A COMPLETION THEOREM FOR FUSION SYSTEMS

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ABSTRACT. We show that the twisted K-theory of the classifying space of a p-local finite group is isomorphic to the completion of the Grothendieck group of twisted representations of the fusion system with respect to the augmentation ideal of the representation ring of the fusion system. We use this result to compute the K-theory of the Ruiz-Viruel exotic 7-local finite groups.

INTRODUCTION

Homotopy invariants of the classifying space of a finite group can often be described in terms of algebraic invariants of the group. An important example is Atiyah's completion theorem [1], which shows that the K-theory of the classifying space is isomorphic to the completion of the representation ring with respect to the augmentation ideal. This result was extended to compact Lie groups in [2] and [3] and to twisted K-theory in [21].

A p-local finite group is an algebraic structure introduced by Broto, Levi and Oliver [9] with the purpose of giving a combinatorial model for the p-completion of the classifying space of a finite group, but also for other spaces that behave similarly. These additional p-local finite groups are called exotic and there are many examples in the literature by now, see for instance [10], [9], [14], [15], [22], [26], [30].

Each p-local finite group $(S, \mathcal{F}, \mathcal{L})$ has a classifying space $|\mathcal{L}|_p^\wedge$ and some of its homotopy invariants have been described in terms of the "algebraic part" of the p-local finite group, which is a saturated fusion system $\mathcal{F}$ over a finite p-group S. For example, the fundamental group of $|\mathcal{L}|_p^\wedge$ is the quotient of S by the hyperfocal subgroup of $\mathcal{F}$, and $H^2(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$ is the set of equivalence classes of central extensions of $\mathcal{F}$ by $\mathbb{Z}/p$ (see [8]).

In this article we give a description of the twisted K-theory of $|\mathcal{L}|_p^\wedge$ in terms of algebraic invariants of $(S, \mathcal{F})$. Namely, we consider the twistings of K-theory classified by the integral third cohomology group and in Section 4, we show that any element α in $H^3(|\mathcal{L}|_p^\wedge)$ is represented by a 2-cocycle of S with values in a cyclic p-group. Our main theorem is the following

Theorem. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and α a 2-cocycle of S with values in a cyclic p-group representing an element in $H^3(|\mathcal{L}|_p^\wedge)$. The completion of $^\alpha R(\mathcal{F})$ with respect to the augmentation ideal of $R(\mathcal{F})$ is isomorphic to $^\alpha K(|\mathcal{L}|_p^\wedge)$.*

Here $^\alpha R(\mathcal{F})$ is the Grothendieck group of the monoid of α-twisted complex representations of S that are $\mathcal{F}$-invariant in a certain sense. For the trivial 2-cocycle, this corresponds to the representation ring $R(\mathcal{F})$ of the fusion system,

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already studied in [11] and [12]. A complex representation ρ of S is $\mathcal{F}$-invariant if $\rho_{|P}$ is equivalent to $\rho_{|f(P)} \circ f$ for any $P \leq S$ and any $f \colon P \to S$ in $\mathcal{F}$.

In Section 2, we adapt Atiyah's proof from [1] to determine the prime ideals of $R(\mathcal{F})$ and compare the $I(S)$-topology and the $I(\mathcal{F})$-topology on $R(S)$, where $I(S)$ and $I(\mathcal{F})$ are the augmentation ideals of $R(S)$ and $R(\mathcal{F})$, respectively, and $R(S)$ can be regarded as an $R(\mathcal{F})$-module via the inclusion $R(\mathcal{F}) \to R(S)$. Both $R(\mathcal{F})$ and the K-theory of $|\mathcal{L}|_p^\wedge$ are computed by stable elements, in the sense that they are limits over the orbit category of $\mathcal{F}$-centric subgroups of the representation ring functor and the K-theory of the classifying space functor. The completion theorem in this case, which is proved in Section 3, follows from the naturality of Atiyah's completion map, the results in Section 2 and the exactness of the completion functor for finitely generated modules.

To study the general case, we see in Section 4 that an element α in $H^3(|\mathcal{L}|_p^\wedge)$ induces a central extension $(S_\alpha, \mathcal{F}_\alpha, \mathcal{L}_\alpha)$ of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ by $\mathbb{Z}/p^n$, where p^n is the order of S. There is a process of turning an α-twisted representation of S into a representation of S_α and we say that the twisted representation is $\mathcal{F}$-invariant when the corresponding twisted representation is $\mathcal{F}_\alpha$-invariant. In this way we obtain that ${}^\alpha R(\mathcal{F})$ is isomorphic to the Grothendieck group $R(\mathcal{F}_\alpha, \mathbb{Z}/p^n)$ of $\mathcal{F}_\alpha$-invariant representations of S_α where $\mathbb{Z}/p^n$ acts by scalar multiplication. Moreover, we show that this group can be computed by stable elements in $\mathcal{F}_\alpha$.

In order to compare this group with the twisted K-theory of $|\mathcal{L}|_p^\wedge$, we note that the classifying space $|\mathcal{L}_\alpha|_p^\wedge$ fits into a principal $B\mathbb{Z}/p^n$-bundle $|\mathcal{L}_\alpha|_p^\wedge \to |\mathcal{L}|_p^\wedge$. We describe in Section 5 an alternative description of the twisted K-theory for spaces X which are the base space of such a principal bundle with respect to the twisting that classifies the bundle. In particular, the twisted K-theory of $|\mathcal{L}|_p^\wedge$ is given as a set of equivariant homotopy classes of $B\mathbb{Z}/p^n$-equivariant maps from $|\mathcal{L}_\alpha|_p^\wedge$ to a certain space of Fredholm operators. The advantage of this characterization is that we can give a stable elements formula in $\mathcal{F}_\alpha$. The completion maps of [21], seen from this point of view, are natural and this allows us to use the same strategy as in the untwisted case to prove the general completion theorem.

In Section 6 we use this theorem for several computations. We compute the K-theory of the p-completion of $B\Sigma_p$ for any odd prime p and the twisted K-theory of the 2-completion of BA_4 with respect to the only nontrivial twisting in the third cohomology group of A_4. We also determine the K-theory of all the classifying spaces for the 7-local finite groups over the extraspecial 7-group of order 343 with more than two elementary abelian centric radical subgroups. This includes in particular the three exotic 7-local finite groups from [30].

1. Preliminaries

We first review the models of twisted K-theory and some of the results in the theory of p-local finite groups that will be used in the paper. This material is included for completeness and to fix some of the notation.

Twisted K-theory with respect to an integral twist $\alpha \in H^3(X; \mathbb{Z})$ was first introduced in [16] and [29]. We will adopt the homotopy theoretical viewpoint, as stated originally in [4], with the point-set topology considerations in [5].

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space. A projective bundle $P \to X$ is a locally trivial bundle with fiber $\mathbb{P}(\mathcal{H})$ and structural group $PU(\mathcal{H})$. We give $PU(\mathcal{H}) = U(\mathcal{H})/S^1$ the quotient topology, where $U(\mathcal{H})$ is given the compactly generated topology generated by the compact-open topology. Since $PU(\mathcal{H})$ is a $K(\mathbb{Z}, 2)$, isomorphism classes of projective bundles over X are classified by elements in $H^3(X)$. However, a twisting will be a projective bundle, which corresponds to a 3-cocycle.

Let $\mathrm{Fred}'(\mathcal{H})$ be the set of pairs (A, B) of bounded operators on $\mathcal{H}$ such that $AB - 1$ and $BA - 1$ are compact operators. Endow $\mathrm{Fred}'(\mathcal{H})$ with the topology induced by the embedding

$$\mathrm{Fred}'(\mathcal{H}) \to B(\mathcal{H}) \times B(\mathcal{H}) \times K(\mathcal{H}) \times K(\mathcal{H})$$

$$(A, B) \mapsto (A, B, AB - 1, BA - 1)$$

where $B(\mathcal{H})$ is the space of bounded operators with the compact-open topology and $K(\mathcal{H})$ is the space of compact operators with the norm topology. The group $PU(\mathcal{H})$ acts on $\mathrm{Fred}'(\mathcal{H})$ by conjugation. Given a projective bundle $P \to X$, we denote by $\mathrm{Fred}(P)$ the associated bundle $P \times_{PU(\mathcal{H})} \mathrm{Fred}'(\mathcal{H})$ over X.

Definition 1.1. The P-twisted K-theory of X is the group of homotopy classes of sections of $\mathrm{Fred}(P) \to X$. We denote it by ${}^P K(X)$.

Twisted K-theory is functorial with respect to maps of spaces and maps of projective bundles. Namely, a map $f: Y \to X$ induces a group homomorphism $f^*: {}^P K(X) \to {}^{f^*P} K(Y)$. And a map of projective bundles $g: P \to Q$ over X induces a group homomorphism $g_*: {}^P K(X) \to {}^Q K(X)$. Hence if two projective bundles are isomorphic, the corresponding twisted K-theory groups are isomorphic, but not canonically. There is also a multiplication

$$ {}^P K(X) \times {}^Q K(X) \to {}^{P \otimes Q} K(X)$$

In particular, ${}^P K(X)$ is a module over $K(X)$.

Now we recall the notion of a p–local finite group introduced by Broto, Levi and Oliver in [9] as a way of modelling combinatorially the p–completion of the classifying space of a finite group. Their definition was based on the concept of saturated fusion system, first described by Puig [27].

Given subgroups P and Q of S we denote by $\mathrm{Hom}_S(P, Q)$ the set of group homomorphisms $P \to Q$ that are conjugations by an element of S and by $\mathrm{Inj}(P, Q)$ the set of monomorphisms from P to Q.

Definition 1.2. A fusion system $\mathcal{F}$ over a finite p–group S is a subcategory of the category of groups whose objects are the subgroups of S and such that the set

of morphisms $\text{Hom}_{\mathcal{F}}(P, Q)$ between two subgroups P and Q satisfies the following conditions:

(a) $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$ for all $P, Q \leq S$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Definition 1.3. Let $\mathcal{F}$ be a fusion system over a p–group S.

- We say that two subgroups $P, Q \leq S$ are $\mathcal{F}$–conjugate if they are isomorphic in $\mathcal{F}$.
- A subgroup $P \leq S$ is fully centralized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which are $\mathcal{F}$–conjugate to P.
- A subgroup $P \leq S$ is fully normalized in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which are $\mathcal{F}$–conjugate to P.
- $\mathcal{F}$ is a saturated fusion system if the following conditions hold:
  (I) Each fully normalized subgroup $P \leq S$ is fully centralized and the group $\text{Aut}_S(P)$ is a p–Sylow subgroup of $\text{Aut}_{\mathcal{F}}(P)$.
  (II) If $P \leq S$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set
  $$N_\varphi = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi P)\},$$
  then there is $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}_{|P} = \varphi$.

The motivating example for this definition is the fusion system of a finite group G. For a fixed Sylow p–subgroup S of G, let $\mathcal{F}_S(G)$ be the fusion system over S defined by setting $\text{Hom}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$. This is a saturated fusion system. Examples of other exotic examples can be found for instance in [10], [9], [14], [15], [22], [26], [30].

Alperin's fusion theorem for saturated fusion systems (Theorem A.10 in [9]) shows that morphisms can be expressed as compositions of inclusions and automorphisms of a particular class of subgroups, called $\mathcal{F}$-centric radical subgroups. This result is often used to reduce limits over the orbit category of $\mathcal{F}$ to the full subcategory whose objects are $\mathcal{F}$-centric subgroups or $\mathcal{F}$-centric radical subgroups.

Definition 1.4. Let $\mathcal{F}$ be a fusion system over a p–group S.

- A subgroup $P \leq S$ is $\mathcal{F}$–centric if P and all its $\mathcal{F}$–conjugates contain their S–centralizers.
- A subgroup $P \leq S$ is $\mathcal{F}$–radical if $\text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$ is p–reduced, that is, it has no nontrivial normal p–subgroup.

The notion of a centric linking system is the extra structure needed in the definition of a p–local finite group to obtain a classifying space which behaves like $BG_p^\wedge$ for a finite group G. For the purposes of this paper, it suffices to say that a centric linking system $\mathcal{L}$ associated to a saturated fusion system is category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi \colon \mathcal{L} \to \mathcal{F}^c$ that satisfies certain properties. See Definition 1.7 in [9].

Definition 1.5. A p–local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p–group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of the p–local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|_p^\wedge$, the p-completion in the sense of Bousfield-Kan [6] of the geometric realization of the category $\mathcal{L}$.

A theorem of Chermak [13] (see also Oliver [25]) states that every saturated fusion system admits a centric linking system and that it is unique up to isomorphism of centric linking systems. In particular, the classifying space of a p–local finite group is determined up to homotopy equivalence by the saturated fusion system. In this paper we need to consider cohomological decompositions of $|\mathcal{L}|_p^\wedge$ over the orbit category. The orbit category of $\mathcal{F}$ is the category whose objects are the subgroups of S and whose morphisms are

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) := \mathrm{Hom}_{\mathcal{F}}(P,Q)/\mathrm{Inn}(Q).$$

Also, $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$–centric subgroups of S. Theorem 4.2 of [11] shows that cohomological invariants of p–local finite groups can be computed by stable elements, in the following sense.

Theorem 1.1. *Let h^* be a generalized cohomology theory. Given a p–local finite group $(S, \mathcal{F}, \mathcal{L})$, there is an isomorphism*

$$h^*(|\mathcal{L}|_p^\wedge) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} h^*(BP).$$

Finally, we would like to say a few words about the theory of central extensions of p–local finite groups in [8]. The center of a fusion system $\mathcal{F}$ over S is the subgroup

$$Z_{\mathcal{F}}(S) = \varprojlim_{\mathcal{F}^c} Z$$

where Z is the functor $P \mapsto Z(P)$. If A is a subgroup of $Z_{\mathcal{F}}(S)$ and $(S, \mathcal{F}, \mathcal{L})$ is a p–local finite group, there is a p–local finite group $(S/A, \mathcal{F}/A, (\mathcal{L}/A)^c)$ with

$$\mathrm{Hom}_{\mathcal{F}/A}(P/A, Q/A) = \mathrm{Im}[\mathrm{Hom}_{\mathcal{F}}(P,Q) \to \mathrm{Hom}(P/A, Q/A)]$$

and such that there is a principal fibration

$$BA \to |\mathcal{L}|_p^\wedge \to |(\mathcal{L}/A)^c|_p^\wedge$$

A central extension of a p–local finite group $(S, \mathcal{F}, \mathcal{L})$ by a finite abelian p–group A is a p–local finite group $(\widetilde{S}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{L}})$ such that $A \leq Z_{\widetilde{\mathcal{F}}}(S)$ and $(S, \mathcal{F}, \mathcal{L})$ is isomorphic to $(\widetilde{S}/A, \widetilde{\mathcal{F}}/A, (\widetilde{\mathcal{L}}/A)^c)$. The following theorem is part of Theorem 6.13 in [8]

Theorem 1.2. *There is a bijective correspondence between*

- *Equivalence classes of central extensions of $(S, \mathcal{F}, \mathcal{L})$ by A.*
- *Equivalence classes of principal fibrations $BA \to X \to |\mathcal{L}|_p^\wedge$.*
- *Elements of $H^2(|\mathcal{L}|_p^\wedge; A)$.*

2. The representation ring of a fusion system

In this section we determine the prime ideals of the representation ring of a saturated fusion system $\mathcal{F}$ over a finite p-group S. The argument is inspired by Section 6 of [1]. We use this information to compare different topologies on representation rings of subgroups of S. We start by recalling the definition of $\mathcal{F}$-invariant representations from [11].

Definition 2.1. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. A representation $\rho\colon S \to U_n$ is $\mathcal{F}$-invariant if the representations $\rho_{|P}$ and $\rho_{|f(P)} \circ f$ are isomorphic for any $P \leq S$ and any $f \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. We will also refer to ρ as a representation of $\mathcal{F}$.

A representation ρ is $\mathcal{F}$-invariant if and only if its character χ_ρ is $\mathcal{F}$-invariant in the sense that $\chi_\rho(s) = \chi_\rho(f(s))$ for any morphism f in $\mathcal{F}$ (See Remark 3.3 in [11]). In particular, the regular representation of S and the trivial representation are $\mathcal{F}$-invariant for any saturated fusion system $\mathcal{F}$ over S. Note that the set $\mathrm{Rep}(\mathcal{F})$ of isomorphism classes of unitary representations of S which are $\mathcal{F}$-invariant is a monoid under the operation induced by direct sum of representations. It is closed under tensor product as well.

Definition 2.2. The representation ring $R(\mathcal{F})$ of $\mathcal{F}$ is the Grothendieck group of $\mathrm{Rep}(\mathcal{F})$.

By Proposition 5.7 in [11], it can be described by stable elements:
$$R(\mathcal{F}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} R(Q)$$

Note that if V is an $\mathcal{F}$-invariant representation which is isomorphic to a direct sum $A \oplus B$ of representations of S, where A is $\mathcal{F}$-invariant, then so is B. Hence any $\mathcal{F}$-invariant representation is isomorphic to a direct sum of $\mathcal{F}$-invariant representations which can not be decomposed as direct sums of nontrivial $\mathcal{F}$-invariant subrepresentations. We will call these irreducible $\mathcal{F}$-invariant representations. Note that an irreducible $\mathcal{F}$-invariant representation may be reducible as a representation of S.

Therefore $R(\mathcal{F})$ is the free abelian group generated by the characters of irreducible $\mathcal{F}$-invariant representations. Let $X_\mathcal{F}$ be the set of $\mathcal{F}$-conjugacy classes of elements of S. Taking an $\mathcal{F}$-invariant representation to its character induces a ring homomorphism $\chi_\mathcal{F} \colon R(\mathcal{F}) \otimes \mathbb{C} \to \mathrm{Map}(X_\mathcal{F}, \mathbb{C})$.

Lemma 2.1. *The map $\chi_\mathcal{F} \colon R(\mathcal{F}) \otimes \mathbb{C} \to \mathrm{Map}(X_\mathcal{F}, \mathbb{C})$ is an isomorphism.*

Proof. Note that $\mathrm{Rep}(\mathcal{F})$ is a submonoid of $\mathrm{Rep}(S)$ by definition, hence the forgetful map $R(\mathcal{F}) \to R(S)$ is an injective ring homomorphism. Since $\mathbb{C}$ is flat over $\mathbb{Z}$, the induced map $R(\mathcal{F}) \otimes \mathbb{C} \to R(S) \otimes \mathbb{C}$ is also injective.

Let X_S the set of S-conjugacy classes of elements of S and consider the following commutative diagram

$$
\begin{array}{ccc}
R(\mathcal{F}) \otimes \mathbb{C} & \xrightarrow{\ \chi_\mathcal{F}\ } & \mathrm{Map}(X_\mathcal{F}, \mathbb{C}) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle r^*} \\
R(S) \otimes \mathbb{C} & \xrightarrow{\ \chi_S\ } & \mathrm{Map}(X_S, \mathbb{C})
\end{array}
$$

where the horizontal maps are induced by sending representations to their characters and $r \colon X_S \to X_\mathcal{F}$ is the quotient map.

It is well known that χ_S is an isomorphism, hence $\chi_\mathcal{F}$ is injective. Given a map $f \colon X_\mathcal{F} \to \mathbb{C}$, we have $r^*(f) = \chi_S(z)$ for some $z \in R(S) \otimes \mathbb{C}$. But this implies
$$z \in \varprojlim_{\mathcal{O}(\mathcal{F}^c)} (R(P) \otimes \mathbb{C})$$

And since $\mathbb{C}$ is flat over $\mathbb{Z}$
$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} (R(P) \otimes \mathbb{C}) = \left(\varprojlim_{\mathcal{O}(\mathcal{F}^c)} R(P) \right) \otimes \mathbb{C} = \mathrm{Im}(j)$$

And then $\chi_\mathcal{F}(z) = f$, that is, $\chi_\mathcal{F}$ is an isomorphism. $\qquad\square$

Corollary 2.2. *The characters of the irreducible $\mathcal{F}$-invariant representations of S form a basis of the vector space of $\mathcal{F}$-invariant maps $S \to \mathbb{C}$. In particular, the number of irreducible $\mathcal{F}$-invariant representations equals the number of $\mathcal{F}$-conjugacy classes of S.*

We denote by $I(G)$ the augmentation ideal of the group G, that is, the kernel of the augmentation map $R(G) \to \mathbb{Z}$. Similarly, we denote by $I(\mathcal{F})$ the kernel of the augmentation map $R(\mathcal{F}) \to \mathbb{Z}$.

Given $P \leq S$, we can regard $R(P)$ as a module over $R(\mathcal{F})$ via the composition $R(\mathcal{F}) \to R(S) \to R(P)$. In this section we prove that the $I(P)$-adic topology of $R(P)$ coincides with the $I(\mathcal{F})$-adic topology of $R(P)$ seen as an $R(\mathcal{F})$-module via this map. By Theorem 6.1 in [1], it suffices to show that the $I(S)$-adic topology of $R(S)$ coincides with its $I(\mathcal{F})$-adic topology.

Let n be the order of S and consider the subring A of $\mathbb{C}$ generated by $\mathbb{Z}$ and the nth roots of unity. Characters of representations of S take values in A, since they are sums of eigenvalues of unitary matrices whose multiplicative orders divide n. Since A is flat over $\mathbb{Z}$, a similar argument to the one used in the proof of Lemma 2.1 shows that $R_A(\mathcal{F}) := R(\mathcal{F}) \otimes A$ has a decomposition

$$R_A(\mathcal{F}) = \varprojlim_{\mathcal{O}(\mathcal{F}^c)} R_A(P)$$

and it can be seen as a subring of $\mathrm{Map}(X_{\mathcal{F}}, A)$.

Lemma 2.3. *Every prime ideal of $R_A(\mathcal{F})$ is the restriction of a prime ideal of $\mathrm{Map}(X_{\mathcal{F}}, A)$ and every prime ideal of $R(\mathcal{F})$ is the restriction of a prime ideal of $R_A(\mathcal{F})$.*

Proof. Since $\mathrm{Map}(X_{\mathcal{F}}, A)$ is a finitely generated abelian group, it follows by the theorem of Cohen-Seidenberg (Theorem 3 in page 257 of [31]) as in Lemma 6.2 of [1]. $\square$

Note that $\mathrm{Map}(X_{\mathcal{F}}, A)$ is a finitely generated free $\mathbb{Z}$-module, in particular it is a Noetherian ring. Given $x \in X_{\mathcal{F}}$ and a prime ideal $\mathfrak{p}$, the set of maps f with $f(x) \in \mathfrak{p}$ is a prime ideal of $\mathrm{Map}(X_{\mathcal{F}}, A)$ and all its prime ideals are of this form. Let us denote

$$P_{\mathfrak{p},x} = \{\chi \in R_A(\mathcal{F}) \mid \chi(x) \in \mathfrak{p}\}$$

where $\chi(x)$ denotes the image of any element of x. By the previous lemma, all the prime ideals of $R_A(\mathcal{F})$ have this form.

If $\mathfrak{p} \neq (0)$ then $\mathfrak{p} \cap \mathbb{Z}$ must be a nontrivial prime ideal of $\mathbb{Z}$, hence $\mathfrak{p} \cap \mathbb{Z} = q(\mathfrak{p})\mathbb{Z}$ for a certain prime number $q(\mathfrak{p})$. Moreover, $\mathfrak{p}$ is a maximal ideal of A and $A/\mathfrak{p}$ is a finite field of characteristic $q(\mathfrak{p})$.

Lemma 2.4. *If $\mathfrak{p} \neq (0)$ and $q(\mathfrak{p}) = p$, then $\chi(s) = \chi(1)$ for any $\chi \in R_A(\mathcal{F})$ and any $s \in S$.*

Proof. It suffices to show this when χ is the character of a representation of S. Since the restriction of this representation to the subgroup generated by s breaks as a direct sum of one-dimensional representations, we can also assume that this representation is one-dimensional and so χ is multiplicative. The order of s is a power of p, hence the multiplicative order of $\chi(s)$ in $A/\mathfrak{p}$ is a power of p. But $A/\mathfrak{p}$ is a finite field of characteristic p, hence $\chi(s) = 1 = \chi(1)$. $\square$

For simplicity, we will use the following notation. Given $x \in X_{\mathcal{F}}$, we will denote

$$x_{\mathfrak{p}} = \begin{cases} 1 & \text{if } \mathfrak{p} \neq (0) \text{ and } q(\mathfrak{p}) = p \\ x & \text{otherwise} \end{cases}$$

where 1 denotes the $\mathcal{F}$-conjugacy class of 1.

Lemma 2.5. *The ideal $P_{\mathfrak{p},x}$ is contained in $P_{\mathfrak{q},y}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$ and $x_{\mathfrak{p}} = y_{\mathfrak{q}}$.*

Proof. Suppose that $\mathfrak{p} \subseteq \mathfrak{q}$ and $x_{\mathfrak{p}} = y_{\mathfrak{q}}$. Let $\chi \in P_{\mathfrak{p},x}$, that is, $\chi(x) \in \mathfrak{p}$. By Lemma 2.4, we have

$$\chi(x) = \chi(x_{\mathfrak{p}}) = \chi(y_{\mathfrak{q}}) = \chi(y)$$

and so $\chi(y) \in \mathfrak{p} \subseteq \mathfrak{q}$. Hence $\chi \in P_{\mathfrak{q},y}$.

On the other hand, $P_{\mathfrak{p},x} \subseteq P_{\mathfrak{q},y}$ implies $\mathfrak{p} = P_{\mathfrak{p},x} \cap A \subseteq P_{\mathfrak{q},y} \cap A = \mathfrak{q}$. Suppose $x_{\mathfrak{p}} \neq y_{\mathfrak{q}}$. The class $y_{\mathfrak{q}}$ is the disjoint union of a finite number of S-conjugacy classes, say of elements $y_1, \ldots, y_n$.

Assume $\mathfrak{q} \neq (0)$ and $q(\mathfrak{q}) = q$. By Lemma 3 in [7] there is character χ_i in $R_A(S)$ that takes values in $\mathbb{Z}$ and such that $\chi_i(y_i) \equiv 1 \mod q$ and $\chi_i(z) = 0$ if z is not S-conjugate to y_i. Let $a_i = \chi_i(y_i)$ and consider the character

$$\chi = \sum_{i=1}^{n} \left(\prod_{j \neq i} a_j \right) \chi_i$$

which again lies in $R_A(S)$. This character also takes values in $\mathbb{Z}$ and it satisfies

$$\chi(y_j) = \prod_{i=1}^{n} a_i \equiv 1 \mod q$$

for all j. Moreover $\chi(z) = 0$ if z is not S-conjugate to y_i for any i. Since this character is constant in each $\mathcal{F}$-conjugacy class, it actually lies in $R_A(\mathcal{F})$. And it satisfies $\chi(y) = \chi(y_{\mathfrak{q}}) \notin \mathfrak{q}$ and $\chi(x) = \chi(x_{\mathfrak{p}}) \in \mathfrak{p}$, which contradicts $P_{\mathfrak{p},x} \subseteq P_{\mathfrak{q},y}$.

If $\mathfrak{q} = 0$, it is well known that there is $\alpha_i \in R(S)$ such that $\alpha_i(y_i) \neq 0$ and $\alpha_i(z) = 0$ if z is not S-conjugate to y_i. Note that $b_i = \alpha_i(y_i)$ belongs to A. We consider

$$\alpha = \sum_{i=1}^{n} \left(\prod_{j \neq i} b_j \right) \alpha_i$$

which is a character in $R_A(S)$. It satisfies

$$\alpha(y_j) = \prod_{i=1}^{n} b_i \neq 0$$

for all j and $\alpha(z) = 0$ if z is not S-conjugate to y_i for any i. Since this character is constant on each $\mathcal{F}$-conjugacy class, it belongs to $R_A(\mathcal{F})$. And it satisfies $\alpha(x) = \alpha(x_{\mathfrak{p}}) = 0 \in (0) = \mathfrak{p}$ and $\alpha(y) = \alpha(y_{\mathfrak{q}}) \notin (0) = \mathfrak{q}$. This is again a contradiction with $P_{\mathfrak{p},x} \subseteq P_{\mathfrak{q},y}$. $\square$

Note that all the prime ideals in A are either (0) or maximal ideals, hence we can conclude:

Proposition 2.6. *The prime ideals of $R_A(\mathcal{F})$ are all of the form $P_{\mathfrak{p},x}$. Two such ideals $P_{\mathfrak{p},x} = P_{\mathfrak{q},y}$ coincide if and only if $\mathfrak{p} = \mathfrak{q}$ and $x_{\mathfrak{p}} = y_{\mathfrak{q}}$. If $\mathfrak{p} \neq (0)$, then $P_{\mathfrak{p},x}$ is a maximal prime ideal, while $P_{(0),x}$ is a minimal prime ideal. The maximal prime*

ideals containing $P_{(0),x}$ are the ideals $P_{\mathfrak{p},x}$ with $\mathfrak{p} \neq (0)$. The minimal prime ideals contained in $P_{\mathfrak{p},x}$ for a certain $\mathfrak{p} \neq (0)$ are the ideals $P_{(0),y}$ with $x_{\mathfrak{p}} = y$.

In short there are the following ideals. There is a $P_{(0),x}$ for each $x \in X_{\mathcal{F}}$, a $P_{\mathfrak{p},x}$ for each $x \in X_{\mathcal{F}}$ and each nontrivial prime ideal $\mathfrak{p}$ of A with $q(\mathfrak{p}) \neq p$, and a $P_{\mathfrak{p},1}$ for each nontrivial prime ideal $\mathfrak{p}$ of A with $q(\mathfrak{p}) = p$.

Corollary 2.7. *The spectra of $R_A(\mathcal{F})$ and $R(\mathcal{F})$ are connected in the Zariski topology.*

Lemma 2.8. *The ideal $P_{(0),1}$ coincides with $I(\mathcal{F}) \otimes A$.*

Proof. See the proof of Lemma 6.4 in [1]. $\square$

Given $y \in X_S$ and a prime ideal $\mathfrak{p}$ of A, we consider

$$Q_{\mathfrak{p},y} = \{\chi \in R_A(S) \mid \chi(y) \in \mathfrak{p}\}$$

and denote by res the maps $R_A(\mathcal{F}) \to R_A(S)$ and $R(\mathcal{F}) \to R(S)$ induced by restriction.

Lemma 2.9. *If $\mathrm{res}^{-1}(Q_{\mathfrak{p},y}) = P_{\mathfrak{q},1}$, then $Q_{\mathfrak{p},y} = Q_{\mathfrak{q},1}$.*

Proof. Note that $\mathrm{res}^{-1}(Q_{\mathfrak{p},y}) = P_{\mathfrak{p},r(y)}$, hence $\mathfrak{p} = \mathfrak{q}$ and $r(y)_{\mathfrak{p}} = 1$. Hence $y_{\mathfrak{p}} = 1$ and so $Q_{\mathfrak{p},y} = Q_{\mathfrak{q},1}$. $\square$

Lemma 2.10. *The prime ideals of $R(S)$ which contain $\mathrm{res}(I(\mathcal{F}))$ are the same as those which contain $I(S)$.*

Proof. The proof is practically the same as that of Lemma 6.7 in [1]. Since the augmentation maps fit into a commutative diagram

$$\begin{array}{ccc} R(\mathcal{F}) & \longrightarrow & \mathbb{Z} \\ \mathrm{res} \downarrow & \nearrow & \\ R(S) & & \end{array}$$

we have $\mathrm{res}(I(\mathcal{F})) \subseteq I(S)$ and so the prime ideals that contain $I(S)$ also contain $\mathrm{res}(I(\mathcal{F}))$.

Let K be a prime ideal of $R(S)$ which contains $\mathrm{res}(I(\mathcal{F}))$. By Lemma 2.3, there is a prime ideal $\mathfrak{p}$ of A and $x \in X_S$ such that $K = R(S) \cap Q_{\mathfrak{p},x}$. Therefore $\mathrm{res}(I(\mathcal{F})) \subseteq Q_{\mathfrak{p},x}$. Recall that $I(\mathcal{F}) \otimes A = P_{(0),1}$ by Lemma 2.8 and so

$$\mathrm{res}(P_{(0),1}) = \mathrm{res}(I(\mathcal{F}) \otimes A) \subseteq \mathrm{res}(I(\mathcal{F})) \otimes A \subseteq Q_{\mathfrak{p},x}$$

and so $P_{(0),1} \subseteq \mathrm{res}^{-1}(Q_{\mathfrak{p},x})$. By Lemma 2.6, we have $\mathrm{res}^{-1}(Q_{\mathfrak{p},x}) = P_{\mathfrak{q},1}$ for a certain prime ideal $\mathfrak{q}$ of A and by Lemma 2.9, $Q_{\mathfrak{p},x} = Q_{\mathfrak{q},1}$. In particular $I(S) \subseteq Q_{\mathfrak{p},x}$ and so $I(S) \subseteq K$. $\square$

Theorem 2.11. *The $I(S)$-adic topology of $R(S)$ coincides with its $I(\mathcal{F})$-adic topology.*

Proof. Just like Theorem 6.1 in [1], it follows from the previous lemma and the fact that in a Noetherian ring, the J-adic topology is the same as the $\mathrm{rad}(J)$-adic topology. $\square$

Corollary 2.12. *Let P be a subgroup of S. Then the $I(P)$-adic topology of $R(P)$ coincides with its $I(\mathcal{F})$-adic topology.*

3. The completion theorem

In this section we show a completion theorem for the K-theory of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. Note that the homomorphism

$$K(|\mathcal{L}|_p^\wedge) \to K(BS)$$

induced by the map $BS \to |\mathcal{L}|_p^\wedge$ coincides with composing the isomorphism

$$K(|\mathcal{L}|_p^\wedge) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} K(BP)$$

from Theorem 4.2 of [11] with the projection to the factor $K(BS)$. In particular, it is injective and we identify $K(|\mathcal{L}|_p^\wedge)$ as the subring of stable elements of $K(BS)$. That is, elements y such that $\operatorname{res}_P^S(y) = f^*(y)$ for all $P \leq S$ and any $f \in \operatorname{Hom}_{\mathcal{F}}(P, S)$. Here res_P^S is the map induced in K-theory by the inclusion of P in S.

Theorem 3.1. *Given a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the completion of $R(\mathcal{F})$ with respect to the ideal $I(\mathcal{F})$ is isomorphic to the K-theory ring of $|\mathcal{L}|_p^\wedge$.*

Proof. The map $\alpha_G \colon R(G) \to K(BG)$ constructed in Section 7 of [1] is natural with respect to group homomorphisms, hence it defines a chain map

$$\alpha_* \colon C^*(\mathcal{O}(\mathcal{F}^c); R) \to C^*(\mathcal{O}(\mathcal{F}^c); K)$$

where R and K are the contravariant functors $\mathcal{O}(\mathcal{F}^c) \to \mathrm{Ab}$ which send P to $R(P)$ and $K(BP)$, respectively. Here $C^*(D; F)$ denotes the cochain complex of the category D with coefficients in a covariant functor $F \colon D \to \mathrm{Ab}$, which can be used to compute the higher limits of the functor F (see Lemma 2 in [24] for instance).

For any morphism $f \colon P \to Q$ in $\mathcal{O}(\mathcal{F}^c)$, we have a commutative diagram

$$
\begin{array}{ccc}
 & R(\mathcal{F}) & \\
\scriptstyle{\operatorname{res}_P^S \operatorname{res}} \Big\downarrow & \searrow \scriptstyle{\operatorname{res}_Q^S \operatorname{res}} & \\
R(P) & \xrightarrow{\ f^*\ } & R(Q)
\end{array}
$$

and therefore f^* is a homomorphism of $R(\mathcal{F})$-modules. So we can regard R as a contravariant functor $\mathcal{O}(\mathcal{F}^c) \to R(\mathcal{F}) - \mathrm{Mod}$ and $C^*(\mathcal{O}(\mathcal{F}^c); R)$ as a complex of $R(\mathcal{F})$-modules. Completion with respect to the ideal $I(\mathcal{F})$ is an additive functor, so we have an isomorphism of complexes

$$C^*(\mathcal{O}(\mathcal{F}^c); R)_{I_{\mathcal{F}}}^\wedge \cong C^*(\mathcal{O}(\mathcal{F}^c); R_{I_{\mathcal{F}}}^\wedge)$$

where $R_{I_{\mathcal{F}}}^\wedge \colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Ab}$ is the contravariant functor that sends P to $R(P)_{I_{\mathcal{F}}}^\wedge$.

The homomorphism α_P induces an isomorphism $R(P)_{I(P)}^\wedge \to K^*(BP)$ by Atiyah's completion theorem and the $I(P)$-adic topology coincides with the $I(\mathcal{F})$-adic topology by Corollary 2.12. Therefore the chain map α_* induces an isomorphism of chain complexes.

$$C^*(\mathcal{O}(\mathcal{F}^c); R_{I_{\mathcal{F}}}^\wedge) \cong C^*(\mathcal{O}(\mathcal{F}^c); K)$$

On the other hand, since $R(\mathcal{F})$ is a Noetherian ring, completion of finitely generated $R(\mathcal{F})$-modules with respect to the ideal $I(\mathcal{F})$ is exact. Therefore we obtain

an isomorphism of abelian groups

$$
\begin{aligned}
K(|\mathcal{L}|_p^\wedge) &\cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} K(BP) \\
&\cong H^0(C^*(\mathcal{O}(\mathcal{F}^c); K)) \\
&\cong H^0(C^*(\mathcal{O}(\mathcal{F}^c); R)_{I_\mathcal{F}}^\wedge) \\
&\cong H^0(C^*(\mathcal{O}(\mathcal{F}^c); R))_{I_\mathcal{F}}^\wedge \\
&\cong \left(\varprojlim_{\mathcal{O}(\mathcal{F}^c)} R(P) \right)_{I_\mathcal{F}}^\wedge \\
&\cong R(\mathcal{F})_{I_\mathcal{F}}^\wedge
\end{aligned}
$$

where the first isomorphism holds by Theorem 4.2 in [11]. This isomorphism is induced by restriction to $K(BS)$ and so the isomorphism from $R(\mathcal{F})_{I_\mathcal{F}}^\wedge$ to $K(|\mathcal{L}|_p^\wedge)$ is induced by the restriction of the ring isomorphism $R(S)_{I_\mathcal{F}}^\wedge \to K(BS)$ to $R(\mathcal{F})_{I_\mathcal{F}}^\wedge$. This shows that in fact it is a ring isomorphism. □

Remark 3.1. It is well known that the topology induced by $I(S)$ coincides with the topology induced by $pI(S)$ and so

$$
\widetilde{K}(BS) \cong I(S)_{I(S)}^\wedge \cong I(S)_{pI(S)}^\wedge \cong J(S)_{J(S)}^\wedge
$$

where $J(S) = I(S) \otimes \mathbb{Z}_p^\wedge$. Hence the induced map $R(S)_p^\wedge \to K(BS; \mathbb{Z}_p^\wedge)$ induces an isomorphism $(R(S)_p^\wedge)_{J(S)}^\wedge \to K(BS; \mathbb{Z}_p^\wedge)$ since

$$
(R(S)_p^\wedge)_{J(S)}^\wedge \cong (\mathbb{Z}_p^\wedge \oplus J(S))_{J(S)}^\wedge \cong \mathbb{Z}_p^\wedge \oplus \widetilde{K}(BS) \cong \mathbb{Z}_p^\wedge \oplus \widetilde{K}(BS; \mathbb{Z}_p^\wedge) = K(BS; \mathbb{Z}_p^\wedge)
$$

The last isomorphism holds because $\widetilde{H}^k(BS; \mathbb{Z}[1/p]) = 0$ and BU is simply connected and so

$$
\widetilde{K}(BS) \cong [BS, BU] \cong [BS, BU_p^\wedge] = \widetilde{K}(BS; \mathbb{Z}_p^\wedge)
$$

by Theorem 1.4 in [23]. The same argument in the proof of the previous theorem shows that the completion of $R(\mathcal{F})_p^\wedge$ at the ideal $J(\mathcal{F}) = I(\mathcal{F}) \otimes \mathbb{Z}_p^\wedge$ is isomorphic to the p-adic K-theory ring $K(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$.

4. TWISTINGS AND TWISTED REPRESENTATIONS

In this section we introduce twisted representations of fusion systems and relate them to representations of central extensions. We begin by determining an appropriate model for the twistings classified by the third integral cohomology group.

Let $|\mathcal{L}|_p^\wedge$ be the classifying space of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$. By Theorem 4.2 in [11], there is an isomorphism

$$
H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z}) \cong \varprojlim_{\mathcal{O}(\mathcal{F})} H^3(BS; \mathbb{Z})
$$

and so this is a finite abelian p-group. In particular

$$
H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z}) \cong H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z})_{(p)} \cong H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z}_{(p)})
$$

On the other hand, from the short exact sequence

$$
0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(p)} \to 0
$$

and the group isomorphisms

$$\mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Z}/p^\infty$$

we obtain

$$H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z}_{(p)}) \cong H^2(|\mathcal{L}|_p^\wedge; \mathbb{Z}/p^\infty)$$

Let p^n be the order of S and consider the short exact sequence

$$0 \to \mathbb{Z}/p^n \overset{i}{\to} \mathbb{Z}/p^\infty \overset{j}{\to} \mathbb{Z}/p^\infty \to 0$$

where i is the standard inclusion and $j(x) = p^n x$. The map $BS \to |\mathcal{L}|_p^\wedge$ induces maps between the associated long exact sequences in cohomology

$$
\begin{array}{ccccc}
H^2(|\mathcal{L}|_p^\wedge; \mathbb{Z}/p^n) & \overset{i_*}{\longrightarrow} & H^2(|\mathcal{L}|_p^\wedge; \mathbb{Z}/p^\infty) & \overset{j_*}{\longrightarrow} & H^2(|\mathcal{L}|_p^\wedge; \mathbb{Z}/p^\infty) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(BS; \mathbb{Z}/p^n) & \overset{i_*}{\longrightarrow} & H^2(BS; \mathbb{Z}/p^\infty) & \overset{j_*}{\longrightarrow} & H^2(BS; \mathbb{Z}/p^\infty)
\end{array}
$$

Since the lower j_* is zero and the vertical arrows are injective, the top j_* is also zero. Hence both maps i_* are surjective. In particular, this shows that any twisting in $H^3(|\mathcal{L}|_p^\wedge; \mathbb{Z})$ comes from an element in $H^2(|\mathcal{L}|_p^\wedge; \mathbb{Z}/p^n)$.

By these considerations and Lemma 6.12 in [8], we can model the twistings of $|\mathcal{L}|_p^\wedge$ by 2-cocycles on the quasicentric linking system $\mathcal{L}^q$ (see Definition 1.9 in [8]) with values in a cyclic p-group A. That is, α is a function from pairs of composable morphisms in the quasicentric linking system $\mathcal{L}^q$ to A such that $\alpha(f, g)$ vanishes if f or g is an identity morphism and for any triple of f, g, h of composable morphisms, the cocycle condition is satisfied:

$$\alpha(g, h) - \alpha(gf, h) + \alpha(f, hg) - \alpha(f, g) = 0$$

The distinguished morphism $\delta_S \colon S \to \mathrm{Aut}_{\mathcal{L}^q}(S)$ together with the inclusion of $\mathrm{Aut}_{\mathcal{L}^q}(S)$ in $\mathcal{L}^q$ define a functor $\mathcal{B}S \to \mathcal{L}^q$. Restriction along this functor define a 2-cocycle for S with values in A which we also denote by α. We will refer to α as a 2-cocycle for $(S, \mathcal{F}, \mathcal{L})$.

Now that we have determined a model for our twistings, we recall how we use 2-cocycles to define twisted representations of groups, following [20]. From now on, A is a cyclic p-group which we regard inside S^1 as the subgroup of corresponding roots of unity.

Definition 4.1. Let α be a 2-cocycle for a group S with values in A. An α-twisted representation of S is a map $\rho \colon S \to U_n$ such that $\rho(1)$ is the identity matrix and

$$\rho(s)\rho(t) = \alpha(s, t)\rho(st).$$

for any s, $t \in S$.

Two α-twisted representations ρ_1, $\rho_2 \colon S \to U_n$ are equivalent if there is a linear isomorphism $f \colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\rho_1 = f\rho_2 f^{-1}$. We denote by $^\alpha\mathrm{Rep}_n(S)$ the set of equivalence classes of α-twisted n-dimensional representations of S.

Remark 4.1. Recall that an $\mathcal{F}$-invariant representation of S is a representation ρ such that $\rho_{|P} = \rho_{|f(P)} \circ f$ in $\mathrm{Rep}(P, U_n)$ for any $f \in \mathrm{Hom}_\mathcal{F}(P, S)$. However, note that j_P, $f \in \mathrm{Hom}_\mathcal{F}(P, S)$ induce maps

$$f^* \colon {}^\alpha\mathrm{Rep}_n(S) \to {}^{f^*\alpha}\mathrm{Rep}_n(P)$$

and

$$j_P^* \colon {}^{\alpha}\operatorname{Rep}_n(S) \to {}^{j_P^*\alpha}\operatorname{Rep}_n(P)$$

But even if $[\alpha] \in H^2(S; A)$ is invariant with respect to the fusion system, it is not necessarily the case that $f^*\alpha = j_P^*\alpha$. In general, they differ by a coboundary and so there is a bijection between the two sets of twisted representations of P, but they may not be equal.

Recall that a 2-cocycle $\alpha \colon S \times S \to A$ determines a central extension of A by S as follows. Let S_α be the set $A \times S$ with the operation

$$(a_1, s_1)(a_2, s_2) = (a_1 + a_2 + \alpha(s_1, s_2), s_1 s_2)$$

This gives S_α the structure of a group which fits into a central extension

$$1 \to A \to S_\alpha \to S \to 1$$

with the obvious homomorphisms. An α-twisted representation $\rho \colon S \to U_n$ defines a representation $\rho_\alpha \colon S_\alpha \to U_n$ by

$$\rho_\alpha(a, s) = a\rho(s)$$

where we see a as a complex number. We refer to ρ_α as the untwisting of ρ.

By Theorem 6.13 in [8] there is a bijective correspondence between the set of equivalence classes of central extensions of $(S, \mathcal{F}, \mathcal{L})$ by A and elements of $H^2(|\mathcal{L}|_p^{\wedge}; A)$. Given a class in $H^2(|\mathcal{L}|_p^{\wedge}; A)$ and a 2-cocycle α for the corresponding class in $H^2(S; A)$, we denote by $(S_\alpha, \mathcal{F}_\alpha, \mathcal{L}_\alpha)$ the corresponding p-local finite group.

Definition 4.2. Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$. We say that an α-twisted representation ρ of S is $\mathcal{F}$-invariant if ρ_α is $\mathcal{F}_\alpha$-invariant.

Let us denote by ${}^{\alpha}\operatorname{Rep}(\mathcal{F})$ the set of isomorphism classes of α-twisted representations of S that are $\mathcal{F}$-invariant. This set is a monoid under direct sum and the following is a natural definition.

Definition 4.3. Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$. The α-twisted representation group of $\mathcal{F}$ is the Grothendieck group of the monoid ${}^{\alpha}\operatorname{Rep}(\mathcal{F})$. We will denote it by ${}^{\alpha}R(\mathcal{F})$.

The abelian group ${}^{\alpha}R(\mathcal{F})$ is an $R(\mathcal{F})$-module with the action

$$R(\mathcal{F}) \times {}^{\alpha}R(\mathcal{F}) \to {}^{\alpha}R(\mathcal{F})$$

$$(V, W) \mapsto V \otimes W$$

Note that representations obtained by untwisting are A-representations in the following sense.

Definition 4.4. We will say that a representation of S_α is an A-representation if A acts by complex multiplication.

Let $\operatorname{Rep}(S_\alpha; A)$ be the monoid of isomorphism classes of A-representations of S_α and $R(S_\alpha; A)$ the corresponding Grothendieck group. By Lemma 1.2 in Chapter 4 of [20], the process of untwisting determines isomorphisms

$$ {}^{\alpha}\operatorname{Rep}(S) \cong \operatorname{Rep}(S_\alpha; A)$$

$$ {}^{\alpha}R(S) \cong R(S_\alpha; A)$$

Let us denote by $\mathrm{Rep}(\mathcal{F}_\alpha; A)$ the monoid of isomorphism classes of A-representations of S_α that are $\mathcal{F}_\alpha$-invariant, that is

$$\mathrm{Rep}(\mathcal{F}_\alpha; A) = \mathrm{Rep}(S_\alpha; A) \cap \mathrm{Rep}(\mathcal{F}_\alpha)$$

and let $R(\mathcal{F}_\alpha; A)$ be the corresponding Grothendieck group. This is an $R(\mathcal{F})$-module with the action

$$R(\mathcal{F}) \times R(\mathcal{F}_\alpha; A) \to R(\mathcal{F}_\alpha; A)$$

$$(V, W) \mapsto V \otimes W$$

where $(a,s)v \otimes w = sv \otimes (a,s)w$.

Lemma 4.1. *Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and let $(S_\alpha, \mathcal{F}_\alpha, \mathcal{L}_\alpha)$ be the corresponding central extension. Then untwisting of α-twisted representations of S induces an isomorphism of $R(\mathcal{F})$-modules*

$$^\alpha R(\mathcal{F}) \to R(\mathcal{F}_\alpha; A)$$

Proof. Consider the restriction of the untwisting isomorphism $^\alpha R(S) \cong R(S_\alpha; A)$ to $^\alpha R(\mathcal{F})$. By definition, ρ is an $\mathcal{F}$-invariant α-twisted representation if and only if ρ_α is $\mathcal{F}_\alpha$-invariant, hence the restriction $^\alpha R(\mathcal{F}) \to R(\mathcal{F}_\alpha; A)$ is an isomorphism of abelian groups. It is straightforward to check that it is an isomorphism of $R(\mathcal{F})$-modules. $\square$

Corollary 4.2. *Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$. Then $^\alpha R(\mathcal{F})$ is a finitely generated module over the representation ring $R(\mathcal{F})$.*

Proof. It follows from the previous lemma and the fact that $R(\mathcal{F}_\alpha; A)$ is finitely generated as an abelian group. $\square$

The following result is mentioned in the proof of Lemma 5.6 (d) of [9] without proof. We include a proof here for completeness.

Lemma 4.3. *Let $\mathcal{F}_\alpha$ be a saturated fusion system over S_α and let A be a subgroup of S_α which is central in $\mathcal{F}_\alpha$. Then there exists a characteristic biset Λ of $\mathcal{F}_\alpha$ such that $ax = xa$ for all $x \in \Lambda$ and all $a \in A$.*

Proof. We know there exists a characteristic biset Ω by Proposition 5.5 of [9]. Consider the conjugation action of A on Ω. Let us see that $\Lambda = \Omega^A$ is a characteristic biset with the desired property. First of all, Λ inherits left and right actions of S_α from Ω because A is central in S_α. And if $x \in \Lambda$ and $a \in A$, then

$$ax = aa^{-1}xa = xa$$

so it satisfies the desired property. It remains to show that it satisfies the three conditions of a characteristic biset from Proposition 5.5 of [9].

(1) The biset Λ is a disjoint union of bisets of the form $S_\alpha \times_{(P,\varphi)} S_\alpha$ with $P \leq S_\alpha$ and $\varphi \colon P \to S_\alpha$ in $\mathcal{F}_\alpha$.

If $[x, y]$ belongs to one the components $S_\alpha \times_{(P,\varphi)} S_\alpha$ in the decomposition of Ω and A is contained in P, then

$$a[x, y]a^{-1} = [ax, ya^{-1}] = [xa, a^{-1}y] = [xa\varphi(a)^{-1}, y] = [xaa^{-1}, y] = [x, y]$$

and therefore $S_\alpha \times_{(P,\varphi)} S_\alpha$ is contained in Λ. If A is not contained in P, let $[x,y] \in (S_\alpha \times_{(P,\varphi)} S_\alpha)^A$ and $a \in A - P$. Then

$$[ax, ya^{-1}] = [x,y]$$

from where $a \in P$. This is a contradiction and so $(S_\alpha \times_{(P,\varphi)} S_\alpha)^A$ is empty. We conclude that the biset Λ is the disjoint union of the bisets of the form $S_\alpha \times_{(P,\varphi)} S_\alpha$ that appear in the decomposition of Ω and that satisfy $A \subseteq P$.

(2) For each $P \leq S_\alpha$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}_\alpha}(P, S_\alpha)$, the (P, S_α)-bisets $\Lambda_{(P,S_\alpha)}$ and $\Lambda_{(\varphi,S_\alpha)}$ are isomorphic.

By Alperin's fusion lemma, it suffices to prove that $\Lambda_{(Q,S_\alpha)}$ and $\Lambda_{(\psi,S_\alpha)}$ are isomorphic (Q, S_α)-bisets for any $\mathcal{F}_\alpha$-centric subgroup Q and any $\psi \colon Q \to S_\alpha$ in $\mathcal{F}_\alpha$. Since $\Omega_{(Q,S_\alpha)}$ and $\Omega_{(\psi,S_\alpha)}$ are isomorphic (Q, S_α)-bisets, there exists a (Q, S_α)-equivariant bijection $\theta \colon \Omega_{(Q,S_\alpha)} \to \Omega_{(\psi,S_\alpha)}$. Given $x \in \Lambda$ and $a \in A$, we have

$$a\theta(x)a^{-1} = \theta(\psi(a)xa^{-1}) = \theta(axa^{-1}) = \theta(x)$$

where the first equality holds because Q is $\mathcal{F}_\alpha$-centric and so it contains A. Hence θ restricts to a (Q, S_α)-equivariant bijection $\Lambda_{(Q,S_\alpha)} \to \Lambda_{(\psi,S_\alpha)}$.

(3) The number $|\Lambda|/|S_\alpha|$ is congruent to 1 mod p.

By the first part, we know we can obtain Λ from Ω by removing the summands $S_\alpha \times_{(P,\varphi)} S_\alpha$ where P does not contain A. Since S_α contains A, all the summands that we are removing satisfy $P \neq S_\alpha$. Then

$$|\Lambda|/|S_\alpha| = |\Lambda/S_\alpha| = |\Omega/S_\alpha| - \sum_{P \neq S_\alpha} |S_\alpha/P| \equiv |\Omega|/|S_\alpha| \mod p$$

and therefore $|\Lambda|/|S_\alpha| \equiv 1 \mod p$. $\qquad\square$

Theorem 4.4. *Let ρ be an A-representation of S_α. Then there exists an A-representation of S_α which is $\mathcal{F}_\alpha$-invariant and that contains ρ as a direct summand.*

Proof. Since A is central in $\mathcal{F}_\alpha$, by Lemma 4.3 there exists a characteristic biset Λ for $\mathcal{F}_\alpha$ such that $ax = xa$ for all $a \in S$ and all $x \in \Lambda$. Following Proposition 3.8 in [11], if ρ is an n-dimensional representation, we consider the action of S_α on $\mathbb{C}[\Lambda] \otimes_{S_\alpha} \mathbb{C}^n$ induced by the left action of S_α on Λ. Since Λ is characteristic, the same argument goes through to show that this representation is $\mathcal{F}_\alpha$-invariant and contains ρ as a direct summand. Moreover, A acts by complex multiplication on this new representation since

$$a(x \otimes v) = ax \otimes v = xa \otimes v = x \otimes \rho(a)(v)$$

and ρ is an A-representation. $\qquad\square$

We now prove an analogue of Proposition 5.7 in [11] for A-representations.

Proposition 4.5. *There is an isomorphism $R(\mathcal{F}_\alpha; A) \cong \varprojlim\limits_{\mathcal{O}(\mathcal{F}_\alpha^c)} R(Q; A)$ of modules over $R(\mathcal{F})$.*

Proof. The obvious map $R(\mathcal{F}_\alpha; A) \to \varprojlim_{\mathcal{O}(\mathcal{F}_\alpha^c)} R(Q; A)$ is clearly a monomorphism of $R(\mathcal{F})$-modules. To see that it is surjective it suffices to show that given an element $([\rho_Q] - [\beta_Q])_Q$ in the limit, there exists $[\rho] - [\beta]$ in $R(\mathcal{F}_\alpha; A)$ such that

$$[\rho] - [\beta] = [\rho_S] - [\beta_S]$$

By Lemma 4.4, there is a $\mathcal{F}_\alpha$-invariant A-representation $\rho_S^{\mathcal{F}_\alpha}$ such that

$$\rho_S^{\mathcal{F}_\alpha} = \rho_S \oplus \rho'$$

Therefore

$$[\rho_S] - [\beta_S] = [\rho_S^{\mathcal{F}_\alpha}] - [\beta_S \oplus \rho']$$

and note that $\beta_S \oplus \rho'$ is $\mathcal{F}_\alpha$-invariant because $\rho_S^{\mathcal{F}_\alpha}$ and the difference are both $\mathcal{F}_\alpha$-invariant. $\qquad\square$

We end this section with the following result, which will be useful in the proof of Theorem 5.4

Lemma 4.6. *Let P be an $\mathcal{F}_\alpha$-centric subgroup of S_α. The $I(\mathcal{F})$-adic and $I(P/A)$-adic topologies coincide for $R_A(P)$.*

Proof. The action of $R(\mathcal{F})$ on $R_A(P)$ factors through the restriction from $R(\mathcal{F})$ to $R(P/A)$. By Corollary 2.12, the $I(P/A)$-adic and $I(\mathcal{F})$-adic topologies coincide for $R(P/A)$ and therefore for $R_A(P)$. $\qquad\square$

5. The twisted completion theorem

In this section we prove the general completion theorem for p-local finite groups. First we determine that the twisted K-theory of the classifying space can be computed by stable elements in a certain sense and use Lahtinen's completion maps in twisted K-theory [21] to induce a completion map for the group of twisted representations of a fusion system. Since this group is also computed by stable elements, we can use the same argument as in Theorem 3.1.

We begin by giving an alternative description of twisted K-theory for some twistings. As before, let A be a cyclic p-group. The inclusion of A in S^1 induces a homomorphism $BA = U(\mathcal{H})/A \to PU(\mathcal{H})$ and therefore BA acts on $\mathrm{Fred}'(\mathcal{H})$. Let $\pi\colon E \to B$ be a principal BA-bundle. We define $K(E;\pi)$ to be the set of equivariant homotopy classes of BA-equivariant maps $\phi\colon E \to \mathrm{Fred}'(\mathcal{H})$, that is, $\phi(e \cdot g) = g^{-1}\phi(e)$ for all $g \in BA$.

Lemma 5.1. *Let $\pi\colon E \to B$ be a principal BA-bundle and let $\alpha\colon B \to K(\mathbb{Z}, 3)$ be the composition of the classifying map and the map $BBA \to BBS^1$ induced by the inclusion of A in S^1. Then there is a bijection*

$$K(E;\pi) \cong {}^\alpha K(B)$$

Proof. Let $P \to B$ be the projective bundle classified by α. Recall that ${}^\alpha K(B)$ is the set of homotopy classes of sections of the associated bundle $P \times_{PU(\mathcal{H})} \mathrm{Fred}'(\mathcal{H}) \to B$. It is straightforward to check that Theorem 8.1 in Chapter 4 of [19] is still valid for homotopy classes of maps, hence this set is in bijective correspondence with the set of equivariant homotopy classes of $PU(\mathcal{H})$-equivariant maps $\phi\colon P \to \mathrm{Fred}'(\mathcal{H})$, in the sense that $\phi(eg) = g^{-1}\phi(e)$.

Since α is the composition of the classifying map $B \to BBA$ for π with the map induced by the inclusion of A in S^1, the principal $PU(\mathcal{H})$-bundle $P \to B$ is

isomorphic to $E \times_{BA} PU(\mathcal{H}) \to B$. Therefore the set of equivariant homotopy classes of $PU(\mathcal{H})$-equivariant maps $P \to \mathrm{Fred}'(\mathcal{H})$ is in bijection with the set of equivariant homotopy classes of BA-equivariant map $E \to \mathrm{Fred}'(\mathcal{H})$. $\qquad\square$

The group structure of $^\alpha K(B)$ is given by fiberwise composition in $\mathrm{Fred}'(H)$. The corresponding operation on $K(E; \pi)$ is therefore also induced by composition in $\mathrm{Fred}'(H)$ and the bijection in this lemma becomes an isomorphism of groups with this structure.

Corollary 5.2. *Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and let $\pi\colon |\mathcal{L}_\alpha|_p^\wedge \to |\mathcal{L}|_p^\wedge$ the corresponding principal BA-bundle. There is an isomorphism*

$$K(|\mathcal{L}_\alpha|_p^\wedge; \pi) \cong {}^\alpha K(|\mathcal{L}|_p^\wedge)$$

Remark 5.1. Note that a commutative diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{\ q\ } & E \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
B' & \longrightarrow & B
\end{array}
$$

where π' and π and principal BA-bundles and q is BA-equivariant, induces a homomorphism $q^*\colon K(E; \pi) \to K(E'; \pi')$. This assignment is clearly functorial in the category of such maps. More generally, given principal BA-bundles $\pi'\colon E' \to B'$ and $\pi\colon E \to B$ that fit into a commutative diagram of spectra

$$
\begin{array}{ccc}
\Sigma^\infty E' & \xrightarrow{\ q\ } & \Sigma^\infty E \\
{\scriptstyle \Sigma^\infty \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma^\infty \pi} \\
\Sigma^\infty B' & \longrightarrow & \Sigma^\infty B
\end{array}
$$

where q is BA-equivariant, we have an induced map $q^*\colon K(E; \pi) \to K(E'; \pi')$ because equivariant homotopy classes of BA-equivariant maps $E \to \mathrm{Fred}'(\mathcal{H})$ are in bijective correspondence with equivariant homotopy classes of BA-equivariant stable maps $\Sigma^\infty E \to \Sigma^\infty \mathrm{Fred}'(\mathcal{H})$. This assignment is also functorial.

Remark 5.2. By Proposition 4.9 of [28], the characteristic biset Λ from Lemma 4.3 determines a characteristic idempotent in $A(S_\alpha, S_\alpha)_p^\wedge$ and the stable summand determined by the corresponding idempotent ω in $\{\Sigma^\infty BS_\alpha, \Sigma^\infty BS_\alpha\}$ is $\Sigma^\infty |\mathcal{L}_\alpha|_p^\wedge$.

Note that the S_α-biset Λ constructed in the proof of Lemma 4.3 is a disjoint union of S_α-bisets of the form $S_\alpha \times_{(P,\varphi)} S_\alpha$ where P contains A. Let $S = S_\alpha/A$. We can construct an S-biset Λ/A by including a copy of $S \times_{(P/A, \varphi/A)} S$ for each copy of $S_\alpha \times_{(P,\varphi)} S_\alpha$ in Λ. Now each S_α-biset $S_\alpha \times_{(P,\varphi)} S_\alpha$ in Λ determines a commutative diagram of stable maps

$$
\begin{array}{ccccc}
\Sigma^\infty BS_\alpha & \xrightarrow{\ \mathrm{tr}_P^{S_\alpha}\ } & \Sigma^\infty BP & \xrightarrow{\ \Sigma^\infty B\varphi\ } & \Sigma^\infty BS_\alpha \\
{\scriptstyle \Sigma^\infty Bq}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma^\infty Bq_{|BP}} & & \big\downarrow{\scriptstyle \Sigma^\infty Bq} \\
\Sigma^\infty BS & \xrightarrow[\ \mathrm{tr}_{P/A}^{S}\ ]{} & \Sigma^\infty B(P/A) & \xrightarrow[\ \Sigma^\infty B(\varphi/A)\ ]{} & \Sigma^\infty BS
\end{array}
$$

where $q\colon S_\alpha \to S$ is the quotient homomorphism and tr denotes the stable transfer map. These two bisets determine stable maps ω and ω/A, given by the sum of these

stable maps above, each summand corresponding to a component of the biset. By the commutativity of the diagrams above, these stable maps fit into commutative diagram

$$
\begin{array}{ccc}
\Sigma^\infty BS_\alpha & \xrightarrow{\ \omega\ } & \Sigma^\infty BS_\alpha \\
{\scriptstyle \Sigma^\infty Bq}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma^\infty Bq} \\
\Sigma^\infty BS & \xrightarrow[\ \omega/A\ ]{} & \Sigma^\infty BS
\end{array}
$$

Theorem 5.3. *Let α be a 2-cocycle for the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and $\pi\colon |\mathcal{L}_\alpha|_p^\wedge \to |\mathcal{L}|_p^\wedge$ the corresponding principal BA-bundle. There is an isomorphism of abelian groups*

$$
K(|\mathcal{L}_\alpha|_p^\wedge; \pi) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_\alpha^c)} K(BQ; \pi_Q)
$$

where $\pi_Q\colon BQ \to B(Q/A)$ is the pullback of π under the composition of the map induced by the inclusion of Q/A in S and the standard map $BS \to |\mathcal{L}|_p^\wedge$.

Proof. By the two previous remarks, one just needs to follow the proof of Theorem 4.2 of [11] with the stable idempotent ω described above.   □

Theorem 5.4. *For any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and any $\alpha \in H^3(|\mathcal{L}|_p^\wedge)$, the completion of $^\alpha R(\mathcal{F})$ with respect to $I(\mathcal{F})$ is isomorphic to $^\alpha K(|\mathcal{L}|_p^\wedge)$ as a module over $K(|\mathcal{L}|_p^\wedge)$.*

Proof. For any subgroup Q of S, the constant map $EQ \to *$ induces a homomorphism

$$
^\alpha R(Q) \to {}^\alpha K_Q(EQ)
$$

and $^\alpha K_Q(EQ) \cong {}^\alpha K(BQ)$ by Proposition 3.2 in [17]. We obtain a homomorphism $^\alpha R(Q) \to {}^\alpha K(BQ)$.

Now let P be an $\mathcal{F}_\alpha$-centric subgroup of S_α. Consider the composition

$$
R(P; A) \cong {}^\alpha R(P/A) \to {}^\alpha K(B(P/A)) \cong K(BP; \pi_P)
$$

where the first isomorphism is given by untwisting, the second map is the map mentioned in the previous paragraph and the last isomorphism comes from Lemma 5.1. The composition sends a representation of P to the homotopy class of the induced map $BP \to \mathbb{Z} \times BU$, hence it determines a natural transformation of functors $\beta\colon \mathcal{O}(\mathcal{F}_\alpha^c) \to \mathrm{Ab}$.

Let $R(\ ; A)$ and $R(\ ; A)_{I_\mathcal{F}}^\wedge$ be the contravariant functors $\mathcal{O}(\mathcal{F}_\alpha^c) \to \mathrm{Ab}$ which send P to $R(P; A)$ and P to $R(P; A)_{I_\mathcal{F}}^\wedge$, respectively. We now follow the same argument from Theorem 3.1. Completion of $R(\mathcal{F})$-modules with respect to the ideal $I(\mathcal{F})$ is an additive functor, so we have isomorphisms of complexes

$$
C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ; A))_{I_\mathcal{F}}^\wedge \cong C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ; A)_{I_\mathcal{F}}^\wedge)
$$

The map β_P induces an isomorphism $R(P; A)_{I(P/A)}^\wedge \to K^*(BP; \pi_P)$ of abelian groups by Theorem 1 in [21]. Since the $I(P/A)$-adic topology coincides with the $I(\mathcal{F})$-adic topology by Lemma 4.6, the chain map β_* induces an isomorphism of chain complexes.

$$
C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ; A)_{I_\mathcal{F}}^\wedge) \cong C^*(\mathcal{O}(\mathcal{F}_\alpha^c); K(\ ; \pi))
$$

where $K(\ ;\pi)$ denotes the contravariant functor $\mathcal{O}(\mathcal{F}_\alpha^c) \to$ Ab which sends P to $K(BP; \pi_P)$. Then we have the following isomorphisms

$$
\begin{aligned}
{}^\alpha K(|\mathcal{L}|_p^\wedge) &\cong K(|\mathcal{L}_\alpha|_p^\wedge; \iota) \\
&\cong \varprojlim_{\mathcal{O}(\mathcal{F}_\alpha^c)} K(BP; \pi_P) \\
&\cong H^0(C^*(\mathcal{O}(\mathcal{F}_\alpha^c); K(\ ;\pi))) \\
&\cong H^0(C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ;A))_{I_\mathcal{F}}^\wedge)
\end{aligned}
$$

where the first isomorphism holds by Corollary 5.2 and the second isomorphism by Theorem 5.3. Since $R(\mathcal{F})$ is a Noetherian ring and the $R(\mathcal{F})$-modules $R(P; A)$ are finitely generated by Lemma 4.2, completion of finitely generated $R(\mathcal{F})$-modules with respect to the ideal $I(\mathcal{F})$ is an exact functor and so

$$
\begin{aligned}
H^0(C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ;A))_{I_\mathcal{F}}^\wedge) &\cong H^0(C^*(\mathcal{O}(\mathcal{F}_\alpha^c); R(\ ;A)))_{I_\mathcal{F}}^\wedge \\
&\cong \left(\varprojlim_{\mathcal{O}(\mathcal{F}_\alpha^c)} R(P; A) \right)_{I_\mathcal{F}}^\wedge \\
&\cong R_A(\mathcal{F}_\alpha)_{I_\mathcal{F}}^\wedge \\
&\cong {}^\alpha R(\mathcal{F})_{I_\mathcal{F}}^\wedge
\end{aligned}
$$

where the third isomorphism follows from Proposition 4.5 and the last isomorphism from Lemma 4.1. Therefore we obtain an isomorphism of abelian groups

$$
{}^\alpha K(|\mathcal{L}|_p^\wedge) \cong {}^\alpha R(\mathcal{F})_{I_\mathcal{F}}^\wedge
$$

This isomorphism is induced by the restriction of the isomorphism of $K(BS)$-modules ${}^\alpha R(S)_{I_\mathcal{F}}^\wedge \to {}^\alpha K(BS)$ to ${}^\alpha R(\mathcal{F})_{I_\mathcal{F}}^\wedge$. Hence it is an isomorphism of $K(|\mathcal{L}|_p^\wedge)$-modules. $\square$

6. Computations

In this final section, we include several computations of twisted and untwisted K-theory of p-local finite groups that were obtained using Theorem 3.1 and Theorem 5.4.

Example 6.1. Let $p \neq 2$ and let $\mathcal{F}$ be the saturated fusion system of Σ_p over the p-Sylow S generated by $\sigma = (1, 2, \ldots, p)$. The irreducible representations of $S \cong \mathbb{Z}/p$ are the tensor powers of the one-dimensional representation ρ that sends σ to $e^{2\pi i/p}$. Its representation ring is given by

$$
R(S) = \mathbb{Z}[\rho]/(\rho^p - 1)
$$

The irreducible representations of $\mathcal{F}$ are given by 1 and $x = \rho + \rho^2 + \ldots + \rho^{p-1}$ since all the nontrivial powers of σ are conjugate in Σ_p. We have

$$
x^2 = \left(\sum_{k=1}^{p-1} \rho^k \right)^2 = (p-1)1 + (p-2)x
$$

hence

$$
R(\mathcal{F}) = \mathbb{Z}[x]/(x^2 - (p-2)x - p + 1)
$$

and its augmentation ideal is generated by $x - p + 1$ as an abelian group. By Theorem 3.1, we have

$$K((B\Sigma_p)_p^\wedge) \cong \left[\mathbb{Z}[x]/(x^2 - (p-2)x - p + 1)\right]_{(x-p+1)}^\wedge$$

and making $y = x - p + 1$, we get

$$\left[\mathbb{Z}[x]/(x^2 - (p-2)x - p + 1)\right]_{(x-2)}^\wedge \cong \left[\mathbb{Z}[y]/(y^2 + py)\right]_{(y)}^\wedge \cong \mathbb{Z}[[y]]/(y^2 + py)$$

Example 6.2. Let $\mathcal{F}$ be the fusion system of A_4 over its 2-Sylow

$$S = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

The nontrivial element α in $H^2(BA_4; \mathbb{Z}/2) \cong \mathbb{Z}/2$ corresponds to the group $SL_2(\mathbb{F}_3)$ as a central extension of A_4 by $\mathbb{Z}/2$. In this example we compute $^\alpha K((BA_4)_2^\wedge)$ as a module over $K((BA_4)_2^\wedge)$.

The central extension $\mathcal{F}_\alpha$ is the fusion system of $SL_2(\mathbb{F}_3)$ over a 2-Sylow S_α, which is isomorphic to Q_8. The only irreducible representation ρ of Q_8 where the center acts by complex multiplication is the 2-dimensional representation ρ coming from the action of Q_8 on the quaternions, which satisfies

$$\chi_\rho(g) = \begin{cases} 2 & \text{if } g = 1 \\ -2 & \text{if } g = -1 \\ 0 & \text{otherwise} \end{cases}$$

Since -1 is central in $SL_2(\mathbb{F}_3)$, this representation is $\mathcal{F}_\alpha$-invariant. Therefore

$$R(\mathcal{F}_\alpha; \mathbb{Z}/2) \cong \mathbb{Z}\rho$$

We need to determine its structure of module over $R(\mathcal{F})$. The group S is normal in A_4 and $\text{Aut}_{A_4}(S) \cong \mathbb{Z}/3$ is generated by conjugation by $(1, 2, 3)$. The set of irreducible representations of S is given by $\{1, 1 \otimes \gamma, \gamma \otimes 1, \gamma \otimes \gamma\}$, where γ is the sign representation of $\mathbb{Z}/2$. The action of $\text{Aut}_{A_4}(S)$ on the three nontrivial irreducible representations is transitive and so the irreducible representations of $\mathcal{F}$ are 1 and $x = 1 \otimes \gamma + \gamma \otimes 1 + \gamma \otimes \gamma$. One can check that $x^2 = 2x + 3$ and so

$$R(\mathcal{F}) \cong \mathbb{Z}[x]/(x^2 - 2x - 3)$$

The action of $R(\mathcal{F})$ on $R(\mathcal{F}_\alpha; \mathbb{Z}/2)$ is induced by tensor product after going through the quotient $S_\alpha \to S$. The composition of the representation x with this quotient results in the representation of S_α with character

$$\chi(g) = \begin{cases} 3 & \text{if } g = 1, -1 \\ -1 & \text{otherwise} \end{cases}$$

and therefore $x \cdot \rho$ is the representation with character

$$\chi(g) = \begin{cases} 6 & \text{if } g = 1 \\ -6 & \text{if } g = -1 \\ 0 & \text{otherwise} \end{cases}$$

that is, $x \cdot \rho = 3\rho$. In particular, the augmentation ideal $I(\mathcal{F})$ acts trivially on $R(\mathcal{F}_\alpha; \mathbb{Z}/2)$ and therefore

$$^\alpha K((BA_4)_2^\wedge) \cong R(\mathcal{F}_\alpha; \mathbb{Z}/2) \cong \mathbb{Z}$$

We can also compute $K((BA_4)_2^\wedge) \cong \mathbb{Z}[[y]]/(y^2 + 4y)$ and since y is the element in K-theory corresponding to the element from $x - 3$ in $R(\mathcal{F})$, we see that y acts trivially on $^\alpha K((BA_4)_2^\wedge)$.

Example 6.3. Consider the extraspecial 7-group of order 7^3 and exponent 7

$$S = \langle a, b, c \mid a^7 = 1, b^7 = 1, c^7 = 1, [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$$

Ruiz and Viruel classified all the saturated fusion systems over S up to equivalence in [30], finding three exotic 7-local finite groups. Following the notation in [32], we use the names RV_1, RV_2 and RV_3 for these exotic 7-local finite groups and their fusion systems, and BRV_i for their classifying spaces. In this example we compute the K-theory of the classifying spaces for the saturated fusion systems over S with more than two elementary abelian $\mathcal{F}$-centric $\mathcal{F}$-radical 7-subgroups, in particular, for the exotic 7-local finite groups RV_i.

By Theorem 5.5.4 in [18], the group S has 49 irreducible 1-dimensional representations and six 7-dimensional irreducible representations. The 1-dimensional representations come from the quotient

$$S \to S/Z(S) \cong \mathbb{Z}/7 \times \mathbb{Z}/7$$

which gives a ring monomorphism

$$\mathbb{Z}[x, y]/(x^7 - 1, y^7 - 1) \cong R(\mathbb{Z}/7 \times \mathbb{Z}/7) \to R(S)$$

and so we denote by $x^i y^j$ the corresponding representations of S. Let ω denote a primitive seventh root of unity. The six remaining irreducible representations z_j for $j = 1, \ldots, 6$ are described by their values on a and b.

$$z_j(a) = \mathrm{diag}(\omega^{6j}, \omega^{5j}, \ldots, \omega^j, 1)$$

and $z_j(b)$ is the linear transformation that sends e_1 to e_7 and e_n to e_{n-1} if $n \geq 2$. Here $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{C}^7$. Note that $z_j(c) = w^j I$.

The outer automorphism group of S is $GL_2(\mathbb{F}_7)$, where the action of a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix}$ is given by

$$a \mapsto a^\alpha b^\gamma \qquad b \mapsto a^\beta b^\epsilon \qquad c \mapsto c^{det(M)}$$

The elementary abelian subgroups of S are the subgroups A_i generated by c and ab^i for $i = 0, \ldots, 6$, and the subgroup A_∞ generated by c and b. These subgroups have a natural action of $GL_2(\mathbb{F}_7)$ by identifying each of them with $\mathbb{F}_7^2$ in such a way that the aforementioned elements form the standard basis.

It is convenient to follow the incremental method of [32], where these fusion systems are divided in two families, and the fusion systems in each family are ordered in such a way that we can form the next by adding new morphisms. For convenience, we include at the end of this example a table with the relevant values of the characters of the nontrivial representations of S that appear in this example.

The first family is given by the fusion systems of the groups $O'N$, $O'N \rtimes \mathbb{Z}/2$, and of the exotic 7-local groups RV_2 and RV_3.

• The O'Nan group $O'N$ and $O'N \rtimes \mathbb{Z}/2$.

We have $\mathrm{Out}_{O'N}(S) \cong D_8 \times \mathbb{Z}/3$. This is generated by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The representations that are invariant under the action of $\mathrm{Out}_{O'N}(S)$ are direct sums of the trivial representation and the representations

$$Z = \sum_{j=1}^{6} \phi_j$$

$$X_1 = \sum_{j=1}^{6} x^j + \sum_{j=1}^{6} y^j$$

$$X_2 = xy + x^6y + x^2y^2 + x^5y^2 + x^4y^4 + x^3y^4$$
$$+ x^6y^6 + xy^6 + x^5y^5 + x^2y^5 + x^3y^3 + x^4y^3$$

and the representation X_3 which is the sum of the representations x^iy^j with $(i,j) \neq (0,0)$ which are not summands of X_1 or of X_2.

Moreover, the groups A_0 and A_1 are centric and radical in $\mathcal{F}_{O'N}(S)$ with $SL_2(\mathbb{F}_7) \rtimes \mathbb{Z}/2$ as group of automorphisms. Therefore c, b and ab are conjugate in $\mathcal{F}_{O'N}(S)$ and so the nontrivial irreducible representations of this fusion system are $A = X_1 + X_2 + 3Z$ and $B = X_3 + 4Z$. Using their characters, we can compute

$$R(\mathcal{F}_S(O'N)) = \mathbb{Z}[A, B]/I$$

where I is the ideal generated by $A^2 - 65A - 66B - 78$, $B^2 - 108A - 107B - 120$ and $AB - 84A - 84B - 72$. Therefore

$$K(B(O'N)^\wedge_7) \cong \mathbb{Z}[[z, w]]/(z^2 + 235z - 66w, w^2 + 277w - 108z, zw + 66w + 108z)$$

where z, w are the elements in K-theory corresponding to $A - 150$ and $B - 192$, respectively. The additional fusion for the group $O'N \rtimes \mathbb{Z}/2$ in $\mathrm{Out}_{O'N \rtimes \mathbb{Z}/2}(S)$, which is isomorphic to SD_{16} comes from the matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

which represents a morphism that maps a to a^6b^6 and b to ab^6. We see that the representations A and B are invariant under the action of $\mathrm{Out}_{O'N \rtimes 2}(S)$ and so $R(\mathcal{F}_S(O'N)) = R(\mathcal{F}_S(O'N \rtimes 2))$. Therefore

$$K(B(O'N)^\wedge_7) \cong K(B(O'N \rtimes \mathbb{Z}/2)^\wedge_7)$$

• The exotic 7-local finite groups RV_2 and RV_3.

In the 7-local finite group RV_2, the group A_2 is centric and radical with $SL_2(\mathbb{F}_7) \rtimes \mathbb{Z}/2$ as group of automorphisms. Therefore c is conjugate to ab^2 in RV_2. Looking at the character table, we find that only nontrivial irreducible representation of RV_2 is $U = A + B = X_1 + X_2 + X_3 + 7Z$. Therefore

$$R(RV_2) \cong \mathbb{Z}[U]/(U^2 - 341U - 342)$$

hence

$$K(BRV_2) \cong \mathbb{Z}[[u]]/(u^2 + 343u)$$

where u is the element coming from $U - 342$. Since the character of U is equal to -1 for all nontrivial elements of S and RV_2 is a subcategory of RV_3, it is also an

irreducible representation of RV_3 and so $R(RV_2) = R(RV_3)$. And therefore

$$K(BRV_2) \cong K(BRV_3)$$

The second family is given by the fusion systems of the groups He, $He \rtimes \mathbb{Z}/2$, Fi'_{24}, Fi_{24} and the exotic 7-local finite group RV_1.

• The Held group He.

We have $\mathrm{Out}_{He}(S) \cong \mathbb{Z}_3 \times \Sigma_3$. This group is generated by the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The representations of S that are invariant under the action of $\mathrm{Out}_{He}(S)$ are direct sums of the trivial representation, the representation Z mentioned above and the representations

$$
\begin{aligned}
Y_1 &= x + x^2 + x^4 + y + y^2 + y^4 \\
Y_2 &= x^3 + x^6 + x^5 + y^3 + y^6 + y^5 \\
Y_3 &= xy + x^2y^2 + x^4y^4 + x^2y^4 + x^4y + xy^2 + x^4y^2 + xy^4 + x^2y \\
Y_4 &= x^3y^3 + x^6y^6 + x^5y^5 + x^6y^5 + x^5y^3 + x^3y^6 + x^5y^6 + x^3y^5 + x^6y^3
\end{aligned}
$$

and the representation Y_5 which is the sum of the representations $x^i y^j$ with $(i,j) \neq 0$ which are not summands of the other Y_i.

The group A_6 is centric and radical in $\mathcal{F}_{He}(S)$ with $SL_2(\mathbb{F}_7)$ as group of automorphisms, hence c and ab^{-1} are conjugate in He. Therefore the nontrivial irreducible representations of $\mathcal{F}_{He}(S)$ are $D_1 = Z + Y_1$, $D_2 = Z + Y_2$, $D_3 = Z + Y_3$, $D_4 = Z + Y_4$ and $D_5 = 3Z + Y_5$. Using their characters we compute

$$R(\mathcal{F}_{He}(S)) \cong \mathbb{Z}[D_1, D_2, D_3, D_4, D_5]/J$$

where J is the ideal generated by the following elements

$$D_1^2 - 7D_1 - 8D_2 - 8D_3 - 6D_4 - 6D_5 - 6$$
$$D_2^2 - 8D_1 - 7D_2 - 6D_3 - 8D_4 - 6D_5 - 6$$
$$D_3^2 - 6D_1 - 6D_2 - 7D_3 - 10D_4 - 8D_5 - 6$$
$$D_4^2 - 6D_1 - 6D_2 - 10D_3 - 7D_4 - 8D_5 - 6$$
$$D_5^2 - 60D_1 - 60D_2 - 60D_3 - 60D_4 - 61D_5 - 72$$
$$D_1D_2 - 7D_1 - 7D_2 - 6D_3 - 6D_4 - 7D_5 - 12$$
$$D_1D_3 - 6D_1 - 6D_2 - 8D_3 - 6D_4 - 8D_5 - 6$$
$$D_1D_4 - 6D_1 - 9D_2 - 6D_3 - 8D_4 - 7D_5 - 6$$
$$D_1D_5 - 21D_1 - 18D_2 - 20D_3 - 22D_4 - 20D_5 - 18$$
$$D_2D_3 - 9D_1 - 6D_2 - 8D_3 - 6D_4 - 7D_5 - 6$$
$$D_2D_4 - 6D_1 - 6D_2 - 6D_3 - 8D_4 - 8D_5 - 6$$
$$D_2D_5 - 18D_1 - 21D_2 - 22D_3 - 20D_4 - 20D_5 - 18$$
$$D_3D_4 - 9D_1 - 9D_2 - 7D_3 - 7D_4 - 7D_5 - 15$$
$$D_3D_5 - 21D_1 - 24D_2 - 20D_3 - 22D_4 - 21D_5 - 18$$
$$D_4D_5 - 24D_1 - 21D_2 - 22D_3 - 20D_4 - 21D_5 - 18$$

and therefore

$$K(BHe_7^\wedge) \cong \mathbb{Z}[[v_1, v_2, v_3, v_4, v_5]]/I$$

where v_i is the element in K-theory corresponding to $D_i - \dim(D_i)$ and I is the ideal generated by the elements

$$v_1^2 + 89v_1 - 8v_2 - 8v_3 - 6v_4 - 6v_5$$
$$v_2^2 - 8v_1 + 89v_2 - 6v_3 - 8v_4 - 6v_5$$
$$v_3^2 - 6v_1 - 6v_2 + 95v_3 - 10v_4 - 8v_5$$
$$v_4^2 - 6v_1 - 6v_2 - 10v_3 + 95v_4 - 8v_5$$
$$v_5^2 - 60v_1 - 60v_2 - 60v_3 - 60v_4 + 227v_5$$
$$v_1v_2 + 41v_1 + 41v_2 - 6v_3 - 6v_4 - 7v_5$$
$$v_1v_3 + 45v_1 - 6v_2 + 40v_3 - 6v_4 - 8v_5$$
$$v_1v_4 + 45v_1 - 9v_2 - 6v_3 + 40v_4 - 7v_5$$
$$v_1v_5 + 123v_1 - 18v_2 - 20v_3 - 22v_4 + 28v_5$$
$$v_2v_3 - 9v_1 + 45v_2 + 40v_3 - 6v_4 - 7v_5$$
$$v_2v_4 - 6v_1 + 45v_2 - 6v_3 + 40v_4 - 8v_5$$
$$v_2v_5 - 18v_1 + 123v_2 - 22v_3 - 20v_4 + 28v_5$$
$$v_3v_4 - 9v_1 - 9v_2 + 44v_3 + 44v_4 - 7v_5$$
$$v_3v_5 - 21v_1 - 24v_2 + 124v_3 - 22v_4 + 30v_5$$
$$v_4v_5 - 24v_1 - 21v_2 - 22v_3 + 124v_4 + 30v_5$$

- The group $He \rtimes \mathbb{Z}/2$.

We have $\text{Out}_{He \rtimes \mathbb{Z}/2}(S) \cong \mathbb{Z}/6 \times \Sigma_3$ and the additional fusion comes from the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which represents a morphisms that sends a to a^6 and b to b^6. There are also additional elements in the group of automorphisms of A_6, but they do not contribute to any conjugacy relationship in $\mathcal{F}_{He \rtimes \mathbb{Z}/2}(S)$. Therefore the nontrivial irreducible representations of $\mathcal{F}_{He \rtimes 2}(S)$ are $M_1 = D_1 + D_2$, $M_2 = D_3 + D_4$ and D_5. Using their characters we compute

$$R(\mathcal{F}_{He \rtimes \mathbb{Z}/2}(S)) \cong \mathbb{Z}[M_1, M_2, D_5]/K$$

where K is the ideal generated by the following elements

$$M_1^2 - 29M_1 - 26M_2 - 26D_5 - 36$$
$$M_2^2 - 30M_1 - 31M_2 - 30D_5 - 42$$
$$D_5^2 - 60M_1 - 60M_2 - 61D_5 - 72$$
$$M_1M_2 - 27M_1 - 28M_2 - 30D_5 - 24$$
$$M_1D_5 - 39M_1 - 42M_2 - 40D_5 - 36$$
$$M_2D_5 - 45M_1 - 42M_2 - 42D_5 - 36$$

and therefore

$$K(B(He \rtimes \mathbb{Z}/2)_7^\wedge) \cong \mathbb{Z}[[t_1, t_2, t_3]]/L$$

where t_1, t_2 and t_3 are the elements in K-theory corresponding to $M_1 - 96$, $M_2 - 102$ and $D_5 - 144$, respectively, and L is the ideal generated by the elements

$$t_1^2 + 163t_1 - 26t_2 - 26t_3$$
$$t_2^2 - 30t_1 + 173t_2 - 30t_3$$
$$t_3^2 - 60t_1 - 60t_2 + 227t_3$$
$$t_1t_2 + 75t_1 + 68t_2 - 30t_3$$
$$t_1t_3 + 105t_1 - 42t_2 + 56t_3$$
$$t_2t_3 - 45t_1 + 102t_2 + 60t_3$$

• The derived Fischer group Fi'_{24} and the Fischer group Fi_{24}.

In the fusion system $\mathcal{F}_{Fi'_{24}}(S)$, the group A_1 is centric and radical with $SL_2(\mathbb{F}_7) \rtimes \mathbb{Z}/2$ as group of automorphisms. Hence c is conjugate to ab in Fi'_{24}. The nontrivial irreducible representations of $\mathcal{F}_{Fi'_{24}}(S)$ are M_1 and $N = M_2 + D_5$. The representation ring is given by

$$R(\mathcal{F}_{Fi'_{24}}(S)) \cong \mathbb{Z}[M_1, N]/I$$

where I is the ideal generated by $M_1^2 - 29M_1 - 26N - 36$, $N^2 - 180M_1 - 175N - 186$ and $M_1N - 66M_1 - 70N - 60$. Therefore

$$K(B(Fi'_{24})_7^\wedge) \cong \mathbb{Z}[[r, s]]/(r^2 + 163r - 26s, s^2 - 180r + 317s, rs + 180r + 26s)$$

where r and s and the elements in K-theory corresponding to $M_1 - 96$ and $N - 246$. In Fi_{24}, the additional fusion in $\mathrm{Out}_{Fi_{24}}(S) \cong \mathbb{Z}/36 \rtimes \mathbb{Z}/2$ is generated by the matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

but a was already conjugate to a^3 in Fi'_{24}, hence this does not add new conjugacy relationships. Therefore $R(\mathcal{F}_{Fi'_{24}}(S)) = R(\mathcal{F}_{Fi_{24}}(S))$ and so

$$K(B(Fi'_{24})_7^\wedge) \cong K(B(Fi_{24})_7^\wedge$$

- The exotic 7-local finite group RV_1.

In RV_1, the subgroup A_0 is centric and radical with $GL_2(\mathbb{F}_7)$ as group of automorphisms. Hence c is conjugate to a and the only nontrivial irreducible representations of RV_1 is the representation U already mentioned above. Therefore

$$R(RV_1) \cong \mathbb{Z}[U]/(U^2 - 341U - 342)$$

hence

$$K(BRV_1) \cong \mathbb{Z}[[u]]/(u^2 + 343u)$$

For reference, we include here a table with some of the values of the characters of the nontrivial representations that appeared in this example. In this table $\omega = e^{2\pi i/7}$ and $\theta = \omega + \omega^2 + \omega^4$.

	1	c	b	a	ab	ab^2	ab^3	ab^4	ab^5	ab^6
$x^i y^j$	1	1	ω^j	ω^i	ω^{i+j}	ω^{i+2j}	ω^{i+3j}	ω^{i+4j}	ω^{i+5j}	ω^{i+6j}
z_k	7	$7\omega^k$	0	0	0	0	0	0	0	0
Z	42	-7	0	0	0	0	0	0	0	0
X_1	12	12	5	5	-2	-2	-2	-2	-2	-2
X_2	12	12	-2	-2	5	-2	-2	-2	-2	5
X_3	24	24	-4	-4	-4	3	3	3	3	-4
A	150	3	3	3	3	-4	-4	-4	-4	3
B	192	-4	-4	-4	-4	3	3	3	3	-4
U	342	-1	-1	-1	-1	-1	-1	-1	-1	-1
Y_1	6	6	$3+\theta$	$3+\theta$	2θ	2θ	-1	2θ	-1	-1
Y_2	6	6	$3+\overline{\theta}$	$3+\overline{\theta}$	$2\overline{\theta}$	$2\overline{\theta}$	-1	$2\overline{\theta}$	-1	-1
Y_3	9	9	3θ	3θ	$\overline{\theta}-1$	$\overline{\theta}-1$	2	$\overline{\theta}-1$	2	2
Y_4	9	9	$3\overline{\theta}$	$3\overline{\theta}$	$\theta-1$	$\theta-1$	2	$\theta-1$	2	2
Y_5	18	18	-3	-3	4	4	-3	4	-3	-3
D_1	48	-1	$3+\theta$	$3+\theta$	2θ	2θ	-1	2θ	-1	-1
D_2	48	-1	$3+\overline{\theta}$	$3+\overline{\theta}$	$2\overline{\theta}$	$2\overline{\theta}$	-1	$2\overline{\theta}$	-1	-1
D_3	51	2	3θ	3θ	$\overline{\theta}-1$	$\overline{\theta}-1$	2	$\overline{\theta}-1$	2	2
D_4	51	2	$3\overline{\theta}$	$3\overline{\theta}$	$\theta-1$	$\theta-1$	2	$\theta-1$	2	2
D_5	144	-3	-3	-3	4	4	-3	4	-3	-3
M_1	96	-2	5	5	-2	-2	-2	-2	-2	-2
M_2	102	4	-3	-3	-3	-3	4	-3	4	4
N	246	1	-6	-6	1	1	1	1	1	1

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