ON TOPOLOGICAL RIGIDITY OF ALEXANDROV 3-SPACES

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ABSTRACT. In this note we prove the Borel Conjecture for closed, irreducible and sufficiently collapsed three-dimensional Alexandrov spaces. We also pose several questions related to the characterization of fundamental groups of three-dimensional Alexandrov spaces, finite groups acting on them and rigidity results.

1. INTRODUCTION AND RESULTS

Alexandrov spaces are inner metric spaces which admit a lower sectional curvature bound in a synthetic sense. They constitute a generalization of the class of complete Riemannian manifolds with a lower sectional curvature bound and since their introduction they have proven to be a natural setting to address geometric-topological questions of a global nature. Therefore, a central problem is to determine whether what is already known in the smooth or topological settings still holds in Alexandrov geometry.

Regarding topological rigidity of spaces, an important conjecture originally formulated in the topological manifold category, is the *Borel conjecture*. It asserts that if two closed, aspherical n-manifolds are homotopy equivalent, then they are homeomorphic. The proof of this conjecture in the three-dimensional case is a consequence of Perelman's resolution of the Geometrization Conjecture [22].

On the other hand, in high dimension (meaning greater or equal than five), the Borel Conjecture for an aspherical manifold with fundamental group G is consequence of the Farrell-Jones Conjecture in Algebraic K- and L-Theory for the group G [14]. A lot of effort in geometric topology and surgery theory has been devoted to prove the Borel conjecture in many cases by these methods, which rely on transversality arguments which are not available for the study of topological rigidity of low dimensional manifolds.

We will present in the following note a series of questions related to generalizations of the Borel conjecture outside of the manifold category. Steps in this direction have been obtained, for example, for $\mathrm{CAT}(0)$-spaces as a consequence of the Farrell Jones-Conjecture [1], and in another direction for certain classes of topological orbifolds [24] as a consequence of classification efforts in three dimensional geometric topology beyond manifolds.

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Negative results concerning the topological rigidity of singular spaces of geometric nature, such as the Coxeter complex have been obtained in [23].

It is therefore natural to inquire whether the Borel conjecture is still valid for closed Alexandrov 3-spaces (cf. [6, Remark 3.12]). These spaces are either topological 3-manifolds or are homeomorphic to quotients of smooth 3-manifolds by smooth orientation-reversing involutions with only isolated fixed points (see [7]).

In this article we address the validity of the Borel Conjecture for the class of *sufficiently collapsed* and *irreducible* closed Alexandrov 3-spaces. The definition of *irreducibility* for a closed Alexandrov 3-space was introduced in [9]. Let us recall that a closed Alexandrov 3-space X is *irreducible* if every embedded 2-sphere in X bounds a 3-ball and, in the case that the set of topologically singular points of X is non-empty, it is further required that every 2-sided $\mathbb{R}P^2$ bound a $K(\mathbb{R}P^2)$, a cone over a real projective plane $\mathbb{R}P^2$. The condition related to collapse is described more precisely by considering the class of spaces $\mathcal{A}^3(D,\varepsilon)$, defined as the class of closed Alexandrov 3-spaces with $\operatorname{curv} \geq -1$, satisfying that $\operatorname{diam} \leq D$ and $\operatorname{vol} < \varepsilon$ for given $D, \varepsilon > 0$. We say that a closed Alexandrov 3-space X is sufficiently collapsed if $X \in \mathcal{A}^3(D,\varepsilon)$ for a sufficiently small ε with respect to D. Our main result is the following.

Theorem A. *For any $D > 0$, there exists $\varepsilon = \varepsilon(D) > 0$ such that, if $X, Y \in \mathcal{A}^3(D,\varepsilon)$ are aspherical and irreducible, then the Borel Conjecture holds for X and Y, that is, if X is homotopy equivalent to Y then X is homeomorphic to Y.*

We point out that a related result was obtained in [18, Theorem 6.1] where the second named author proved the Borel Conjecture for closed Alexandrov 3-spaces admitting an isometric circle action. The proof of Theorem A is based on two points: the Borel conjecture in the 3-manifold case and the following result.

Theorem B. *For any $D > 0$, there exists $\varepsilon = \varepsilon(D) > 0$ such that, if $X \in \mathcal{A}^3(D,\varepsilon)$ is irreducible and aspherical, then X is homeomorphic to a 3-manifold.*

The classification of closed collapsing Alexandrov 3-spaces due to Mitsuishi-Yamaguchi is a key tool in the proof of Theorem B. The classification of closed Alexandrov 3-spaces admitting isometric (local) circle actions [10], [18] obtained by Galaz-García and the second named author also plays a role. The strategy of proof resembles that of [9, Theorem A]. In fact, without assuming that the spaces in question are sufficiently collapsed or irreducible, the analysis of Section 3.2 implies the following result.

Corollary C. *Let X be a closed, non-orientable Alexandrov 3-space with fundamental group $\mathbb{Z}$, $\mathbb{Z} \rtimes \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Then $\pi_2(X) \neq 0$. In particular, X is not aspherical.*

For arbitrary dimension we observe the following result, which is an immediate consequence of a result of Mitsuishi (see [16, Corollary 5.7] and Theorem 3.1 below) and Theorem 3.2 (stated below).

Corollary D. *Let X be a closed, aspherical Alexandrov n-space such that its universal cover is compact. Then X is homeomorphic to a closed n-manifold.*

In light of Theorem B and Corollary D we propose the following natural conjecture (cf. Remark 3.4).

Conjecture E. *Every closed, aspherical Alexandrov* 3*-space is a* 3*-manifold*

The organization of the article is the following. In Section 2, we briefly recall the basic structure of Alexandrov 3-spaces following Galaz-García-Guijarro [7] and the classification of collapsing Alexandrov 3-spaces of Mitsuishi-Yamaguchi [17]. In Section 3, we prove Theorem B which yields as a consequence the validity of Theorem A. In section 4, we state some questions related to the fundamental groups and groups which can act on Alexandrov 3-spaces, in analogy with similar results obtained in connection with the topological rigidity of aspherical manifolds.

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2. Preliminaries

We will assume that the reader is familiar with the general theory of Alexandrov spaces of curvature bounded below and refer to [3] for a more detailed introduction. In this section we will briefly recall some results concerning the structure of closed Alexandrov 3-spaces. All spaces are considered to be connected throughout the article.

2.1. **Alexandrov** 3**-spaces.** Let X denote a closed Alexandrov 3-space and for each $x \in X$, let $\Sigma_x X$ be the space of directions at x. The space $\Sigma_x X$ is a closed Alexandrov 2-space with $\operatorname{curv} \Sigma_x X \geq 1$ (see [3, Theorem 10.8.6]). This implies, via the Bonnet-Myers Theorem (see [3, Theorem 10.4.1]) and the classification of closed surfaces that the homeomorphism type of $\Sigma_x X$ is that of a sphere $\mathbb{S}^2$ or that of a real projective plane $\mathbb{R}P^2$. A point $x \in X$ such that $\Sigma_x X$ is homeomorphic to $\mathbb{S}^2$ is called *topologically regular*, while a point such that $\Sigma_x X$ is homeomorphic to $\mathbb{R}P^2$ is called *topologically singular*. We let $S(X)$ be the subset of X consisting of topologically singular points. Then $X \setminus S(X)$ is open and dense in X (see [3, Theorem 10.8.5]). Furthermore, the Conical Neighborhood Theorem of Perelman [21] states that each point $x \in X$ has a neighborhood which is pointed-homeomorphic to the cone over $\Sigma_x X$. As a consequence, $S(X)$ is a finite set.

Topologically, a closed Alexandrov 3-space X can be described as a compact 3-manifold M having a finite number of $\mathbb{R}P^2$-boundary components with a cone over $\mathbb{R}P^2$ attached on each boundary component. In the case that $S(X) \neq \varnothing$ there is an alternative topological description of X as quotient of a closed, orientable, topological 3-manifold $\tilde{X}$ by an orientation-reversing involution $\iota : \tilde{X} \to \tilde{X}$ having only isolated fixed points. The 3-manifold $\tilde{X}$ is called the *branched orientable double cover of* X (see [7, Lemma 1.7]). It is possible to lift the Alexandrov metric on X to an Alexandrov metric on $\tilde{X}$ having the same lower curvature bound in such a way that ι is an isometry. In particular, ι is equivalent to a smooth involution on $\tilde{X}$ regarded as a smooth 3-manifold (a detailed description of this construction can be found in [7, Lemma 1.8], [4, Section 2.2] and [11, Section 5]).

2.2. **Collapsing Aleandrov 3-spaces.** Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed Alexandrov 3-spaces with diameters uniformly bounded above by $D > 0$ and $\operatorname{curv} X_i \geq k$ for some $k \in \mathbb{R}$. Gromov's Precompactness Theorem implies that (possibly after passing to a subsequence), there exists an Alexandrov space Y with diameter bounded above by D and $\operatorname{curv} Y \geq k$ such that $X_i \xrightarrow{GH} Y$. In the case in which $\dim Y < 3$, the sequence X_i is said to *collapse to* Y. Similarly, a closed Alexandrov 3-space X is a *collapsing Alexandrov* 3-*space* if there exists a sequence of Alexandrov metrics d_i on X such that the sequence $\{(X, d_i)\}_{i=1}^{\infty}$ is a collapsing sequence.

The topological classification of closed collapsing Alexandrov 3-spaces was obtained by Mitsuishi-Yamaguchi in [17]. We now give a brief summary of the classification. We denote the boundary of an Alexandrov space Y by ∂Y.

In the case in which $\dim Y = 2$ (cf. [17, Theorems 1.3, 1.5]), for sufficiently big i, X_i is homeomorphic to a *generalized Seifert fibered space* $\operatorname{Seif}(Y)$ (see [17, Definition 2.48]). In the case in which $\partial Y \neq \varnothing$, $\operatorname{Seif}(Y)$ is attached with a finite number of *generalized solid tori and Klein bottles* (see [17, Definition 1.4]).

In the event that $\dim Y = 1$ and $\partial Y = \varnothing$ (cf. [17, Theorem 1.7]), for big enough i, X_i is homeomorphic to an F-fiber bundle over $\mathbb{S}^1$, where F is homeomorphic to one of the spaces T^2, K^2, $\mathbb{S}^2$ or $\mathbb{R}P^2$. On the other hand, if $\partial Y \neq \varnothing$ (cf. [17, Theorem 1.8]), X_i is homeomorphic to a union of two spaces B and B' with one boundary component, glued along their homeomorphic boundaries, where ∂B is one of the spaces T^2, K^2, $\mathbb{S}^2$ or $\mathbb{R}P^2$. The pieces B and B' are determined as follows:

(i) If $\partial B \cong \mathbb{S}^2$ then B and B' are homeomorphic to one of: a 3-ball D^3, a 3-dimensional projective space with the interior of a 3-ball removed $\mathbb{R}P^3 \setminus \operatorname{int} D^3$ or $\mathrm{B}(S_2)$, a space homeomorphic to a small metric ball of an $\mathbb{S}^2$-soul of an open non-negatively curved Alexandrov space $L(\mathbb{S}^2; 2)$ (cf. [17, Corollary 2.56]).
(ii) If $\partial B \cong \mathbb{R}P^2$ then B and B' are homeomorphic to a closed cone over a projective plane $K_1(\mathbb{R}P^2)$.
(iii) If $\partial B \cong T^2$ then B and B' are homeomorphic to one of $\mathbb{S}^1 \times D^2$, $\mathbb{S}^1 \times \mathrm{Mo}$, the orientable non-trivial I-bundle over K^2, $K^2 \tilde{\times} I$ or $\mathrm{B}(S_4)$, a space homeomorphic to a small metric ball of an $\mathbb{S}^2$-soul of an open non-negatively curved Alexandrov space $L(\mathbb{S}^2; 4)$ (cf. [17, Corollary 2.56]).
(iv) If $\partial B \cong K^2$ then B and B' are homeomorphic to one of $\mathbb{S}^1 \tilde{\times} D^2$ the non-orientable D^2-bundle over $\mathbb{S}^1$, $K^2 \hat{\times} I$ the non-orientable non-trivial I-bundle over K^2, the space $\mathrm{B}(\mathrm{pt})$ defined in [17, Example 1.2], or $\mathrm{B}(\mathbb{R}P^2)$, a space homeomorphic to a small metric ball of an $\mathbb{R}P^2$-soul of an open non-negatively curved Alexandrov space $L(\mathbb{R}P^2; 2)$ (cf. [17, Corollary 2.56]).

Finally, if $\dim Y = 0$ (cf. [17, Theorem 1.9]), then for i sufficiently big, X_i is homeomorphic to either a generalized Seifert fiber space $\operatorname{Seif}(Z)$, (where Z is a 2-dimensional Alexandrov space with $\operatorname{curv} Z \geq 0$), a space appearing in the cases in which $\dim Y = 1, 2$ or a non-negatively curved Alexandrov space with finite fundamental group.

In order to provide information on the homotopy groups of some of the spaces appearing in the previous classification we now recall the celebrated Soul Theorem for Alexandrov spaces due to Perelman [20, §6]

Theorem 2.1 (Soul Theorem)**.** *Let X be a compact Alexandrov space of* $\operatorname{curv} \geq 0$ *with $\partial X \neq \varnothing$. Then there exists a totally convex, compact subset $S \subset X$, called the soul of X with $\partial S = \varnothing$ which is a strong deformation retract of X.*

The spaces $\mathrm{B}(S_2)$, $\mathrm{B}(S_4)$, $\mathrm{B}(\mathrm{pt})$ and $\mathrm{B}(\mathbb{R}P^2)$ admit Alexandrov metrics of $\operatorname{curv} \geq 0$. Therefore the Soul Theorem can be applied. Moreover, given that the soul is a strong deformation retract of the space, in particular we have a homotopy equivalence. Whence, $\pi_k(\mathrm{B}(S_2)) \cong \pi_k(\mathbb{S}^2)$, $\pi_k(\mathrm{B}(S_4)) \cong \pi_k(\mathbb{S}^2)$, while $\pi_k(\mathrm{B}(\mathrm{pt})) = 0$ and $\pi_k(\mathrm{B}(\mathbb{R}P^2)) \cong \pi_k(\mathbb{R}P^2)$ for all $k \geq 1$.

3. Proofs

We proceed to prove Theorem B. As stated in the Introduction, this result readily implies our main result, Theorem A.

Proof of Theorem B. We will proceed by contradiction. Let us suppose that the result in question does not hold. Then, there exists a sequence of closed, irreducible and aspherical Alexandrov 3-spaces $\{X_i\}_{i=1}^{\infty}$ of $\operatorname{curv} X_i \geq -1$, satisfying that $\operatorname{diam} X_i \leq D$ and $\operatorname{vol} X_i \xrightarrow{i\to\infty} 0$ which are not homeomorphic to 3-manifolds. Therefore by Gromov's precompactness Theorem we can assume (possibly passing to a non-relabeled subsequence) that X_i collapses in the Gromov-Hausdorff topology to a closed Alexandrov space Y of dimension < 3. We will split the proof in three cases depending on whether $\dim Y = 0, 1, 2$ and obtain a contradiction in each case. Observe that by our contradiction assumption in the following analysis we will exclude any Alexandrov 3-spaces appearing in the classification [17] that are homeomorphic to 3-manifolds.

3.1. 2-dimensional limit space.

In the case that $\dim Y = 2$, we need to address two further cases depending on whether X_i contains singular fibers with neighborhoods of type $\mathrm{B}(\mathrm{pt})$ or not. In the case in which X_i does not contain fibers of type $\mathrm{B}(\mathrm{pt})$, then by [10, Corollary 6.2], the collapse $X_i \xrightarrow{GH} Y$ is equivalent to the one obtained by collapsing along the orbits of a local circle action on X_i. Therefore, by [10, Theorem B] X_i is homeomorphic to a connected sum of the form $M \# \operatorname{Susp}(\mathbb{R}P^2) \# \cdots \# \operatorname{Susp}(\mathbb{R}P^2)$, where M is a closed 3-manifold admitting a local circle action. It now follows from [9, Lemma 5.1] that either X_i is homeomorphic to M or to $\operatorname{Susp}(\mathbb{R}P^2)$. Since we are assuming X_i is not homeomorphic to a 3-manifold, we conclude that X_i is homeomorphic to $\operatorname{Susp}(\mathbb{R}P^2)$. However, $\operatorname{Susp}(\mathbb{R}P^2)$ is not aspherical, as a combination of the suspension isomorphism and the Hurewicz Theorem yields that $\pi_2(\operatorname{Susp}(\mathbb{R}P^2))$ is isomorphic to $\mathbb{Z}_2$.

We move on to the case in which X_i contains fibers with tubular neighborhoods of the form $\mathrm{B}(\mathrm{pt})$. Let us consider the branched orientable double cover $\tilde{X}_i$ of X_i.

We now recall that, by [9, Corollary 3.2] X_i collapses if and only if the sequence of branched orientable double covers $\tilde{X}_i$ collapses. Then, by [10, Corollary 6.2], the connected sum decomposition in [10, Theorem B], and taking into account the orientability of $\tilde{X}_i$ we have that $\tilde{X}_i$ is homeomorphic to a connected sum of the form

$$(\underset{\varphi}{\#}\,\mathbb{S}^2\times\mathbb{S}^1)\#(\overset{n}{\underset{j=1}{\#}}\,L(\alpha_j,\beta_j)) \tag{3.1}$$

where $L(\alpha_j,\beta_j)$ denotes a lens space determined by the Seifert invariants (α_j,β_j) (see [19, Section 1.7]).

It was proved in [9, Page 14] that, in this situation, the connected sum (3.1) cannot contain both lens spaces and copies of $\mathbb{S}^2\times\mathbb{S}^1$, that is, either X_i is homeomorphic to a connected sum of lens spaces or to a connected sum of copies of $\mathbb{S}^2\times\mathbb{S}^1$. However, the irreducibility assumption implies, as in [9, Case 5.6], that the expression (3.1) cannot contain $\#_{j=1}^n L(\alpha_j,\beta_j)$ as a connected summand and therefore only copies of $\mathbb{S}^2\times\mathbb{S}^1$ appear in the connected sum 3.1. Therefore X_i is homeomorphic to $\#_\varphi\,\mathbb{S}^2\times\mathbb{S}^1$. Furthemore, it follows from the irreducibility of X_i as in [9, Case 5.7] that $\tilde{X}_i$ is homeomorphic to $\mathbb{S}^2\times\mathbb{S}^1$. Hence, it suffices to consider the case in which X_i is a quotient of $\mathbb{S}^2\times\mathbb{S}^1$ by an orientation reversing involution $\iota:\mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ having only isolated fixed points.

Let us consider $\mathbb{S}^2\times\mathbb{S}^1$ as a subspace of $\mathbb{R}^3\times\mathbb{C}$ and denote its points by $((x,y,z),w)$. The classification [25] of involutions on $\mathbb{S}^2\times\mathbb{S}^1$ yields that the involution ι_i on $\tilde{X}_i$ satisfying that $\tilde{X}_i/\iota_i\cong X_i$ is equivalent to the involution defined by

$$((x,y,z),w)\mapsto((-x,-y,z),\overline{w})$$

which has four fixed points. The quotient space of this involution is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)\#\,\mathrm{Susp}(\mathbb{R}P^2)$ (see incise (a) of Case (2) in Page 7 of [7]). Then it follows, as in the proof of [18, Theorem 6.1], that X_i is not aspherical.

3.2. 1**-dimensional limit space.** In this case, for sufficiently big i, X_i is homeomorphic to a gluing of two pieces B and B' appearing in the classification [17], along their isometric boundaries. As X_i is not homeomorphic to a manifold, at least one of the following pieces must appear in the decomposition: $\mathrm{B}(S_2)$, $K_1(\mathbb{R}P^2)$, $\mathrm{B}(S_4)$, $\mathrm{B}(\mathrm{pt})$ and $\mathrm{B}(\mathbb{R}P^2)$. We will show in this section that every possible space X_i having one of these pieces cannot be aspherical.

Case 3.1 (One of the pieces is $\mathrm{B}(S_2)$)**.** The possible pieces in this case are D^3, $\mathbb{R}P^3\smallsetminus\mathrm{int}D^3$ and $\mathrm{B}(S_2)$. By [17, Remark 2.62], $B(S_2)$ is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)\smallsetminus D^3$. Whence the possible spaces arising by gluing with this piece are homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$, $\mathbb{R}P^3\#\mathrm{Susp}(\mathbb{R}P^2)$ and $\mathrm{Susp}(\mathbb{R}P^2)\#\mathrm{Susp}(\mathbb{R}P^2)$. All of these spaces have $\mathrm{Susp}(\mathbb{R}P^2)$ as a connected summand, and therefore the arguments of [18, Theorem 6.1] show that they are not aspherical, a contradiction.

Case 3.2 (One of the pieces is $K_1(\mathbb{R}P^2)$)**.** The only piece is the closed cone $K_1(\mathbb{R}P^2)$ and therefore, the space X_i in this case is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$.

As previously mentioned, this space is not aspherical and then a contradiction is ensued.

As a first step to deal with the remaining cases, we observe that $H_2(X_i) \neq 0$ whenever X_i contains a piece of the form $\mathrm{B}(S_4)$ or $\mathrm{B}(\mathrm{pt})$ a fact that follows from the following result due to Mitsuishi (see [16, Corollary 5.7]). The result is originally stated for the more general class of NB-spaces (see [16, Definition 1.6]). For simplicity we restate it here for Alexandrov spaces only.

Theorem 3.1 (Mitsuishi)**.** *Let X be a closed, connected Alexandrov n-space. If X is non-orientable then the torsion subgroup of $H_{n-1}(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$ and, in particular, $H_{n-1}(X;\mathbb{Z})$ is non-zero.*

In order to obtain that $\pi_2(X_i) \neq 0$ using the information that $H_2(X_i) \neq 0$ we will use the following classical theorem proved by Hopf in [12, Theorem a), Page 257].

Theorem 3.2 (Hopf)**.** *Let X be a CW-complex with finitely many cells. Then, there exists an exact sequence*

$$\pi_2(X) \to H_2(X) \to H_2(B\pi_1(X);\mathbb{Z}) \to 0. \tag{3.2}$$

Here, $B\pi_1(X)$ denotes a model for the classifying space of the fundamental group $\pi_1(X)$, which is characterized up to homotopy by being an aspherical CW-complex having the same fundamental group as X. As $\pi_1(X)$ depends on the pieces B and B' we will split the following analysis to go over every possibility.

Case 3.3 (One of the pieces is $\mathrm{B}(S_4)$)**.** The possible pieces B and B' with $X_i \cong B \cup B'$ in this case are $S^1 \times D^2$, $S^1 \times \mathrm{Mo}$, $K^2 \tilde{\times} I$ and $B(S_4)$. We assume that $B' = B(S_4)$ is fixed.

Case 3.3.1 ($B = B(S_4)$)**.** If both pieces of the decomposition of X_i are homeomorphic to $B(S_4)$, then Van Kampen's Theorem readily implies that $\pi_1(X_i) = 0$. Moreover, by Theorem 3.1, $H_2(X_i) \neq 0$. Therefore, by Hurewicz's Theorem, $\pi_2(X_i) \neq 0$, which contradicts the asphericity of X_i.

Case 3.3.2 ($B = S^1 \times \mathrm{Mo}$)**.** The fundamental group of X_i in this case is $\mathbb{Z} \oplus \mathbb{Z}$, as calculated from Van Kampen's Theorem. A model for the classifying space $B(\mathbb{Z} \oplus \mathbb{Z})$ is the torus T^2. Hence, the sequence (3.2) becomes

$$\pi_2(X_i) \to H_2(X_i) \to \mathbb{Z} \to 0. \tag{3.3}$$

Here, we have used the fact that $H_2(T^2) \cong \mathbb{Z}$ (as a consequence of the orientability of T^2). Therefore, if $\pi_2(X_i) = 0$ we would obtain that $H_2(X_i) \cong \mathbb{Z}$, in particular yielding that $H_2(X_i)$ is torsion-free. This contradicts Theorem 3.1. Therefore, $\pi_2(X_i) \neq 0$ which is a contradiction to the asphericity of X_i.

Case 3.3.3 ($B = S^1 \times D^2$)**.** In this case, a computation via the Van Kampen's Theorem yields that $\pi_1(X_i) \cong \mathbb{Z}$. Now, a model for $B\mathbb{Z}$ is the circle S^1. Hence, Hopf's exact sequence (3.2) takes the form

$$\pi_2(X_i) \to H_2(X_i) \to 0 \to 0. \tag{3.4}$$

Therefore, the morphism $\pi_2(X_i) \to H_2(X_i) \neq 0$ is surjective, implying that $\pi_2(X_i) \neq 0$. This contradicts the assumption that X_i is aspherical.

Case 3.3.4 ($B = K^2 \tilde{\times} I$)**.** As in the previous cases, using Van Kampen's theorem it follows that the fundamental group of X_i is that of $K^2 \tilde{\times} I$. Since I is contractible, $\pi_1(K^2 \tilde{\times} I) \cong \pi_1(K^2) \cong \mathbb{Z} \rtimes \mathbb{Z}$. We now note that a model for $B(\mathbb{Z} \rtimes \mathbb{Z})$ is the Klein bottle. Therefore by the non-orientability of K^2, the Hopf's sequence (3.2) becomes

$$\pi_2(X_i) \to H_2(X_i) \to 0 \to 0. \tag{3.5}$$

Whence, as in the previous case, $\pi_2(X_i) \neq 0$.

Case 3.4 (One of the pieces is $\mathrm{B}(\mathrm{pt})$)**.** In this case the possible pieces taking the role of B and B' are $S^1 \tilde{\times} D^2$, $K^2 \hat{\times} I$, $\mathrm{B}(\mathrm{pt})$ and $\mathrm{B}(\mathbb{R}P^2)$. We will exclude the space $\mathrm{B}(\mathrm{pt}) \cup \mathrm{B}(\mathbb{R}P^2)$ in this case as it will be considered below. Let us observe that by Van Kampen's Theorem, the possible fundamental groups of X_i in this case are the same that appear in the case that one of the pieces is $\mathrm{B}(S_4)$. Therefore, a contradiction to the asphericity of X_i is obtained for $\mathrm{B}(\mathrm{pt}) \cup \mathrm{B}(\mathrm{pt})$ as in Case 3.3.1, for $\mathrm{B}(\mathrm{pt}) \cup K^2 \hat{\times} I$ as in Case 3.3.4, and for $\mathrm{B}(\mathrm{pt}) \cup S^1 \tilde{\times} D^2$ as in Case (3.3.3).

We now address the remaining case in which one of the pieces in the decomposition of X_i is $\mathrm{B}(\mathbb{R}P^2)$.

Case 3.5 (One of the pieces is $\mathrm{B}(\mathbb{R}P^2)$)**.** Under this assumption, the possible pieces B and B' forming X_i are $S^1 \tilde{\times} D^2$, $K^2 \hat{\times} I$, $\mathrm{B}(\mathrm{pt})$ and $\mathrm{B}(\mathbb{R}P^2)$. As some possibilities overlap with the previous Case (3.4), we only consider the spaces $S^1 \tilde{\times} D^2 \cup \mathrm{B}(\mathbb{R}P^2)$, $K^2 \hat{\times} I \cup \mathrm{B}(\mathbb{R}P^2)$, $\mathrm{B}(\mathrm{pt}) \cup \mathrm{B}(\mathbb{R}P^2)$ and $\mathrm{B}(\mathbb{R}P^2) \cup \mathrm{B}(\mathbb{R}P^2)$ here. To address the question of asphericity of these spaces we apply the following result (see [13, Lemma 4.1])

Theorem 3.3. *The fundamental group of an aspherical finite-dimensional CW-complex is torsion-free.*

An analysis via Van Kampen's Theorem yields that the possible fundamental groups of X_i in this case are $\mathbb{Z} *_{\pi_1(K^2)} \mathbb{Z}_2$, $(\mathbb{Z} \rtimes \mathbb{Z}) *_{\pi_1(K^2)} \mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2 *_{\pi_1(K^2)} \mathbb{Z}_2$. It is immediate to check that these groups have non-zero torsion, as at least one of the factors in the amalgamated products has non-zero torsion. Therefore Theorem 3.3 yields that X_i cannot be aspherial, a contradiction.

3.3. 0**-dimensional limit space.** In this case, if X_i is a generalized Seifert fibered space then the contradiction is obtained as in Section 3.1. If X_i is homeomorphic to a space appearing in the 1-dimensional limit case, then the contradiction is obtained as in Section 3.2. The remaining cases are non-negatively curved (non-manifold) Alexandrov 3-spaces with finite fundamental group. In these cases, if $\pi_1(X_i) = 0$, Theorem 3.1, implies that X_i is not aspherical. Furthermore, if $\pi_1(X_i)$ is non-trivial then 3.3 yields that X_i is not aspherical. Hence, in every case we obtain a contradiction to asphericity and the result is settled. $\square$

Remark 3.4. In light of Theorem B, a natural conjecture would be that a closed, aspherical Alexandrov 3-space X is homeomorphic to a 3-manifold. This is indeed the case whenever X is simply-connected as a consequence of incise (2) of [16, Corollary 5.7] and the Hurewicz Theorem.

4. QUESTIONS ON FUNDAMENTAL GROUPS AND ACTIONS ON ALEXANDROV 3-SPACES

A lot of effort has been devoted in geometric topology to the development of characterization of fundamental groups of manifolds and spaces which are topologically rigid. On the other hand, a similar fruitful effort has been devoted to the characterization of groups which act on geometrically defined classes of manifolds.

We will now consider a series of questions inspired by the study of topological rigidity of manifolds and their extrapolation to (possibly singular) Alexandrov 3-spaces.

These questions evolve from topological or geometric rigidity results such as the Borel Conjecture for the fundamental group G of an aspherical manifold, into the characterization of fundamental groups of such spaces through concepts of geometric group theory or group cohomology.

Specifically related to group cohomology, it is an open conjecture originally posed by Wall [26] that every poincaré duality group of dimension 3 is the fundamental group of a three dimensional manifold.

See [5], [8] for a modern discussions on the subject. Here, we present the following question:

Question 4.1. Let G be a Poincaré duality group which is the fundamental group of an aspherical Alexandrov 3-space. Is it the fundamental group of an orientable three dimensional manifold?

Question 4.1 follows readily from Conjecture E.

In the direction of characterizations of groups acting on manifolds, it is a classical result by Wall [26], that the finite groups acting on three-dimensional Poincaré complexes have periodic cohomology of period 4. It is known that the symmetric group on three letters Σ_3 cannot be realized by any honest manifold [15].

Question 4.2. Which finite groups act by homeomorphisms on Alexandrov 3-spaces?

The study of geometric and large scale geometric properties of fundamental groups of three dimensional manifolds in connection with topological rigidity has been oriented in recent times to the characterization of the map involved in the homotopy equivalence referred to in the statement. Consider as an example the following question.

Question 4.3. Let $f : M \to N$ be a map between three dimensional, aspherical manifolds with boundary inducing an epimorphism in fundamental groups. Under which conditions is f homotopic to a homeomorphism.?

This problem has been studied using simplicial volume, specifically degree theorems, and more recently, Agol's solution to the virtually fibering conjecture by Boileau and Friedl [2].

(The epimorphic condition comes from the fact that a degree one map induces such an epimorphism due to Poincaré Duality and the loop theorem).

4.1. **Consequences of Agol's virtually fibering Theorem.** The following theorem was proved by Boileau and Friedl, [2].

Theorem F. *Let $f : M \to N$ be a proper map between aspherical manifolds with either toroidal or empty boundary. Assume that N is not a closed graph manifold, and that f induces an epimorphism on the fundamental group.Then f is homotopic to a homeomorphism if any of the following two conditions are met:*

- *For each H finite index, subnormal subgroup of $\pi_1(N)$, the ranks of H and $f^{-1}(H)$ agree.*
- *For each finite cover $\widetilde{N}$ of N, the Heegard genus of $\widetilde{N}$ and $\widetilde{M}$ agree.*

Question 4.4. Let $f : M \to N$ be an oriented map between oriented, aspherical Alexandrov spaces. What are the conditions on f, M, and N for f be homotopic to homemomorphism?

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