the vanishing result.

Topological K-theory and K-homology. We begin with a recollection of the building blocks of the action, as well as their Bredon Cohomology groups.

$\mathbb{R}$ *with the action of* $\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Let $X = \mathbb{R}$ with the action of the group $G = \mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ where the semidirect product is taken with respect to the action given by multiplication with -1, $-1 : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}$. Notice that X is a model for $\underline{E}G$.

The space X has a G-CW-complex structure with two 0-cell orbits $\{0, 1/2\}$ both with isotropy groups isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and one 1-cell orbit $[0, 1/2]$ with isotropy group $\mathbb{Z}/2\mathbb{Z}$. The Bredon cellular complex takes the form

$$0 \to R(\mathbb{Z}/4\mathbb{Z}) \oplus R(\mathbb{Z}/4\mathbb{Z}) \to R(\mathbb{Z}/2\mathbb{Z}) \to 0.$$

Where $R(\mathbb{Z}/4\mathbb{Z})$ is the representation ring of the finite cyclic group of order 4, $(\mathbb{Z}/4\mathbb{Z})$. This is a polynomial algebra isomorphic to $\mathbb{Z}[\eta]/\eta^4-1$, where η is the generator.

The Bredon cohomology groups of X with respect to the family of finite subgroups $\mathcal{FIN}(\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$ (we denote by $\mathcal{F}$) with coefficients in representations can be easily calculated from it and they are concentrated in degree 0, with $H^0_{\mathcal{F}}(X, \mathcal{R}) = \mathbb{Z}^{\oplus 6}$. We calculate also some quotients of $H^0_{\mathcal{F}}(X, \mathcal{R})$ that we need in the following. From the calculations of the cohomology groups of X we know that we have an exact sequence of projective $R(\mathbb{Z}/4\mathbb{Z})$-modules

$$0 \to H^0_{\mathcal{F}}(X, \mathcal{R}) \to (R(\mathbb{Z}/4\mathbb{Z}))^2 \to R(\mathbb{Z}/2\mathbb{Z}) \to 0.$$

As all $R(\mathbb{Z}/4\mathbb{Z})$-modules in the above exact sequence are projective, hence we can apply Lemma 2.23 with the ideal $I = \langle \eta^2 - 1 \rangle$ contained in $R(\mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}[\eta]/\langle \eta^4 - 1 \rangle$, obtaining the exact sequence

$$0 \to H^0_{\mathcal{F}}(X, \mathcal{R})/I{\cdot}H^0_{\mathcal{F}}(X, \mathcal{R}) \to$$
$$(R(\mathbb{Z}/4\mathbb{Z}))^2/I \cdot (R(\mathbb{Z}/4\mathbb{Z})^2) \to R(\mathbb{Z}/2\mathbb{Z}) \to 0.$$

From the last exact sequence counting ranks we obtain

$$H^0_{\mathcal{F}}(X, \mathcal{R})/I \cdot H^0_{\mathcal{F}}(X, \mathcal{R}) \cong \mathbb{Z}^{\oplus 2}.$$

Finally, if we denote by J the ideal $\langle \eta - 1 \rangle$, as X/G is path-connected,

$$H^0_{\mathcal{F}}(X, \mathcal{R})/J \cdot H^0_{\mathcal{F}}(X, \mathcal{R}) \cong \mathbb{Z}.$$

$\mathbb{R}^2$ *with the action of* $\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}$. Let $Y = \mathbb{R}^2$ with the action of the group $H = \mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}$ where the semidirect product is taken with respect to the action $i : \mathbb{Z}/4\mathbb{Z} \to Gl_2(\mathbb{Z})$ given by multiplication by i. Note that Y is a model for $\underline{E}H$. Y is an H-CW-complex with three 0-cell orbits $(0,0)$, $(1/2, 0)$ and $(1/2, 1/2)$, two 1-cell orbits a_0 and a_1 and one 2-cell orbit T.

The Bredon cellular chain complex takes the form

$$0 \to R(\mathbb{Z}/4\mathbb{Z}) \oplus R(\mathbb{Z}/4\mathbb{Z}) \oplus R(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

Or in an equivalent way

$$0 \to R(\mathbb{Z}/4\mathbb{Z}) \oplus R(\mathbb{Z}/4\mathbb{Z}) \oplus R(\mathbb{Z}/4\mathbb{Z})/I \to$$
$$R(\mathbb{Z}/4\mathbb{Z})/J \oplus R(\mathbb{Z}/4\mathbb{Z})/J \to R(\mathbb{Z}/4\mathbb{Z})/J \to 0.$$

The Bredon cohomology groups of Y respect to the family of finite subgroups $\mathcal{FIN}(\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$ (we denote by $\mathcal{F}'$) with coefficients in representations are concentrated in degree 0 and 2, with $H^0_{\mathcal{F}'}(Y, \mathcal{R}) = \mathbb{Z}^{\oplus 8}$ and $H^2_{\mathcal{F}'}(Y, \mathcal{R}) = \mathbb{Z}$. Note that it is compatible with the results in

[Lüc05] in section 2, since the group $\mathbb{Z}^2 \rtimes_i \mathbb{Z}/4\mathbb{Z}$ satisfies hypotheses M and NM (Lemma 2.2 in page 1648).

From the calculations of the cohomology groups of Y we know that there is an exact sequence of projective $R(\mathbb{Z}/4\mathbb{Z})$-modules

$$0 \to H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to (R(\mathbb{Z}/4\mathbb{Z}))^2 \oplus R(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}^2 \to 0.$$

As we do for X, we calculate also the quotients $H^*_{\mathcal{F}'}(Y, \mathcal{R})/I \cdot H^*_{\mathcal{F}'}(Y, \mathcal{R})$. Taking the quotient by I from the last exact sequence we obtain

$$0 \to H^0_{\mathcal{F}'}(Y, \mathcal{R})/I \cdot H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to$$
$$(R(\mathbb{Z}/4\mathbb{Z}))^2/I \cdot (R(\mathbb{Z}/4\mathbb{Z})^2) \oplus R(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}^2 \to 0.$$

And counting ranks we have

$$H^0_{\mathcal{F}'}(Y, \mathcal{R})/I \cdot H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}^{\oplus 2}.$$

As Y/H is path-connected

$$H^0_{\mathcal{F}'}(Y, \mathcal{R})/J \cdot H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}.$$

Finally, since $\eta^2 - 1 = 0$ and $\eta - 1 = 0$ in $H^2_H(Y, \mathcal{R})$ we obtain

$$H^2_{\mathcal{F}'}(Y, \mathcal{R})/I \cdot H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong H^2_{\mathcal{F}'}(Y, \mathcal{R})/J \cdot H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}.$$

$\mathbb{R}^6$ *with the action of* $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$. We proceed to calculate the Bredon cohomology groups of the space $X^2 = X \times X$ with the action of the group $G \times_{\mathbb{Z}/4\mathbb{Z}} G$. Theorem 2.21 gives us an isomorphism

$$H^0_{\mathcal{F} \times_{\mathbb{Z}/4\mathbb{Z}} \mathcal{F}}(X^2, \mathcal{R}) \cong H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}).$$

From the calculations of the cohomology groups of X, we know that we have an exact sequence of projective $R(\mathbb{Z}/4\mathbb{Z})$-modules

$$(3.2) \quad 0 \quad \to \quad H^0_{\mathcal{F}}(X, \mathcal{R}) \quad \to \quad (R(\mathbb{Z}/4\mathbb{Z}))^2 \quad \to \quad R(\mathbb{Z}/2\mathbb{Z}) \quad \to \quad 0,$$

as $R(\mathbb{Z}/2) \cong R(\mathbb{Z}/4)/I$ by tensoring this sequence with $H^0_{\mathcal{F}}(X, \mathcal{R})$ we obtain the exact sequence

$$(3.3) \quad H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \to (H^0_{\mathcal{F}}(X, \mathcal{R}))^2 \to$$
$$H^0_{\mathcal{F}}(X, \mathcal{R})/I \cdot H^0_{\mathcal{F}}(X, \mathcal{R}) \to 0.$$

The rank of $(H^0_{\mathcal{F}}(X, \mathcal{R}))^2$ is 12, as can be seen from counting ranks in sequence 3.2

and the rank of $H^0_{\mathcal{F}}(X, \mathcal{R})/I \cdot H^0_{\mathcal{F}}(X, \mathcal{R})$ is 2 then

$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \cong \mathbb{Z}^{\oplus 10}.$$

For simplicity we denote the group $H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R})$ by A. Dividing the sequence 3.3 by the ideal I, respectively, by J we obtain $A/I \cdot A \cong \mathbb{Z}^{\oplus 2}$ and $A/J \cdot A \cong \mathbb{Z}$.

We continue calculating

$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}).$$

From the calculations of the cohomology groups of Y we know that there is an exact sequence of projective $R(\mathbb{Z}/4\mathbb{Z})$-modules

$$(3.4) \qquad 0 \to H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to (R(\mathbb{Z}/4\mathbb{Z}))^2 \oplus R(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}^2 \to 0.$$

by tensoring this sequence with $A = H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R})$, we obtain

$$(3.5) \quad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to A^2 \oplus (A/I \cdot A) \to (A/J \cdot A)^2 \to 0,$$

then

$$(3.6) \qquad\qquad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}^{\oplus 20}.$$

If we divide 3.5 by I, respectively by J, we obtain

$$(3.7) \qquad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})/I \cdot (A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})) \cong \mathbb{Z}^{\oplus 6}$$

and

$$(3.8) \qquad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})/J \cdot (A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})) \cong \mathbb{Z}.$$

On the other hand as $H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong R(\mathbb{Z}/4\mathbb{Z})/J$ we have

$$(3.9) \ \ H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong \ A/J \cdot A \cong \mathbb{Z}.$$

We continue calculating

$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}).$$

Tensoring the sequence 3.5 with $H^0_{\mathcal{F}'}(Y, \mathcal{R})$ we obtain an exact sequence

$$(3.10) \quad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to A^2 \oplus (A/I \cdot A) \to (A/J \cdot A)^2 \to 0.$$

If we tensor this sequence by $H^0_{\mathcal{F}'}(Y, \mathcal{R})$ we obtain

$$A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})^{\otimes 2}$$
$$\to A^2 \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \oplus (A/I \cdot A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})$$
$$\to (A/J \cdot A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R})^2 \to 0.$$

On the other hand by 3.6 we know $A^2 \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}^{\oplus 40}$, by 3.7 we know $(A/I \cdot A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}^{\oplus 6}$, and by 3.8 $(A/I \times A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}$. Then the exact sequence 3.10 is isomorphic to

$$A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \to \mathbb{Z}^{\oplus 46} \to \mathbb{Z}^{\oplus 2} \to 0,$$

then we have

$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \cong \mathbb{Z}^{\oplus 44}.$$

Now, Theorem 2.21 implies

$$(3.11) \qquad\qquad H^0_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong \mathbb{Z}^{\oplus 44}$$

On the other hand, Theorem 2.21 gives us an isomorphism

$$H^2_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong$$

$$(H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}))^2.$$

Tensoring the sequence 3.10 by $H^2_{\mathcal{F}'}(Y, \mathcal{R})$, we obtain an exact sequence

$$(3.12) \quad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \to$$
$$A^2 \oplus (A/I \cdot A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \to$$
$$(A/J \cdot A)^2 \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \to 0.$$

But by 3.9 we have that

$$A^2 \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong (A/J \cdot A)^2 \cong \mathbb{Z}^{\oplus 2}$$

and

$$(A/I \cdot A) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong ((A/(J \cdot A)/(I \cdot A)) \cong A/J \cdot A \cong \mathbb{Z}.$$

Then the sequence 3.12 is isomorphic to

$$(3.13) \quad A \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \to$$
$$Z^{\oplus 2} \oplus \mathbb{Z} \to \mathbb{Z}^{\oplus 2} \to 0.$$

And we obtain

$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong$$
$$\cong A/J \cdot A \cong \mathbb{Z}.$$

Then,

$$(3.14) \qquad\qquad H^2_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Finally, Theorem 2.21 gives us an isomorphism

$$(3.15) \quad H^4_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong$$
$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}).$$

Tensoring 3.9 by $H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong R(\mathbb{Z}/4\mathbb{Z})/J$ we obtain

(3.16)
$$H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^0_{\mathcal{F}}(X, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \otimes_{R(\mathbb{Z}/4\mathbb{Z})} H^2_{\mathcal{F}'}(Y, \mathcal{R}) \cong$$
$$A/J \cdot A \cong \mathbb{Z}.$$

We summarize these results in

Theorem 3.17. *Let Γ be the Vafa-Witten Group $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$ 1.6 acting on $\mathbb{R}^6$ as is described in Section 1. The Bredon cohomology groups of $\mathbb{R}^6$ are given as follows:*

- $H^0_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong \mathbb{Z}^{\oplus 44}$,
- $H^2_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$,
- $H^4_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) \cong \mathbb{Z}$, *and*
- $H^k_{\mathcal{FIN}(\Gamma)}(\mathbb{R}^6, \mathcal{R}) = 0$, *for $k \neq 0, 2, 4$.*

Proof. Recall the multiple pullback structure

$$(\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}).$$

The result follows from 3.11, 3.14 and 3.15. □

As the Bredon cohomology groups are concentrated in even degrees and all differentials in the Atiyah-Hirzebruch spectral sequence are defined from even degrees to odd degrees or from odd degrees to even degrees one has that all differentials are zero then it spectral sequence collapses at the E_2 term, and we get

Theorem 3.18 (Equivariant K theory of $\underline{E}\Gamma$). *Let Γ denote the group $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$ acting on the model for $\underline{E}\Gamma$ given by $\mathbb{R}^6$ as it is described in Section 1. The equivariant K-theory groups satisfy*

- $K^0_\Gamma(\underline{E}\Gamma) \cong \mathbb{Z}^{\oplus 47}$ *and*
- $K^1_\Gamma(\underline{E}\Gamma) = 0$

Recall the universal coefficient theorem for Bredon cohomology with coefficients in complex representations, Theorem 1.13 in [BV14], which we quote here for the sake of completeness:

Theorem (Universal Coefficient Theorem for Bredon Cohomology). Let X be a proper, finite G-CW complex. Let $M^?$ and $M_?$ be the complex representation ring with contravariant, respectively covariant functoriality. Then, there exists a short exact sequence of abelian groups involving Bredon Homology with coefficients in $M^?$ and Bredon homology with coefficients in $M_?$

$$0 \to Ext_{\mathbb{Z}}(H^{\mathcal{F}}_{n-1}(X, M_?), \mathbb{Z}) \to H^n_{\mathcal{F}}(X, M^?) \to Hom_{\mathbb{Z}}(H^{\mathcal{F}}_n(X, M_?), \mathbb{Z}) \to 0$$

We conclude that the Bredon homology groups above are isomorphic on each degree to the equivariant Bredon cohomology groups.

Now consider the Atiyah Hirzebruch spectral sequence for computing Equivariant K-homology groups. The E_2 term consists of the Bredon cohomology groups $H_*^{\mathcal{F}}(X, M_?) = H_{\mathcal{F}}^*(X, M^?)$, which are concentrated on even degrees. Since all differentials in the (homological!) Atiyah-Hirzebruch spectral sequence are zero, the edge homomorphism identifies the zeroth equivariant K-homology group with the sum $\bigoplus_{N=0,1,2} H_{2n}^{\mathcal{F}}(X, M_?)$ and the first equivariant K-homology group with 0.

On the other hand, the Baum-Connes assembly map $K_*^{\Gamma}(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma))$ is an isomorphism due to results of Higson-Kasparov [HK01]. Putting all this together, we obtain the following computation of the reduced C^*-algebra of the group Γ.

Corollary 3.19 (Equivariant K-Homology of $\underline{E}\Gamma$). *Let Γ denote the group $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$ acting on the model for $\underline{E}\Gamma$ given by $\mathbb{R}^6$ as is described in Section 1.*

- $K_0^{\Gamma}(\underline{E}\Gamma) \cong K_0^*(C_r^*(\Gamma)) \cong \mathbb{Z}^{\oplus 47}$ *and*
- $K_1^{\Gamma}(\underline{E}\Gamma) \cong K_1^*(C_r^*(\Gamma)) = 0$

Negative Algebraic K-Theory. The success of the Eilenberg-Moore method in the previous computations of Bredon cohomology with respect to the family of finite subgroups relies on the structure lemma 2.10. For the family of virtually cyclic subgroups, there is no such decomposition. The following result, however, identifies some restrictions for a subgroup in Γ in order to be virtually cyclic.

Proposition 3.20. *Let Γ be a group obtained as a pullback of the type*

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ p_2\ } & G \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\
H & \xrightarrow{\ \pi_2\ } & K
\end{array} \quad ,
$$

where p_1 and p_2 are surjective maps. Given a virtually cyclic subgroup $V \leq \Gamma$, the groups $p_1(V)$, $p_2(V)$ are virtually cyclic.

We define the following family of subgroups of Γ

$$\mathcal{VC}(G) \times_K \mathcal{VC}(H) =$$

$$\{V_1 \times_{\pi_1(V_1)} V_2 \mid V_1 \in \mathcal{VC}(G) \text{ and } V_2 \in \mathcal{VC}(H)\}$$

The family $\mathcal{VC}(G) \times_K \mathcal{VC}(H)$ does not agree with the family of virtually cyclic subgroups of Γ. However, every virtually cyclic subgroup in Γ is contained in an element of the family.

Thus, a strategy to the classification of virtually cyclic subgroups of the Group Γ consists of using the iterated pullback decomposition 3, the several projections to the components, as well as classification results for the family of virtually cyclic subgroups of the components, take the pullback family and verify whether the groups appearing there are virtually cyclic.

Proposition 3.21. *The virtually cyclic subgroups of the Vafa-Witten Group are, up to isomorphism, as follows:*

- *Finite groups 0, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$,*
- *the infinite cyclic group $\mathbb{Z}$,*
- *$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$, and*
- *$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$, D_∞, $D_\infty \times \mathbb{Z}/2\mathbb{Z}$, $D_\infty \times \mathbb{Z}/4\mathbb{Z}$.*

Proof. The finite groups are readily realizable. We obtain the groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$ inside the product $\mathbb{Z}^4 \rtimes_{-1} \mathbb{Z}/4\mathbb{Z}$

On the other hand, the infinite virtually cyclic subgroups of the group $\mathbb{Z}^2 \rtimes_i \mathbb{Z}/4\mathbb{Z}$ have been classified in Lemma 3.7, page 1656 of [Lüc05], which are either cyclic or D_∞.

From the product family for the pullback

$$\mathbb{Z}^2 \rtimes_i \mathbb{Z}/4\mathbb{Z} \times_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}^4 \rtimes_{-1} \mathbb{Z}/4\mathbb{Z},$$

and from the group

$$\mathbb{Z} \rtimes_{-1} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$$

we obtain the virtually cyclic subgroups

$$D_\infty, \quad D_\infty \times \mathbb{Z}/2\mathbb{Z}, \quad D_\infty \times \mathbb{Z}/4\mathbb{Z},$$

and the group $\mathbb{Z} \rtimes_{-1} \mathbb{Z}/4\mathbb{Z}$, which is isomorphic to the amalgam

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}.$$

$\square$

The validity of the Farrell-Jones isomorphism for Γ is a well established fact, see for example [LS00] this means that the *assembly map* in 1.4 is an isomorphism. Thus, the algebraic K-theory groups of $R[\Gamma]$ are isomorphic to the equivariant homology groups

$$\mathbb{H}_i^\Gamma(E_{\mathcal{VC}}(\Gamma), \mathbb{K}^{-\infty}(R)) \text{ for all } i \in \mathbb{Z}.$$

In order to compute these groups there is an Atiyah-Hirzebruch, [DL98] spectral sequence converging to them with second page given by

$$E_{p,q}^2 \cong H_p(B_{\mathcal{VC}}; \{\mathcal{K}_q(R[V])\}),$$

where the above are homology groups with local coefficients in the algebraic K-theory groups of $R[V]$ and V in the family of virtually

cyclic subgroups of Γ. Let us analyze the coefficients in the above homology groups $H_p(B_{\mathcal{VC}}; \{\mathcal{K}_q(R[V])\})$ for $p + q \leq -1$. First observe that from the work of Carter [Carb, Theorem 1], we have that the group $K_{-1}(\mathbb{Z}[G])$ vanishes for the finite groups of our list above and by [Cara, Theorem 3], $K_{-i}(\mathbb{Z}[G]) = 0$ for all $i > 1$ and all finite groups G. By the work of T. Farrell and L. Jones [FJ, Theorem 2.1 (b)] and the generalizations in [JP03], $K_{-1}(R[V])$ vanishes for V infinite virtually cyclic subgroup of our list above and by [FJ, Theorem 2.1 (a)] and the generalizations in [JP03], $K_{-i}(R[V])$ also vanish for $i \geq 2$ and for all virtually cyclic groups V. Hence the above spectral sequences consists of zero terms in the range $p + q \leq -1$. Hence we have:

Theorem 3.22. *Let R be a ring of algebraic integers. Let $i \leq -1$. Then, the algebraic K-Theory groups $K_i(R\Gamma)$ vanish.*

References

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In C^*-*algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[Bou97] Serge Bouc. *Green functors and G-sets*, volume 1671 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[Bre67] Glen E. Bredon. *Equivariant cohomology theories*. Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin, 1967.

[BV14] Noé Bárcenas and Mario Velásquez. Twisted equivariant K-theory and K-homology of $\mathrm{Sl}_3\mathbb{Z}$. *Algebr. Geom. Topol.*, 14(2):823–852, 2014.

[Cara] David W. Carter. Localization in lower algebraic k-theory. *Comm. Algebra*, 8(7):603–622.

[Carb] David W. Carter. Lower k-theory of finite groups. *Comm. Algebra*, 8(20):1927–1937.

[CK90] Frank Connolly and Tadeusz Koźniewski. Rigidity and crystallographic groups. I. *Invent. Math.*, 99(1):25–48, 1990.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[DL13] James F. Davis and Wolfgang Lück. The topological K-theory of certain crystallographic groups. *J. Noncommut. Geom.*, 7(2):373–431, 2013.

[FJ] Francis Tom Farrell and Lowell Jones. The lower algebraic k-theory of virtually infinite cyclic groups. *K-Theory*, 9(1):13–30.

[FO12] Daniel Farley and Ivonne Ortiz. The lower algebraic k-theory of split three dimensional chrystallographic groups. Preprint, ArXiv:1211.2024 [KT], 2012.

[HK01] Nigel Higson and Gennadi Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001.

[HR62]   Alexander Heller and Irving Reiner. On groups with finitely many inde-
         composable integral representations. *Bull. Amer. Math. Soc.*, 68:210–212,
         1962.

[JP03]   Daniel Juan-Pineda. On the lower algebraic K-theory of virtually cyclic
         groups. In *High-dimensional manifold topology*, pages 301–314. World Sci.
         Publ., River Edge, NJ, 2003.

[LL12]   Martin Langer and Wolfgang Lück. Topological K-theory of the group
         C^*-algebra of a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ for a free conjugation
         action. *J. Topol. Anal.*, 4(2):121–172, 2012.

[LS00]   Wolfgang Lück and Roland Stamm. Computations of K- and L-theory of
         cocompact planar groups. *K-Theory*, 21(3):249–292, 2000.

[Lüc05] Wolfgang Lück. K- and L-theory of the semi-direct product of the dis-
         crete 3-dimensional Heisenberg group by $\mathbb{Z}/4$. *Geom. Topol.*, 9:1639–1676
         (electronic), 2005.

[McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cam-
         bridge Studies in Advanced Mathematics*. Cambridge University Press,
         Cambridge, second edition, 2001.

[Tsa95] Georgios Tsapogas. On the K-theory of crystallographic groups. *Trans.
         Amer. Math. Soc.*, 347(8):2781–2794, 1995.

[VW95]   Cumrun Vafa and Edward Witten. On orbifolds with discrete torsion. *J.
         Geom. Phys.*, 15(3):189–214, 1995.

Centro de Ciencias Matemáticas., UNAM Campus Morelia, Ap.Postal
61-3 Xangari, Morelia, Michoacán. MÉXICO 58089
   E-mail address: barcenas@matmor.unam.mx
   URL: http://www.matmor.unam.mx/~barcenas

Centro de Ciencias Matemáticas., UNAM Campus Morelia, Ap.Postal
61-3 Xangari, Morelia, Michoacán. MÉXICO 58089
   E-mail address: daniel@matmor.unam.mx

Centro de Ciencias Matemáticas., UNAM Campus Morelia, Ap.Postal
61-3 Xangari, Morelia, Michoacán. MÉXICO 58089

Departamento de Matemáticas., Pontificia Universidad Javeriana,
Cra. 7 No. 43-82 - Edificio Carlos Ortz 5to piso, Bogotá D.C, Colom-
bia
   E-mail address: mavelasquezm@gmail.com
   URL: https://sites.google.com/site/mavelasquezm/