SEGAL'S SPECTRAL SEQUENCE IN TWISTED EQUIVARIANT K-THEORY FOR PROPER AND DISCRETE ACTIONS

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Abstract. We use a spectral sequence developed by Graeme Segal in order to understand the twisted G-equivariant K-theory for proper and discrete actions. We show that the second page of this spectral sequence is isomorphic to a version of Bredon cohomology with local coefficients in twisted representations. We furthermore explain some phenomena concerning the third differential of the spectral sequence, and we recover known results when the twisting comes from finite order elements in discrete torsion.

Introduction

One of the tools for calculating generalized cohomology groups is the Atiyah-Hirzebruch spectral sequence which was originally developed in [4] in order to study K-theory. Many generalizations of this spectral sequence have been developed for studying cohomology theories in the equivariant context and we will pay specific attention to the spectral sequence developed by Segal in [26].

Twisted equivariant K-theory was defined by Atiyah and Segal in [2] using bundles of Fredholm operators and was extended to the context of proper actions by M. Joachim and the first three authors in [6]. Due to the relation of the Verlinde Algebra to the twisted equivariant K-theory of a compact Lie group acting on itself by conjugation stablished by Freed, Hopkins and Teleman in [16], computational methods have become necessary in order to calculate the twisted equivariant K-theory groups. Specialized to the case of the conjugation action, the Künneth Spectral sequence [9], and the Rothenberg-Steenrod spectral sequence for twisted equivariant K-homology [11] have been successfully used to determine the twisted equivariant K-theory groups. Theoretical tools like the completion theorem of Lahtinen [19], together with the previously described methods, define the group of ideas used for the computation in [18] of the twisted K-theory of the loop space of the classifying space of a simply connected and simple compact lie group.

Besides from notable specific examples explained in [15], and the case when the twisting comes from discrete torsion [12] (where a method is used to reduce the construction of the spectral sequence to an untwisted version as done in [10]), a systematic study of a spectral sequence for computing twisted equivariant K-theory under the presence of a generic twist has not been carried out. This is the main objective of this work.

In [6] the twisted equivariant K-theory for a proper G-ANR X was defined given a Stable equivariant Projective Unitary bundle; these bundles were shown in [6] to be classified by elements of the degree three Borel cohomology group $H^3(X\times_G EG;\mathbb{Z})$. In this note we use the explicit construction of the universal stable equivariant projective unitary bundle done in [6] in order to determine the first two pages of Segal's spectral sequence converging to the twisted equivariant K-theory groups.

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For this purpose we develop a twisted version of Bredon cohomology, cohomology which turns out to determine the E_2-page of Segal's spectral sequence once it is applied to an equivariantly contractible cover.

The construction of the spectral sequence extends and generalizes previous work of C. Dwyer [12], who only treated the twistings which are classified by cohomology classes of finite order which lie in the image of the canonical map $H^3(BG,\mathbb{Z}) \to H^3(X\times_G EG;\mathbb{Z})$; these twistings take the name of discrete torsion twistings.

The main result of this note, which is Theorem 4.7, relies on the construction and the properties of the universal stable equivariant projective unitary bundle carried out in [6]. Since this work can be seen as a continuation of what has been done in [6], we will use the notation, the definitions and the results of that paper. We will not reproduce any proof that already appears in [6], instead we will give appropriate references whenever a definition or a result of [6] is used.

We emphasize that the topological issues that may appear when working with the Projective Unitary Group have all been resolved in [22, Section 15] when it is endowed with the norm topology. We therefore assume in this work that we are working with the norm topology when discussing topological properties of operator spaces.

This note is organized as follows. In Section 1 a version of Bredon cohomology associated to an equivariant cover of a space is constructed. In Section 2 the basics of Transformation Groups and Parametrized Homotopy Theory needed for the construction are quickly reviewed. This is used to construct a version of Bredon cohomology with local coefficients. In Section 3 the construction of twisted equivariant K-theory for proper and discrete actions given in [6] is reviewed. In Section 4 , the Bredon cohomology with local coefficients in twisted representations is shown to be isomorphic to the second page of a spectral sequence converging to twisted equivariant K-theory. Some phenomena concerning the third differential of this spectral sequence is also analyzed. In Section 5 some simple examples are given including the case of discrete torsion which was developed by Dwyer in [12].

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Contents

1. Bredon cohomology associated to a cover

We introduce first the formalism of modules and spaces over a category (see [10] for details).

Definition 1.1. Let $\mathcal{C}$ be a small category. A contravariant $\mathcal{C}$-space is a contravariant functor $\mathcal{C} \longrightarrow$ SPACES to the category of compactly generated spaces.

Definition 1.2. Let X and Y be $\mathcal{C}$-spaces of the same variance. Their mapping space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the space of natural transformations between the functors X and Y, endowed with the subspace topology of the product of the spaces of pointed maps $\Pi_{c\in \mathrm{Obj}(\mathcal{C})}\mathrm{Map}(X(c), Y(c))$, where $\mathrm{Map}(X(c), Y(c))$ has the compact-open topology for any $c \in \mathrm{Obj}(\mathcal{C})$.

Let I be the constant functor with value the interval $[0, 1]$. A $\mathcal{C}$-homotopy between two $\mathcal{C}$-maps $f_0, f_1 : X \to Y$ is a natural transformation $H : X \times I \to Y$ such that the composition $H \circ i_k$ with the inclusions $i_k : X \to X \times I$ for $k = 0, 1$ are equal to f_k. The set of $\mathcal{C}$-homotopy classes of maps between two spaces will be denoted by $[X, Y]_{\mathcal{C}}$

Definition 1.3. Let X be a contravariant, pointed $\mathcal{C}$-space over $\mathcal{C}$ and let Y be a covariant $\mathcal{C}$-space over $\mathcal{C}$. Their tensor product $X \otimes_{\mathcal{C}} Y$ is the space defined by

$$\coprod_{c\in \mathrm{Obj}(\mathcal{C})} X(c) \times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(X(\phi)(x), y) \sim (x, Y(\phi)(y))$ for all morphisms $\phi : c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$.

Definition 1.4. Let $\mathcal{C}$ be a small category. A free $\mathcal{C}$-CW complex is a contravariant $\mathcal{C}$-space together with a filtration

$$X_0 \subset X_1 \subset \ldots = X$$

such that $X = \mathrm{colim}_n X_n$ and each X_n is obtained from the X_{n-1} by a pushout consisting of maps of $\mathcal{C}$-spaces of the form

$$\begin{array}{ccc} \coprod_{i\in I_n} \mathrm{Mor}_{\mathcal{C}}(?, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} \mathrm{Mor}_{\mathcal{C}}(?, c_i) \times D^n & \longrightarrow & X_n \end{array}$$

where the c_i's are objects in $\mathcal{C}$ and the spaces $\mathrm{Mor}_{\mathcal{C}}(?, c_i)$ carry the discrete topology.

Definition 1.5. Let $\mathcal{C}$ be a small category and R be a commutative ring. A contravariant $R\mathcal{C}$-module is a contravariant functor from $\mathcal{C}$ to the category of R-modules. A contravariant $R\mathcal{C}$-chain complex is a functor from $\mathcal{C}$ to the category of R-chain complexes.

We write $\mathcal{C}$-module for a $\mathbb{Z}\mathcal{C}$-module.

An R$\mathcal{C}$-module F is free if it is isomorphic to an $R\mathcal{C}$-module of the form

$$F(?) = \bigoplus_{i\in I} R[\mathrm{Mor}_{\mathcal{C}}(?, c_i)]$$

for some index set I and objects $c_i \in \mathcal{C}$.

Given two $R\mathcal{C}$-modules A, B of the same variance, the R-module

$$\mathrm{Hom}_{R\mathcal{C}}(A, B)$$

is the module of natural transformations of functors from $\mathcal{C}$ to R-modules.

Definition 1.6. Given a category $\mathcal{C}$ and an object c in $\mathcal{C}$, the category over c, $\mathcal{C}\downarrow c$ is the category where the objects are morphisms $\varphi : c_0 \to c$ and a morphism between $\varphi_0 : c_0 \to c$ and $\varphi_1 : c_1 \to c$ is a morphism $\psi : c_0 \to c_1$ in $\mathcal{C}$ such that $\varphi_0 = \varphi_1 \circ \psi$.

Dually, the category under an object c, denoted $c \downarrow \mathcal{C}$ is the category where the objects are morphisms $\varphi : c \to c_0$ and a morphism between $\varphi_0 : c \to c_0$ and $\varphi : c \to c_1$ is a morphism in $\mathcal{C}$, $\psi : c_0 \to c_1$ such that $\varphi_1 = \psi \circ \varphi_0$.

Fix an object c, and denote by B$\mathcal{C}\downarrow c$ the classifying space of the category over c and by B$c\downarrow\mathcal{C}$ the classifying space of the category under c.

The contravariant, free $\mathbb{Z}\mathcal{C}$-chain complex $C_*^{\mathbb{Z}}(\mathcal{C})$ is defined on every object as the cellular $\mathbb{Z}$-chain complex of B$\mathcal{C}\downarrow c$.

Definition 1.7. Let M be a contravariant $\mathcal{C}$-module. The cohomology of $\mathcal{C}$ with coefficients in M, $H^n(\mathcal{C}, M)$ is defined to be the cohomology groups of the cochain complex of natural transformations between the $\mathcal{C}$-modules $C_*^{\mathbb{Z}}(\mathcal{C})$ and M,

$$H^n(\mathcal{C}, M) := H^n\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(C_*^{\mathbb{Z}}(\mathcal{C}), M).$$

We now specialize to the categories relevant to twisted K-theory and Bredon cohomology with local coefficient systems.

Let G be a group and X be a proper G-ANR. Let $\mathcal{U} = \{U_i\}_{i\in\Sigma}$ be a countable covering of X by open, G-invariant sets $X = \bigcup_{i\in\Sigma} U_i$. Given a subset $\sigma \subset \Sigma$, define $U_\sigma = \bigcap_{i\in\sigma} U_i$. We will assume that for all σ, the open set U_σ is G-homotopy equivalent to an orbit G/H_σ for a finite group $H_\sigma \subset G$. The existence of such a cover, sometimes known as *contractible slice cover*, is guaranteed for proper G-ANR's by an appropriate version of the slice Theorem (see [1]).

The category associated to $\mathcal{U}$, denoted by $\mathcal{N}_G\mathcal{U}$, has for objects $\bigsqcup_{\sigma\subset\Sigma} U_\sigma$ and for morphism the inclusions $U_\tau \to U_\sigma$ whenever there is an inclusion of sets $\sigma \subset \tau$.

A coefficient system with values on R-modules is a contravariant functor $\mathcal{N}_G\mathcal{U} \to R-\mathrm{Mod}$.

Definition 1.8. Let X be a proper G-space with a contractible slice cover $\mathcal{U}$, and let M be a coefficient system. Define the Bredon cohomology groups with respect to $\mathcal{U}$ as the cohomology groups of the category $\mathcal{N}_G\mathcal{U}$ with coefficients in M,

$$H_G^n(X,\mathcal{U}; M) := H^n(\mathcal{N}_G\mathcal{U}, M).$$

Whenever we have a refinement $\mathcal{V} \to \mathcal{U}$ of the G-invariant cover, we get a group homomorphism

$$H_G^n(X,\mathcal{U}; M) \to H_G^n(X, \mathcal{V}; M')$$

where the functor M' is obtained by the composition of the functor $\mathcal{N}_G\mathcal{V} \to \mathcal{N}_G\mathcal{U}$ with the functor M.

Remark 1.9. For more general spaces than a proper G-ANR, a version of Čech cohomology might be constructed by taking the inverse limit over open covers $\mathcal{U}$ of the space X:

$$\check{H}^n_G(X; M) := \lim_{\mathcal{U}} H^n(\mathcal{N}_G\mathcal{U}, M).$$

Details are provided in [24]. Other approaches to Čech versions of Bredon cohomology include [17].

2. Parametrized equivariant topology

The orbit category was introduced by Bredon for the definition of cohomological invariants of spaces with an action. We introduce now a formalism for taking into account also twisting data.

Definition 2.1. Let G be a discrete group. The orbit category $\mathcal{O}^P_G$, with respect to the family of finite subgroups, has as objects

$$\mathrm{Obj}(\mathcal{O}^P_G) = \{G/H \mid H \text{ is a finite subgroup of } G\}$$

and as morphisms G-maps

$$\mathrm{Mor}_{\mathcal{O}^P_G}(G/H, G/K) = \mathrm{Map}(G/H, G/K)^G.$$

Given a G-space X, the *fixed point set system* of X, denoted by ΦX, is the $\mathcal{O}^P_G$-space defined by:

$$\Phi X(G/H) := \mathrm{Map}(G/H, X)^G = X^H$$

and if $\theta : G/H \to G/K$ corresponds to $gK \in (G/K)^H$ then

$$\Phi X(\theta)(x) := gx \in X^H$$

whenever $x \in X^K$. The functor Φ becomes a functor from the category of proper G-spaces to the category of $\mathcal{O}^P_G$-spaces.

If $\mathcal{X}$ is a contravariant functor from $\mathcal{O}^P_G$ to spaces, and ∇ is the covariant functor from $\mathcal{O}^P_G$ to spaces which assigns to an orbit G/H the homogenous space G/H, one can define the G-space

$$\hat{\mathcal{X}} := \bigsqcup_{c \in \mathrm{Obj}(\mathcal{O}^P_G)} \mathcal{X}(c) \times \nabla(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(\mathcal{X}(\phi)(x), y) \sim (x, \nabla(\phi)(y))$ for all morphisms $\phi : c \to d$ in $\mathcal{O}^P_G$ and points $x \in \mathcal{X}(d)$ and $y \in \nabla(c)$, and the G-action comes from the left translation action on G/H.

For G a discrete group [10, lemma 7.2] the functors Φ and $\_ \times_{\mathcal{O}^P_G} \nabla$ are adjoint, i.e. for a $\mathcal{O}^P_G$ space $\mathcal{X}$ and a G-space Y there is a natural homeomorphism

$$\mathrm{Map}(\mathcal{X} \times_{\mathcal{O}^P_G} \nabla, Y)^G \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}^P_G}(\mathcal{X}, \Phi Y),$$

and moreover, the adjoint of the identity map on ΦY under the above adjunction, is a natural G-homeomorphism

$$(\Phi Y) \times_{\mathcal{O}^P_G} \nabla \xrightarrow{\cong} Y.$$

A model for the homotopical version of the previous construction is defined as follows. Consider the topological category $(\mathcal{X}, \nabla)$ whose objects are

$$\mathrm{Obj}((\mathcal{X}, \nabla)) = \bigsqcup_{c \in \mathrm{Obj}(\mathcal{O}^P_G)} \mathcal{X}(c) \times \nabla(c)$$

and whose morphisms consist of all triples (x, ϕ, y) where $\phi : c \to d$ is a morphism in $\mathcal{O}_G^P$ and $x \in \mathcal{X}(d)$ and $y \in \nabla(c)$, with source$(x, \phi, y) = (\mathcal{X}(\phi)(x), y)$ and target$(x, \phi, y) = (x, \nabla(\phi)(y))$. Define the space $\hat{\mathcal{X}}^h$ as the geometric realization of the category $(\mathcal{X}, \nabla)$. The space $\hat{\mathcal{X}}^h$ is provided with a map $\hat{\mathcal{X}}^h \to \hat{\mathcal{X}}$ which is a model for the map from the homotopy colimit to the colimit. This map is a G-homotopy equivalence if $\mathcal{X}$ is a free $\mathcal{O}_G^P$-complex.

We recall now results on the homotopy theory of spaces with an action of a group G and $\mathcal{O}_G^P$-spaces.

Definition 2.2. Let G be a discrete group. Given a family $\mathcal{F}$ of subgroups of G, which is closed under conjugation and taking subgroups.

- A map $f\colon X \to Y$ of G-spaces is called an $\mathcal{F}$-equivalence if for every finite subgroup $H \leq G$, the map $f^H\colon X^H \to Y^H$ is a weak equivalence of topological spaces.
- A map $f\colon X \to Y$ of G-spaces is called an $\mathcal{F}$-fibration if for every finite subgroup $H \leq G$, the map $f^H\colon X^H \to Y^H$ is a Serre fibration of topological spaces.
- A map $f\colon X \to Y$ of G-spaces is called an $\mathcal{F}$-cofibration if it has the left lifting property with respect to any map which is $\mathcal{F}$-equivalence and $\mathcal{F}$-fibration.

The qf-model structure on $\mathcal{O}_G^P$-spaces, with levelwise weak equivalences and cofibrations having the left homotopy extension property is Quillen equivalent to the homotopy category of compactly generated, weak Hausdorff G-spaces, with the above mentioned model category structure for the family $\mathcal{F} = \mathcal{ALL}$ of all subgroups of G [14]. In the Appendix we prove a parametrized version of this result and we show some more facts concerning the homotopy category of both G-spaces and $\mathcal{O}_G^P$-spaces.

We now introduce the category of $\mathcal{O}_G^P$-spectra. Recall that a spectrum is a sequence of pointed spaces $\{E_n\}_{n\in\mathbb{Z}}$ with structure maps $E_n \wedge S^1 \to E_{n+1}$.

A (strong) map of spectra $f : E \to F$ is a sequence of maps compatible with the structure maps.

Finally, recall that a spectrum is called an Ω-spectrum if the adjoint of the structure maps $E_n \to \Omega E_{n+1}$ are weak homotopy equivalences.

Definition 2.3. Let G be a discrete group. An $\mathcal{O}_G^P$-spectrum is a contravariant functor $E : \mathcal{O}_G^P \to$ SPECTRA to the category of spectra and strong maps.

Given an $\mathcal{O}_G^P$-space $\mathcal{X}$, we denote by

$$\Sigma\mathcal{X} = \mathcal{X}_+ \wedge S^1$$

the space given on each object G/H as the reduced suspension $\mathcal{X}_+(G/H) \wedge S^1$, together with the structure maps given by smashing with the identity map.

The n-th suspension $\Sigma^n\mathcal{X}$ is the space defined on objects as $\mathcal{X}(G/H)_+ \wedge S^n$.

Definition 2.4. Let $\mathcal{X}$ be an $\mathcal{O}_G^P$-space. The naive $\mathcal{O}_G^P$-suspension spectrum of $\mathcal{X}$, denoted by $\Sigma^\infty\mathcal{X}$, is defined on each object G/H as the n-th suspension space $\Sigma^\infty_{\mathcal{O}_G^P}\mathcal{X}(n) = \Sigma^n\mathcal{X} = \mathcal{X}_+ \wedge S^n$ with the $\mathcal{O}_G^P$-structure maps obtained by smashing the $\mathcal{O}_G^P$-maps of $\mathcal{X}$ with the identity map $S^n \to S^n$ and spectra structure maps given by the homeomorphisms $S^n \wedge S^1 \to S^{n+1}$.

We now introduce parametrized versions of the constructions defined in the orbit category.

Definition 2.5. Fix a contravariant $\mathcal{O}_G^P$-space $\mathcal{B}$. A $\mathcal{O}_G^P$-space over $\mathcal{B}$ is a contravariant $\mathcal{O}_G^P$ space $\mathcal{X}$ endowed with a natural transformation of $\mathcal{O}_G^P$-spaces $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{B}$; this map is usually called projection.

A map of $\mathcal{O}_G^P$-spaces over $\mathcal{B}$ is a map of $\mathcal{O}_G^P$-spaces $F : \mathcal{X} \to \mathcal{Y}$, which in addition is compatible with projections in the sense that $p_{\mathcal{Y}} \circ F = p_{\mathcal{X}}$.

The space of maps over $\mathcal{B}$, denoted by $\mathrm{Hom}_{\mathcal{O}_G^P}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}}$, is defined as the subspace of the $\mathcal{O}_G^P$-mapping space consisting of $\mathcal{O}_G^P$-maps which are compatible with the projection maps:

$$\mathrm{Hom}_{\mathcal{O}_G^P}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}} := \{F \in \mathrm{Hom}_{\mathcal{O}_G^P}(\mathcal{X}, \mathcal{Y}) \mid p_{\mathcal{Y}} \circ F = p_{\mathcal{X}}\}.$$

We denote the set of homotopy classes of maps over $\mathcal{B}$ by

$$_{\mathcal{O}_G^P}[\mathcal{X}, \mathcal{Y}]_{\mathcal{B}} := \pi_0 \left(\mathrm{Hom}_{\mathcal{O}_G^P}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}} \right).$$

3. Twisted equivariant K-theory and local coefficient versions of Bredon cohomology

3.1. **Twisted equivariant K-theory.** Twisted equivariant K-theory for proper actions of discrete groups was introduced in [6]. In what follows we will recall its definition using Fredholm bundles and its properties following [6] and the classification of equivariant principal bundles done in [6, 22].

Definition 3.1. Let X be a proper G-space with the homotopy type of a proper G-ANR. Let $\mathcal{H}$ be a separable complex Hilbert space and

$$\mathcal{U}(\mathcal{H}) = \{U : \mathcal{H} \to \mathcal{H} \mid U \circ U^* = U^* \circ U = \mathrm{Id}\}$$

be the unitary group endowed with the norm topology. The group $P\mathcal{U}(\mathcal{H}) = \mathcal{U}(\mathcal{H})/S^1$ with the quotient topology is the group of projective unitary operators.

A projective unitary, stable G-equivariant bundle is a right $P\mathcal{U}(\mathcal{H})$, principal bundle

$$P\mathcal{U}(\mathcal{H}) \to P \to X$$

endowed with a left G action lifting the action on X such that:

- the left G-action commutes with the right $P\mathcal{U}(\mathcal{H})$ action, and
- for all $x \in X$ there exists a G-neighborhood V of x and a G_x-contractible slice U of x with V equivariantly homeomorphic to $U \times_{G_x} G$ with the action

  $$G_x \times (U \times G) \to U \times G, \qquad k \cdot (u, g) = (ku, gk^{-1}),$$

  together with a local trivialization

  $$P|_V \cong (P\mathcal{U}(\mathcal{H}) \times U) \times_{G_x} G$$

  where the action of the isotropy group is:

  $$\begin{array}{rcl} G_x \times [(P\mathcal{U}(\mathcal{H}) \times U) \times G] & \to & (P\mathcal{U}(\mathcal{H}) \times U) \times G \\ k \cdot [(F, y), g] & \mapsto & [(f_x(k)F, ky), gk^{-1}] \end{array}$$

  with $f_x : G_x \to P\mathcal{U}(\mathcal{H})$ a fixed *stable* homomorphism, in the sense that the unitary representation $\mathcal{H}$ induced by the homomorphism $\widetilde{f_x} : \widetilde{G_x} = f_x^* \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ contains each of the irreducible representations of $\widetilde{G}_x$ on which S^1 acts by multiplication an infinite number of times.

Let X be a G-space and $P \to X$ a projective unitary stable G-equivariant bundle over X. Recall [2, 6] that the space of Fredholm operators is endowed with

a continuous right action of the group $P\mathcal{U}(\mathcal{H})$ by conjugation, therefore we can take the associated bundle over X

$$\mathrm{Fred}(P) := P \times_{P\mathcal{U}(\mathcal{H})} \mathrm{Fred}(\mathcal{H}),$$

where $\mathrm{Fred}(\mathcal{H})$ is the space of Fredholm operators with the norm topology, and with the induced G action given by

$$g \cdot [(\lambda, A)] := [(g\lambda, A)]$$

for g in G, λ in P and A in $\mathrm{Fred}(\mathcal{H})$.

Denote by

$$\Gamma(X; \mathrm{Fred}(P))$$

the space of sections of the bundle $\mathrm{Fred}(P) \to X$ and choose as base point in this space the section which chooses the identity operator on each fiber. This section exists because the $P\mathcal{U}(\mathcal{H})$-action on $\mathrm{Id}_{\mathcal{H}}$ is trivial, and therefore

$$X \cong P/P\mathcal{U}(\mathcal{H}) \cong P \times_{P\mathcal{U}(\mathcal{H})} \{\mathrm{Id}_{\mathcal{H}}\} \subset \mathrm{Fred}(P);$$

let us denote this *identity section* by s.

The proof of Bott periodicity done in [5, Theorem 5.1] shows the homotopy equivalence $\Omega^2(\mathrm{Fred}(\mathcal{H})) \simeq \mathrm{Fred}(\mathcal{H})$. This proof can be carried without changes whenever a compact Lie group K acts in $\mathcal{H}$ with infinitely many representations for each irreducible representation appearing in $\mathcal{H}$. Taking equivariant Fredholm operators $\mathrm{Fred}(\mathcal{H})^K$ we obtain the homotopy equivalence $\Omega^2(\mathrm{Fred}(\mathcal{H})^K) \simeq \mathrm{Fred}(\mathcal{H})^K$. Therefore we obtain Bott periodicity for the twisted and equivariant case an we may define the twisted G-equivariant K-theory groups as follows.

Definition 3.2. Let X be a connected G-space and P a projective unitary stable G-equivariant bundle over X. The *twisted G-equivariant K-theory* groups of X twisted by P are defined as the homotopy groups of the G-equivariant sections

$$K^p_G(X; P) := \pi_0\left(\Gamma(X; \mathrm{Fred}(P))^G, s\right) \qquad \text{whenever } p \text{ is even}$$
$$K^p_G(X; P) := \pi_1\left(\Gamma(X; \mathrm{Fred}(P))^G, s\right) \qquad \text{whenever } p \text{ is odd}$$

where s denotes the identity section.

3.2. **Universal projective unitary stable equivariant bundle.** In [6, Section 3.2] it was constructed the universal projective unitary stable equivariant bundle by gluing the universal bundles over each orbit type. Let us recall how this bundle is assembled since we need this information in order to define the Bredon cohomology with local coefficients.

The base of this universal bundle was constructed from the $\mathcal{O}^P_G$-space $|\mathcal{C}|$ which at each orbit type G/K assigns the space $|\mathcal{C}_{G/K}|$; this space is the geometric realization of the groupoid

$$\mathcal{C}_{G/K} = [\mathrm{Funct}_{st}(G \ltimes G/K, P\mathcal{U}(\mathcal{H}))/\mathrm{Map}(G/K, P\mathcal{U}(\mathcal{H}))]$$

whose objects are functors $\mathrm{Funct}_{st}(G \ltimes G/K, P\mathcal{U}(\mathcal{H}))$ from the category defined by the left G action on G/K, denoted by $G \ltimes G/K$, and the category defined by the group $P\mathcal{U}(\mathcal{H})$ whose restriction to $\mathrm{Hom}(K, P\mathcal{U}(\mathcal{H}))$ are stable homomorphisms, and whose morphisms are given by natural transformations $\mathrm{Map}(G/K, P\mathcal{U}(\mathcal{H}))$.

In the category of $\mathcal{O}^P_G$-spaces, a classifying map for the bundle $\Phi P \to \Phi X$ is obtained by map $\mu : \Phi X \to |\mathcal{C}|$ assembling the maps $\mu_{G/K} : X^K \to |\mathcal{C}_{G/K}|$, with the property that

$$(\mu_{G/K})^* |\mathcal{D}_{G/K}| \cong P|_{X^K}$$

where $|\mathcal{D}_{G/K}| \to |\mathcal{C}_{G/K}|$ is the universal projective unitary stable $N_G(K)$-equivariant bundle over $|\mathcal{C}_{G/K}|$ and which is defined as follows [6, Def. 4.1]: the morphisms $\mathrm{Mor}(\mathcal{D}_{G/K})$ are

$$\mathrm{Funct}_{st}(G \ltimes G/K, P\mathcal{U}(\mathcal{H})) \times P\mathcal{U}(\mathcal{H}) \times \mathrm{Map}(G/K, P\mathcal{U}(\mathcal{H}))$$

and the objects $\mathrm{Obj}(\mathcal{D}_{G/K})$

$$\mathrm{Funct}_{st}(G \ltimes G/K, P\mathcal{U}(\mathcal{H})) \times P\mathcal{U}(\mathcal{H}),$$

with structural maps $\mathrm{source}(\psi, F, \sigma) = (\psi, F)$, $\mathrm{target}(\psi, F, \sigma) = (\sigma F^{-1}\psi F \sigma^{-1}, \sigma([K]))$ and composition $\mathrm{comp}((\psi, F, \sigma), (\sigma F^{-1}\psi F\sigma^{-1}, \sigma([K]), \delta) = (\psi, F, \delta\sigma([K])^{-1}\sigma)$. The fucntor $\mathcal{D}_{G/K} \to \mathcal{C}_{G/K}$ forgets the $P\mathcal{U}(\mathcal{H})$ component, and the map $|\mathcal{D}_{G/K}| \to |\mathcal{C}_{G/K}|$ denotes the map of the geometric realizations.

Denote by $\mathrm{Fred}(|\mathcal{D}|))$ the $\mathcal{O}_G^P$-space over $|\mathcal{C}|$ defined on the orbit type G/K by

$$\mathrm{Fred}(|\mathcal{D}_{G/K}|))^K := \left(|\mathcal{D}_{G/K}| \times_{P\mathcal{U}(\mathcal{H})} \mathrm{Fred}(\mathcal{H})\right)^K$$

and denote by $p : \mathrm{Fred}(|\mathcal{D}|) \to |\mathcal{C}|$ the projection map which is the assembly of the canonical projection maps

$$p_{G/K} : \mathrm{Fred}(|\mathcal{D}_{G/K}|)^K \to |\mathcal{C}_{G/K}|.$$

Since the identity operator $\mathrm{Id}_{\mathcal{H}}$ on the Hilbert space $\mathcal{H}$ is invariant under the conjugation action of $P\mathcal{U}(\mathcal{H})$, then the projection map p has a canonical section

$$s : |\mathcal{C}| \to \mathrm{Fred}(|\mathcal{D}|)$$

which assigns to every point the operator $\mathrm{Id}_{\mathcal{H}}$.

Alternatively, we could define the twisted equivariant K-theory groups in the category of $\mathcal{O}_G^P$-spaces in the following way. For a proper G-CW complex X endowed with a map of $\mathcal{O}_G^P$-spaces $\mu : \Phi X \to |\mathcal{C}|$, we can alternatively define the twisted equivariant K-theory groups of the pair $(\Phi X; \mu)$ as the homotopy groups of the pointed space

$$\left(\mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathrm{Fred}(|\mathcal{D}|))_{|\mathcal{C}|}, s \circ \mu\right)$$

namely

$$K_G^p(\Phi X; \mu) := \pi_0\left(\mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathrm{Fred}(|\mathcal{D}|))_{|\mathcal{C}|}, s \circ \mu\right) \quad \text{whenever } \ p \ \text{ is even}$$

$$K_G^p(\Phi X; \mu) := \pi_1\left(\mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathrm{Fred}(|\mathcal{D}|))_{|\mathcal{C}|}, s \circ \mu\right) \quad \text{whenever } \ p \ \text{ is odd.}$$

Remark 3.3. We would like to note here that an alternative, and homotopically equivalent, construction of the universal projective unitary stable equivariant bundle was done in [22, Section 15]. There all topological issues regarding the existence of local sections were resolved.

3.3. **Bredon cohomology with local coefficients.** The local coefficients for the Bredon cohomology that we are going to define in this section are constructed from the fiberwise homotopy groups of the fiber bundles

$$p_{G/K} : \mathrm{Fred}(|\mathcal{D}_{G/K}|)^K \to |\mathcal{C}_{G/K}|.$$

In order to have an explicit definition of these local coefficients, we need to recall some properties of the previous fibration.

The only non trivial homotopy groups of the spaces $|\mathcal{C}_{G/K}|$ exist in degree 0, 1 and 3. We know by [6, Theorem 1.9] that $\pi_0(|\mathcal{C}_{G/K}|) \cong \mathrm{Ext}(K, S^1)$ is the set of isomorphism classes of S^1-central extensions of K, and that the fundamental group of each connected component of $|\mathcal{C}_{G/K}|$ is isomorphic to $\mathrm{Hom}(K, S^1)$. Let us

denote by $|\mathcal{C}_{G/K}|_{\widetilde{K}}$ the connected components of $|\mathcal{C}_{G/K}|$ associated the S^1-central extension $\widetilde{K}$; hence

$$|\mathcal{C}_{G/K}| = \bigcup_{\widetilde{K}\in \mathrm{Ext}(S^1,K)} |\mathcal{C}_{G/K}|_{\widetilde{K}}.$$

Now, for any point $x \in |\mathcal{C}_{G/K}|_{\widetilde{K}}$ there is associated a specific stable homomorphism $\alpha : K \to P\mathcal{U}(\mathcal{H})$ with $\widetilde{K} \cong \alpha^*(\mathcal{U}(\mathcal{H}))$ and lift $\widetilde{\alpha} : \widetilde{K} \to \mathcal{U}(\mathcal{H})$ such that the fiber

$$p^{-1}_{G/K}(x) \subset \left(|\mathcal{D}_{G/K}| \times_{P\mathcal{U}(\mathcal{H})} \mathrm{Fred}(\mathcal{H})\right)^K$$

is isomorphic to the space of K-invariant Fredholm operators

$$\mathrm{Fred}(\mathcal{H})^{\widetilde{\alpha}} := \{F \in \mathrm{Fred}(\mathcal{H}) \mid \widetilde{\alpha}(k)F = F\widetilde{\alpha}(k) \text{ for all } k \in \widetilde{K}\}.$$

The index map

$$\begin{aligned} \mathrm{ind} : \mathrm{Fred}(\mathcal{H})^{\widetilde{\alpha}} &\to R_{S^1}(\widetilde{K}) \\ F &\mapsto [\ker(F)] - [\mathrm{coker}(F)] \end{aligned}$$

is a homomorphism of groups that induces an isomorphism of groups at the level of the connected components

$$\mathrm{ind} : \pi_0(p^{-1}_{G/K}(x)) = \pi_0(\mathrm{Fred}(\mathcal{H})^{\widetilde{\alpha}}) \xrightarrow{\cong} R_{S^1}(\widetilde{K}); \tag{3.4}$$

here $R_{S^1}(\widetilde{K})$ denotes the Grothendieck group of isomorphism classes of $\widetilde{K}$ representations where $\ker(\widetilde{K} \to K)$ acts by multiplication of scalars. Hence we have that the connected components of the fibers of the map

$$p_{G/K} : \mathrm{Fred}(|\mathcal{D}_{G/K}|)^K|_{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \to |\mathcal{C}_{G/K}|_{\widetilde{K}}$$

are all isomorphic to the group $R_{S^1}(\widetilde{K})$ via the *index* map.

Definition 3.5. Consider the $\mathcal{O}^P_G$-space $\mathfrak{TR}_0$ over $|\mathcal{C}|$ which at each orbit type G/K is defined by

$$(\mathfrak{TR}_0)_{G/K} := \bigsqcup_{\widetilde{K}\in \mathrm{Ext}(K,S^1)} \overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \times_{\mathrm{Hom}(K,S^1)} R_{S^1}(\widetilde{K})$$

where $\overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}}$ is the universal cover of $|\mathcal{C}_{G/K}|_{\widetilde{K}}$, the action of $\mathrm{Hom}(K, S^1)$ on the left hand side is given by an explicit isomorphism $\pi_1(|\mathcal{C}_{G/K}|_{\widetilde{K}}) \cong \mathrm{Hom}(K, S^1)$ and the action on the right hand side is given by

$$\mathrm{Hom}(K, S^1) \times R_{S^1}(\widetilde{K}) \to R_{S^1}(\widetilde{K}), \qquad (\rho, V) \mapsto \overline{\rho} \otimes_{\mathbb{C}} V$$

where $\overline{\rho}$ is understood as the 1-dimensional representation of $\widetilde{K}$ that the homomorphism ρ defines. Denote by $t : |\mathcal{C}_{G/K}| \to (\mathfrak{TR}_0)_{G/K}$ the 0-section.

Note that the definition of the explicit isomorphism $\pi_1(|\mathcal{C}_{G/K}|_{\widetilde{K}}) \cong \mathrm{Hom}(K, S^1)$ is based on the following construction. The first two homotopy groups of $|\mathcal{C}_{G/K}|$ come from the first two homotopy groups of $\mathrm{Hom}_{st}(K, P\mathcal{U}(\mathcal{H}))$, the space of stable homomorphisms. Denote by $\mathrm{Hom}_{st}(K, P\mathcal{U}(\mathcal{H}))_{\widetilde{K}}$ the connected component that defines $\widetilde{K}$ and let $\mathrm{Hom}_{S^1}(\widetilde{K}, \mathcal{U}(\mathcal{H}))$ be the space of homomorphisms such that $\ker(\widetilde{K} \to K)$ acts on $\mathcal{H}$ by multiplication. Then the projection map

$$\mathrm{Hom}_{S^1}(\widetilde{K}, \mathcal{U}(\mathcal{H})) \to \mathrm{Hom}_{st}(K, P\mathcal{U}(\mathcal{H}))_{\widetilde{K}}$$

is a principal $\mathrm{Hom}(K, S^1)$-bundle where $\mathrm{Hom}(K, S^1)$ acts on $\mathrm{Hom}_{S^1}(\widetilde{K}, \mathcal{U}(\mathcal{H}))$ by multiplication [22, Prop. 15.7], and therefore the projection map is a universal cover for the base.

For a stable homomorphism $\alpha : K \to P\mathcal{U}(\mathcal{H})$ such that $\alpha^*\mathcal{U}(\mathcal{H}) \cong \widetilde{K}$, we choose a lift $\widetilde{\alpha} : \widetilde{K} \to \mathcal{U}(\mathcal{H})$ in order to define the index map

$$\begin{aligned} \mathrm{ind}^{\widetilde{\alpha}} : \mathrm{Fred}(\mathcal{H})^{\alpha} &\to R_{S^1}(\widetilde{K}) \\ F &\mapsto [\ker(F)] - [\mathrm{coker}(F)] \end{aligned}$$

Whenever we choose another lift $\widetilde{\alpha}' = \widetilde{\alpha} \cdot \rho$ with $\rho : K \to S^1$, we have that $\mathrm{ind}^{\widetilde{\alpha}'}(F) = \mathrm{ind}^{\widetilde{\alpha}}(F) \cdot \overline{\rho}$, and since the structural group of $\mathrm{Fred}(|\mathcal{D}_{G/K}|)^K$ is connected, we have that the fiberwise index map

$$\begin{array}{ccc} \mathrm{Fred}(|\mathcal{D}_{G/K}|)^K & \xrightarrow{\mathrm{ind}} & (\mathfrak{TR}_0)_{G/K} \\ {\scriptstyle p_{G/K}}\downarrow\uparrow{\scriptstyle s_{G/K}} & & {\scriptstyle q_{G/K}}\downarrow\uparrow{\scriptstyle t_{G/K}} \\ |\mathcal{C}_{G/K}| & \xrightarrow[=]{} & |\mathcal{C}_{G/K}| \end{array} \tag{3.6}$$

is a well defined map of fiber bundles, and that it induces an isomorphism of the connected components of the fibers

$$\pi_0(p_{G/K}^{-1}(x)) \cong R_{S^1}(\widetilde{K}) \tag{3.7}$$

for every point $x \in |\mathcal{C}_{G/K}|_{\widetilde{K}}$ and every S^1-central extension $\widetilde{K}$. Assembling these maps we obtain an index map at the level of the $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$

$$\begin{array}{ccc} \mathrm{Fred}(|\mathcal{D}|) & \xrightarrow{\mathrm{ind}} & \mathfrak{TR}_0 \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle q}\downarrow\uparrow{\scriptstyle t} \\ |\mathcal{C}| & \xrightarrow[=]{} & |\mathcal{C}| \end{array} \tag{3.8}$$

which induces an isomorphism on the connected components of the fibers.

To construct the Bredon cohomology with coefficients in twisted representations, we perform a construction similar to the one done in Definition 3.5 but we replace the group of twisted representations $R_{S^1}(\widetilde{K})$ by $HR_{S^1}(\widetilde{K})$, the Eilenberg-MacLane spectrum of the abelian group $R_{S^1}(\widetilde{K})$.

Denote by $HR_{S^1}(\widetilde{K})$ the Eilenberg-MacLane spectrum associated to the group $R_{S^1}(\widetilde{K})$, i.e. at level $n \geq 0$ we have $(HR_{S^1}(\widetilde{K}))_n = K(R_{S^1}(\widetilde{K}), n)$ where $K(R_{S^1}(\widetilde{K}), n)$ is a functorial model for the Eilenberg-MacLane space whose only non-trivial homotopy group is $R_{S^1}(\widetilde{K})$ in degree n, and which comes endowed with weak homotopy equivalences $\Omega K(R_{S^1}(\widetilde{K}), n) \simeq K(R_{S^1}(\widetilde{K}), n+1)$.

Definition 3.9. For $n \geq 0$ consider the $\mathcal{O}_G^P$-space $\mathfrak{TR}_n$ over $|\mathcal{C}|$ of twisted representations, such that on the orbit type G/K we have

$$(\mathfrak{TR}_n)_{G/K} := \bigsqcup_{\widetilde{K} \in \mathrm{Ext}(K, S^1)} \overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \times_{\mathrm{Hom}(K,S^1)} K(R_{S^1}(\widetilde{K}), n)$$

where $\overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}}$ is the universal cover of $|\mathcal{C}_{G/K}|_{\widetilde{K}}$, the action of $\mathrm{Hom}(K, S^1)$ on the left hand side is given by an explicit isomorphism $\pi_1(|\mathcal{C}_{G/K}|_{\widetilde{K}}) \cong \mathrm{Hom}(K, S^1)$ and the action on the right hand side is the one induced on the Eilenberg-MacLane space $K(R_{S^1}(\widetilde{K}), n)$ by the action

$$\mathrm{Hom}(K, S^1) \times R_{S^1}(\widetilde{K}) \to R_{S^1}(\widetilde{K}), \qquad (\rho, V) \mapsto \overline{\rho} \otimes_{\mathbb{C}} V.$$

Denote by $r_n : \mathfrak{TR}_n \to |\mathcal{C}|$ the natural projection map and by $\sigma_n : |\mathcal{C}| \to \mathfrak{TR}_n$ the section which chooses the base point in $K(R_{S^1}(\widetilde{K}), n)$. For $n < 0$ let $\mathfrak{TR}_n := |\mathcal{C}|$ with $r_n = \sigma_n = \mathrm{Id}_{|\mathcal{C}|}$.

The weak homotopy equivalences $\Omega K(R_{S^1}(\widetilde{K}), n) \simeq K(R_{S^1}(\widetilde{K}), n+1)$ induce weak homotopy equivalences $\Omega\mathfrak{TR}_n \simeq \mathfrak{TR}_{n+1}$ in the category of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$. Assembling the spaces $\mathfrak{TR} = \{\mathfrak{TR}_n\}_{n\in\mathbb{Z}}$ we obtain the following lemma.

Lemma 3.10. *$\mathfrak{TR} = \{\mathfrak{TR}_n\}_{n\in\mathbb{Z}}$ is a Ω-spectrum in the category of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$.*

We are now ready to define the twisted Bredon cohomology associated to twisted representations.

Definition 3.11. Let X be a proper G-ANR endowed with a fixed map of $\mathcal{O}_G^P$-spaces $\xi : \Phi X \to |\mathcal{C}|$. The Bredon cohomology groups with local coefficients in twisted representations associated to the pair $(\Phi X; \xi)$ are defined as the connected components of the based spaces $\mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathfrak{TR}_n)_{|\mathcal{C}|}$, i.e.

$$\mathbb{H}_G^p(\Phi X, \xi) := \pi_0\left(\mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathfrak{TR}_p)_{|\mathcal{C}|}; \sigma_p \circ \xi\right).$$

Alternatively, in the category of $\mathcal{O}_G^P$-spectra over $|\mathcal{C}|$ we have:

$$\mathbb{H}_G^p(\Phi X, \xi) := \pi_p\left(\mathrm{Hom}_{\mathcal{O}_G^P}(\Sigma^\infty \Phi X, \mathfrak{TR})_{|\mathcal{C}|}; \sigma \circ \xi\right).$$

These cohomology groups satisfy the axioms of a parametrized G-equivariant cohomology theory and the proof follows the same lines as the one for the twisted equivariant K-theory groups which can be found in [6, Chapter 5]; we will not reproduce its proof here.

Remark 3.12. Other approaches to Bredon cohomology with local coefficients include [8], where methods from the theory of crossed complexes and their classifying spaces are used to produce a classifying object for Bredon cohomology with local coefficients.

4. Segal's spectral sequence for twisted equivariant K-theory

We will use Segal's method [26] to obtain a filtration of the homotopy theoretically defined twisted equivariant K-theory, as well as a version of Bredon cohomology associated to a cover to handle the homotopical version of Bredon cohomology described in the previous section. We describe first the local coefficient system associated to twisted equivariant K-theory.

Definition 4.1 (Local coefficient system of twisted equivariant K-theory)**.** Consider a projective unitary stable bundle P over a proper G-space X and a G-invariant and countable cover $\mathcal{U}$ for which each open set U_σ is equivariantly contractible, i.e. G-homotopic to G/H_σ for some finite subgroup H_σ depending on the set U_σ. We can define local coefficient systems by the functors

$$\begin{aligned}\mathcal{K}_G^p(?, P_?) : \mathcal{N}_G\mathcal{U} &\to \mathbb{Z} - \mathrm{Mod}\\ U_\tau \subset U_\sigma &\mapsto K_G^p(U_\sigma; P|_{U_\sigma}) \to K_G^p(U_\tau; P|_{U_\tau})\end{aligned}$$

Proposition 4.2. *Let X be a proper compact G-ANR and P a projective unitary stable equivariant bundle. Then Segal's spectral sequence applied to $K_G^*(X, P)$ and associated to the locally finite and equivariantly contractible cover $\mathcal{U}$, has as second page $E_2^{p,q}$ the cohomology of $\mathcal{N}_G\mathcal{U}$ with coefficients in the functor $\mathcal{K}_G^0(?, P|_?)$ whenever q is even, i.e.*

$$E_2^{p,q} := H_G^p(X, \mathcal{U}; \mathcal{K}_G^0(?, P|_?)) \tag{4.3}$$

and is trivial if q is odd. Its higher differentials

$$d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$$

vanish for r even.

Proof. Since the cover consists of equivariantly contractible spaces we know that the groups $K_G^q(U_\sigma; P|_{U_\sigma})$ are periodic and trivial for q odd. Therefore the fact that the second page of Segal's spectral sequence is isomorphic to $H_G^p(X,\mathcal{U}; \mathcal{K}_G^q(?, P|_?))$ follows directly from Segal's original proof. Bott's isomorphism implies that $K_G^{2n}(U_\sigma; P|_{U_\sigma}) \cong K_G^0(U_\sigma; P|_{U_\sigma})$ and therefore we have that the even differentials vanish. □

We are mainly interested in the second page of the spectral sequence, and to understand it we need to elaborate on the cohomology of $\mathcal{N}_G\mathcal{U}$ with coefficients in the functor $\mathcal{K}_G^0(?, P|_?))$ and compare it with the homotopy theoretic definition given in Section 3. This comparison will be done in the category of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$.

We claim the following result:

Theorem 4.4. *Let $\mathcal{U}$ be a locally finite cover of G-invariant sets of X such that each non-trivial intersection of sets in the cover is equivariantly contractible. Then for any map $\mu : \Phi X \to |\mathcal{C}|$ the second page of Segal's spectral sequence applied to the groups $K_G^*(\Phi X; \mu)$ is isomorphic to the Bredon cohomology groups with local coefficients in twisted representations $\mathbb{H}_G^p(\Phi X, \mu)$; i.e. for q even*

$$E_2^{p,q} \cong \mathbb{H}_G^p(\Phi X, \mu).$$

Proof. Applying Segal's spectral sequence to $\mathbb{H}_G^p(\Phi X, \mu)$ with the cover $\mathcal{U}$ we get that the second page of this spectral sequence is

$$\bar{E}_2^{p,q} = H_G^p(\Phi X, \mathcal{U}; \mathbb{H}_G^q(?, \mu|_?)).$$

Since the open sets U_σ are equivariantly contractible we have that for $q \neq 0$

$$\mathbb{H}_G^q(\Phi U_\sigma, \mu|_{\Phi U_\sigma})) = 0$$

and therefore $\bar{E}_2^{p,q} = 0$ for $q \neq 0$. Therefore the spectral sequence collapses at the second page and this page becomes

$$\bar{E}_2^{p,0} = H_G^p(\Phi X, \mathcal{U}; \mathbb{H}_G^0(?, \mu|_?)) \cong \mathbb{H}_G^p(\Phi X, \mu)$$

where $\mathbb{H}_G^0(?, \mu|_?)$ is the local coefficient system defined by

$$\mathbb{H}_G^0(?, \mu|_?) : \mathcal{N}_G\mathcal{U} \to \mathbb{Z} - \text{Mod}$$
$$U_\tau \subset U_\sigma \mapsto \mathbb{H}_G^0(\Phi U_\sigma, \mu|_{\Phi U_\sigma}) \to \mathbb{H}_G^0(\Phi U_\tau, \mu|_{\Phi U_\tau}).$$

Now we need to show that there is a canonical way to assign isomorphisms

$$\phi_\sigma : K_G^0(\Phi U_\sigma; \mu|_{\Phi U_\sigma}) \xrightarrow{\cong} \mathbb{H}_G^0(\Phi U_\sigma,\ \mu|_{\Phi U_\sigma}) \tag{4.5}$$

which commute with the restriction maps on each side; the existence of such isomorphisms would induce a canonical isomorphism between the complexes defined in the first page of the spectral sequences

$$E_1^{p,0} \xrightarrow{\cong} \bar{E}_1^{p,0}$$

and therefore would induce an isomorphism at the second pages

$$E_2^{p,0} \xrightarrow{\cong} \bar{E}_2^{p,0}.$$

The existence of the isomorphisms described in (4.5) and which are compatible with the inclusions, follow from the explicit maps described in diagrams (3.6), (3.8) and from the isomorphisms of equations (3.7). Since U_σ is equivariantly contractible to a point, by equation (3.4) we know that the index map

$$\text{Hom}_{\mathcal{O}_G^P}(\Phi U_\sigma, \text{Fred}(|\mathcal{D}|))_{|\mathcal{C}|} \to \text{Hom}_{\mathcal{O}_G^P}(\Phi U_\sigma, \mathfrak{TM}_0)_{|\mathcal{C}|}$$
$$f \mapsto \text{ind} \circ f$$

induces an isomorphism on connected components, and hence a canonical isomorphism

$$\phi_\sigma : K^0_G(\Phi U_\sigma; \mu|_{\Phi U_\sigma}) \overset{\cong}{\to} \mathbb{H}^0_G(\Phi U_\sigma; \mu|_{\Phi U_\sigma}).$$

The inclusion $U_\tau \subset U_\sigma$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}^P_G}(\Phi U_\sigma, \mathrm{Fred}(|\mathcal{D}|))_{|\mathcal{C}|} & \xrightarrow{\mathrm{ind}} & \mathrm{Hom}_{\mathcal{O}^P_G}(\Phi U_\sigma, \mathfrak{TR}_0)_{|\mathcal{C}|} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}^P_G}(\Phi U_\sigma, \mathrm{Fred}(|\mathcal{D}|))_{|\mathcal{C}|} & \xrightarrow{\mathrm{ind}} & \mathrm{Hom}_{\mathcal{O}^P_G}(\Phi U_\sigma, \mathfrak{TR}_0)_{|\mathcal{C}|} \end{array}$$

which implies that the isomorphisms ϕ_σ are compatible with restrictions, i.e. we have the commutative diagram

$$\begin{array}{ccc} K^0_G(\Phi U_\sigma; \mu|_{\Phi U_\sigma}) & \xrightarrow[\cong]{\phi_\sigma} & \mathbb{H}^0_G(\Phi U_\sigma; \mu|_{\Phi U_\sigma}) \\ \downarrow & & \downarrow \\ K^0_G(\Phi U_\tau; \mu|_{\Phi U_\tau}) & \xrightarrow[\cong]{\phi_\tau} & \mathbb{H}^0_G(\Phi U_\tau; \mu|_{\Phi U_\tau}). \end{array}$$

The isomorphisms ϕ_σ induce the desired isomorphism $E_1^{p,0} \overset{\cong}{\to} \bar{E}_1^{p,0}$, and since they are compatible with restrictions, they induced an isomorphism of complexes thus preserving the first differential. This implies that $E_2^{p,0} \overset{\cong}{\to} \bar{E}_2^{p,0}$. Bott periodicity implies that there are canonical isomorphisms $E_1^{p,q} \cong E_1^{p,0}$ for q even, which are compatible with the restrictions. We conclude that

$$E_2^{p,q} \cong \mathbb{H}^p_G(\Phi X; \mu)$$

whenever q is even and $E_2^{p,q} = 0$ whenever q is odd. □

4.1. **The third differential.** The third differential on Segal's spectral sequence

$$d_3 : E_2^{p,q} \to E_2^{p+3,q-2},$$

together with the isomorphism of Theorem 4.4, induce a degree three map

$$d_3 : \mathbb{H}^p_G(\Phi X, \mu) \to \mathbb{H}^{p+3}_G(\Phi X, \mu)$$

on the Bredon cohomology with local coefficients in twisted representations which we will denote with the same symbol d_3.

The purpose of this section is to evidence some particular phenomena concerning this differential d_3.

4.1.1. *G-Invariant cohomology class.* Consider the trivial subgroup $\{1\} \subset G$ and recall that the bundle $|\mathcal{D}_{G/\{1\}}| \to |\mathcal{C}_{G/\{1\}}|$ is a universal projective unitary bundle thus having that $|\mathcal{C}_{G/\{1\}}|$ is a $K(\mathbb{Z}, 3)$. Hence for any map $\mu : \Phi X \to |\mathcal{C}|$ which classifies a projective equivariant stable unitary bundle over X, the map $\mu_{G/\{1\}} : X \to |\mathcal{C}_{G/\{1\}}|$ encodes the information of the projective unitary bundle once the G-action is forgotten. The map $\mu_{G/\{1\}}$ defines a degree 3 cohomology class $\eta \in H^3(X, \mathbb{Z})$ which is moreover G-invariant.

In cohomological terms we know that the bundle $P \to X$ is classified by an element $\overline{\eta} \in H^3(X \times_G EG, \mathbb{Z})$. Denoting by η the restriction of $\overline{\eta}$ to any fiber of the Serre fibration $X \to X \times_G EG \to BG$, and restricting it further to the fixed point set of the group K, we get a class

$$\eta_K := \eta|_{X^K} \in H^3(X^K; \mathbb{Z}).$$

This class η_K is precisely the class defined by the the composition $X^K \overset{\mu_{G/K}}{\to} |\mathcal{C}_{G/K}| \overset{\kappa_{G/K}}{\to} |\mathcal{C}_{G/\{1\}}|$, and it is furthermore $N_G(K)/K$-invariant.

Since the groups $R_{S^1}(\widetilde{K})$ are free $\mathbb{Z}$-modules, there is an induced structure at the level of the Eilenberg-MacLane spaces

$$|\mathcal{C}_{G/\{1\}}| \times K(R_{S^1}(\widetilde{K}), n) \to K(R_{S^1}(\widetilde{K}), n+3)$$

which is $N_G(K)/K$-equivariant, compatible with restrictions and which recovers the cup product by a degree 3 cohomology class. Composing with the canonical maps $\kappa_{G/K} : |\mathcal{C}_{G/K}| \to |\mathcal{C}_{G/\{1\}}|$ we obtain maps

$$\varepsilon_{G/K} : |\mathcal{C}_{G/K}| \times K(R_{S^1}(\widetilde{K}), n) \to K(R_{S^1}(\widetilde{K}), n)$$

which are $\mathrm{Hom}(K, S^1)$ equivariant, and therefore they define maps

$$(\mathfrak{TR}_n)_{G/K} \to (\mathfrak{TR}_{n+3})_{G/K}$$

over $|\mathcal{C}_{G/K}|$ which can be assembled into a map $\mathfrak{TR}_n \to \mathfrak{TR}_{n+3}$ over $|\mathcal{C}|$. At the level of based maps we have an induced map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathfrak{TR}_n)_{|\mathcal{C}|} & \to & \mathrm{Hom}_{\mathcal{O}_G^P}(\Phi X, \mathfrak{TR}_{n+3})_{|\mathcal{C}|} \\ F & \mapsto & \widetilde{F} \end{array}$$

with $\widetilde{F}_{G/K}(x) := \varepsilon_{G/K}(\mu_{G/K}(\kappa_{G/K}(x)), F(x))$, such that it induces a degree three homomorphism

$$\eta\cup : \mathbb{H}_G^n(\Phi X, \mu) \to \mathbb{H}_G^{n+3}(\Phi X, \mu).$$

Remark 4.6. The procedure described above defines in general a $H^*(X, \mathbb{Z})^G$-module structure on $\mathbb{H}_G^*(\Phi X, \mu)$ by the cup product. Therefore we could say that the degree three homomorphism $\eta\cup$ is equivalent to performing the cup product with the class η.

If the group G is trivial, the class $\eta \in H^3(X, \mathbb{Z})$ classifies the projective unitary bundles over X, and it was proven by Atiyah and Segal [3] that the third differential of Segal's spectral sequence was equivalent to the homomorphism $Sq^3_{\mathbb{Z}} - \eta\cup$.

Theorem 4.7. *Consider the Segal's spectral sequence defined in Theorem 4.4 and the isomorphism of its second page with the Bredon cohomology with coefficients in twisted representations*

$$E_2^{p,q} \cong \mathbb{H}_G^p(\Phi X, \mu)$$

whenever q is even. Then the third differential of the spectral sequence $d_3 : E_2^{p,q} \to E_2^{p+3,q-2}$ is a natural transformation in Bredon cohomology with local coefficients in twisted representations.

Proof. The result follows from Brown's representability theorem (see section 6.4 in the Appendix for a discussion of Brown representability in the parametrized and equivariant setting). Since the third differential is a homomorphism

$$\mathbb{H}_G^p(\Phi X, \mu) \to \mathbb{H}_G^{p+3}(\Phi X, \mu)$$

which is functorial and only depends on the map $\mu : \Phi X \to |\mathcal{C}|$, the third differential is thus given by a map of $\mathfrak{TR}_p \to \mathfrak{TR}_{p+3}$ of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$. □

Note that a map from $(\mathfrak{TR}_p)_{G/K} \to (\mathfrak{TR}_{p+3})_{G/K}$ over $|\mathcal{C}_{G/K}|$ is determined by a $\mathrm{Hom}(K, S^1)$-equivariant map

$$\overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \times K(R_{S^1}(\widetilde{K}), n) \to K(R_{S^1}(\widetilde{K}), n+3).$$

The assembly of these maps produces a map $\mathfrak{TR}_p \to \mathfrak{TR}_{p+3}$ of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$.

In the case that the acting group is trivial, Atiyah and Segal have proved [3, Prop. 4.6] that the map of Eilenberg-MacLane spaces

$$K(\mathbb{Z}, 3) \times K(\mathbb{Z}, p) \to K(\mathbb{Z}, p+3)$$

is given by the operation $(\eta, b) \mapsto Sq^3_{\mathbb{Z}} b - \eta \cup b$.

Equivariantly, the situation is much more involved. A complete description of natural transformations in Bredon cohomology with local coefficients is not available in the literature.

Even untwisted, the expression for the third differential in Bredon cohomology turns out to be considerably different. One could expect that a version of Steenrod cubes defined as follows should cover the natural transformations.

For any S^1 central extension $\widetilde{K}$, the group of twisted representations $R_{S^1}(\widetilde{K})$ is a free $\mathbb{Z}$-module generated by the irreducible representations of $\widetilde{K}$ on which S^1 acts by scalar multiplication. Therefore we have the short exact sequence of coefficients

$$0 \to R_{S^1}(\widetilde{K}) \xrightarrow{\times 2} R_{S^1}(\widetilde{K}) \xrightarrow{\mathrm{mod}_2} R_{S^1}(\widetilde{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to 0 \tag{4.8}$$

and we can consider the composition of maps of Ω-spectra

$$HR_{S^1}(\widetilde{K}) \xrightarrow{\mathrm{mod}_2} H(R_{S^1}(\widetilde{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^2} \Sigma^2 H(R_{S^1}(\widetilde{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2) \xrightarrow{\beta} \Sigma^3 HR_{S^1}(\widetilde{K})$$

where the first map is the reduction modulo 2 map, the second is the Steenrod square defined over each $\mathbb{Z}/2$-module generated by irreducible representations, and the third map is the Bockstein map induced by the short exact sequence of (4.8).

Denote the composition

$$\mathrm{Sq}^3_{\widetilde{K}} = \beta \circ \mathrm{Sq}^2 \circ \mathrm{mod}_2 : HR_{S^1}(\widetilde{K}) \to \Sigma^3 HR_{S^1}(\widetilde{K})$$

and note that it is compatible with the $N_G(K)/K$-action on $HR_{S^1}(\widetilde{K})$ and with the restriction maps. At the level of the $\mathcal{O}^P_G$-spaces over $|\mathcal{C}|$ we see that the maps $\mathrm{Sq}^3_{\widetilde{K}}$ induce maps

$$\begin{array}{c} \bigsqcup_{\widetilde{K} \in \mathrm{Ext}(K,S^1)} \overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \times_{\mathrm{Hom}(K,S^1)} K(R_{S^1}(\widetilde{K}), n) \\ \Big\downarrow \mathrm{Sq}^3_{\widetilde{K}} \\ \bigsqcup_{\widetilde{K} \in \mathrm{Ext}(K,S^1)} \overline{|\mathcal{C}_{G/K}|_{\widetilde{K}}} \times_{\mathrm{Hom}(K,S^1)} K(R_{S^1}(\widetilde{K}), n+3) \end{array}$$

which can be assembled into a map that we denote

$$\mathrm{Sq}^3 : \mathfrak{TR}_n \to \mathfrak{TR}_{n+3},$$

which furthermore assembles into a map of $\mathcal{O}^P_G$-spectra over $|\mathcal{C}|$ which we denote

$$\mathrm{Sq}^3 : \mathfrak{TR} \to \Sigma^3 \mathfrak{TR}.$$

At the level of Bredon cohomology with local coefficients in twisted representations, the map Sq^3 induces a degree three homomorphism

$$\mathrm{Sq}^3 : \mathbb{H}^p_G(\Phi X, \mu) \to \mathbb{H}^{p+3}_G(\Phi X, \mu) \tag{4.9}$$

which will be denoted by the same symbol in order to simplify the notation.

The Steenrod cube over twisted representation vanishes on zero dimensional Bredon cohomology classes. The coincidence of the third differential for the spectral sequence with this cohomology operation would imply that the edge homomorphism

$$K^0_G(X) \to E^{0,2r}_\infty \to E^{0,2r}_2 \cong H^0_G(X; R(?))$$

is surjective. However, evidence in specific computations [21, Ex. 5.2 pp. 614] and [20, Lemma 3.3 pp. 6] shows that this is not the case. The first author thanks Dieter Degrijse and Justin Noel for conversations on this issue leading to a precision on the first version of this note.

5. Applications

5.1. **Equivariant K-theory.** When Segal's spectral sequence is applied to non-twisted equivariant K-theory, it is known that the second page of the spectral sequence is isomorphic to the Bredon cohomology with coefficients in representations

$$E_2^{p,q} = \mathbb{H}^p_G(X, \mathcal{R}(?))$$

where $\mathcal{R}(G/K) = R(K)$ is the representation ring of K.

5.2. **The case of $\eta = 0$.** If the restriction of the class $\overline{\eta} \in H^3(X \times_G EG; \mathbb{Z})$ to $H^3(X; \mathbb{Z})$ is zero, then we have that all the higher differentials of Segal's spectral sequence vanish if we tensor the spectral sequence with the rationals. This follows from the fact that the operations on the Eilenberg-MacLane spectrum are all torsion operations. In this case Segal's spectral sequence tensored with the rationals collapses at the second page, and therefore the twisted equivariant K-theory is isomorphic to the Bredon cohomology with local coefficients in twisted representations after tensoring both cohomology groups with the rationals.

5.3. **Twisted equivariant K-theory for trivial G-spheres.** We know from [6, Theorem 4.8] that the twistings are classified by $H^3(X \times_G EG; \mathbb{Z})$. In the case that X is a trivial G-space, we have that the group G is finite and the Borel cohomology group satisfies

$$H^3(X \times_G EG; \mathbb{Z}) \cong H^3(X \times BG; \mathbb{Z}),$$

and if X has torsion free integral cohomology, by the Künneth isomorphism we obtain

$$H^3(X \times BG; \mathbb{Z}) \cong \bigoplus_{i=0}^{3} H^i(X; \mathbb{Z}) \otimes H^{3-i}(BG; \mathbb{Z}).$$

In the case that $X = S^1$, given

$$\alpha = [P] \in H^3(S^1 \times_G EG; \mathbb{Z}) \cong H^2(BG; \mathbb{Z}) \oplus H^3(BG; \mathbb{Z}),$$

the class α can be decomposed as $\alpha = \gamma \oplus \beta$, with $\gamma \in H^2(BG; \mathbb{Z}) \cong \mathrm{Hom}(G, S^1)$ and $\beta \in H^3(BG; \mathbb{Z}) \cong \mathrm{Ext}(G, S^1)$. To the homomorphism $\gamma : G \to S^1$ one can associate the linear 1-dimensional representation ρ_γ, and let $1 \to S^1 \to \widetilde{G} \to G \to 1$ be the S^1-central extension of G associated to β.

If U and V are two open contractible subsets of S^1, with $U \cup V = S^1$ and $U \cap V \simeq S^0$, then the Mayer-Vietoris sequence for $K^*_G(S^1; P)$ is given by the following six-terms exact sequence

$$\begin{array}{ccccc} K^0_G(S^1;P) & \longrightarrow & K^0_G(U;P|_U) \oplus K^0_G(V;P|_V) & \longrightarrow & K^0_G(U\cap V;P|_{U\cap V}) \\ \uparrow & & & & \downarrow \\ K^1_G(U\cap V;P|_{U\cap V}) & \longleftarrow & K^1_G(U;P|_U) \oplus K^1_G(V;P|_V) & \longleftarrow & K^1_G(S^1;P). \end{array}$$

On the other hand (cf. [6, Section 5.3.4]), the isomorphisms $K^0_G(U; P|_U) \cong R_{S^1}(\widetilde{G}) \cong K^0_G(V; P|_V)$ and $K^0_G(U \cap V; P|_{U\cap V}) \cong R_{S^1}(\widetilde{G}) \oplus R_{S^1}(\widetilde{G})$ fit in the following commutative diagram

$$\begin{array}{ccc} K^0_G(U;P|_U) \oplus K^0_G(V;P|_V) & \longrightarrow & K^0_G(U\cap V;P|_{U\cap V}) \\ \downarrow \cong & & \downarrow \cong \\ R_{S^1}(\widetilde{G}) \oplus R_{S^1}(\widetilde{G}) & \xrightarrow{j^*} & R_{S^1}(\widetilde{G}) \oplus R_{S^1}(\widetilde{G}) \end{array},$$

where the bottom morphism $j^* : (a, b) \mapsto (a - b, a - \rho_\gamma \cdot b)$ is induced by the inclusions $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$; thus we obtain the exact sequence

$$0 \longrightarrow K^0_G(S^1; P) \longrightarrow R_{S^1}(\widetilde{G}) \xrightarrow{\times(1-\rho_\gamma)} R_{S^1}(\widetilde{G}) \longrightarrow K^1_G(S^1; P) \longrightarrow 0$$

which implies that the K-theory groups are respectively the invariants and the coinvariants of the operator ρ_γ, i.e.

$$K^0_G(S^1; P) \cong R_{S^1}(\widetilde{G})^{\rho_\gamma} \quad \text{and} \quad K^1_G(S^1; P) \cong R_{S^1}(\widetilde{G})/(1 - \rho_\gamma)R_{S^1}(\widetilde{G}).$$

For the 2-dimensional sphere, the Borel cohomology is given by

$$H^3(S^2 \times_G EG; \mathbb{Z}) \cong H^3(BG; \mathbb{Z})$$

by Künneth formula and the fact $H^2(S^2; \mathbb{Z}) \otimes H^1(BG; \mathbb{Z}) = 0$, since G is finite. So, in this case there is only discrete torsion and

$$K^*_G(S^2; P) \cong K^*(S^2) \otimes R_{S^1}(\widetilde{G}).$$

where $\widetilde{G}$ is the S^1-central extension associated to $[P]$.

For $X = S^3$ with a trivial G-action, we have

$$H^3(S^3 \times_G EG; \mathbb{Z}) \cong H^3(S^3; \mathbb{Z}) \oplus H^3(BG; \mathbb{Z}),$$

thus every cohomology class $\alpha \in H^3_G(S^3; \mathbb{Z})$ can be decomposed as $n\gamma \oplus \beta$, where $\gamma \in H^3(S^3; \mathbb{Z})$ is the generator and $\beta \in H^3(BG; \mathbb{Z})$. Take a projective unitary stable bundle P over S^3 which is classified by the class $n\gamma \oplus \beta$. Then in this case the second page of Segal's spectral sequence is isomorphic to

$$H^*(S^3) \otimes_{\mathbb{Z}} R_{S^1}(\widetilde{G})$$

and the third differential is given by cupping with the class $n\gamma \otimes 1$. Therefore we get that for $n \neq 0$

$$K^0_G(S^3; P) = 0 \quad \text{and} \quad K^1_G(S^3; P) \cong \mathbb{Z}/n \otimes_{\mathbb{Z}} R_{S^1}(\widetilde{G}).$$

5.4. **Discrete torsion.** One of the first versions of twisted equivariant K-theory were defined with the information of a 2-cocycle $Z^2(G, S^1)$ whenever the group was finite (see [27, 23] and references therein); these cocycles were called *discrete torsion*. Using the fact that the group $H^2(G, S^1)$ classifies isomorphism classes of S^1-central extensions of the group G, this definition of the twisted equivariant K-theory was generalized to the context of proper and discrete actions in [12], under the additional hypothesis that the class $\eta \in H^2(G, S^1)$ is a finite order element. With our setup we can recover the twisted equivariant K-theory groups associated to discrete torsion, as well as the spectral sequence developed in [12].

Let G be a countable discrete group and let $1 \to S^1 \to \widetilde{G} \to G \to 1$ be a S^1-central extension of G which is classified by the cohomology class $\alpha \in H^2(G, S^1)$. Consider $L^2(\widetilde{G})$, the square integrable complex functions on $\widetilde{G}$, and endow it with the natural $\widetilde{G}$-action given by composition $(g \cdot f)(h) := f(hg^{-1})$. Let

$$V(\widetilde{G}) := \{f \in L^2(\widetilde{G}) \mid f(hx) = f(h)x \text{ for all } h \in \widetilde{G} \text{ and } x \in S^1\}$$

be the subspace on which S^1 acts by multiplication and let $\mathcal{H} := V(\widetilde{G}) \otimes L^2([0, 1])$ be the $\widetilde{G}$-Hilbert space on which kernel $\widetilde{G} \to G$ acts also by multiplication. Let $\mathcal{U}(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$ and note that the $\widetilde{G}$-action on $V(\widetilde{G})$ defines a homomorphism

$$\widetilde{\rho} : \widetilde{G} \to \mathcal{U}(\mathcal{H})$$

whose projectivisation $\rho : G \to P\mathcal{U}(\mathcal{H})$ makes the following diagram commutative

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\widetilde{\rho}} & \mathcal{U}(\mathcal{H}) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\rho} & P\mathcal{U}(\mathcal{H}). \end{array}$$

For every orbit type G/K with K finite, define the functor

$$\rho_{G/K} := \mathrm{Funct}_{st}(G \ltimes G/K, P\mathcal{U}(\mathcal{H}))$$

by the equation $\rho_{G/K}(g, h[K]) := \rho(g)$, and note that this assignment is functorial since any G-equivariant map $\psi : G/K \to G/H$ induces a functor $G \rtimes G/K \to G \rtimes G/H, (g, h[K]) \mapsto (g, \psi(h[K]))$, and therefore the first coordinate stays fixed. Moreover, since $L^2(\widetilde{K}) \subset L^2(\widetilde{G})$, where $\widetilde{K}$ denotes the S^1-central extension of K induced by $\widetilde{G}$ and the inclusion $K \subset G$, then we know by Peter-Weyl's theorem that $\mathcal{H}$ includes all irreducible representations of $\widetilde{K}$ on which the circle acts by multiplication, an infinitely number of times; therefore we know that $\rho_{G/K}$ is a stable functor, since its restriction to the group K is a stable homomorphism (see Definition 3.1), and therefore it defines a point in $|\mathcal{C}_{G/K}|$.

For every proper G-CW-complex X we can associate the map of $\mathcal{O}_G^P$-spaces

$$\overline{\rho}^X : \Phi X \to |\mathcal{C}|$$

such that for every orbit type we get the constant map

$$\overline{\rho}^X_{G/K} : X^K \mapsto |\mathcal{C}_{G/K}|, \qquad x \mapsto \rho_{G/K}.$$

In this way we get that the twisted G-equivariant K-theory groups $K^*_G(\Phi X; \overline{\rho}^X)$ realize the twisted G-equivariant K-theory groups associated to the S^1-central extension $\widetilde{G}$ defined by Dwyer in [12]. Now, since the map $\overline{\rho}^X$ is constant on each orbit type and only depends on the central extension $\widetilde{G}$ defined by α, we could define the contravariant $\mathcal{O}_G^P$-module $\mathcal{R}_\alpha(?)$ with $\mathcal{R}_\alpha(G/K) := R_{S^1}(\widetilde{K})$ thus obtaining a canonical isomorphism

$$\mathbb{H}^*_G(\Phi X, \overline{\rho}^X) \cong \check{H}^*_G(X; \mathcal{R}_\alpha(?))$$

between the Bredon cohomology of the map $\overline{\rho}^X$ and the Bredon cohomology with coefficients in the twisted representations $\mathcal{R}_\alpha(?)$.

The groups $\check{H}^*_G(X; \mathcal{R}_\alpha(?))$ are the ones shown in [12] to be isomorphic to the second page of the Atiyah-Hirzebruch spectral sequence that converges to the twisted equivariant K-theory groups $K^*_G(\Phi X; \overline{\rho}^X)$.

The methods developed in the present work have been successfully applied in [7] for the explicit calculation of the twisted $Sl_3(\mathbb{Z})$-equivariant K-theory and K-homology of the space $\underline{E}Sl_3(\mathbb{Z})$. In this case, the calculations are done using an universal coefficients theorem for α-twisted Bredon cohomology, and the fact that the spectral sequence constructed in this work collapses at the second page.

6. Appendix: Brown representability

The content of this appendix is based on Chapter 7 of [25]. We assume the reader is familiar with the qf-model category structure defined in [25, Section 6.2].

6.1. **Based G-CW-complexes.** Let B denote a fixed proper G-CW-complex. A based proper G-CW-complex is a pair $(X; x)$ with X a G-CW-complex, $X - \{x\}$ a proper G-CW-complex and x a G-fixed point.

A based G-space over B is a triple $X = (X, p, s)$ where $p : X \to B$ and $s : B \to X$ are G-maps and $p \circ s = \mathrm{id}_B$. A map $X \to X'$ of based G-spaces over B is a map of based G-spaces that commute with projections and sections. We denote the space of such maps by $\mathrm{Hom}^0_{G,B}(X, X')$ and by ${}_G[X, X']^0_B$ the corresponding set of homotopy classes.

Let (X, p) be a G-space over B. We use the notation $(X, p)_+$ for the union $X \coprod B$ of a based G-space (X, p) over B with a disjoint section, i.e. $(X, p)_+ = (X \coprod B, p \coprod \mathrm{id}, i)$, where $i : B \to X \coprod B$ is the natural inclusion.

If (X, p) is a G-space over B and Z is a based G-space, then let $X \times_B Z$ be the G-space $X \times Z$ with projection the product of the projections $p : X \to B$ and $Z \to *$. Define $X \wedge_B Z$ to be the quotient of $X \times_B Z$ obtained by taking fiberwise smash products, so that $(X \wedge_B Z)_b = X_b \wedge Z$; the basepoints of fibers prescribe the section.

For G-spaces (X, p) and (Y, q) over B, $X \times_B Y$ is the pullback of the projections $p : X \to B$ and $q : Y \to B$, with the evident G-projection $X \times_B Y \to B$. When X and Y have G-equivariant sections s and t, their pushout $X \vee_B Y$ specifies the coproduct, or wedge, of X and Y in the category of based proper G-spaces, and s and t induce a G-map $X \vee_B Y \to X \times_B Y$ over B that sends x and y to $(x, tp(x))$ and $(sq(y), y)$. Then $X \wedge_B Y$ is the pushout in the category of compactly generated spaces over B, displayed in the diagram

$$\begin{array}{ccc} X \vee_B Y & \longrightarrow & X \times_B Y \\ \downarrow & & \downarrow \\ *_B & \longrightarrow & X \wedge_B Y. \end{array}$$

This implies that $(X \wedge_B Y)_b = X_b \wedge Y_b$, and the section and projection are evident maps.

We denote by $\Sigma_B X$ the G-space $S^1 \wedge_B X$ over B, where S^1 has the trivial G-action.

6.2. **qf-model category structure for G-CW-complexes over B.** Let n be a natural number. Let I^G be the set of all maps of the form $G/H_+ \times i$, where H is a finite subgroup of G and i runs through the set of based inclusions $i : S^{n-1}_+ \to D^n_+$ (where S^{-1} is empty). Analogously, let J^G be the set of all maps of the form $G/H_+ \times i_0$, where H is a finite subgroup of G and i_0 runs through the set of based maps $i_0 : D^n_+ \to (D^n \times I)_+$.

Given maps $i : (X, p) \to (Y, q)$ and $d : (Y, q) \to B$ of based G-CW-complexes, the composition $d \circ i : (X, p) \to B$ defines i as a map over B. We write $i(d)$ for this map over B. Let I^G_B be the set of all such maps $i(d)$ with $i \in I^G$, and denote by J^G_B the set of all such maps $j(d)$ with $j \in J^G$.

In order to define the qf-model category structure on proper G-CW-complexes over B we need to recall the definition of q-fibration.

Proposition 6.1. [25, Prop. 6.2.2] *The following conditions on a map of compactly generated spaces $p : E \to Y$ are equivalent. If they are satisfied, then p is called a q-fibration.*

(i) *The map p satisfies the covering homotopy property with respect to disks D^n; that means there is a lift in the following diagram*

$$\begin{array}{ccc} D^n & \xrightarrow{\alpha} & E \\ \downarrow & \nearrow & \downarrow p \\ D^n \times I & \xrightarrow{h} & Y. \end{array}$$

(ii) *If h is a homotopy relative to the boundary S^{n-1} in the diagram above, then there is a lift that is a homotopy relative to the boundary.*

(iii) *The map p has the relative lifting property (RLP) with respect to the inclusion $S^n_+ \to D^{n+1}$ of the upper hemisphere into the boundary S^n of D^{n+1}; that is, there is a lift in the diagram*

$$\begin{array}{ccc} S^n_+ & \xrightarrow{\alpha} & E \\ \downarrow & \nearrow & \downarrow p \\ D^{n+1} & \xrightarrow{\bar{h}} & Y. \end{array}$$

Definition 6.2. A map g of spaces over B is an f-cofibration if it satisfies the fiberwise homotopy extension property (HEP), that is, if it has the left lifting property (LLP) with respect to the maps $p_0 : \mathrm{Map}_B(I, X) \to X$.

A map $d : D^n \to B$ of compactly generated spaces is said to be an f-disk if $i(d) : S^{n-1} \to D^n$ is an f-cofibration. An f-disk $d : D^{n+1} \to B$ is said to be a relative f-disk if the lower hemisphere S^n_- is also an f-disk, so that the restriction $i(d) : S^{n-1} \to S^n_-$ is an f-cofibration.

A map $f : (X, p, s) \to (Y, q, t)$ of based G-spaces over B is called a q-equivalence if $f : X \to Y$ is a G-equivariant weak equivalence of spaces (forgetting the based structure over B). Define I^f_B to be the set of inclusions $i(d) : S^{n-1} \to D^n$ where $d : D^n \to B$ is an f-disk. Define J^f_B to be the set of inclusions $i(d) : S^n_+ \to D^{n+1}$ of the upper hemisphere into a relative f-disk $d : D^{n+1} \to B$. A map of compactly generated spaces over B is said to be

(i) a qf-fibration if it has the RLP with respect to J^f_B and
(ii) a qf-cofibration if it has the LLP with respect to all q-acyclic qf-fibrations, that is, with respect to those qf-fibrations that are q-equivalences.

Now we proceed equivariantly. Let $\mathcal{O}_G^{\mathcal{ALL}}$ denote the set of all orbits G/H.

Definition 6.3. A set $\mathcal{C}$ of proper G-CW-complexes that contains the orbits G/K with $K \in \mathcal{FIN}(G)$ and is closed under products with elements in $\mathcal{O}_G^{\mathcal{ALL}}$ is called a generating set. It is closed if it is closed under finite products.

Let $\mathcal{C}$ be a generating set.

(i) Let $I^f_B(\mathcal{C})$ be the set of maps

$$(\mathrm{id} \times i)(d) \coprod \mathrm{id} : C \times S^{n-1} \coprod B \to C \times D^n \coprod B$$

such that $C \in \mathcal{C}$, $d : C \times D^n \to B$ is a G-map, i is the boundary inclusion, and the associated map $\widetilde{i}$ over $\mathrm{Map}_G(C, B)$ is a generating qf-cofibration in the category of compactly generated spaces over $\mathrm{Map}_G(C, B)$.

(ii) Let $J^f_B(\mathcal{C})$ consist of the maps

$$(\mathrm{id} \times i)(d) \coprod \mathrm{id} : C \times S^n_+ \coprod B \to C \times D^{n+1} \coprod B$$

such that $C \in \mathcal{C}$, $d : C \times D^{n+1} \to B$ is a G-map, i is the inclusion, and the associated map $\widetilde{i}$ over $\mathrm{Map}_G(C, B)$ is a generating acyclic qf-cofibration in the category of compactly generated spaces over $\mathrm{Map}_G(C, B)$.

Fixing a generating set $\mathcal{C}$, we define a qf-type model structure based on $\mathcal{C}$, called the $qf(\mathcal{C})$-model structure. Its weak equivalences are the weak equivalences of proper G-CW-complexes. We define now the $qf(\mathcal{C})$-fibrations.

Definition 6.4. A G-map over B is a $qf(\mathcal{C})$-fibration if $\mathrm{Map}_G(C, f)$ is a qf-fibration in the category of compactly generated spaces over $\mathrm{Map}_G(C, B)$, for all $C \in \mathcal{C}$.

In [25, Section 5.5, p.90], the notion of *well grounded* model category is introduced. There it is established that the category of based proper G-CW-complexes can be endowed with a structure of a well grounded model category.

Theorem 6.5. [25, Th. 7.2.8] *For any generating set $\mathcal{C}$ the category of based proper G-CW-complexes over B is a well grounded model category. The weak equivalences are the based weak G-homotopy equivalences and fibrations are $qf(\mathcal{C})$-fibrations. The sets $I^f_B(\mathcal{C})$ and $J^f_B(\mathcal{C})$ are the generating $qf(\mathcal{C})$-cofibrations and the generating acyclic $qf(\mathcal{C})$-cofibrations.*

We define a qf-fibration in the category of based G-spaces over B as a map which is a qf-fibration when regarded as a map of G-spaces over B, and similarly for qf-cofibrations.

Let (Y, q, t) is a based G-space over B and $f : A \to B$ be a G-map, we define f^*Y as the based G-space over A obtained from the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ f^*Y & \longrightarrow & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

On the other hand if (X, s, p) is a based G-space over A and $f : A \to B$, define f_*X and its structure maps q and t by means of the map of retracts in the following diagram on the left, where the top square is a pushout and the bottom square is defined by the universal property of pushouts and the requirement that $q \circ t = id$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ X & \longrightarrow & f_!X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

Recall that an adjoint pair of functors (T, U) between model categories is a *Quillen adjoint pair*, or a *Quillen adjunction*, if the left adjoint T preserves cofibrations and acyclic cofibrations or, equivalently, the right adjoint U preserves fibrations and acyclic fibrations. It is a *Quillen equivalence* if the induced adjunction on homotopy categories is an adjoint equivalence.

The following base change result will be useful in the following:

Proposition 6.6. . *If $f : Z_1 \to Z_2$ is a G-map, then the pair (f_*, f^*) is a Quillen adjunction for the model category structures defined above. Morever, if f is a weak*

$\mathcal{F}$-equivalence with respect to a family of subgroups $\mathcal{F}$, then (f_*, f^*) is a Quillen equivalence

Proof. This is the content of Propositions 7.3.4 and 7.3.5 in [25]. Note that f_* is denoted by $f_!$ in [25].

□

Theorem 6.7. *Let G be a discrete group. For a G-CW complex B, there exists a ziz-zag of Quillen equivalences between the category of G-spaces over B with the qf-model structure and $\mathcal{O}_G^P$-spaces over ΦB with the levelwise model structure.*

Proof. Let Φ be the fixed point functor, which associates an $\mathcal{O}_G^P$-complex to a G-CW complex X. Notice that the additional section or projection data $p : X \to B$, and $s : B \to X$ restrict to fixed points.

Let J be a functorial cellular approximation functor in the category of $\mathcal{O}_G^P$-spaces (See Theorem 3.7 in [10]), in the sense that $J(\mathcal{X})$ is a free $\mathcal{O}_G^P$-CW complex for every $\mathcal{O}_G^P$-space $\mathcal{X}$, and $J(\mathcal{X}) \to \mathcal{X}$ is a weak $\mathcal{O}_G^P$-equivalence .

The cellular approximation functor defines a map $\epsilon : J(\Phi B) \times_{\mathcal{O}_G^P} \nabla \to B$, which is a G-homotopy equivalence. The base change functors (ϵ_*, ϵ^*) satisfy the hypothesis of Proposition 6.6 thus giving a Quillen equivalence pair between the categories of G-spaces over B and over $J(\Phi B) \times_{\mathcal{O}_G^P} \nabla$.

Analogously, the mentioned cellular approximation defines a map of $\mathcal{O}_G^P$-spaces $\epsilon' : J\Phi(B) \to \Phi(B)$ giving a Quillen equivalence ϵ'_*, ϵ'^* between the categories of $\mathcal{O}_G^P$-spaces over ΦB and over $J(\Phi B)$.

Finally, the $\mathcal{O}_G^P$ map $\Phi B \to J(\Phi B)$ determines a Quillen equivalence pair between the categories of $\mathcal{O}_G^P$ spaces over ΦB and G-spaces over $J(\Phi B) \times_{\mathcal{O}_G^P} \nabla$.

□

6.3. **Generating sets in the category of proper G-CW-complex over B.** Let $\mathfrak{H}$ be a category with weak colimits, denoted by $\operatorname{hocolim} Y_n$, we say that an object X of $\mathfrak{H}$ is compact if

$$\operatorname{colim} \mathfrak{H}(X, Y_n) \cong \mathfrak{H}(X, \operatorname{hocolim}(Y_n))$$

for any sequence of maps $Y_n \to Y_{n+1}$ in $\mathfrak{H}$.

Definition 6.8. A set $\mathcal{D}$ of objects in a pointed category $\mathfrak{H}$ is a generating set if a map $f : X \to Y$ such that $f_* : \mathfrak{H}(D, X) \to \mathfrak{H}(D, Y)$ is a bijection for all $D \in \mathcal{D}$ is an isomorphism.

For $n > 0$, $b \in B$, and $H \subset G_b$, let $S_H^{n,b}$ be the based G-space over B given by $(G/H_+ \wedge S^n) \vee_b B$, where the wedge is taken with respect to the standard basepoint of $G/H_+ \wedge S^n$ and the basepoint $b \in B$. The inclusion of B gives the section and the projection maps $G/H_+ \wedge S^n$ to the point b and maps B by the identity map.

Let $\mathbb{D}_B^c$ be the set of all such based G-spaces $S_H^{n,b}$ over B, with $n > 0$. Then, from [25, Lemma 7.5.13-14] it follows the next result.

Lemma 6.9. *$\mathbb{D}_B^c$ is a generating set in the homotopy category of based G-connected spaces over B. Moreover, each element in $\mathbb{D}_B^c$ is a compact object in that category.*

We want to use the following abstract Brown representability theorem.

Theorem 6.10. [25, Th. 7.5.7] *Let $\mathfrak{H}$ be a category with coproducts and weak pushouts. Assume that $\mathfrak{H}$ has a generating set of compact objects. Let $k : \mathfrak{H} \to \mathsf{SETS}$ be a contravariant functor that takes coproducts to products and weak pushouts to weak pullbacks. Then there is an object $Y \in \mathfrak{H}$ and a natural isomorphism $k(X) \cong \mathfrak{H}(X, Y)$ for $X \in \mathfrak{H}$.*

Given a model category $\mathfrak{T}$, it is possible to construct the *homotopy category* $\mathfrak{H}$. For its definition see [13].

6.4. **Equivariant parametrized homotopy theories and Brown representability.** Let us recall that a functor $\mathcal{H}_G^B$ defined on the category of based proper G-CW-complexes over B with values in $\mathbb{Z}$-modules is a proper reduced generalized cohomology theory over B if it satisfies a parametrized version of the Eilenberg-Steenrod axioms (except the reduced dimension axiom). For definitions of axioms see for example [6, Section 4.3.2] or [25, Definition 20.1.2].

Theorem 6.11. *Let $\mathcal{H}_G^*$ be a reduced proper G-equivariant parametrized cohomology theory over B. Then, there exist a sequence of proper G-CW-complexes $\mathcal{R}_n$ over B and natural transformations such that*

$$\mathcal{H}_G^n(X,p,s) \cong {}_G[X,\mathcal{R}_n]_B^0$$

for every based G-connected proper G-CW-complex X over B.

Proof. Given a G-equivariant parametrized cohomology theory $\mathcal{H}_G^*$ over B, since the category of proper G-CW-complexes has a compact generating set $\mathbb{D}_B^c$, one applies Theorem 6.10 for the parametrized cohomology theory $\mathcal{H}_G^*$, and we obtain a Brown Representability Theorem for reduced proper G-equivariant parametrized cohomology theories. □

We can apply the above theorem for Bredon cohomology associated to a cover. Note that this functor $\mathbb{H}_G^p(\Phi(-),\mu)$ is an equivariant parametrized cohomology theory over $|\mathcal{C}|$. Therefore any operation in cohomology

$$\mathbb{H}_G^p(\Phi X,\mu) \to \mathbb{H}_G^{p+q}(\Phi X,\mu)$$

which is functorial and only depends on the map $\mu : \Phi X \to |\mathcal{C}|$, must be obtained by a map $\mathfrak{TR}_p \to \mathfrak{TR}_{p+q}$ of $\mathcal{O}_G^P$-spaces over $|\mathcal{C}|$.

References

[1] S. A. Antonyan and E. Elfving. The equivariant homotopy type of G-ANR's for proper actions of locally compact groups. In *Algebraic topology—old and new*, volume 85 of *Banach Center Publ.*, pages 155–178. Polish Acad. Sci. Inst. Math., Warsaw, 2009.

[2] M. Atiyah and G. Segal. Twisted K-theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004.

[3] M. Atiyah and G. Segal. Twisted K-theory and cohomology. In *Inspired by S. S. Chern*, volume 11 of *Nankai Tracts Math.*, pages 5–43. World Sci. Publ., Hackensack, NJ, 2006.

[4] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I., 1961.

[5] M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators. *Inst. Hautes Études Sci. Publ. Math.*, (37):5–26, 1969.

[6] N. Bárcenas, J. Espinoza, M. Joachim, and B. Uribe. Universal twist in equivariant K-theory for proper and discrete actions. *Proc. Lond. Math. Soc. (3)*, 108(5):1313–1350, 2014.

[7] N. Barcenas and M. Velasquez. Twisted equivariant K-theory and K-homology of $\mathrm{Sl}_3(\mathbb{Z})$. *Algebraic and Geometric Topology*, 14(2):823–852, 2014. DOI: 10.2140/agt.2014.14.823.

[8] S. Basu and D. Sen. Representing Bredon cohomology with local coefficients. *J. Pure Appl. Algebra*, 219(9):3992–4015, 2015.

[9] V. Braun. Twisted K-theory of Lie groups. *J. High Energy Phys.*, (3):029, 15 pp. (electronic), 2004.

[10] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[11] C. L. Douglas. On the twisted K-homology of simple Lie groups. *Topology*, 45(6):955–988, 2006.

[12] C. Dwyer. Twisted equivariant K-theory for proper actions of discrete groups. *K-Theory*, 38(2):95–111, 2008.

[13] W. G. Dwyer and J. Spalinski. *Homotopy theories and model categories*. Handbook of algebraic topology, Elsevier, 1995.

[14] A. D. Elmendorf. Systems of fixed point sets. *Trans. Amer. Math. Soc.*, 277(1):275–284, 1983. MR MR690052 (84f:57029).

[15] D. S. Freed, M. J. Hopkins, and C. Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[16] D. S. Freed, M. J. Hopkins, and C. Teleman. Loop groups and twisted K-theory III. *Ann. of Math. (2)*, 174(2):947–1007, 2011.
[17] H. Honkasalo. The equivariant Serre spectral sequence as an application of a spectral sequence of Spanier. *Topology Appl.*, 90(1-3):11–19, 1998.
[18] D. Kneezel and I. Kriz. Completing Verlinde algebras. *J. Algebra*, 327:126–140, 2011.
[19] A. Lahtinen. The Atiyah-Segal completion theorem in twisted K-theory. *Algebraic and Geometric Topology*, 12(4):1925–1940, 2012. DOI: 10.2140/agt.2012.12.1925.
[20] I. J. Leary and D. Degrijse. Equivariant vector bundles over classifying spaces for proper actions. *arXiv: 1504.07358 [math.AT]*, 2015.
[21] W. Lück and B. Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology*, 40:585–616, 2001.
[22] W. Lück and B. Uribe. Equivariant principal bundles and their classifying spaces. *Algebraic and Geometric Topology*, 14(4):1925–1995, 2014. DOI: 10.2140/agt.2014.14.1925.
[23] E. Lupercio and B. Uribe. Gerbes over orbifolds and twisted K-theory. *Comm. Math. Phys.*, 245(3):449–489, 2004.
[24] T. Matumoto. Equivariant CW complexes and shape theory. *Tsukuba J. Math.*, 13(1):157–164, 1989.
[25] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.
[26] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.
[27] E. Witten. D-branes and K-theory. *J. High Energy Phys.*, (12):Paper 19, 41 pp. (electronic), 1998.

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