TWISTED GEOMETRIC K-HOMOLOGY FOR PROPER ACTIONS OF DISCRETE GROUPS.

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Abstract. We define Twisted Equivariant K-Homology groups using geometric cycles. We compare them with analytical approaches using Kasparov KK-Theory and (twisted) C^*-algebras of groups and groupoids.

1. Introduction

In this work, we introduce a geometric notion of twisted equivariant K-Homology groups for proper G-CW complexes.

Computational evidence [7], [15], as well as analytical methods related to the Dirac-Dual Dirac method for the Baum-Connes conjecture [11], [14] have suggested the relation of a Twisted Equivariant K-Homology Theory to the topological K-Theory of twisted group C^*-algebras.

However, the definition of such an invariant has been only given in particular cases [11], [13] using Kasparov KK-Theory and rather ad-hoc methods inspired in ideas related the solution of the Connes-Kasparov conjecture by Chabert, Echterhoff and Nest [12].

We describe twisted equivariant K-homology groups using geometric cycles for proper actions of discrete groups on G-CW complexes. We extend and generalize results from [9], [8] to the twisted, respectively equivariant case.

The main methods in this note are a combination of consequences of the pushforward homomorphism constructed in [3], and ideas ultimately originated in geometric pictures of K-homology [9], [8].

We compare the cycle version using an index map to an equivariant Kasparov KK-Theory group. We also introduce some computational tools, notably a spectral sequence to compute the equivariant K-homology groups described here and we compare the twisted equivariant K-homology invariants of the classifying space for proper actions to the topological K-theory of twisted versions of the reduced C^* algebra of a discrete group.

This work is organized as follows:

In section 2, we recall definitions, methods, and results related to twisted equivariant K-Theory for proper actions [5], [4], [3].

In section 3, geometric cycles for twisted equivariant K-homology are introduced, as well as appropriate equivalence relations (bordism, isomorphism and vector bundle modification). It is proved in Theorem 3.9, that the geometric Twisted Equivariant K-Homology groups satisfy dual homological properties to the twisted equivariant K-Theory groups in [5]. In that section computational methods, including a spectral sequence abutting to Twisted Equivariant K-Homology are addressed.

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In section 4, the construction is compared to equivariant Kasparov KK-groups via the definition of an index map, which is proved in Theorem 4.6 to give an isomorphism between both constructions in the case of a proper, finite G-CW complex. On the way, several results, including versions of KK-Theoretical Poincaré Duality [14], [13] and consequences of the Thom isomorphisms and Pushforward Maps for Twisted Equivariant K-Theory of [3] are discussed.

Finally, the relation of the construction depicted below with the K-Theory of twisted crossed products [16], [27] is established in section 5.

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Contents

2. Review on Twisted Equivariant K-Theory and Pushforward

Twisted Equivariant K-Theory for proper actions of discrete groups was introduced in [5]. We will recall definitions, results and methods from [5] and [3].

Let $\mathcal{H}$ be a separable Hilbert space and

$$\mathcal{U}(\mathcal{H}) := \{U : \mathcal{H} \to \mathcal{H} \mid U \circ U^* = U^* \circ U = \mathrm{Id}\}$$

the group of unitary operators on $\mathcal{H}$ with the $*$-strong topology.

Consider the group $P\mathcal{U}(\mathcal{H})$ with the topology determined by the exact sequence

$$1 \to S^1 \to \mathcal{U}(\mathcal{H}) \to P\mathcal{U}(\mathcal{H}) \to 1.$$

Let $\mathcal{H}$ be a Hilbert space. A continuous homomorphism a defined on a Lie group G, $a : G \to P\mathcal{U}(\mathcal{H})$ is called stable if the unitary representation $\mathcal{H}$ induced by the homomorphism $\widetilde{a} : \widetilde{G} = a^*\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ contains each of the irreducible representations of $\widetilde{G}$

Definition 2.1. Let X be a proper G-CW complex. A projective unitary G-equivariant stable bundle over X is a principal $P\mathcal{U}(\mathcal{H})$-bundle

$$P\mathcal{U}(\mathcal{H}) \to P \to X$$

where $P\mathcal{U}(\mathcal{H})$ acts on the right, endowed with a left G-action lifting the action on X such that:

- the left G-action commutes with the right $P\mathcal{U}(\mathcal{H})$ action, and
- for all $x \in X$ there exists a G-neighborhood V of x and a G_x-contractible slice U of x with V equivariantly homeomorphic to $U \times_{G_x} G$ with the action
$$G_x \times (U \times G) \to U \times G, \quad k \cdot (u, g) = (ku, gk^{-1}),$$
together with a local trivialization
$$P|_V \cong (P\mathcal{U}(\mathcal{H}) \times U) \times_{G_x} G$$
where the action of the isotropy group is:
$$\begin{array}{rcl} G_x \times ((PU(\mathcal{H}) \times U) \times G) & \to & (P\mathcal{U}(\mathcal{H}) \times U) \times G \\ (k, ((F, y), g)) & \mapsto & ((f_x(k)F, ky), gk^{-1}) \end{array}$$
with $f_x : G_x \to PU(\mathcal{H})$ a fixed stable homomorphism.

Projective unitary G-equivariant stable bundles are the main tool to introduce twisted equivariant K-Theory as sections of Fredholm bundles. Other topologies have been used on the relevant operator spaces [5], [6]. However, they are all easily seen to agree ([6], Section 1.1. See also [31]).

Projective unitary G-equivariant stable bundles have been classified in [5], Theorem 4.8 in page 1341. The isomorphism classes of projective unitary stable G-equivariant bundles over a proper G-CW complex X are in correspondence with the third Borel Cohomology Group $H^3(X \times_G EG, \mathbb{Z})$. There exists an intrinsic abelian group structure on the isomorphism classes of projective unitary stable G-equivariant bundles for which this correspondence is an isomorphism of abelian groups.

Let X be a proper G-CW complex. A G-Hilbert bundle over X is a locally trivial bundle $E \to X$ with fiber on a Hilbert space $\mathcal{H}$ and structural group the group of unitary operators $\mathcal{U}(\mathcal{H})$ with the $*$-strong operator topology.

The Bundle of Hilbert-Schmidt operators with the $*$-strong topology between Hilbert Bundles E and F will be denoted by $L_{HS}(E, F)$.

Let P be a projective unitary stable G-equivariant bundle. Consider the projective unitary stable G-equivariant bundle $-P$ associated to the additive inverse of P using the isomorphism from Theorem 4.8 of [5] to the Borel cohomology group and form similarly the class $P + -P$. The classification of Projective Unitary stable G-equivariant bundles [5] gives:

Proposition 2.2. *There exists a Hilbert bundle E over X such that the zero class $P + -P$ admits a representative given by the principal bundle associated to the Bundle $L_{HS}(E_P, {E_P}^*)$ of Hilbert-Schmidt homomorphisms of E_P to the dual bundle ${E_P}^*$.*

Definition 2.3. Let X be a connected G-space and P a projective unitary stable G-equivariant bundle over X, with associated Hilbert space $\mathcal{H}$.

Denote by $\mathrm{Fred}(\mathcal{H})$ the space of $*$-strong continuous, Fredholm operators on $\mathcal{H}$ and recall that the group of projective unitary operators $P\mathcal{U}(\mathcal{H})$ in the *-strong topology acts continuously by conjugation on $\mathrm{Fred}(\mathcal{H})$. The twisted equivariant K-Theory groups denoted by $K_G^{-p}(X; P)$ are defined in [5] as the $p-th$ homotopy groups (based a at a suitable section s) of the space of G-invariant sections of the bundle with fibre $\mathrm{Fred}(\mathcal{H})$ and structural group $P\mathcal{U}(\mathcal{H})$ associated to P. In symbols,

$$K_G^{-p}(X;P) := \pi_p\left(\Gamma(X;\mathrm{Fred}(P)_{s^*})^G, s\right).$$

Given a G-CW pair (X, A), one defines the relative twisted equivariant K- Theory groups $K_G^{-p}(X, A; P)$ in an analogous way.

Let G be a discrete group. A proper, G-compact manifold is a smooth, orientable manifold with an orientation preserving, proper and smooth action of the group G, in such a way that the quotient space G/M is compact.

Given a proper G-compact manifold M of dimension n, consider the tangent bundle TM. It is a real G-vector bundle over X of rank n. Let $F(M)$ be the associated $\mathrm{SO}(n)$-principal, G-equivariant bundle.

A G-spin structure on M is a homotopy class of a reduction of the bundle $F(M)$ to a $\mathrm{Spin}(c)$-principal, G-equivariant bundle. Given a choice of the $\mathrm{Spin}(c)$-structure for M we denote by $-M$ the manifold with the opposite $\mathrm{Spin}(c)$-structure.

This is the homotopy class of the reduction of the bundle $F(M)$ to the $\mathrm{Spin}(c)$-principal, G-equivariant bundle, where the right action of $\mathrm{Spin}(c)$ is twisted by an automorphism $\epsilon : \mathrm{Spin}(c) \to \mathrm{Spin}(c)$ induced from the orientation reversing automorphism $-\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$.

We recall one of the main results from [3], obtained using a version of the Thom Isomorphism and suitable deformation index maps in Twisted Equivariant K-theory. It is stated there (Proposition 5.18) in the more general context of Metalinear structures, but, as observed there, the proof readily applies to the $\mathrm{Spin}(c)$-case.

Theorem 2.4 (Pushforward morphism for proper G-manifolds)**.** *Let X, Y be two proper, smooth* $\mathrm{Spin}(c)$*- G- manifolds. Let P be a projective unitary stable G-bundle over Y. Let $f : X \longrightarrow Y$ be a map such the G-vector bundle $TX \oplus f^*TY$ is an equivariant* $\mathrm{Spin}(c)$*-vector bundle of even rank. Then, there exists a natural pushforward homomorphism*

$$f! : K^0_G(X, f^*(P)) \to K^0_G(Y, P).$$

3. Geometric Approach to twisted equivariant K-Homology

We describe now a geometric version of Twisted Equivariant K-Homology. The results in this section extend and generalize methods by Baum-Higson-Schick and Baum-Carey-Wang [9], [32], [8].

Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1\leq n} X_n$ with $X_{-1} = \emptyset$, $X_0 = A$ and every space is inductively obtained from the previous one by attaching cells with pushout diagrams

$$\begin{array}{ccc} \coprod_{\lambda\in I_n} S^{n-1}\times G/H_\lambda & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in I_n} D^{n}\times G/H_\lambda & \longrightarrow & X_{n}. \end{array}$$

The set I_n is called the set of n-cells. A G-CW complex is finite if it is obtained out of finitely many orbits of cells $S^n \times G/H$ in a finite number of dimensions n.

This is equivalent to the fact that the quotient G/X is compact (sometimes called G-compactness or co-compactness), which is also equivalent to X being compact.

Let G be a discrete group. Recall that given a G-CW complex X, the G-action on X is proper exactly when all point stabilizers are finite. This is equivalent to slice conditions due to the slice theorem [28], [1].

Definition 3.1. [Geometric Twisted Equivariant K-Homology]

Let G be a discrete group and let X be a proper G-CW complex, let P be a projective, unitary stable G-equivariant bundle.

A geometric cycle is a triple (M, f, σ) consisting of

- A G-compact, proper orientable, Spin(c) manifold without boundary M.
- A G-equivariant map $f : M \to X$.
- A homotopy class of a G-invariant section $\sigma : M \to \text{Fred}(-f^*(P))$ representing an element on the twisted equivariant K-theory group
$$K^0_G(M, -f^*(P)).$$

We will impose some conditions on geometric cycles in order to obtain geometric Twisted Equivariant K-Homology.

(i) Isomorphism. The cocycle (M, f, σ) is isomorphic to (M', f', σ') if there exists a G-equivariant, Spin(c)-structure preserving diffeomorphism $\psi : M \to M'$, which makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[dr, "f"'] & & M' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

(ii) Bordism. The cycle $(M_0, f_0, \sigma_0)$is bordant to $(M_1, f_1, , \sigma_1)$ if there exists a cycle (W, f, σ), where W is a G-compact, orientable and Spin(c)-manifold with boundary, $f : W \to X$ is a G-equivariant map, and $\sigma : W \to \text{Fred}(-f^*(P))$ is the homotopy class of a G-invariant section, with the property that
$$(\partial W, f\mid_{\partial W}, \sigma) = (M_0, f_0, \sigma_0) \coprod (-M_1, f_1, \sigma_1)$$
Where $(-M_1, f_1, \sigma_1)$ has the reversed orientation with respect to M_1.

(iii) Vector bundle modification. Given a geometric cycle (M, f, σ), and an even dimensional real vector bundle with a G-equivariant Spin(c)-structure $F \to M$, denote by $\pi : S(F \oplus \mathbb{R}) \to M$ the unit sphere bundle of $F \oplus \mathbb{R}$. We consider the map $S_{M,F} : M \to S(F \oplus \mathbb{R})$ given by the 1-section and let $S_{M,F}! : K^0_G(M, -f^*(P)) \longrightarrow K^0_G(S(F \oplus \mathbb{R}), S^*_{M,F}(-f^*(P)))$ be the pushforward homomorphism.

Then, the geometric cycles (M, f, σ) and $S(F \oplus \mathbb{R}), \pi \circ f, S_{M,F}!(\sigma))$ are equivalent.

We will introduce the notation $E\bigstar(M, f, \sigma)$ for the class of the modification of the geometric cycle (M, f, σ) with respect to the Spin(c)-vector bundle $E \to M$.

Similarly, we will denote by $E\bigstar M$ the manifold given as the total space of the spherical bundle $S(E \oplus \mathbb{R})$ over M.

The following result, generalization of the "Composition Lemma" on page 81 of [8] in the *non-equivariant* setting, will be useful to study interactions between bordism and vector bundle modification:

Lemma 3.2. *Let M be a proper, G-compact, smooth manifold, consider a G-equivariant,* $\mathrm{Spin}(c)$*, smooth real vector bundle of even dimensional fibers, $\pi : E \to M$. Let $s : M \to S(E \oplus \mathbb{R})$ be the unit section.*

Given any real, even dimensional smooth vector bundle F with a $\mathrm{Spin}(c)$*-structure over $S(E \oplus \mathbb{R})$, the proper G-manifolds*

$$F\bigstar(E\bigstar M),$$

and

$$(s^*(F) \oplus E)\bigstar M$$

are bordant.

Proof. Pick up a G-invariant, hermitian metric on $E \oplus \mathbb{R}$. This is possible because the action on M is proper. Let $D(E \oplus \mathbb{R})$ be the unit Ball bundle therein. Notice that the boundary of $D(E \oplus \mathbb{R})$ is $E\bigstar M$.

Consider the map $i : M \to D(F \oplus \mathbb{R})$ given by the composition of the 1-section $M \to S(E \oplus \mathbb{R})$ and the 0-section $S(E \oplus \mathbb{R}) \to D(F \oplus \mathbb{R})$.

Let ν be the normal bundle to this inclusion, and consider the ball bundle inside the normal bundle, denoted by $D_\epsilon(\nu)$ for $\epsilon < \frac{1}{4}$. Then, the sphere bundle of $s^*(F) \oplus E \oplus \mathbb{R}$ is G-equivariantly diffeomorphic to the boundary of the Ball bundle in ν. Consider the manifold with boundary $W = D(E \oplus \mathbb{R}) - D_\epsilon(\nu)$. This is a proper, $\mathrm{Spin}(c)$-manifold with boundary, which is a bordism between $F\bigstar(E\bigstar M)$ and $(s^*(F) \oplus E)\bigstar M$.

□

Corollary 3.3. *Given a geometric cycle (M, f, σ) and real orientable, and* $\mathrm{Spin}(c)$*-vector bundles $\pi : E \to M$ and $F \to S(E \oplus \mathbb{R})$,*

Denote by $\rho_E : E\bigstar M \to M$ and $\rho_F : F\bigstar(E\bigstar M) \to E\bigstar M$ the projection maps. Let $S_{M,E} : M \to E\bigstar M$ and $S_{M,S^*(F)\oplus\mathbb{R}} : M \to S^*F \oplus E\bigstar M$ be the 1-sections into the spherical bundles.

Then, the geometric cycles

$$\big(F\bigstar(E\bigstar M), f \circ \rho_E \circ \rho_F, S_{M,E}! \circ S_{E\bigstar M,F}!(\sigma)\big)$$

and

$$\big((S^*(F) \oplus E)\bigstar M, f \circ \rho_{S^*(F)\oplus F}, S_{M,S^*(F)\oplus\mathbb{R}}!(\sigma)\big)$$

are equivalent.

Addition on the set of geometric cycles is defined as disjoint union:

$$(M, \sigma, f) + (M'\sigma', f') = (M, \sigma, f) \coprod (M'\sigma', f'),$$

and turns the equivalence classes into an abelian group.

Definition 3.4. Let X be a proper G-CW complex and a projective unitary stable G-equivariant bundle P over X. We will denote by $K_G^{\mathrm{geo}}(X, P)$ the abelian group of geometric cycles on X, divided by the equivalence relation generated by Isomorphism, Bordism and $\mathrm{Spin}(c)$-modification.

Denote by $K_{G,0}^{\mathrm{geo}}(X, P)$ the subgroup generated by those cycles (M, f, σ) for which each component of M has even dimension. Similarly, denote by $K_{G,1}^{\mathrm{geo}}(X, P)$ the subgroup generated by cycles (M, f, σ) for which each component of M has odd dimension.

We introduce now a version for G-CW pairs.

Given a proper G-CW pair (X, A) and a G-equivariant, projective unitary stable bundle P over X, consider the triples (M, σ, f) consisting of

- A G-compact, proper manifold M (possibly with boundary).
- A G-equivariant map $f : M \to X$ with $f(\partial M) \subset A$).
- A homotopy class of a G-invariant section $\sigma : M \to \mathrm{Fred}(-f^*(P))$ representing an element on the twisted equivariant K-theory group
$$K^0_G(M, -f^*(P)),$$

Define on the set of geometric cycles for (X, A) and P an equivalence relation generated by

- Isomorphism.
- Bordism.
- Spin(c)-vector bundle modification.

Denote by $K^{\mathrm{geo}}_{G,*}(X, A, P)$ the set of such equivalence classes.

As before, disjoint union defines an abelian group structure on $K^{\mathrm{geo}}_{G,*}(X, A, P)$. Notice that the decomposition

$$K^{\mathrm{geo}}_{G,*}(X, A, P) = K^{\mathrm{geo}}_{G,0}(X, A, P) \bigoplus K^{\mathrm{geo}}_{G,1}(X, A, P)$$

holds, where the summands 0 and 1 correspond to the subgroup generated by geometric cycles for (X, A) and P given by manifolds of even, respectively odd dimensions.

There exists a boundary map

$$\delta : K^{\mathrm{geo}}_{G,j}(X, A, P) \longrightarrow K^{\mathrm{geo}}_{G,j-1}(A, P),$$

which is defined as follows.

Given a geometric cycle (M, σ, f) for (X, A) and P, the cycle $\delta(M, \sigma, f)$ has as underlying manifold ∂M. The map $\partial M \to A$ is given as the composition $\partial M \xrightarrow{j} M \xrightarrow{f} X$, where j denotes the inclusion of ∂M into M. The section $\partial M \to \mathrm{Fred}(-f^*(P))$ is given as the composition of the maps $j : \partial M \xrightarrow{j} M \xrightarrow{f} \mathrm{Fred}(-f^*(P)) \xrightarrow{|j} \mathrm{Fred}(-f^*(P))$, where the map $\mid j$ restricts a bundle with fiber $\mathrm{Fred}(-f^*(P))$ over M to a bundle over ∂M.

We introduce an equivalence relation on geometric cycles, which is more technically convenient. This is a generalization of several constructions appearing in the literature [18], [8] under the name of normal bordism.

Definition 3.5. Let M be a proper, G- compact Manifold together with a G-equivariant map $f : M \to X$. Let TM be the tangent bundle A normal bundle for TM with respect to f is a real, G-equivariant vector bundle $NM \to M$ such that $TM \oplus NM$ is isomorphic to a pullback $f^*(F)$, where F is a G-equivariant, Spin(c)-vector bundle over X.

The following result concerns the existence of normal bundles:

Lemma 3.6. *Let X be a finite, proper G-CW complex. Let M be a proper, G-compact manifold with a G-equivariant map $f : M \to X$. Then, there exists a normal bundle for TM with respect to f.*

Proof. This follows from Lemma 3.7 in page 599 of [24], after noticing that the manifold M has a proper G-CW structure, which consists of finitely many cells. $\square$

Definition 3.7. [Normal Bordism for cycles] Two cycles

$$(M, f, \sigma),$$

and

$$(M^{'}, f^{'}, \sigma')$$

for $K^{\mathrm{geo}}_{G,*}(X, A, P)$ are said to be normally bordant if there exist finite dimensional, real orientable vector bundles for TM with respect to f, denoted by $E \to M$, and a normal bundle for $TM^{'}$ with respect to $f^{'}$ denoted by $E^{'} \to M^{'}$,

such that the cycles

$$\big(E \bigstar M, \pi_E \circ f, S_{M,E}!(\sigma)\big)$$

and

$$\big(E^{'} \bigstar M^{'}, \pi_{E'} \circ f^{'}, S_{M',E'}!(\sigma^{'})\big)$$

are bordant.

Lemma 3.8. *Two cycles are normal bordant if and only if they are equivalent.*

Proof. The implication from normal bordism to equivalence in the sense of definition 3.1 is clear.

For the converse, suppose that (W, f, σ) is a bordism from the cycle (M_0, f_0, σ_0) to (M_1, f_1, σ_1). Choose a normal bundle NW for TW with respect to f in a way that $NW \mid_{\partial W}$ is a normal bundle for the boundary. Then $NW \bigstar W$ gives a normal bordism between Spin(c)-modifications along normal bundles for the boundary components.

The vector bundle modification procedure also implies normal bordism. Let $E \to M$ be a real, Spin(c), G-equivariant vector bundle. Let $F \to S(E \oplus \mathbb{R})$ be a Spin(c), G-equivariant real vector bundle, and let N be a normal bundle with respect to the map $S(E \oplus \mathbb{R}) \to M \to X$.

Consider the map $s : M \to F$ given as the composition of the 1-section $s : M \to S(E \oplus \mathbb{R})$, and the 0-section $S(E \oplus \mathbb{R}) \to F$.

Now, consider the bundle $s^*(N) \oplus F \mid M$, and notice that it is a normal bundle for M with respect to the map f.

After lemma 3.2, these total spaces are bordant, proper G- Spin(c)-manifolds. The bordism determines a bordism of geometric cycles. $\square$

We now prove the usual homological properties of twisted equivariant K-homology for proper actions of a discrete group.

Theorem 3.9 (Homological Properties)**.** *The groups $K^{\mathrm{geo}}_{G,j}(X, A, P)$ satisfy the following properties.*

(i) (Functoriality). A G-equivariant map of pairs $\psi : (X, A) \to (Y, B)$ induces a group homomorphism $\psi^* : K^{\mathrm{geo}}_{G,j}(X, A, \psi^* P) \to K^{\mathrm{geo}}_{G,j}(Y, B, P)$

(ii) (Homotopy invariance). Two G-equivariantly homotopic maps $\psi_0 : (X, A) \to (Y, B)$ and $\psi_1 : (X, A) \to (Y, B)$ induce the same group homomorphism $\psi := \psi_0 = \psi_1 : K^{\mathrm{geo}}_{G,j}(X, A, P) \to K^{\mathrm{geo}}_{G,j}(Y, B, \psi^*(P))$.

(iii) (Six term exact sequence). For a proper G-CW pair (X, A), the following sequence is exact:

$$\begin{tikzcd}
K^{\mathrm{geo}}_{G,0}(X,A,P) \arrow[r, "\delta"'] & K^{\mathrm{geo}}_{G,1}(A,P) \arrow[d, "i^*"'] \\
K^{\mathrm{geo}}_{G,0}(X,P) \arrow[u, "j_*"'] & K^{\mathrm{geo}}_{G,1}(X,P) \arrow[d, "j_*"'] \\
K^{\mathrm{geo}}_{G,0}(A,i^*P) \arrow[u, "i_*"'] & K^{\mathrm{geo}}_{G,1}(X,A,P) \arrow[l, "\delta"']
\end{tikzcd} ,$$

where the maps i_*, j_* are induced by the maps of proper G-CW pairs $i : A \to X$ and $j : X := (X, \emptyset) \to (X, A)$.

(iv) (Excision). Given a G-CW pair (X, A) and a G-invariant, open set U with the property that the closure of U is contained in the interior of A, the inclusion $l : (X - U, A - U) \to (X, A)$ induces an isomorphism of abelian groups

$$K^{\mathrm{geo}}_{G,j}(X,A,P) \to K^{\mathrm{geo}}_{G,j}(X-U, A-U, l^*(P))$$

(v) (Disjoint union Axiom). Let $X = \coprod_{\alpha\in A} X_\alpha$ be a proper G-CW complex which is obtained as the disjoint union of the Proper G-CW complexes X_α. Then, the inclusions $X_\alpha \to X$ induce an isomorphism of abelian groups

$$\bigoplus_{\alpha\in A} K^{\mathrm{geo}}_{G,j}(X_\alpha, P \mid X_\alpha) \cong K^{\mathrm{geo}}_{G,j}(X,P)$$

Proof. (i) Associate to the geometric cycle (M, σ, f) for the pair (X, A) the cycle $(M, \psi^*(\sigma), \psi \circ f)$.

(ii) It follows from the bordism relation.

(iii) We will check exactness of the sequence at $K^{\mathrm{geo}}_{G,0}(X,P)$. Let (M, f, σ) be a geometric cycle for the pair (X, A) which maps to 0 under the map δ. According to Lemma 3.2, there exists a real, orientable and Spin(c) G-vector bundle E of even dimension over ∂M such that the geometric cycle $E\bigstar(\partial M, f\mid_{\partial M}, \sigma\mid_{\partial M})$ is twisted null bordant. We can assume that E is defined on all of M, otherwise using lemma 3.7 of [24], page 599.

Consider a geometric cycle (W, F, Σ), where W is such that $\partial W = E\bigstar\partial M$, $F\mid_{\partial W} = \pi_E \circ f$, $\Sigma\mid_{\partial W} = \sigma\mid_{\partial M}$.

Form the manifold $\hat{W} := W \cup_{\partial W = \partial M} E\bigstar M$, where the inclusion of ∂M into the second factor is given as composition of the inclusion $\partial M \to M$ together with the unit section $M \to E\bigstar M$. The manifold $\hat{W}$ together with the map $F \cup_{\partial M} f$, and the class $\hat{\Sigma} = \Sigma \cup_{\partial M} \sigma$ give a geometric cycle in $K^{\mathrm{geo}}_{G,*}(X,P)$ which maps to the given cycle for (X, A).

Exactness at other points of the sequence is verified analogously.

(iv) We will construct an inverse to the map induced by the inclusion l. Recall the extension of Morse Theory to proper actions [19], Theorems A and B in page 94. It can be concluded that given a geometric cycle (M, f, σ)

for (X, A), there exists a smooth G-invariant function $\epsilon : M \to \mathbb{R}$ with a number of critical orbits, which are all generic.

We will assume that the function ϵ separates $f^{-1}(\overline{U})$ and $M - f^{-1}(A)$. Then, there exist real numbers $a < b$ such that $a = \sup_{x \in M - f^{-1}(A)} \epsilon(x)$, and $b = \inf_{x \in f^{-1}(\overline{U})} \epsilon(x)$.

For a regular value $c \in (a, b)$, $\epsilon^c = \epsilon^{-1}(-\infty, c])$ can be furnished with a smooth structure. The boundary component $\epsilon^{-1}(c) \cup \epsilon^{-1}(-\infty, c]) \cap \partial M$ is mapped under f to $X - U$.

Then, the restriction of the cycle to M_1 gives a geometric cycle

$$(M_1, f\mid_{M_1}, \sigma\mid_{M_1})$$

for the pair $(X - U, A - U)$. This construction gives an inverse to the map l_*.

(v) This follows easily from manipulations on the components of the manifolds involved in the geometric cocycles.

□

Given discrete groups H and G, a group homomorphism $\alpha : H \to G$, and a proper H-CW pair (X, A), the induced space $\mathrm{ind}_\alpha X$, is defined to be the G-CW complex defined as the quotient space obtained from $G \times X$ by the right H-action given by $(g, x) \cdot h = (g\alpha(h), h^{-1}x)$. The following result summarizes the behaviour of the geometric model for twisted equivariant K-homology with respect to induction and restriction procedures on spaces. Recall the existence of the induction homomorphisms in Borel cohomology $\mathrm{ind}_\alpha : H^3(\mathrm{ind}_\alpha X \times_G EG, \mathbb{Z}) \to H^3(X \times_H EH, \mathbb{Z})$ and in Twisted Equivariant K-Theory $\mathrm{ind}_\alpha : K^*_G(\mathrm{ind}_\alpha X, \mathrm{ind}_\alpha P) \to K^*_H(X, P)$ [5], (Lemma 5.4 in page 1347).

Theorem 3.10. *There exist natural group homomorphisms*

$$\mathrm{ind}_\alpha : K^{\mathrm{geo}}_{H,j}(X, A) \longrightarrow K^{\mathrm{geo}}_{G,j}(\mathrm{ind}_\alpha(X, A), \mathrm{ind}_\alpha P)$$

associated with a homomorphism $\alpha : H \to G$ which satisfy the following conditions

(i) ind_α *is an isomorphism whenever* $\ker \alpha$ *acts freely on* X.

(ii) For any j, $\partial^G_n \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial^H_n$.

(iii) For any group homomorphism $\beta : G \to K$ *such that* $\ker \beta \circ \alpha$ *acts freely on* X, *one has*

$$\mathrm{ind}_{\alpha\circ\beta} = K^{\mathrm{geo}}_{K,j}(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : K^{\mathrm{geo}}_{H,j}(X, A) \to K^{\mathrm{geo}}_{K,n}(\mathrm{ind}_{\beta\circ\alpha}(X, A))$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ *is the canonical* G*-homeomorphism.*

(iv) For any $j \in \mathbb{Z}$, *any* $g \in G$, *the homomorphism*

$$\mathrm{ind}_{c(g):G\to G} K^{\mathrm{geo}}_{G,j}(X, A) \to K^{\mathrm{geo}}_{G,j}(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X, A}))$$

agrees with the map $\mathcal{H}^G_n(f_2)$, *where* $f_2 : (X, A) \to \mathrm{ind}_{c(g):G\to G}(X, A)$ *sends* x *to* $(1, g^{-1}x)$ *and* $c(g)$ *is the conjugation isomorphism in* G.

Proof. Given a geometric cycle for (M, f, σ) for $K^{\mathrm{geo}}_H(X, P)$, form the cycle

$$(\mathrm{ind}_\alpha M, \mathrm{ind}_\alpha f, \Sigma),$$

where $\Sigma : \mathrm{ind}_\alpha X \to \mathrm{Fred}(P)$ is the map determined by $(g,x) \mapsto g\sigma(x)$. This is compatible with the equivalence relations defining the geometric K-homology groups. The homomorphism is seen to satisfy i-iv.

□

3.1. **The spectral sequence.** In order to proof the coincidence of geometric twisted equivariant K-homology, we will introduce a spectral sequence based on G-equivariantly contractible covers. This is the dual (homological instead of cohomological) counterpart of the spectral sequence constructed in [4] for twisted equivariant K-theory. Due to the poor properties of Čech versions of homology (particularly its failure to satisfy exactness and disjoint union axiom in general), we will make the following assumption on the space X.

Condition 3.11. [Condition CMP] The G-space X is said to satisfy condition CMP if X is a Compact, Metrizable Proper G-absolute neighbourhood retract (ANR).

These conditions are certainly met in the case of smooth, proper and compact G-manifolds with isometric actions, as well as spaces with a proper G-CW structure with a finite number of cells.

The input for the spectral sequence is given in terms of Bredon homology with local coefficients associated to a contractible cover. We will introduce some notation relevant to this.

A detailed description of the homological algebra involved in Bredon cohomology, as well as computations related to twisted equivariant K-theory and Bredon cohomology can be found in [7], [4].

Let $\mathcal{U} = \{U_\sigma \mid \sigma \in I\}$ be an open cover of the proper G-space X satisfying hypothesis 3.11. Assume that each open set U_σ is G-equivariantly homotopic to an orbit $G/H_\sigma \subset U_\sigma$ for a finite subgroup H_σ, and each finite intersection is either empty or G-equivariantly homotopic to an orbit.

The existence of such a cover, sometimes known as *contractible slice cover*, is guaranteed for proper G-ANR's by an appropriate version of the Slice Theorem (see [1]).

Let $\mathcal{N}_G\mathcal{U}$ be the category where the objects are non-empty finite intersections of $\mathcal{U}$, and where a morphism is an inclusion.

Definition 3.12. A covariant system with values on R-Modules is a covariant functor $\mathcal{N}_G\mathcal{U} \to R-\mathrm{Mod}$.

Definition 3.13. The nerve associated to the cover $\mathcal{U}$, denoted by $\mathcal{N}_G\mathcal{U}^*$ is the simplicial set where the set of 0-simplices $\mathcal{N}_G\mathcal{U}^0$ is given by elements of the cover $\mathcal{U}$. The 1 -simplices in $\mathcal{N}_G\mathcal{U}^1$ are non-empty intersections $U_{\sigma_1} \cap U_{\sigma_2}$ an in general, for any natural number n, the n-simplices, $\mathcal{N}_G\mathcal{U}^n$ are non-empty intersections of elements in the cover $\mathcal{U}$ of length $n+1$, $U_{\sigma_1} \cap \ldots \cap U_{\sigma_2}$.

There are face $\delta_i : \mathcal{N}_G\mathcal{U}^i \to \mathcal{N}_G\mathcal{U}^{i-1}$ and degeneracy maps $s_i : \mathcal{N}_G\mathcal{U}^{i-1} \to \mathcal{N}_G\mathcal{U}^i$ coming from the associativity, respectively idempotence of intersection.

Definition 3.14. Let X be a proper G-space satisfying condition 3.11. Given a contractible slice cover $\mathcal{U}$, and a covariant coefficient system M, the Bredon equivariant homology groups with respect to $\mathcal{U}$, denoted by $H_n^G(X,\mathcal{U};M)$, are the homology groups of the chain complex defined as the R-module

$$C_*(\mathcal{N}_G\mathcal{U}, M)$$

Defined as follows. On a given degree n, $C_n(\mathcal{N}_G\mathcal{U}, M)$ is the R-module

$$\bigoplus_{U_\sigma = U_{\sigma_1}\cap\ldots\cap U_{\sigma_n}\in\mathcal{N}_G\mathcal{U}^n} M(U_\sigma)\otimes_R R[U_\sigma],$$

where $R[U_\sigma]$ denotes a free R module of rank one and U_σ denotes a non empty length n intersection.

The boundary maps of the simplicial set $\mathcal{N}_G\mathcal{U}$ tensored with the R-module homomorphisms given by the functoriality of M give the differential

$$d_n : \bigoplus_{U_\sigma = U_{\sigma_1}\cap\ldots\cap U_{\sigma_{n+1}}\in\mathcal{N}_G\mathcal{U}^n} M(U_\sigma)\otimes_R R[U_\sigma] \longrightarrow \bigoplus_{U_\tau = U_{\tau_1}\cap\ldots\cap U_{\tau_n}\in\mathcal{N}_G\mathcal{U}^{n-1}} M(U_\tau)\otimes_R R[U_\tau].$$

We will analyze now the coefficient system associated to twisted equivariant K-homology.

Let X be a space satisfying condition 3.11. $i_\sigma : G/H_\sigma \to U_\sigma \to X$ be the inclusion of a proper G-orbit into the contractible slice $U_\sigma \subset X$ and consider the Borel cohomology group $H^3(EG\times_G G/H_\sigma, \mathbb{Z})$. Given a class $P\in H^3(EG\times_G X,\mathbb{Z})$, we will denote by $\widetilde{H_{P_\sigma}}$ the central extension $1\to S^1\to \widetilde{H_{P_\sigma}}\to H_\sigma\to 1$ associated to the class given by the image of P under the maps

$$\omega_\sigma : H^3(EG\times X,\mathbb{Z}) \overset{i_\sigma^*}{\to} H^3(EG\times_G G/H_\sigma,\mathbb{Z}) \overset{\cong}{\to} H^3(BH_\sigma,\mathbb{Z}) \overset{\cong}{\to} H^2(BH_\sigma, S^1).$$

Denote by $R(\widetilde{H_{P_\sigma}})$ the abelian group of isomorphism classes of complex representations of the group $\widetilde{H_{P_\sigma}}$. Let $R_{S^1}(\widetilde{H_{P_\sigma}})$ be the subgroup generated by the complex representations of $\widetilde{H_{P_\sigma}}$ on which the central subgroup S^1 acts by complex multiplication.

Lemma 3.15 (Coefficients on equivariant contractible covers)**.** *The restriction of the functors $K^{\mathrm{geo}}_{G,0}(X,P)$ and $K^{\mathrm{geo}}_{G,1}(X,P)$ to the subsets U_σ gives covariant functors defined on the category $\mathcal{N}_G\mathcal{U}$. As abelian groups, the functors $K^{\mathrm{geo}}_{G,j}(X,P)$ satisfy:*

$$K^{\mathrm{geo}}_{G,j}(U_\sigma,P)\mid_{U_\sigma} = \begin{cases} R_{S^1}(\widetilde{H_{P_\sigma}}) \text{ If } j=0 \\ 0 \text{ If } j=1 \end{cases}.$$

In degree 0, a map $K^{\mathrm{geo}}_{G,0}(U_\sigma,P)\mid_{U_\sigma}\to K^{\mathrm{geo}}_{G,0}(U_\tau,P)\mid_{U_\sigma}$ is given by induction of representations along group inclusions $\widetilde{H}_\sigma\to\widetilde{H}_\tau$.

Proof. Notice the group isomorphism $K^{\mathrm{geo}}_{G,j}(U_\sigma,P)\mid_{U_\sigma}\cong K^{\mathrm{geo}}_{G,j}(G/H_\sigma,P)\mid_{U_\sigma}$ due to homotopy invariance, part ii in Theorem 3.9. The induction structure gives an isomorphism

$$K^{\mathrm{geo}}_{G,j}(G/H_\sigma, P\mid_{U_\sigma}) \cong K^{\mathrm{geo}}_{H,j}(\{\bullet\},\omega_\sigma(P)).$$

Consider the H_σ-trivial Space $\{\bullet\}$, and a geometric cycle

$$\big(M, f: M\to\{\bullet\}, s: M\to \mathrm{Fred}(f^*(\omega_{P_\sigma}))\big),$$

representing an element in the geometric equivariant K-homology group

$$K^{\mathrm{geo}}_{H,j}(\{\bullet\},\omega_\sigma(P)).$$

The pushforward map $f_!(s) : K^j_{H_\sigma}(M, f^*(\omega_{P_\sigma})) \to K^j_{H_\sigma}(\{\bullet\}, \omega_{P_\sigma})$ defined in [3] associated to the map $f : M \to \{\bullet\}$ gives an isomorphism to the group $K^j_{H_\sigma}(\{\bullet\}, \omega_{P_\sigma})$. On the other hand, this group has been verified to be isomorphic to either 0 if j is odd or the group $R_{S^1}(\widetilde{H_{P_\sigma}})$ if j is even in [5], 5.3.4.

For the description of the induced morphism, notice the compatibility of the induction structure in the sense of 3.10 with the representation theoretical induction structure of $R_{S^1}(\widetilde{H_{P_\sigma}})$. This follows from the naturality of the equivariant topological index in the context of finite groups [2].

□

We will now give a description of Segal's spectral sequence [29], used to compute Twisted Equivariant Geometric K-homology. The main application will be Theorem 4.6 below, relating these groups with analytical approaches to twisted equivariant K-homology. The construction of the spectral sequence is completely analogous to that of [4].

Proposition 3.16. *Let X be a G-space satisfying condition 3.11. There exists a spectral sequence associated to a locally finite and equivariantly contractible cover $\mathcal{U}$. It converges to $K^{\mathrm{geo}}_G(X, P)$, has for second page $E^2_{p,q}$ the homology of $\mathcal{N}_G\mathcal{U}$ with coefficients in the functor $\mathcal{K}^G_0(?, P|_?)$ whenever q is even, i.e.*

$$E^2_{p,q} := H^G_p(X, \mathcal{U}; \mathcal{K}^G_{\mathrm{geo},0}(\mathcal{U}, P|_{\mathcal{U}})) \tag{3.17}$$

and is trivial if q is odd. Its higher differentials

$$d_r : E^r_{p,q} \to E^{r+1}_{p-r,q+r-1}$$

vanish for r even.

Proof. The cover consists of open spaces, which are G-equivariantly homotopic to an orbit. Hence, we know that the groups $K^G_{q,\mathrm{geom}}(U_\sigma; P|_{U_\sigma})$ are periodic and trivial for q odd. Hence, the E^2 term is isomorphic to $H^G_p(X, \mathcal{U}; \mathcal{K}^q_G(?, P|_?))$.

□

4. Analytic K-homology and the Index Map.

Twisted Versions of Equivariant K-theory have been considered in the literature using a number of analytical constructions involving KK theory [11], [30].

We will use the coincidence of these approaches, previously studied in the literature [30], [17], as well as duality results to define an analytical index map with image on an equivariant KK-theory group, which is verified to be an isomorphism for proper finite G-CW complexes.

We shall recall notations for equivariant KK-groups. References for further study include [10], [25].

Definition 4.1. Let A and B be graded G-C^* algebras. The set $\mathcal{E}_G(A, B)$ of equivariant Kasparov modules consists of triples (E, ϕ, T) of a countably generated, graded Hilbert B-module E with a continuous action of G, an equivariant graded $*$-homomorphism to the adjointable operators of the Hilbert Module E $\phi : A \to \mathbb{B}(E)$, and a G-continuous operator F of degree 1 in $\mathbb{B}(E)$, with the following properties:

- The commutators $[F, \phi(a)]$ are in the algebra of compact, adjoinable operators $\mathbb{K}(E)$ for all $a \in A$.
- $(F^2 - 1)\phi(a) \in \mathbb{K}(E)$.

- $(F - F^*)\phi(a) \in \mathbb{K}(E)$.
- $(gF - F)\phi(a) \in \mathbb{K}(E)$.

Here $\mathbb{K}(E)$ denotes the compact adjoinable operators of the Hilbert Module E. Direct sum and homotopy turn KK_G into a group, [10] 20.2 in page 206. Usings Kasparov's Bott periodicity, [10] 20.2. 5, these groups are extended to $\mathbb{Z}/2$-graded groups $KK^*_G(\quad,\quad)$, which are covariant in the second variable, respectively contravariant in the first one.

Definition 4.2. Let X be a proper G-CW complex. Given a projective unitary stable G-equivariant bundle P defined with respect to a stable Hilbert space $\mathcal{H}$, consider the Space $\mathcal{K}$ of compact operators in $\mathcal{H}$. Let the group $P\mathcal{U}(\mathcal{H})$ (with the $*$-strong topology), act by conjugation on the space $\mathcal{K}$ of compact operators on $\mathcal{H}$ with the norm topology (which agrees with the $*$-topology.

The continuous trace Bundle associated to P is the bundle with structural group $P\mathcal{U}(\mathcal{H})$ and fiber $\mathcal{K}$ associated to P, denoted by $\mathcal{K}_P = P \times_{P\mathcal{U}(\mathcal{H})} \mathcal{K}$.

The following result summarizes some properties of the continuous trace bundle associated to a twist P.

Proposition 4.3. *Let P be a projective unitary stable G-equivariant bundle P defined over a G-CW complex X. Then,*

(i) The bundle $\mathcal{K}_P$ is locally trivial.
(ii) The sections vanishing at infinity of $\mathcal{K}_P$ form a G-Hilbert Module over the C^*-algebra $C_0(X)$, denoted by A_P.

Proof. We will verify the statements.

(i) It follows from the local triviality of the projective unitary stable G-Bundle P.
(ii) This follows from local triviality.

□

Theorem 4.4. *Let X be a proper, G- compact . There exist group isomorphisms*

$$A : K^*_G(X, P) \longrightarrow \mathcal{R}KK^*_{G,X}(C_0(X), A_P)$$

$$B : \mathcal{R}KK^*_{G,X}(C_0(X), A_P) \longrightarrow KK(\mathbb{C}, A_P \rtimes G)$$

Where the right hand side of the first equation, respectively the right hand side of the second one is representable KK-theory [21] *with respect to $C_0(X)$.*

Proof.

- This is a consequence of Theorem 3.14 in page 872 and proposition 6.10 in page 900 of [30]. However, we will collect the needed results. First notice that the action groupoid $\Gamma := X \rtimes G$ is proper and locally compact. The continuous trace Algebra bundle A_P carries the structure of a Fell bundle over Γ. Tu-Xu-Laurent construct in [30] the reduced C^* algebra associated to a Fell bundle and denote it by $C^*_r(\Gamma, E)$.

  Specializing Proposition 6.11 in page 900 of [30] identifies the Kasparov KK-group
  $$KK^*_\Gamma(C_0(X), C_0(X, \mathcal{K}_P))$$
  with the K-Theory groups $K_*(C^*_r(\Gamma, \mathcal{K}_P))$. Theorem 3.14 in [30] identifies this K-Theory group with the twisted equivariant K-theory groups

$K^*_G(X,P)$ as introduced in [5] in terms of homotopy groups of spaces of sections of Fredholm operators. On the other hand, the groups $KK^*_\Gamma(C_0(X), C_0(X,\mathcal{K}_P)$ are isomorphic to the groups $\mathcal{R}KK^*_{G,X}(C_0(M), A_P)$ of representable, equivariant KK- theory with respect to X. This gives isomorphism A.

- For the isomomorphism B, consider the composition

$$\mathcal{R}KK^*_{G,X}(C_0(M), A_P) \longrightarrow KK(C_0(X) \rtimes G, A_P \rtimes G) \to KK(\mathbb{C}, A_P \rtimes G),$$

where the first map is Kasparov's descent homomorphism and the second is given by tensoring with the Mishchenko-line bundle for X. It is proved in [14], Theorem 2.6 in page 15, that this map is an isomorphism.

□

General instances of the following theorem have been proved in [14] (Theorem 1.2 in page 3), as well as [17], Section 7.3.

Theorem 4.5. *Let G be a discrete Group. Let M be a proper, cocompact* $\mathrm{Spin}(c)$ *smooth G-manifold. Let $P \in H^3(X \times_G, EG, \mathbb{Z})$ be a class representing a projective unitary stable G-equivariant bundle. Denote by A_P the continuous trace algebra associated to it and let A_{-P} be the continuous trace algebra asssociated to the additive inverse of P.*

There exists an isomorphism of Kasparov KK-groups

$$PD : \mathcal{R}KK^*_{G,X}(A_P, \mathbb{C}) \longrightarrow KK^*_G(\mathbb{C}, A_{-P})$$

Proof. The proof consists of two KK-equivalences.

$$\mathcal{R}KK^*_{G,X}(A_P, \mathbb{C}) \cong \mathcal{R}KK^*_{G,X}(\mathbb{C}, A_{-P}) \cong KK^*_G(\mathbb{C}, A_{-P})$$

The first isomorphism comes from Theorem 1.2 in [14]. The last equivalence follows from the compacity assumption on M.

□

Theorem 4.6. *Let X be a finite proper G-CW complex. There exists an isomorphism*

$$\mathrm{Index} : K^{\mathrm{geo}}_{G,*}(X,P) \longrightarrow KK^*_G(\mathbb{C}, A_P)$$

Proof. Consider a geometric cycle (M, f, σ), and recall that it consists of a G-compact, proper orientable, $\mathrm{Spin}(c)$ manifold without boundary M, a G-equivariant map $f : M \to X$, as well as homotopy class of a G-invariant section $\sigma : M \to \mathcal{K}_{-f^*(P)}$ representing an element on the twisted equivariant K-theory group $K^0_G(M, -f^*(P))$. Consider the Poincaré duality isomorphism 4.5 and the equivalence to the KK-approach 4.4. They give isomorphims fitting in the following commutative diagram:

$$K^*_G(X, -f^*(P)) \overset{A}{\cong} \mathcal{R}_X KK^*_G(A_{-f^*(P)}, \mathbb{C}) \overset{PD}{\cong} KK^*_G(\mathbb{C}, A_{f^*(P)})$$

Define the map Index on the cycle (M, f, σ) as the element $f_* \circ PD \circ A(\sigma)$, where f_* denotes the map induced by f on equivariant KK-theory groups.

Since Both functors satisfy the axioms of an equivariant homology theory, there exist spectral sequences $E^r_{p,q}$ converging to $K^{\mathrm{geo}}_*(X,P)$, as indicated in 3.16. Using the usual homological properties of KK-theory, one constructs out of the cover for the spectral sequence in 3.16, a spectral sequence $F^r_{p,q}$ converging to $KK^*_G(\mathbb{C}, A_P)$.

Moreover, the Index map is natural and gives a natural transformation between the second terms of the spectral sequences, which consists of isomorphisms for a compact=finite=cocompact G-CW complex X. □

5. Relation to twisted crossed products

We state briefly the relation to twisted crossed products, studied previously in [16], [27].

Let G be a discrete, countable group.

Definition 5.1. A model for the classifying space for proper actions $\underline{E}G$ is a proper G-CW complex $\underline{E}G$ with the property that, given any other proper G-CW complex X, there exists up to G-equivariant homotopy a unique map $X \to \underline{E}G$.

Models for $\underline{E}G$ always exists and are unique up to G-equivariant homotopy [23], Theorem 1.9 in page 275. Moreover, $\underline{E}G$ can be characterized as a G-CW Complex with isotropy in the family of finite subgroups and such that the fixed point set X^H is either contractible if H is finite or empty otherwise.

Notice in particular that $\underline{E}G$ is contractible after forgetting the group action.

Hence, the Borel construction $\underline{E}G \times_G EG$ is a model for BG, the base space of the universal principal G-bundle $G \to EG \to BG$. This gives an isomorphism

$$\omega : H^3(\underline{E}G \times_G EG, \mathbb{Z}) \to H^3(G, \mathbb{Z}),$$

where $H^3(G, \mathbb{Z})$ denotes group cohomology with constant coefficients, which is isomorphic to Borel-Moore cohomology [26] since G is discrete.

Let S^1 be the abelian group of complex numbers of norm one.

There exists an isomorphism $\omega : H^3(G, \mathbb{Z}) \to H^2(G, S^1)$, which is given as a connection homomorphism of the long exact sequences produced by the coefficient sequence $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$.

Given a cohomology class $P \in H^3(\underline{E}G \times_G EG, \mathbb{Z})$, we will denote by $\alpha_\omega(P)$ a cocycle representative $\alpha_{\omega(P)} : G \times G \to S^1$ of the class associated to $\omega(P) \in H^2(G, S^1) = Z^2(G, S^1)/B^2(G, S^1)$.

Given such a cocycle $\alpha_{\omega(P)}$, one can form the reduced and full $\alpha_{\omega(P)}$-twisted group C^*-algebras, as follows.

Definition 5.2. Let G be a discrete group. Let $\alpha : G \times G \to S^1$ be a 2-cocycle. The α-twisted convolution algebra $l^1(G, \alpha)$ is the space of all summable complex functions on F with convolution and involution given by

$$f *_\alpha g(s) = \sum_{t \in G} f(t) g(t^{-1}s) \alpha(t, t^{-1}s),$$

$$f^*(s) = \alpha(s, s^{-1}) f(s^{-1}).$$

The full twisted group algebra $C^*(G, \alpha)$ is defined as the enveloping C^* algebra of $l^1(G, \alpha)$.

The reduced twisted group C^*-algebra $C^*_r(G, \alpha)$ is the closure of the *α-regular representation* $L_\alpha : G \to U(l^2(G))$ given by

$$(L_\alpha(s)\xi)(t) = \alpha(s, s^{-1}t)\xi(s^{-1}t),\ \xi \in l^2(G), s,\, t \in G$$

In a similar fashion, given a discrete, countable group G, a cocycle $\alpha : G \times G \to S^1$, a C^* algebra A and a map $U : G \to Aut(A)$, satisfying $U(g_1 g_2) = \alpha(g_1, g_2) U(g_1) U_(g_2)$, one forms the twisted reduced and full crossed product, denoted by $A \rtimes^r_\alpha G$ respectively $A \rtimes_\alpha G$. See [27] for details.

The following result relates the Continuous trace Algebra A_P with a twisted crossed product construction.

Proposition 5.3. *The $C_0(\underline{E}G)$ algebras A_P and $K \rtimes^r_{\alpha_{\omega(P)}} G \otimes C_0(\underline{E}G)$ are Morita equivalent.*

Proof. Consider the action groupoid $\Gamma := \underline{E}G \rtimes G$. Both $C_0(\underline{E}G)$-algebras are section algebras of elementary C^* bundles over the space of objects $\underline{E}G$ of Γ.

Theorem 16 in page 229 of [22] identifies the Morita equivalence of such bundles in terms of classes in a cohomology group isomorphic to the Borel construction $H^2(\underline{E}G \times_G EG, S^1)$. For the bundles in question, the classes are representatives of the class of $\omega(P)$, which is the same as the one of $\alpha_{\omega(P)}$. See also proposition 1.5 in [15] □

We will assume now that the group G has a model for $\underline{E}G$ which is constructed out of finitely many equivariant cells. Some examples of such groups include fundamental groups of hyperbolic manifolds (where the universal cover is a model), Gromov Hyperbolic groups (where the Rips complex with specific parameters provides a model), Mapping class groups of surfaces (the Teichmüller space gives a model for $\underline{E}G$), and groups acting properly and cocompacly on $CAT(0)$-spaces.

Theorem 5.4. *Let G be a discrete group with a model for $\underline{E}G$ with finitely many cells. Then, there exists a commutative diagram*

$$\begin{tikzcd}
K_*(\mathcal{K} \rtimes^r_{\alpha_{\omega(P)}} G) \arrow[rr, "PR"] & & K_*(C^*_r(G, \alpha_{\omega(P)})). \\
\mathcal{R}KK^*_{\underline{E}G}(C_0(\underline{E}G), C_0(\underline{E}G) \otimes \mathcal{K} \rtimes^r_{\alpha_{\omega(P)}} G) \arrow[u, "BC"] & & \\
\mathcal{R}KK^*_{G,\underline{E}G}(C_0(\underline{E}G), A_P) \arrow[u, "C"] & & K^{\mathrm{geo}}_{G,*}(\underline{E}G, P) \arrow[ll, "B^{-1}\circ \mathrm{Index}"']
\end{tikzcd}$$

Proof. The map PR is an isomorphism due to the Packer-Raeburn Stabilization Trick, Theorem 3.4 in page 299 of [27].

The map BC, the Baum-Connes Assembly map with coefficients in $\mathcal{K} \rtimes_{\alpha_{\omega(P)}} G$ is an isomorphism because of Theorem 5.4, page 180 of [20].

The map C is an isomorphism because the $C_0(\underline{E}G)$-Morita equivalence of 5.3 induces an isomorphism in the representable KK-theory group.

The last map is the Index isomorphism map of 4.6 composed with the isomorphism B^{-1}.

□

References

[1] S. A. Antonyan and E. Elfving. The equivariant homotopy type of G-ANR's for proper actions of locally compact groups. In *Algebraic topology—old and new*, volume 85 of *Banach Center Publ.*, pages 155–178. Polish Acad. Sci. Inst. Math., Warsaw, 2009.

[2] M. F. Atiyah and G. B. Segal. The index of elliptic operators. II. *Ann. of Math. (2)*, 87:531–545, 1968.

[3] N. Bárcenas, P. Carrillo, and M. Velásquez. Multiplicative structures and the twisted baum-connes assembly map. Arxiv:1501.05255, 2015.

[4] N. Bárcenas, J. Espinoza, B. Uribe, and M. Velasquez. Segal's spectral sequence in twisted equivariant k-theory for proper and discrete actions. preprint, ArXiv:1307.1003 [Math.KT], 2013.

[5] N. Bárcenas, M. Espinoza, Jesus Joachim, and B. Uribe. Universal twist in equivariant K-theory for proper and discrete actions. *Proc. Lond. Math. Soc. (3)*, 108(5):1313–1350, 2014.

[6] N. Bárcenas and m. Velásquez. The completion theorem in twisted equivariant K- theory for proper actions. ArXiv:1408.2404.

[7] N. Bárcenas and M. Velásquez. Twisted equivariant K-theory and K-homology of $\mathrm{Sl}_3\mathbb{Z}$. *Algebr. Geom. Topol.*, 14(2):823–852, 2014.

[8] P. Baum, A. Carey, and B.-L. Wang. K-cycles for twisted K-homology. *J. K-Theory*, 12(1):69–98, 2013.

[9] P. Baum, N. Higson, and T. Schick. A geometric description of equivariant K-homology for proper actions. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 1–22. Amer. Math. Soc., Providence, RI, 2010.

[10] B. Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[11] J. Chabert and S. Echterhoff. Twisted equivariant KK-theory and the Baum-Connes conjecture for group extensions. *K-Theory*, 23(2):157–200, 2001.

[12] J. Chabert, S. Echterhoff, and R. Nest. The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups. *Publ. Math. Inst. Hautes Études Sci.*, (97):239–278, 2003.

[13] S. Echterhoff. The K-theory of twisted group algebras. In *C^*-algebras and elliptic theory II*, Trends Math., pages 67–86. Birkhäuser, Basel, 2008.

[14] S. Echterhoff, H. Emerson, and H. J. Kim. KK-theoretic duality for proper twisted actions. *Math. Ann.*, 340(4):839–873, 2008.

[15] S. Echterhoff, W. Lück, N. C. Phillips, and S. Walters. The structure of crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. *J. Reine Angew. Math.*, 639:173–221, 2010.

[16] S. Echterhoff and D. P. Williams. Central twisted transformation groups and group C^*-algebras of central group extensions. *Indiana Univ. Math. J.*, 51(6):1277–1304, 2002.

[17] H. Emerson and R. Meyer. Equivariant representable K-theory. *J. Topol.*, 2(1):123–156, 2009.

[18] H. Emerson and R. Meyer. Dualities in equivariant Kasparov theory. *New York J. Math.*, 16:245–313, 2010.

[19] N. Hingston. Equivariant Morse theory and closed geodesics. *J. Differential Geom.*, 19(1):85–116, 1984.

[20] G. Kasparov and G. Skandalis. Groups acting properly on "bolic" spaces and the Novikov conjecture. *Ann. of Math. (2)*, 158(1):165–206, 2003.

[21] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.

[22] A. Kumjian. On equivariant sheaf cohomology and elementary C^*-bundles. *J. Operator Theory*, 20(2):207–240, 1988.

[23] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[24] W. Lück and B. Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology*, 40(3):585–616, 2001.

[25] G. Mislin and A. Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[26] C. C. Moore. Extensions and low dimensional cohomology theory of locally compact groups. I, II. *Trans. Amer. Math. Soc.*, 113:40–63, 1964.

[27] J. A. Packer and I. Raeburn. On the structure of twisted group C^*-algebras. *Trans. Amer. Math. Soc.*, 334(2):685–718, 1992.

[28] R. S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

[29] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[30] J.-L. Tu, P. Xu, and C. Laurent-Gengoux. Twisted K-theory of differentiable stacks. *Ann. Sci. École Norm. Sup. (4)*, 37(6):841–910, 2004.

[31] B. Uribe and J. Espinoza. Topological properties of the unitary group. ArXiv:1407.1869, 2014.

[32] B.-L. Wang. Geometric cycles, index theory and twisted K-homology. *J. Noncommut. Geom.*, 2(4):497–552, 2008.

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