THE IDEAL-VALUED INDEX OF FIBRATIONS WITH TOTAL SPACE A G_2 FLAG MANIFOLD

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ABSTRACT. Using the cohomology of the G_2-flag manifolds $G_2/U(2)_\pm$, and their structure as a fiber bundle over the homogeneous space $G_2/SO(4)$, we compute their Borel cohomology and the $\mathbb{Z}_2$ Fadell-Husseini index of such fiber bundles, for the $\mathbb{Z}_2$ action given by complex conjugation.

Considering the orthogonal complement of the tautological bundle γ over $\widetilde{G}_3(\mathbb{R}^7)$, we compute the $\mathbb{Z}_2$ Fadell-Husseini index of the pullback bundle of $s\gamma^\perp$ along the composition of the embedding between $G_2/SO(4)$ and $\widetilde{G}_3(\mathbb{R}^7)$, and the fiber bundle $G_2/U(2)_\pm \to G_2/SO(4)$. Here $s\gamma^\perp$ means the associated sphere bundle of the orthogonal bundle $\gamma^\perp$. Furthermore, we derive a general formula for the n-fold product bundle $(s\gamma^\perp)^n$ for which we make the same computations.

1. Introduction.

A *generalized flag manifold* is an homogeneous space of the form $G/C(T)$, where G is a semisimple, compact and connected Lie group, and $C(T)$ is the centralizer of a torus $T \subset G$. In case that T is the maximal torus, then $T = C(T)$ and we call G/T a *complete flag manifold*.

The group in which we want to focus is the exceptional Lie group G_2, which is the automorphism group of the $\mathbb{R}$-algebra homomorphisms of the octonions $\mathbb{O}$. From all the possible G_2 flag manifolds, we are particularly interested in the spaces $G_2/U(1) \times U(1)$ and $G_2/U(2)_\pm$. In the following diagram of fiber bundles we appreciate how they are related:

$$\begin{tikzcd}
 & G_2/U(1)\times U(1) \arrow[dl, "\rho_4"'] \arrow[dd, "\rho_3"] \arrow[dr, "\rho_5"] & \\
G_2/U(2)_+ \arrow[dr, "\rho_1"'] & & G_2/U(2)_- \arrow[dl, "\rho_2"] \\
 & G_2/SO(4) &
\end{tikzcd} \tag{1.1}$$

In fact, those flag manifolds are precisely the only three twistor spaces of the homogeneous space $G_2/SO(4)$, see [15, Sec. 2.3].

Previous to this work, several authors studied the integral cohomology of all the homogeneous spaces appearing in diagram 1.1. In sections 3 and 4 we calculate their cohomology with $\mathbb{F}_2$ coefficients, and the Stiefel-Whitney classes of the fiber bundles ρ_1 and ρ_2. Moreover, because of these calculations, in section 4 we show that the cohomology rings with $\mathbb{F}_2$ coefficients of $G_2/U(2)_+$ and $G_2/U(2)_-$ are

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isomorphic. For that reason we will not distinguish between them and we will just write $G_2/U(2)_{\pm}$.

Considering the action of $\mathbb{Z}_2$ on $G_2/U(2)_{\pm}$ by complex conjugation, in section 4.3 we prove our first main result.

Theorem 4.5. *The Borel cohomology of $G_2/U(2)_{\pm}$ is given by*

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_{\pm}; \mathbb{F}_2) = H^*(B\mathbb{Z}_2 \times G_2/SO(4); \mathbb{F}_2) \Big/ \langle t^3 + u_2 t + u_3 \rangle,$$

where $H^*(\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[t]$ with $\deg(t) = 1$. Consequently, the Fadell-Husseini index of ρ_1 and ρ_2 is given by

$$\mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_1; \mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_2; \mathbb{F}_2) = \langle t^3 + u_2 t + u_3 \rangle.$$

On the other hand, let us consider the tautological bundle over the oriented Grassmann manifold $\widetilde{G}_k(\mathbb{R}^n)$:

$$\gamma_k^d = \big(E(\gamma_k^d),\ \widetilde{G}_k(\mathbb{R}^d),\ E(\gamma_k^d) \xrightarrow{\pi} \widetilde{G}_k(\mathbb{R}^d),\ \mathbb{R}^k\big),$$

and the n-fold product associated to sphere bundle $s\gamma_k^d$:

$$(s\gamma_k^d)^n = \big(E(s\gamma_k^d)^n,\ \widetilde{G}_k(\mathbb{R}^d)^n,\ E(s\gamma_k^d)^n \xrightarrow{(s\pi)^n} \widetilde{G}_k(\mathbb{R}^d)^n,\ (S^{k-1})^n\big).$$

Given the pullback bundle of $(s\gamma_k^d)^n$ along the diagonal map Δ_n:

$$\begin{tikzcd} E\Big(\Delta_n^*\big((s\gamma_k^d)^n\big)\Big) \arrow[r] \arrow[d] & E\big(s\gamma\big)^n \arrow[d, "(s\pi)^n"] \\ \widetilde{G}_k(\mathbb{R}^d) \arrow[r, "\Delta_n"] & \widetilde{G}_k(\mathbb{R}^d)^n, \end{tikzcd}$$

the total space of $\Delta_n^*\big((s\gamma_k^d)^n\big)$ has been an excellent candidate to be the configuration spaces of several geometric problems which uses the *configuration space/test map scheme*. This means that, the more we know about the pullback bundle $\Delta_n^*\big((s\gamma_k^d)^n\big)$, the better chance we have of solving any related problem.

Motivated by this situation, and by some calculations over Grassmann manifolds presented in [3] and [4], in this work we are going to study the pullback bundle over $G_2/U(2)_{\pm}$ induced by ρ_j and γ_3^7, with $j \in \{1, 2\}$. Let us consider the orthogonal complement of the tautological bundle $\gamma := \gamma_3^7$ over $\widetilde{G}_3(\mathbb{R}^7)$:

$$\gamma^{\perp} = \big(E(\gamma^{\perp}),\ \widetilde{G}_3(\mathbb{R}^7),\ E(\gamma^{\perp}) \xrightarrow{\pi} \widetilde{G}_3(\mathbb{R}^7),\ \mathbb{R}^4\big).$$

Since there is an embedding i between $G_2/SO(4)$ and $\widetilde{G}_3(\mathbb{R}^7)$, we are interested in the pullback bundle of $(s\gamma^{\perp})^n$ along the map $\Delta_n \circ i \circ \rho_j$:

$$\zeta_n = \big(E(\zeta_n) = \mathcal{S}^n_{\gamma^{\perp}},\ G_2/U(2)_{\pm},\ \mathcal{S}^n_{\gamma^{\perp}} \xrightarrow{\phi_n} G_2/U(2)_{\pm},\ (S^3)^n\big).$$

An example of this kind of constructions appears in [4]. The total space $\mathcal{S}^n_{\gamma^{\perp}}$ considers now collections of unitary vectors inside 4-dimensional subspaces of $\mathbb{R}^8$, which we will call *(non) complex-coassociative subspaces*. The specific details about the construction of $\mathcal{S}^n_{\gamma^{\perp}}$, as well as some topological properties of the bundle ζ_n, are discussed further in section 4.4. We will prove then the following results:

Theorem 4.7. *The cohomology $H^*(\mathcal{S}^n_{\gamma^{\perp}}; \mathbb{F}_2)$ is described as follows:*

$$H^*(\mathcal{S}^n_{\gamma^{\perp}}; \mathbb{F}_2) = \Big[\langle 1, x\rangle \otimes \mathcal{I}\Big] \oplus \Big[\langle y, y^2, xy, xy^2\rangle \otimes H^*\big((S^3)^n; \mathbb{F}_2\big) \Big/ \langle y^2, xy^2\rangle \otimes \mathcal{I}\Big],$$

where

$$\mathcal{I} = \left\langle \sum_{j=1}^{m} z_{a_1 \cdots \hat{a_j} \cdots a_m} \middle| a_i \in \{1, 2, \ldots, n\},\ a_i \neq a_j\ \forall\ i \neq j \ \text{ and }\ 1 \leq m \leq n \right\rangle.$$

Theorem 4.11. *Consider the action of $\mathbb{Z}_2^{n+1}$ on $S^n_{\gamma^\perp}$ where the first summand acts on $G_2/U(2)_\pm$ by complex conjugation, and the others n summands acts antipodally on the unitary elements. Then the Fadell-Husseini index of $\phi_n \colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_\pm$ is given by*

$$\mathrm{Index}^{G_2/U(2)_\pm}_{\mathbb{Z}_2^{n+1}}(\phi_n; \mathbb{F}_2) = \langle y^2 + yt_2^2 + t_2^4, \ldots, y^2 + yt_{n+1}^2 + t_{n+1}^4 \rangle,$$

where $H^*(\mathbb{Z}_2^{n+1}; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_{n+1}]$ with $\deg(t_1) = \cdots = \deg(t_{n+1}) = 1$.

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2. The exceptional Lie group G_2.

In this section we will recall three equivalent definitions of the real form of the exceptional Lie group G_2. Let us fix now the notation. General references for the upcoming discussion include [2] and [8].

2.1. **Octonions algebra.** Given a normed division algebra A, the Cayley-Dickinson construction creates a new algebra $A^{'}$ with elements $(a, b) \in A^2$ and conjugation $(a, b)^* = (\bar{a}, -b)$. The addition in $A^{'}$ is done component-wise, and multiplication goes like

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb),$$

where juxtaposition indicates multiplication in A. An equivalent way to define the new algebra $A^{'}$ is to add an independent square root of -1, i, that multiples the second named element on each pair (a, b). Now the conjugation in $A^{'}$ uses the original conjugation of A and $i^* = -i$. Then the construction becomes an algebra of elements $a + ib$ for some $a, b \in A$.

Starting with the real numbers $\mathbb{R}$, the complex numbers are defined via the Cayley-Dickinson construction to be pairs of reals $a+ib$. Similarly, the quaternions are generated as a real algebra by $\{1, i, j, k\}$, subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. Having this in mind, we get the following definition

Definition 2.1. The octonions are generated as a quaternionic algebra via the Cayley-Dickinson construction as $\mathbb{H} \oplus \mathbb{H}[l]$, where l denotes an independent square root of -1.

Even if the Cayley-Dickinson construction is an excellent method to produce other normed division algebras, we lost very nice properties in the process. The multiplication in $\mathbb{O}$ turns out to be non-commutative and non-associative. As a real vector space, $\mathbb{O}$ is generated by $\{1, i, j, k, l, li, lj, lk\} = \{1, e_1, \ldots, e_7\}$, where

$e_1, e_2, \ldots e_7$ are imaginary units, which square to -1, switch sign under complex conjugation and anticommute. They span the seven dimensional real subspace which we will denote as the purely imaginary part of the octonions, $\mathrm{Im}\,\mathbb{O}$.

Denote by $\cdot$ the product furnishing the octonions with the structure of real division algebra. The product $\cdot$ determines a cross product in $\mathbb{O}$, expressed as

$$x \times y := \frac{1}{2} x \cdot y - y \cdot x,$$

which contains all the non-trivial algebraic information about the octonions, and turns $\mathrm{Im}\,\mathbb{O}$ into an algebra. As will be discussed below, the algebra $(\mathrm{Im}\,\mathbb{O}, \times)$ provides a new definition of the group G_2.

2.2. **The exceptional Lie group G_2: three equivalent definitions.** Now that we are familiar with the $\mathbb{R}$-algebra $\mathbb{O}$, we are ready to introduce the exceptional Lie group G_2.

Definition 2.2. The exceptional Lie group G_2 is defined to be the group of all $\mathbb{R}$-algebra automorphism of the octonions.

Let us consider an alternative but equivalent definition of G_2: Given two orthogonal and unit imaginary octonions x and y, the cross product $x \times y$ is orthogonal to both of them, and the subalgebra generated by $\{1, x, y, x \times y\}$ is isomorphic to $\mathbb{H}$. If we consider another unit imaginary octonion z orthogonal to the subspace generated by $\{1, x, y, x \times y\} \cong \mathbb{H}$, then the subalgebra over the quaternions $\mathbb{H} \oplus \mathbb{H}[z]$ is isomorphic to the octonions. This has the consequence that an element $g \in G_2$ can be characterized by prescribing three unit imaginary octonions $\{x, y, z\}$, with z orthogonal to x, y and $x \times y$. The element g is then the unique automorphism of the imaginary part of the octonions, which sends $\{x, y, z\}$ to $\{i, j, l\}$. Then the exceptional Lie group G_2 is defined to be the automorphism group of the algebra $(\mathrm{Im}\,\mathbb{O}, \times)$.

On $\mathrm{Im}\,\mathbb{O}$ one can define a cross product three-form on the generators e_i, e_j, e_k as

$$\phi(e_i, e_j, e_k) = f^{ijk},$$

where $e_i \times e_j = \sum_k f^{ijk} e_k$. If we consider the dual basis $\{w^i := e_i^*\}$ of the generators $\{e_i\}$ given above, then the three-form ϕ is given by

$$\phi = w^{123} - w^{145} - w^{167} - w^{246} + w^{257} - w^{347} - w^{356},$$

where the notation w^{ijk} denotes the wedge product $e^i \wedge e^j \wedge e^j$. Notice that ϕ encodes the multiplicative structure of cross product in the imaginary part of the octonions. Then, using the above considerations, we obtain a third equivalent definition of the exceptional Lie group G_2 as follows:

$$G_2 = \{g \in GL(\mathrm{Im}\,\mathbb{O}) \mid g^* \phi = \phi\}.$$

In [6, Theorem 1] Bryant presented some other facts about G_2.

Proposition 2.3. *The Lie group G_2 is a compact subgroup of $SO_7(\mathbb{R})$, of dimension 14, which is connected, simple and simply connected.*

The following result summarizes the discussion of basic facts of the real form of G_2.

Theorem 2.4. *The real form of the group G_2 has dimension 14. It is simple, simply connected, and is isomorphic to one and hence to all of the following Lie groups:*

(i) The group of all $\mathbb{R}$-algebra automorphisms of the octonions.

(ii) The automorphism group of the subalgebra $(\operatorname{Im}\mathbb{O}, \times)$.
(iii) The subgroup of SO_7 *which preserves the cross product three form.*

Also, as a smooth manifold, G_2 *is diffeomorphic to the following closed subset of* $(\mathbb{R}^7)^3$

$$\big\{(x_1, x_2, x_3) \mid \langle x_i, x_j\rangle = \delta_{i,j} \text{ and} \langle x_1 \times x_2, x_3\rangle = 0\big\}$$

3. G_2 flag manifolds

The flag manifolds on G_2 have received recently a lot of attention from several viewpoints. From riemannian geometry ([6], [15]), algebraic topology relevant to global analysis [1], complex and Kähler geometry ([11], [13]), and from the classical computations of their characteristic classes via representation theory of Borel and Hirzebruch ([5], [10]). The combination of the above facts, excellently produced in [11], aroused our interest in the subject and gave us the original idea to calculate the Fadell-Husseini index of some fiber bundles over $G_2/SO(4)$ with total space a G_2-flag manifold.

In this section we introduce some flag manifolds associated to the exceptional Lie group G_2. Since we are particularly interested on the G_2-flag manifolds which fibre over $G_2/SO(4)$, we are going to stat introducing the G_2-homogeneous space $G_2/SO(4)$ in order to define the fiber bundles for which we want to calculate their Fadell-Husseini indexes.

3.1. **The space of associative subspaces $G_2/SO(4)$.** Now that we have introduced the exceptional Lie group G_2 with some equivalent definitions, we are ready to study the G_2-homogeneous space that we are going to use.

Definition 3.1. A 3-dimensional subspace $\xi \subset \operatorname{Im}\mathbb{O}$ is said to be *associative* if it is the imaginary part of a subalgebra isomorphic to the quaternions.

Notice that the subspace ξ acquires a canonical orientation from that of the quaternions. On the other hand, in terms of the three-form defined in 2.2, we can also define an associative subspace as the 3-dimensional real subspace of $\mathbb{R}^7 \cong_{\mathbb{R}} \operatorname{Im}\mathbb{O}$ in which the three form ϕ agrees with the volume form $\operatorname{Vol}(\xi)$.

The group G_2 stabilizes 1, and since the scalar product is the real part of the octonionic multiplication $\cdot$, it acts by isometries on $\operatorname{Im}\mathbb{O}$. Since also preserves the vector product, it preserves orientation so that

$$G_2 \subset SO(\operatorname{Im}\mathbb{O}) \cong SO(7).$$

Every associative subspace ξ admits an orthonormal basis $\{e_1, e_2, e_3\}$ with $e_1 \times e_2 = e_3$. Given such a triple $\{e_1, e_2, e_3\}$, there exists a unique automorphism in $\operatorname{Im}\mathbb{O}$ taking the ordered triple $\{i, j, l\}$ to it. This induces a transitive action of G_2 on the Grassmannian of associative 3-dimensional subspaces of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$, with stabilizer $SO(4)$. Then $G_2/SO(4)$ is the set of all associative 3-dimensional subspaces of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. Since every associative subspace has a canonical orientation, there is an embedding of $G_2/SO(4)$ in the 3-dimensional oriented Grassmannian manifold $\widetilde{G}_3(\mathbb{R}^7)$, with $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$.

Borel and Hirzebruch studied in [5] the cohomology with $\mathbb{F}_2$-coefficients and the Stiefel-Whitney classes of $G_2/SO(4)$. They proved the following result:

Proposition 3.2. *The homogeneous space of associative* 3*-dimensional real subspaces, denoted by* G_2/SO_4*, is an* 8*-dimensional manifold for which* $H^*(G_2/SO(4); \mathbb{F}_2)$ *is generated by two elements* u_2, u_3 *of degrees* 2 *and* 3 *respectively, with the relations*

$$u_2^3 = u_3^2 \quad \text{and} \quad u_3 u_2^2 = 0.$$

Also, the Stiefel-Whitney classes of $G_2/SO(4)$ are non-zero only in dimensions $0, 4, 6$ *and* 8.

On the other hand, Shi and Zhou in [14, Section 10], Thung in [11, Section 2], and Akbulut and Kalafat [1, Theorem 10.3], studied the integral cohomology of $G_2/SO(4)$, and the relations between the corresponding generators to write it as a truncated polynomial algebra.

Proposition 3.3. *The integral cohomology of $G_2/SO(4)$ is a truncated polynomial algebra generated by two elements a and b, of degree 3 and 4 respectively, subject to the relations*

$$\{2b = 0,\ b^3 = 0,\ a^3 = 0,\ ab = 0\}.$$

3.2. **The six dimensional sphere $G_2/SU(3)$.** We are going to start with a G_2-flag manifold that, even if it do not fibre over $G_2/SO(4)$, it is a good first example of this type of spaces.

Remember that, by theorem 2.4[part ii], G_2 can be characterized by triples $\{x, y, z\}$ of mutually orthogonal unitary vectors in $\operatorname{Im}\mathbb{O}$, where z is also orthogonal to $x \times y$. This means that the six dimensional sphere, which can be seen as the unitary vectors in the seven dimensional real subspace $\operatorname{Im}\mathbb{O}$, carries a transitive action of G_2.

To determine the isotropy group of an element $l \in S^6$, consider the subspace V defined as the orthogonal complement of l inside $\operatorname{Im}\mathbb{O}$, and the complex structure on V given by the left multiplication by l. The identification with $\mathbb{C}^3$, equipped with its stardand Hermitian scalar product, induces a scalar product on V defined as

$$\langle v, w\rangle_V = (v, w) + l(v, lw),$$

where $(-, -)$ denote the standard real product. Since $g \in (G_2)_l$ preserves $(-, -)$, it also preserves $\langle -, -\rangle_V$, and $(G_2)_l \subset U(V) \cong U(3)$. Calculating the determinant of $g \in (G_2)_l$, can be proved that actually the isotropy group is isomorphic to $SU(3)$. We have obtained the following result:

Proposition 3.4. *There is a transitive action of G_2 on S^6 with isotropy group isomorphic to $SU(3)$, i.e. $S^6 \cong G_2/SU(3)$.*

3.3. **The space of complex coassociative 2-planes $G_2/U(2)_+$.** We will consider now the complexification of the purely imaginary subspace of the octonions, which is isomorphic as complex vector space $\mathbb{C}^7$, in symbols $\operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \cong_{\mathbb{C}} \mathbb{C}^7$.

Definition 3.5. We call a complex 2-dimensional subspace $W \subset \mathbb{C}^7$ *coassociative* if $v \times w = 0$ for all $v, w \in W$

Notice that the coassociative subspace W is automatically isotropic. Also, the associative three-form $\phi \otimes \mathrm{id}$ defines a complex three-form $\phi_{\mathbb{C}}$ on the complex space $\mathbb{C}^7$. So, in terms of the three-form defined in 2.2, a complex vector subspace $W \subset \mathbb{C}^7$ is called coassociative if the complexified form $\phi_{\mathbb{C}}$ vanishes on V. We will consider coassociative complex lines (1-dimensional complex subspaces) and coassociative planes (2-dimensional complex subspaces).

On the other hand, let J be an orthogonal complex structure on $\xi^\perp$ with $(1, 0)$-space W, where W is the eigenspace associated to the eigenvalue i of J. Choose now an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\xi^\perp$ with $J(e_1) = e_2$ and $J(e_3) = e_4$. We say that J, or the corresponding $(1, 0)$-space W, is positive (resp. negative) according as the basis of $\xi^\perp$ is positive (resp. negative).

The following result, which connects the notion of complex-coassociative and positive subspaces, is proved in [15, Lemma 2.2, page 293].

Lemma 3.6. *Let ξ be an associative 3-dimensional subspace in $\mathrm{Im}\,\mathbb{O}$, and let $W \subset \xi^{\perp} \otimes \mathbb{C}$ be a maximally isotropic subspace. Then W is positive if and only if it is complex coassociative.*

Consider the space of all complex coassociative 2-dimensional subspaces of $\mathbb{C}^7$. Since such space has a transitive action of G_2 with stabilizer isomorphic to $U(2)$, then we have the following definition.

Definition 3.7. The space of complex coassociative 2-planes in $\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}$ will be denoted by $G_2/U(2)_+$.

The reason for the notation comes from lemma 3.6. Moreover, there is a map

$$\rho_1 : G_2/U(2)_+ \to G_2/SO_4,$$

given by $W \mapsto (W \oplus \bar{W})^{\perp}$, which exhibits $G_2/U(2)_+$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. By lemma 3.6 the fiber of any $\xi \in G_2/SO(4)$ is all positive maximally isotropic subspaces of $\xi^{\perp} \otimes \mathbb{C}$, equivalently, all posible orthogonal complex structures on $\xi^{\perp}$. Actually, by [15, section 4.4] there is an identification of $G_2/U(2)_+$ with a quaternionic twistor space of G_2/SO_4.

As is exposed in [11, Prop. 3], the cohomology of $G_2/U(2)_+$ with integral coefficients is generated by classes $g_i \in H^{2i}(G_2/U(2)_+, \mathbb{Z})$, for $i = 1, \ldots, 5$, and the multiplicative structure is determined by the relations

$$g_1^2 = 3g_2,\ g_1 g_2 = 2g_3,\ g_2^2 = 2g_4,\ g_1 g_4 = g_5.$$

The cohomology with coefficients in $\mathbb{F}_2$ will be calculated in subsection 4.3.

3.4. The space of complex non-coassociative 2-planes $\mathbf{G_2/U(2)_-}$.

Definition 3.8. The space of all 2-planes in $\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}$ which are not complex coassociative will be denoted by $G_2/U(2)_-$.

Similarly to $G_2/U(2)_+$, the reason for the notation comes from lemma 3.6. A 2-dimensional real subspace W, which is a $(1,0)$-space for a negative complex structure in $\xi^{\perp}$, is said to be negative and is still a maximally isotropic subspace of $\xi^{\perp} \otimes \mathbb{C}$.

Alternatively, we can think the space $G_2/U(2)_-$ as follows: Let Q^5 be the complex quadric $\{(z_1 : z_2 : \ldots : z_6) \in \mathbb{C}P^6 \mid \sum_{i=0}^{5} z_i^2 = 0\}$ consisting of all 1-dimensional isotropic subspaces of $\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}$. There is a G_2-equivariant isomorphism from Q^5 to $G_2/U(2)_-$ given by $\ell \mapsto \ell^{\perp} \cap \ell^a$, with inverse $W \mapsto W \times W$, where $\ell^a = \{L \in \mathrm{Im}\,\mathbb{O} \otimes \mathbb{C} \mid L \times \ell = 0\}$.

Since $\widetilde{G}_2(\mathbb{R}^7)$ is also diffeomorphic to the complex quadric Q^5, we can identify every element in $G_2/U(2)_-$ as an oriented plane P with oriented, orthonormal basis $\{x, y\}$. Then there is a well defined map $\rho_6 : G_2/U(2)_- \to S^6$ which sends $P \mapsto x \times y = xy$. Actually, the oriented plane P can be identify, via the almost complex structure of S^6, with a complex line in $T_{xy}S^6$. On the other hand, given an element $k \in S^6$, there is a oriented orthonormal basis $\{x, y\}$ of a complex line in $T_k S^6$ that satisfies $k = xy$. This means that the fiber over an element $v \in S^6$ is precisely $\mathbb{P}(T_v S^6)$. This exhibits $G_2/U(2)_-$ as $\mathbb{P}(TS^6)$ with ρ_6 as the base point projection. We can also think of ρ_6 as a fibration with fibers diffeomorphic to $\mathbb{C}P^2$.

Finally, the isomorphism $TS^6 \cong T^*S^6$ as real vector bundles induces a diffeomorphism $\mathbb{P}(TS^6) \cong \mathbb{P}(T^*S^6)$. The following result resumes all the diffeomorphic definitions of $G_2/U(2)_-$. See [11, Prop. 8].

Proposition 3.9. *The following 10-dimensional manifolds are all diffeomorphic to each other:*

(i) *The space of all* 2*-planes in* $\operatorname{Im}\mathbb{O}\otimes\mathbb{C}$ *which are not complex coassociative.*
(ii) *The Grassmannian* $\widetilde{G}_2(\mathbb{R}^7)$ *of oriented* 2*-planes in* $\mathbb{R}^7$*.*
(iii) *The complex quadric*

$$Q^5 = \{(z_1 : z_2 : \ldots : z_6) \in \mathbb{C}P^6 \mid \sum_{i=1}^{7} z_i^2 = 0\}.$$

(iv) *The projectivization of the tangent and cotangent bundle* $\mathbb{P}(TS^6)$ *and* $\mathbb{P}(T^*S^6)$ *for any almost complex structure in* S^6*.*

The cohomology with integral coefficients of $G_2/U(2)_-$ is calculated in [11, Prop. 11]. It is the quotient of a polynomial algebra with generators x in degree 6, and y in degree 2, satisfying the relations $x^2 = 0$ and $y^3 = -2x$. Unlike $G_2/U(2)_+$, using the bundle ρ_6 we can calculate the cohomology ring of $G_2/U(2)_-$ with coefficients in $\mathbb{F}_2$ as follows:

Proposition 3.10. *The cohomology of* $G_2/U(2)_-$ *with coefficients in* $\mathbb{F}_2$ *is given by*

$$H^*(G_2/U(2)_-, \mathbb{F}_2) = \mathbb{F}_2[x,y]/\langle x^2, y^3\rangle,$$

where $deg(y) = 2$*,* $deg(x) = 6$*.*

Proof. Consider the fiber bundle

$$\mathbb{CP}^2 \hookrightarrow G_2/U(2)_- \to S^6$$

and the corresponding Serre spectral sequence. Since S^6 is simply connected, there are no local coefficients and the E_2-term is given by

$$E_2^{p,q} = H^p\big(S^6; H^q(\mathbb{CP}^2;\mathbb{F}_2)\big).$$

Let us denote the cohomology of the fiber $\mathbb{CP}^2$ and the base space S^6 as follows:

$$H^*(S^6;\mathbb{F}_2) = \mathbb{Z}_2[x]/\langle x^2\rangle \quad\text{and}\quad H^*(\mathbb{CP}^2;\mathbb{F}_2) = \mathbb{F}_2[y]/\langle y^3\rangle.$$

For an illustration of the E_2-term see Figure 1. Then, since all the possible differentials are trivial, $E_2^{p,q} \cong E_\infty^{p,q}$ and

$$H^n(G_2/U(2)_-, \mathbb{F}_2) = \mathbb{F}_2[x,y]/\langle x^2, y^3\rangle.$$

$\square$

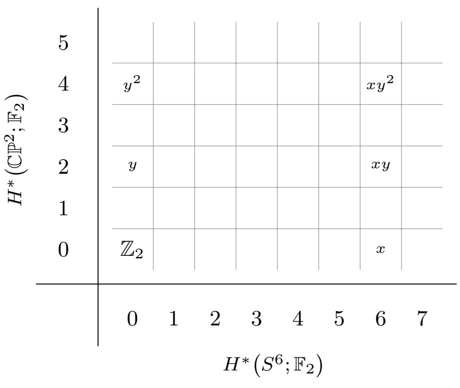

FIGURE 1. $E_2^{p,q} = H^p\big(S^6; H^q(\mathbb{CP}^2;\mathbb{F}_2)\big) \Rightarrow H^*(G_2/U(2)_-;\mathbb{F}_2)$.

Similar to $G_2/U(2)_+$, there exists a map

$$\rho_2 \colon G_2/U(2)_- \to G_2/SO_4,$$

given by $W \mapsto (W \oplus \bar{W})^\perp$, which exhibits $G_2/U(2)_-$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. By lemma 3.6, the fiber at any $\xi \in G_2/SO(4)$ consist of all negative maximally isotropic subspaces of $\xi^\perp \otimes \mathbb{C}$, or equivalently, all negative orthogonal complex structures on $\xi^\perp$. Let us describe now the induced homomorphism ρ_2^*.

Lemma 3.11. *The induced homomorphism in cohomology, denoted by ρ_2^*, with $\mathbb{F}_2$ coefficients, maps the class u_2^2 to y^2.*

Proof. Let us start analyzing the Gysin exact sequence associated to the sphere bundle ρ_2,

$$\cdots \to H^{i-3}(G_2/SO(4);\mathbb{F}_2) \xrightarrow{\smile w_3} H^i(G_2/SO(4);\mathbb{F}_2) \to$$
$$\xrightarrow{\rho_2^*} H^i(G_2/U(2)_-;\mathbb{F}_2) \to H^{i-2}(G_2/SO(4);\mathbb{F}_2) \to \cdots .$$

Since $H^1(G_2/SO_4, \mathbb{F}_2) = 0$, by the exactness of the sequence,

$$\rho_2^* \colon H^4(G_2/SO(4);\mathbb{F}_2) \to H^i(G_2/U(2)_-;\mathbb{F}_2)$$

is a monomorphism. This means that $\rho_2^*(u_2) = y$ and $\rho_2^*(u_2^2) = y^2$. $\square$

3.5. **The full flag manifold $\mathbf{G_2/U(1) \times U(1)}$.** Given a complex isotropic line $\ell \subset \operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, consider the annihilator ℓ^a, that is the subspace of $\operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ described as $\{x \in \operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \mid x \times \ell = 0\}$. Notice that this is a complex 3-dimensional isotropic subspace of $\operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. Since the maximal torus in G_2 is of rank 2, the complete flag manifold $G_2/U(1) \times U(1)$ is a smooth complex manifold of dimension 6 that can be described as follows: The space of pairs (ℓ, D) where ℓ is a complex isotropic line, D is a 2-plane containing ℓ, and both are contained in ℓ^a.

For every pair (ℓ, D) in $G_2/U(1) \times U(1)$, we write $D = \ell \oplus q$, where q is the orthogonal complement of ℓ. Then we get a fibration

$$\rho_3 \colon G_2/U(1) \times U(1) \to G_2/SO(4)$$

given by $(\ell, D) \mapsto \xi$, where $\xi \otimes_{\mathbb{R}} \mathbb{C} = q \oplus \bar{q} \oplus (q \times \bar{q})$. The map ρ_3 factors through a fibration over $G_2/U(2)_\pm$, which sends (ℓ, D) to the positive (resp. negative) maximally isotropic subspace of $\xi^\perp \otimes_{\mathbb{R}} \mathbb{C}$ which contains ℓ, and ρ_1 (resp. ρ_2) defined above.

$$\begin{tikzcd}
 & & G_2/U(1) \times U(1) \arrow[dll, "\rho_4"'] \arrow[dd, "\rho_3"] \arrow[drr, "\rho_5"] & & & \\
G_2/U(2)_+ \arrow[drr, "\rho_1"'] & & & & G_2/U(2)_- \cong Q_5 \arrow[dll, "\rho_2"] \arrow[dr, "\rho_6"] & \\
 & & G_2/SO(4) & & & S^6
\end{tikzcd} \tag{3.12}$$

All the maps ρ_i in diagram 3.12 are locally trivial smooth fibrations, with fiber diffeomorphic to $\mathbb{C}P^1 \approx S^2$ (except ρ_3 and ρ_6 which fiber is diffeomorphic to $\mathbb{C}P^2$). This means that, using the map ρ_5, we can also calculate the cohomology of $G_2/U(1) \times U(1)$ with $\mathbb{F}_2$ coefficients.

Proposition 3.13. *The cohomology of $G_2/U(1)\times U(1)$ with $\mathbb{F}_2$ coefficients is given by*

$$H^*(G_2/U(1)\times U(1),\mathbb{F}_2)=\mathbb{F}_2[x,y,z]/\langle x^2,y^3,z^2\rangle$$

where $deg(y)=2$, $deg(x)=6$ and $deg(z)=2$.

Proof. Consider the fiber bundle

$$S^2\hookrightarrow G_2/U(1)\times U(1)\xrightarrow{\rho_5} G_2/U(2)_-,$$

and the corresponding Serre spectral sequence, where the cohomology of the fiber is described as $H^*(S^2;\mathbb{F}_2)=\mathbb{F}_2[z]/z^2$. Since $\pi_1(G_2/U(2)_-)$ is trivial, there are no local coefficients and the E_2-term is given by

$$E_2^{p,q}=H^p\big(G_2/U(2)_-;H^q(S^2;\mathbb{F}_2)\big).$$

For an illustration of the E_2-term see Figure 2. The rest of the proof is similar to the one in 3.10, since all the possible differentials are trivial. $\square$

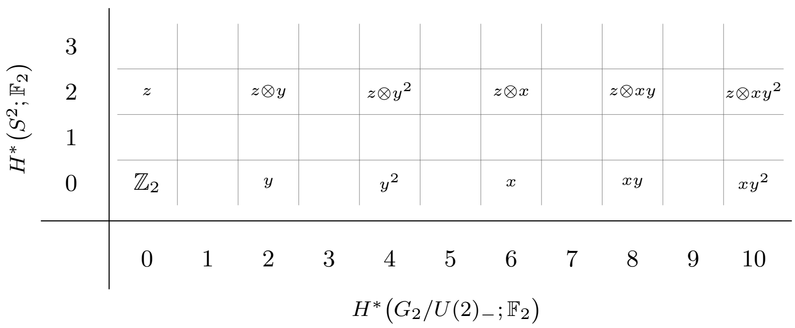

FIGURE 2. $E_2^{p,q}=H^p\big(G_2/U(2)_-;H^q(S^2;\mathbb{F}_2)\big)\Rightarrow H^*(G_2/U(1)\times U(1);\mathbb{F}_2)$.

4. Calculations of Fadell-Husseini Index.

Before we make some calculations, we briefly recall the definition and some basic properties of the Fadell-Husseini index.

4.1. **The Fadell-Husseini Index.** Let G be a finite group, and let R be a commutative ring with unit. For a $G-$ equivariant map $p\colon X\to B$, the *Fadell-Husseini index* of p with coefficients in R is defined to be the kernel ideal of the following induced map

$$\begin{aligned}\mathrm{Index}_G^B(p;R)&=\ker\big(p*\colon H^*(EG\times_G B;R)\longrightarrow H^*(EG\times_G X;R)\big)\\&=\ker\big(p*\colon H^*_G(B;R)\longrightarrow H^*_G(X;R)\big).\end{aligned}$$

Here $H^*_G(\cdot)$ stands for the Borel cohomology defined as the Čech cohomology of the Borel construction $EG\times_G\cdot$. The basic properties of the index are:

- *Monotonicity:* If $p\colon X\to B$ and $q\colon Y\to B$ are $G-$equivariant maps, and $f\colon X\to Y$ is a $G-$equivariant map such that $p=q\circ f$, then
$$\mathrm{Index}_G^B(p;R)\supseteq\mathrm{Index}_G^B(q;R).$$
- *Additivity:* If $(X_1\bigcup X_2,X_1,X_2)$ is an excisive triple of $G-$spaces and $p\colon X_1\bigcup X_2\to B$ is a $G-$equivariant map, then
$$\mathrm{Index}_G^B(p|_{X_1};R)\cdot\mathrm{Index}_G^B(p|_{X_2};R)\subseteq\mathrm{Index}_G^B(p;R).$$

- *General Borsuk-Ulam-Bourgin-Yang theorem:* Let $p\colon X \to B$ and $q\colon Y \to B$ be $G-$equivariant maps, and let $f\colon X \to Y$ be a $G-$equivariant map such that $p = q \circ f$. If $Z \subseteq Y$ then

$$\mathrm{Index}_G^B(p|_{f^{-1}(Z)}; R) \cdot \mathrm{Index}_G^B(q|_{Y\setminus Z}; R) \subseteq \mathrm{Index}_G^B(p; R).$$

In the case when B is a point we simplifly notation and write $\mathrm{Index}_G^B(p; R) = \mathrm{Index}_G(X; R)$. From here on we will be considering $\mathbb{F}_2$ coefficients. For more details see [9]

4.2. **Dold's argument.** In [7] Dold presented a very useful tool to deduce some differentials of the Serre spectral sequence associated to a particular kind of fiber bundles. Let us start with the general argument.

Let $E \xrightarrow{\pi} B \xleftarrow{\pi'} E'$ be vector bundles of fiber-dimension n, m over the same paracompact space B, and let $f\colon S(E) \longrightarrow E'$ be an odd map ($f(-x) = -f(x)$), where $S(E)$ is the total space of the sphere bundle associated to π, such that

$$\begin{tikzcd} S(E) \arrow[rr, "f"] \arrow[dr, "s\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes. Let us define $Z_f = \{x \in S(E) \mid f(x) = 0\}$, where 0 stands for the zero section of π', and the projection maps

$$S(E) \to \bar{S}(E) = S(E)/\mathbb{Z}_2 \quad \text{and} \quad Z \to \bar{Z} = Z/\mathbb{Z}_2,$$

where we are considering the antipodal action.

Cohomology H^* is understood in the Čech sense with mod 2 coefficients, and $H^*(B)[t]$ is the polynomial ring over $H^*(B)$ in one indeterminate t of degree 1. Since the antipodal action is fixed point free in $S(E)$ and Z, the projection maps $S(E) \to \bar{S}(E)$ and $Z \to \bar{Z}$ are 2-sheeted covering maps. Their characteristic classes are denoted by u, resp. $u \mid \bar{Z}$, which can be replaced by the indeterminate t and obtain an homomorphism of $H^*(B)$-algebras

$$\sigma\colon H^*(B)[t] \longrightarrow H^*(\bar{S}(E)) \longrightarrow H^*(\bar{Z})$$

given by $t \longmapsto u \longmapsto u \mid \bar{Z}$. Dold proved the following result.

Theorem 4.1. *If $q(t) \in H^*(B)[t]$ is such that $\sigma(q(t)) = 0$, then*

$$q(t)W(\pi'; t) = W(\pi; t)q'(t)$$

for some $q'(t) \in H^*(B)[t]$, where $W(\pi; t) = \sum_{j=0}^{n} w_j(\pi) \otimes t^{n-j}$.

The last theorem means that, under the last conditions, $W(\pi; t)$ divides $q(t)W(\pi'; t)$. We show the effectiveness of this theorem in the following remark.

Remark 4.2. Under the same hypothesis, consider the fiber bundles

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] & S(E) \arrow[d, "s\pi"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} \{0\} \arrow[r, hook] & B \times \{0\} \arrow[d, "\mathrm{proj}_1"] \\ & B \end{tikzcd}$$

where $\mathbb{Z}_2$ acts antipodally on $S(E)$ and trivial on B. Let $f\colon S(E) \longrightarrow B \times \{0\}$ be a $\mathbb{Z}_2$-equivariant map given by $f(e) = (\pi(e), 0)$, such that the following diagram commutes:

$$\begin{tikzcd} SE \arrow[rr, "f"] \arrow[dr, "s\pi"'] & & B \times \{0\} \arrow[dl, "\mathrm{proj}_1"] \\ & B & \end{tikzcd}$$

Since $Z_f = S(E)$ and $W(proj_1, t) = 1$, if we consider $q(t)$ as the image of the transgression map $d_n^{0,n-1}$ of the Serre spectral sequence associated to the sphere bundle

$$S^{n-1} \hookrightarrow E\mathbb{Z}_2 \times_{\mathbb{Z}_2} E \to B\mathbb{Z}_2 \times B,$$

by theorem 4.1,

$$q(t) = d_n^{0,n-1}(z) = \sum_{j=0}^{n} w_j(\pi) \otimes t^{n-j},$$

where $H^*(S^{n-1}) = \mathbb{F}_2[z]/\langle z^2\rangle$.

We are going to use Dold's argument in some of the computations that we present.

4.3. **The Fadell-Husseini index of ρ_1 and ρ_2.** Following the idea presented in section 4.2, we are going to compute the Stiefel-Whitney classes of ρ_1 and ρ_2 to deduce their corresponding Serre spectral sequences. Since we already know the cohomology of $G_2/U(2)_-$, we are going to start with the characteristic classes of ρ_2.

Lemma 4.3. *The Stiefel-Whitney classes of the fiber bundle* $\rho_2 \colon G_2/U(2)_- \longrightarrow G_2/SO(4)$ *are non-zero in dimensions* $0, 2$ *and* 3.

Proof. Consider the Gysin exact sequence applied to the sphere bundle ρ_2

$$\dots \to H^{i-3}(G_2/SO(4); \mathbb{F}_2) \xrightarrow{\smile w_3} H^i(G_2/SO(4); \mathbb{F}_2) \xrightarrow{\rho_2^*} H^i(G_2/U(2)_-; \mathbb{F}_2) \to$$
$$\to H^{i-2}(G_2/SO(4); \mathbb{F}_2) \to \dots$$

Since $H^3(G_2/U(2)_-; \mathbb{F}_2) = 0$, we get an epimorphism

$$H^0(G_2/SO(4); \mathbb{F}_2) \twoheadrightarrow H^3(G_2/SO(4); \mathbb{F}_2)$$

that by proposition 3.2 can be seen as $\mathbb{Z}_2 \twoheadrightarrow \langle u_3\rangle$. Hence, $w_3(\rho_2) = u_3$. Considering the definition of the Stiefel-Whitney classes, this means that

$$\phi^{-1} \circ Sq^3(th) \neq 0,$$

where th is the Thom class associated to ρ_2. Then, since

$$Sq^3 = Sq^1 \circ Sq^2,$$

$Sq^2(th) \neq 0$ and consequently $w_2 \neq 0$. Is not hard to see that $w_2(\rho_2) = u_2$. $\square$

Consider now the Serre spectral sequence associated to the sphere bundle

$$S^2 \hookrightarrow G_2/U(1) \times U(1) \xrightarrow{\rho_4} G_2/U(2)_+.$$

Since $H^2(G_2/U(1) \times U(1); \mathbb{F}_2) = \langle z, y\rangle$ and $H^3(G_2/U(1) \times U(1); \mathbb{F}_2) = 0$, we get that $H^3(G_2/U(2)_+; \mathbb{F}_2) = 0$. Because of that, using the same arguments in lemma 4.3, ρ_1 has the same Stiefel-Whitney classes than ρ_2.

Now that we know the Stiefel-Whitney classes of ρ_1, we can calculate the cohomology of $G_2/U(2)_+$ with $\mathbb{F}_2$ coefficients as follows: Consider the Serre spectral sequence associated to ρ_1, with E_2-term

$$E_2^{p,q} = H^p\big(G_2/SO(4); H^q(S^2; \mathbb{F}_2)\big).$$

For an illustration of the E_2-term see Figure 3. Notice that, for the same reason exposed in 3.10, there are no local coefficients. Then, applying the Gysin sequence to ρ_1 [12, Example 5.C], and by lemma 4.3, we get that

$$d_3^{0,2}(z) = w_3(\rho_1) = u_3,$$

where $H^2(S^2;\mathbb{F}_2) = \langle z\rangle$. Finally, using the Leibniz rule, $d_3^{0,2}$ determinen the complete cohomology of $G_2/U(2)_+$. We conclude then the following result.

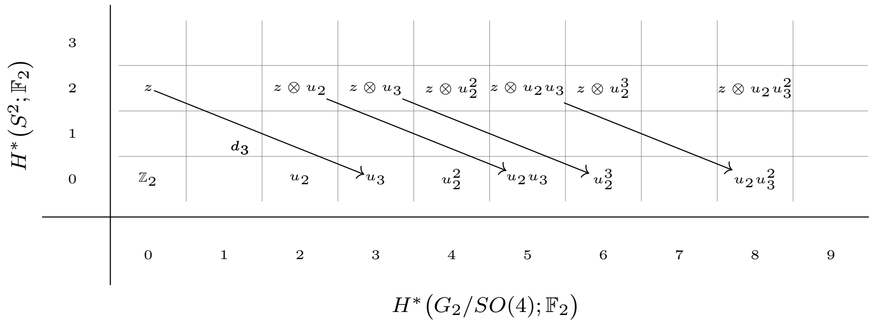

FIGURE 3. $E_2^{p,q} = H^p\big(G_2/SO(4);H^q(S^2;\mathbb{F}_2)\big) \Rightarrow H^*(G_2/U(2)_+;\mathbb{F}_2)$.

Corollary 4.4. *The G_2 flag manifolds $G_2/U(2)_+$ and $G_2/U(2)_-$ have isomorphic cohomology rings when we consider $\mathbb{F}_2$ coefficients. Moreover, the result in lemma 3.11 applies in the same way to ${\rho_1}^*$.*

From now on, since ρ_1 and ρ_2 have the same topological properties considering $\mathbb{F}_2$ coefficients, we will not make any distinction between them and we will write $G_2/U(2)_\pm$ and ρ_j. As shown in [11], the similarity mentioned in corollary 4.4 does not happens if we consider the cohomology rings with integer coefficients.

Consider the action of $\mathbb{Z}_2$ on $G_2/U(2)_\pm$ by complex conjugation. Since $\rho_j(W) = \rho_j(\bar{W})$ for every coassociative subspace W, and $j \in \{1,2\}$, then both maps are $\mathbb{Z}_2$-equivariant, with $\mathbb{Z}_2$ acting trivial on $G_2/SO(4)$. This means that we can ask for the Fadell-Husseini index of such bundles. Before we prove our first main result, we fix the notation for the cohomology of the group $\mathbb{Z}_2$ as

$$H^*(B\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[t],$$

with $\deg(t) = 1$.

Theorem 4.5. *The Borel cohomology of $G_2/U(2)_\pm$ is given by*

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm;\mathbb{F}_2) = H^*(B\mathbb{Z}_2 \times G_2/SO(4);\mathbb{F}_2)\Big/\langle t^3 + u_2t + u_3\rangle.$$

Consequently, the Fadell-Husseini index of ρ_1 and ρ_2 with respect to the introduced $\mathbb{Z}_2$ action is given by

$$\mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_1;\mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_2;\mathbb{F}_2) = \langle t^3 + u_2t + u_3\rangle.$$

Proof. Consider the Borel construction of the bundle ρ_i, for $i \in \{1,2\}$,

$$\begin{tikzcd} S^2 \arrow[r, hook] & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm \arrow[d, "\rho_i"] \\ & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/SO(4), \end{tikzcd}$$

where the base space can be written as $B\mathbb{Z}_2 \times G_2/SO(4)$. Since the fundamental group

$$\pi_1(B\mathbb{Z}_2 \times G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2) \times \pi_1(G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2) \cong \mathbb{Z}_2$$

acts trivially on the cohomology of the fiber, then the E_2-terms of the associated Serre spectral sequence looks as follows:

$$E_2^{p,q} = H^p\big(B\mathbb{Z}_2 \times G_2/SO(4); H^q(S^2; \mathbb{F}_2)\big).$$

Using theorem 4.1 and lemma 4.3, we get that the transgression map $d_3^{0,2}$ is given by

$$d_3^{0,2}(z) = t^3 + u_2 t + u_3,$$

where $H^2(S^2; \mathbb{F}_2) = \langle z \rangle$. Finally, by the Leibniz rule, $d_3^{0,2}$ determines the complete cohomology of $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm$ and the Fadell-Husseini index of ρ_1 and ρ_2 with coefficients in $\mathbb{F}_2$. □

4.4. **The Fadell-Husseini index of the pullback bundle ζ_n.** Let us consider the tautological bundle of the oriented Grassmann manifold $\widetilde{G}_3(\mathbb{R}^7)$:

$$\gamma = \big(E(\gamma),\ \widetilde{G}_3(\mathbb{R}^7),\ E(\gamma) \xrightarrow{\pi} \widetilde{G}_3(\mathbb{R}^7),\ \mathbb{R}^3\big),$$

and the sphere bundle associated to the orthogonal complement of γ:

$$s\gamma^\perp = \big(E(s\gamma^\perp),\ \widetilde{G}_3(\mathbb{R}^7),\ E(s\gamma^\perp) \xrightarrow{s\pi} \widetilde{G}_3(\mathbb{R}^7),\ S^3\big).$$

Using the embedding $i : G_2/SO(4) \hookrightarrow \widetilde{G}_3(\mathbb{R}^7)$, and the fiber bundle ρ_j defined in 3.3 and 3.4, we construct the following commutative squares:

$$\begin{tikzcd}
\mathcal{S}^1_{\gamma^\perp} \arrow[r] \arrow[d, "\phi_1"'] & E\big(i^*(s\gamma^\perp)\big) \arrow[r] \arrow[d] & E(S\gamma^\perp) \arrow[d, "s\pi"] \\
G_2/U(2)_\pm \arrow[r, "\rho_j"] & G_2/SO(4) \arrow[r, "i"] & \widetilde{G}_3(\mathbb{R}^7),
\end{tikzcd}$$

where the map on the left is the pullback bundle of $s\gamma^\perp$ along the map $i \circ \rho_j$,

$$\zeta_1 = \big(E(\zeta_1) = \mathcal{S}^1_{\gamma^\perp},\ G_2/U(2)_-,\ \mathcal{S}^1_{\gamma^\perp} \xrightarrow{\phi_1} G_2/U(2)_-,\ S^3\big).$$

To be more precise, the total space of ζ_1 is given by

$$\begin{aligned}
\mathcal{S}^1_{\gamma^\perp} &:= E(\zeta_1) \\
&= \{(W; \xi, v) \mid W \in G_2/U(2)_\pm,\ \rho_j(W) = \xi,\ v \in \xi^\perp \text{ and } \deg(v_1) = 1\}.
\end{aligned}$$

The construction of the bundle ζ_1 can be easily generalized as follows: Consider the the n-fold product bundle $(s\gamma^\perp)^n$, and the pullback along the diagonal map $\Delta_n : \widetilde{G}_3(\mathbb{R}^7) \to \widetilde{G}_3(\mathbb{R}^7)^n$:

$$\begin{tikzcd}
E\big(\Delta_n^*(s{\gamma^\perp}^n)\big) \arrow[r] \arrow[d] & E\big(s\gamma^\perp\big)^n \arrow[d, "(s\pi)^n"] \\
\widetilde{G}_3(\mathbb{R}^7) \arrow[r, "\Delta_n"] & \widetilde{G}_3(\mathbb{R}^7)^n.
\end{tikzcd}$$

Applying the last construction to the fiber bundle $\Delta_n^*(s{\gamma^\perp}^n)$, we get a new bundle

$$\zeta_n = \big(E(\zeta_n) = \mathcal{S}^n_{\gamma^\perp},\ G_2/U(2)_-,\ \mathcal{S}^n_{\gamma^\perp} \xrightarrow{\phi_n} G_2/U(2)_\pm,\ (S^3)^n\big).$$

where

$$\begin{aligned} \mathcal{S}^n_{\gamma^\perp} &:= E(\zeta_n) \\ &\cong \{(W;\xi,v_1,\dots,v_n) \mid W\in G_2/U(2)_\pm,\ \rho_2(W)=\xi, \\ &\qquad v_1,\dots,v_n\in\xi^\perp \text{ and } \deg(v_1)=\cdots=\deg(v_n)=1\}. \end{aligned}$$

Notice that, since W is contained in $\xi^\perp\otimes_{\mathbb{R}}\mathbb{C}$, we can actually consider the unitary elements inside the (non) complex-coassociative subspace. Because of that we rewrite the space $\mathcal{S}^n_{\gamma^\perp}$ as follows:

$$\mathcal{S}^n_{\gamma^\perp}\cong\{(W;v_1,\dots,v_n)\mid W\in G_2/U(2)_\pm,\ v_1,\dots,v_n\in W \text{ and } \deg(v_1)=\cdots=\deg(v_n)=1\}.$$

To describe the cohomology of $\mathcal{S}^n_{\gamma^\perp}$, the first step is to calculate the Stiefel-Whitney classes of ζ_1. Notice that since $H^1(G_2/U(2)_\pm;\mathbb{F}_2)=H^3(G_2/U(2)_\pm;\mathbb{F}_2)=0$, we are just looking for $w_2(\zeta_1)$ and $w_4(\zeta_1)$. Then we have the following result.

Lemma 4.6. *The Stiefel-Whitney classes of the pullback bundle ζ_1 are non-zero only in the dimensions $0,2$ and 4. Moreover, $w_2(\zeta_1)=y$ and $w_4(\zeta_1)=y^2$.*

Proof. By the naturality of the Stiefel-Whitney classes is enough to describe the image of $w_2(\gamma^\perp)$ and $w_4(\gamma^\perp)$ along the composition

$$G_2/U(2)_\pm \xrightarrow{\rho_j} G_2/SO(4) \xrightarrow{i} \widetilde{G}_3(\mathbb{R}^7).$$

Akbulut and Kalafat in [1, Theorem 2.11] proved that the embedding $i: G_2/SO_4\to\widetilde{G}_3(\mathbb{R}^7)$ maps the euler class of $\gamma^\perp$, which reduces to $w_4(\gamma^\perp)$, to the integral generator in dimension 4, which corresponds to u_2^2. Also, since $w_4(\gamma^\perp_+)=w_2(\gamma^\perp_+)^2$, i^* maps $w_2(\gamma^\perp_+)$ to u_2.

On the other hand, by lemma 3.11, $\rho_j^*(u_2)=y$ and $\rho_j^*(u_2^2)=y^2$. This means that

$$w_2(\zeta_1)=(i\circ\rho_j)^*(w_2(\gamma^\perp_+))=y$$

and

$$w_4(\zeta_1)=(i\circ\rho_j)^*(w_4(\gamma^\perp_+))=y^2,$$

for $j\in\{1,2\}$. □

This lead us to our second main result. For the cohomology of $(S^3)^n$ we fix the following notation

$$\begin{aligned} H^*\big((S^3)^n;\mathbb{F}_2\big) &\cong H^*(S^3;\mathbb{F}_2)^{\otimes n} \\ &\cong \mathbb{F}_2[z_1]/\langle z_1^2\rangle\otimes\cdots\otimes\mathbb{F}_2[z_n]/\langle z_n^2\rangle \\ &\cong \mathbb{F}_2[z_1,\dots,z_n]/\langle z_1^2,\dots,z_n^2\rangle, \end{aligned}$$

where $\deg(z_1)=\cdots=\deg(z_n)=1$.

Theorem 4.7. *The cohomology ring $H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2)$ is described as follows:*

$$H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2)=\Big[\langle 1,x\rangle\otimes\mathcal{I}\Big]\oplus\Big[\langle y,y^2,xy,xy^2\rangle\otimes H^*\big((S^3)^n;\mathbb{F}_2\big)\Big/\langle y^2,xy^2\rangle\otimes\mathcal{I}\Big],$$

where

$$\mathcal{I}=\left\langle \sum_{j=1}^m z_{a_1\cdots\hat{a_j}\cdots a_m}\,\middle|\, a_i\in\{1,2,\dots,n\},\ a_i\neq a_j\ \forall\, i\neq j \ \text{ and }\ 1\le m\le n\right\rangle.$$

Proof. Consider the sphere bundle

$$S^3\hookrightarrow\mathcal{S}^1_{\gamma^\perp}\xrightarrow{\phi_1}G_2/U(2)_\pm,$$

and the corresponding Serre spectral sequence with E_2-term given by

$$E_2^{p,q}(\phi_1)=H^p(G_2/U(2)_\pm;\mathcal{H}^q(S^3;\mathbb{F}_2)).$$

Since the fundamental group $\pi_1(G_2/U(2)_\pm)$ is trivial, the E_2-term simplifies and become

$$E_2^{p,q}(\phi_1) = H^p(G_2/U(2)_\pm; \mathbb{F}_2) \otimes H^q(S^3; \mathbb{F}_2).$$

Notice that the first non-trivial differential appears on page E_4. Also, as a consecuence of the Gysin exact sequence applied to ϕ_j [12, Example 5.C], the trasgression map on E_4 is given by $d_4^{0,3}(z_1) = w_4(\zeta_1) = y^2$.

Let $g_i \colon \mathcal{S}^n_{\gamma^\perp} \to \mathcal{S}^1_{\gamma^\perp}$ be the projection map given by $(W; \xi, v_1, \dots, v_n) \mapsto (W; \xi, v_i)$, for $i = 1, \dots, n$, which induces a morphism of bundles

$$\begin{tikzcd}
\mathcal{S}^n_{\gamma^\perp} \arrow[r] \arrow[d, "\phi_n"'] & \mathcal{S}^1_{\gamma^\perp} \arrow[d, "\phi_1"] \\
G_2/U(2)_\pm \arrow[r, "\mathrm{id}"] & G_2/U(2)_\pm .
\end{tikzcd}$$

This morphism of bundles induces a morphism between the corresponding Serre spectral sequences, which on the zero column is a monomorphism. Then, by the commutativity of the differentials, the map $d_4^{0,3}$ in $E_4^{*,*}(\phi_n)$ is given by

$$d_4^{0,3}(z_1) = \cdots = d_4^{0,3}(z_n) = y^2.$$

Moreover, by the the Leibniz rule we can prove that all the differentials in $E_4^{*,*}(\phi_n)$ are trivial except for

- $d_4^{0,3m}(z_{a_1\cdots a_m}) = y^2 \otimes \left(\sum_{j=1}^m z_{a_1\cdots\hat{a_j}\cdots a_m}\right)$,
- $d_4^{6,3m}(x \otimes z_{a_1\cdots a_m}) = xy^2 \otimes \left(\sum_{j=1}^m z_{a_1\cdots\hat{a_j}\cdots a_m}\right)$,

for $1 \le m \le n$, where $z_{a_1\cdots a_m} = z_{a_1} \cdots z_{a_m}$, and $a_i \in \{1, 2, \dots, n\}$ with $a_i \neq a_j$ for all $i \neq j$. Here $\hat{a_j}$ means that we are removing the element a_j, and $z_{\hat{a_j}} = 1$. For an ilustration of the page E_4 see Figure 4. Since there are no more differentials, the term E_5 describes the cohomology of $S^n_{\gamma^\perp}$. $\square$

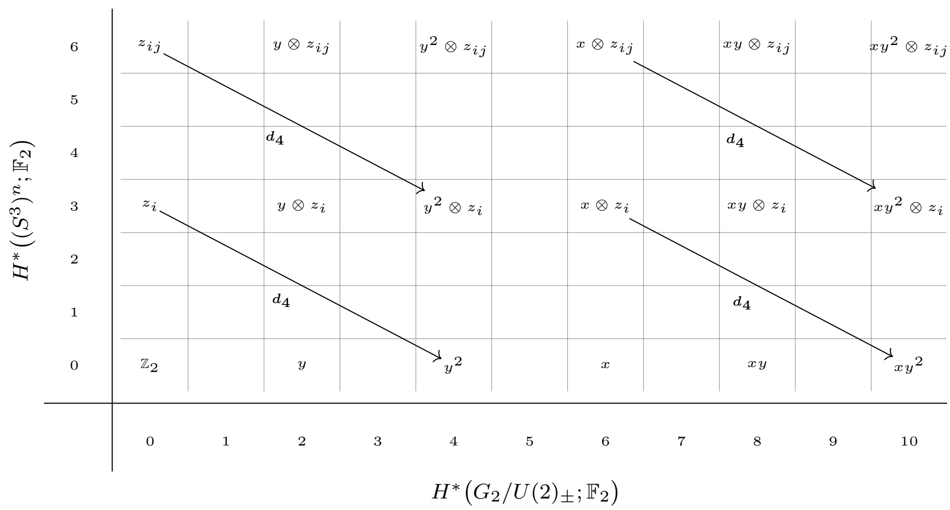

FIGURE 4. $E_4^{p,q} = E_2^{p,q} = H^p(G_2/U(2)_\pm; \mathbb{F}_2) \otimes H^q\big((S^3)^n; \mathbb{F}_2\big)$.

A more pleasant description of $H^*(\mathcal{S}^n_{\gamma^\perp}; \mathbb{F}_2)$ happens when $n = 2$.

Corollary 4.8. *The cohomology $H^*(\mathcal{S}^2_{\gamma^\perp}; \mathbb{F}_2)$ is described as follows:*

n=0	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*
$\mathbb{Z}_2$	0	$\langle y\rangle$	$\langle z_1+z_2\rangle$	0	$\langle y\otimes z_1, y\otimes z_2\rangle$
n=6	*n=7*	*n=8*	*n=9*	*n=10*	*n=11*
$\langle x\rangle$	$\langle y^2\otimes z_1\rangle$	$\langle xy, y\otimes z_1z_2\rangle$	$\langle x\otimes(z_1+z_2)\rangle$	$\langle y^2\otimes z_1z_2\rangle$	$\langle xy\otimes z_1, xy\otimes z_2\rangle$
n=12	*n=13*	*n=14*	*n=15*	*n=16*	
0	$\langle xy^2\otimes z_1\rangle$	$\langle xy\otimes z_1z_2\rangle$	0	$\langle xy^2\otimes z_1z_2\rangle$	

There is an action of $\mathbb{Z}_2^{n+1}$ on $\mathcal{S}^n_{\gamma^\perp}$ defined as follows: The first copy of $\mathbb{Z}_2$ acts on $G_2/U(2)_\pm$ by complex conjugation, while the other n copies of $\mathbb{Z}_2$ acts antipodally on the corresponding sphere S^3. To be more precise, let $(\beta_0,\beta_1,\dots,\beta_n)\in\mathbb{Z}_2^{n+1}$ and $(W;\xi,v_1,\dots,v_n)\in\mathcal{S}^n_{\gamma^\perp}$, then

$$(\beta_0,\beta_1,\dots,\beta_n)\cdot(W;\xi,v_1,\dots,v_n)=(c^{\beta_0}(W);\xi,(-1)^{\beta_1}v_1,\dots,(-1)^{\beta_n}v_n),$$

where $c^0=id$ and c^1 represent the complex conjugation. Considering the action of $\mathbb{Z}_2^{n+1}$ on $G_2/U(2)_\pm$ by complex conjugation with the first copy of $\mathbb{Z}_2$, the projection map ϕ_n is $\mathbb{Z}_2^{n+1}$-equivariant and we can ask for its Fadell-Husseini index. For the cohomology of the group $\mathbb{Z}_2^{n+1}$ we fix the notation

$$H^*(B\mathbb{Z}_2^{n+1};\mathbb{F}_2)=\mathbb{F}_2[t_1,\dots,t_{n+1}],$$

where $\deg(t_1)=\ldots=\deg(t_{n+1})=1$. We will start with the case $n=1$.

Proposition 4.9. *Consider the action of $\mathbb{Z}_2^2$ on $\mathcal{S}^1_{\gamma^\perp}$ introduced before. Then the Borel cohomology of $S^1_{\gamma^\perp}$ is described as follows:*

$$H^*(E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}S^1_{\gamma^\perp};\mathbb{F}_2)=H^*(E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}G_2/U(2)_\pm;\mathbb{F}_2)\Big/\langle y^2+yt_2^2+t_2^4\rangle.$$

Consequently, the Fadell-Husseini index of $\phi_1\colon\mathcal{S}^1_{\gamma^\perp}\longrightarrow G_2/U(2)_\pm$ is given by

$$\mathrm{Index}^{G_2/U(2)_\pm}_{\mathbb{Z}_2^2}(\phi_1;\mathbb{F}_2)=\langle y^2+yt_2^2+t_2^4\rangle.$$

Proof. Consider the morphism of bundles

$$\begin{tikzcd} E\mathbb{Z}_2\times_{\mathbb{Z}_2}\mathcal{S}^1_{\gamma^\perp} \arrow[r] \arrow[d, "id\times_{\mathbb{Z}_2}\phi_1"'] & E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}\mathcal{S}^1_{\gamma^\perp} \arrow[d, "id\times_{\mathbb{Z}_2^2}\phi_1"] \\ E\mathbb{Z}_2\times_{\mathbb{Z}_2}G_2/U(2)_\pm \arrow[r] & E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}G_2/U(2)_\pm, \end{tikzcd} \tag{4.10}$$

induced by the inclusion into the second factor $i_2\colon\mathbb{Z}_2\longrightarrow\mathbb{Z}_2^2$. This morphism, actually of Borel constructions, induces a morphism of the corresponding Serre spectral sequences which on the zero column of the E_2-term is an isomorphism. Notice that, since

$$E\mathbb{Z}_2\times_{\mathbb{Z}_2}G_2/U(2)_\pm\cong B\mathbb{Z}_2\times G_2/U(2)_\pm,$$

and $\pi_1(B\mathbb{Z}_2\times G_2/U(2)_\pm)\cong\mathbb{Z}_2$ acts trivially on the fiber, by theorem 4.1 and lemma 4.6,

$$d_4^{0,3}(z)=y^2+yt_2^2+t_2^4.$$

For an illustration of the E_4-term of the Serre spectral sequence associated to $id\times_{\mathbb{Z}_2}\phi_1$ see Figure 5.

Consider now the Serre spectral sequence associated to the bundle

$$S^3\hookrightarrow E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}\mathcal{S}^1_{\gamma^\perp}\xrightarrow{id\times_{\mathbb{Z}_2^2}\phi_1}E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}G_2/U(2)_\pm,$$

with E_2-term given by

$$E_2^{p,q}=H^p\big(E\mathbb{Z}_2^2\times_{\mathbb{Z}_2^2}G_2/U(2)_\pm;\mathcal{H}^q(S^3;\mathbb{F}_2)\big).$$

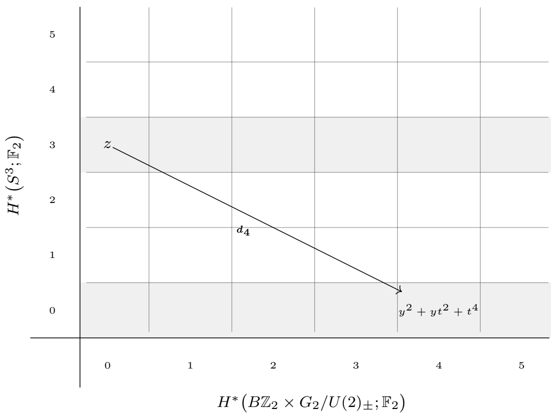

FIGURE 5. $E_4^{p,q} = E_2^{p,q} = H^p\big(B\mathbb{Z}_2 \times G_2/U(2)_\pm; H^q(S^3;\mathbb{F}_2)\big)$

Since the fundamental group $\pi_1(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_\pm) \cong \mathbb{Z}_2^2$ acts trivially on the cohomology of the fiber, then the E_2-terms simplify and looks as follows:

$$E_2^{p,q} = H^p\big(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_\pm; H^q(S^3;\mathbb{F}_2)\big).$$

Also, by theorem 4.5, since the second summand of $\mathbb{Z}_2^2$ acts trivially on $G_2/U(2)_\pm$, and $\mathbb{Z}_2^2$ also acts trivially on $G_2/SO(4)$,

$$H^*(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_\pm;\mathbb{F}_2) = H^*(B\mathbb{Z}_2^2 \times G_2/SO(4);\mathbb{F}_2)\Big/\langle t_1^3 + u_2 t_1 + u_3\rangle.$$

Notice that the first differential of the spectral sequence associated to $id \times_{\mathbb{Z}_2^2} \phi_1$ appears on the E_4-term. Then, by the commutativity of the differentials between $E_4^{*,*}(id \times_{\mathbb{Z}_2} \phi_1)$ and $E_4^{*,*}(id \times_{\mathbb{Z}_2^2} \phi_1)$, $d_4^{0,3}$ is given by

$$d_4^{0,3}(\xi) = y^2 + yt_2^2 + t_2^4.$$

Using the Leibniz rule, this describes the complete cohomology of $E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} S^1_{\gamma^\perp}$ and the Fadell-Husseini index of $\phi_1 \colon \mathcal{S}^1_{\gamma^\perp} \longrightarrow G_2/U(2)_\pm$. $\square$

Using the projection of $\mathcal{S}^n_{\gamma^\perp}$ on $\mathcal{S}^1_{\gamma^\perp}$, and proposition 4.9, we can prove now our third main result.

Theorem 4.11. *Consider the action of $\mathbb{Z}_2^{n+1}$ on $\mathcal{S}^n_{\gamma^\perp}$ introduced before. Then the Fadell-Husseini index of $\phi_n \colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_\pm$ is given by*

$$\mathrm{Index}^{G_2/U(2)_\pm}_{\mathbb{Z}_2^{n+1}}(\phi_n;\mathbb{F}_2) = \langle y^2 + yt_2^2 + t_2^4, \dots, y^2 + yt_{n+1}^2 + t_{n+1}^4\rangle.$$

Proof. Let $i_k \colon \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^{n+1}$ be the inclusion into the first and kth summand, with $2 \le k \le n+1$. The map i_k induces a morphism between Borel constructions

$$\begin{tikzcd}
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^1_{\gamma^\perp} \arrow[d, "id\times_{\mathbb{Z}_2^{n+1}}\phi_1"'] & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} \mathcal{S}^1_{\gamma^\perp} \arrow[l] \arrow[d, "id\times_{\mathbb{Z}_2^2}\phi_1"] \\
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_\pm & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_\pm, \arrow[l]
\end{tikzcd}$$

where $\mathbb{Z}_2^{n+1}$ acts on $\mathcal{S}^1_{\gamma^\perp}$ as follows: The first summand acts on $G_2/U(2)_\pm$ by complex conjugation, and the kth summand acts antipodally on the unitary element. We will refer to this action as the one induced by i_k. This morphism of Borel constructions induces a morphism between the corresponding Serre spectral sequences, which on the zero column of the E_2-term is an isomorphism. Then, by proposition 4.9, the transgression map $d_4^{0,3}$ of the spectral sequence associated to $id \times_{\mathbb{Z}_2^{n+1}} \phi_1$ is given by

$$d_4^{0,3}(z) = y^2 + yt_k^2 + t_k^4,$$

for every $2 \leq k \leq n+1$.

Consider now the projection $p_k \colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow \mathcal{S}^1_{\gamma^\perp}$ given by

$$p_k(W; \xi, v_1, \dots v_n) = (W; \xi, v_{k-1}),$$

for $2 \leq k \leq n+1$. With respect to the $\mathbb{Z}_2^{n+1}$-action on $\mathcal{S}^1_{\gamma^\perp}$ induced by i_k, the map p_k induces a morphism of Borel constructions

$$\begin{tikzcd}
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^n_{\gamma^\perp} \arrow[r, "E(id)\times_{\mathbb{Z}_2^{n+1}} p_k"] \arrow[d, "id\times_{\mathbb{Z}_2^{n+1}}\phi_n"'] & E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^1_{\gamma^\perp} \arrow[d, "id\times_{\mathbb{Z}_2^{n+1}}\phi_1"] \\
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_\pm \arrow[r, "E(id)\times_{\mathbb{Z}_2^{n+1}} id"] & E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_\pm .
\end{tikzcd}$$

This morphism of Borel constructions, in turn, induces a morphism between the corresponding Serre spectral sequences, which on the zero column of the E_2-term is a monomorphism. For an illustration of the morphism between the corresponding Serre spectral sequences see Figure 6. Then, by the commutativity of the differentials, the map $d_4^{0,3}$ of the spectral sequences associated to $id \times_{\mathbb{Z}_2^{n+1}} \phi_n$ is given by

$$d_4^{0,3}(z_{k-1}) = d_4^{0,3}(p_k^*(z)) = \mathrm{id}(d_4^{0,3}(z)) = y^2 + yt_k^2 + t_k^4,$$

for every generator z_{k-1} in $H^3((S^3)^n; \mathbb{F}_2)$, with $2 \leq k \leq n+1$. Then, by the Leibnitz rule, this describes the Fadell-Husseini index of $\phi_n \colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_\pm$.

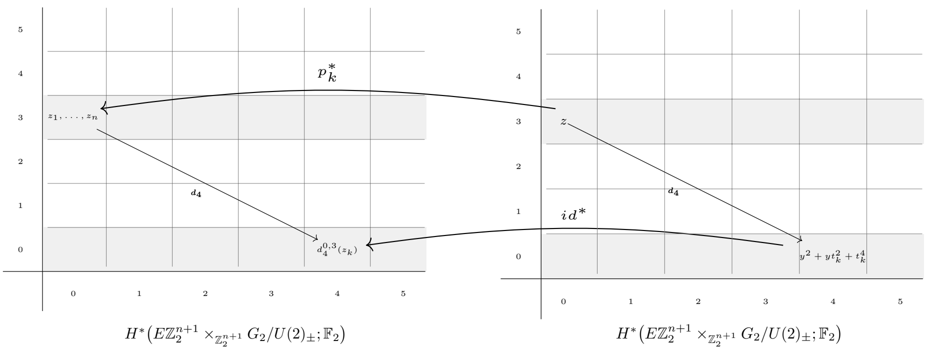

FIGURE 6. Morphism of spectral sequences induced by p_k.

$\square$

Finally, following the same idea in 4.11, we get the Borel cohomology of the total space of the fiber bundle ϕ_2.

Corollary 4.12. *The cohomology of $E\mathbb{Z}_2^3 \times_{\mathbb{Z}_2^3} \mathcal{S}^2_{\gamma^\perp}$ is given by*

$$H^*_{\mathbb{Z}_2^3}(S^2_{\gamma^\perp}; \mathbb{F}_2) = H^*_{\mathbb{Z}_2^3}(G_2/U(2)_\pm; \mathbb{F}_2) \Big/ \langle y^2 + yt_2^2 + t_2^4, y^2 + yt_3^2 + t_3^4 \rangle.$$

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