A PROBABILISTIC APPROACH TO EXHAUSTION IN THE INFINITE-GENUS CASE

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1. Introduction

In the last few years, probabilistic methods became an important tool for the study of problems of geometric nature. These problems have their origin in the study of several models of random groups [26], [11] [14] and they try to include a rich variety of arguments from the theory of random graphs [6] [22], [23].

Stochastic Algebraic Topology, and specifically the study of several models of random simplicial complexes has emerged as an important field of research in the intersection of Geometric Group Theory, Measurable Group Theory, Algebraic Topology and Probability. Relevant for this work is the introduction by A. Costa and M. Farber of the vast generalization of a multiparametric random simplicial complex [8], [9], [10].

The curve complex of an orientable surface S, first introduced by W.J. Harvey [15] and denoted by $\mathcal{C}(S)$, is the abstract simplicial complex whose vertices are the isotopy classes of essential simple closed curves in S, and whose simplices are defined by disjointness. See Section 2 for more details. It is clearly a flag complex, and its 1-skeleton is known as the *curve graph*.

It is well-known that the curve complex of S has the following properties:

(i) It is connected.
(ii) The vertices of the curve graph have infinite degree.
(iii) If S is a closed surface of genus $g \geq 2$, the curve graph has clique number bounded above by $3g - 3$.
(iv) If S is a surface of finite topological type (i.e. its fundamental group is finitely generated), then both the curve graph and the curve complex have infinite diameter and are Gromov-hyperbolic as metric spaces.

The *extended mapping class group* of S, denoted by $\mathrm{MCG}^*(S)$, is the group of isotopy classes of self-homeomorphisms of S. There is a natural action of $\mathrm{MCG}^*(S)$ on $\mathcal{C}(S)$ by simplicial automorphisms, with representation $\Psi : \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{C}(S))$ given by $[h] \mapsto ([\alpha] \mapsto [h(\alpha)])$. This representation is an isomorphism for most of the surfaces. See [12], [19], [21],[24], [25], [20], [18], [3], [7].

In the last couple of decades Ivanov's result for the curve complex has been generalised to more general simplicial/graph morphisms. Of particular interest for this work is the work of J. Aramayona and C.J. Leininger on rigid subgraphs of the curve graph; see [1] and [2]. Here we examine a question of geometric nature, motivated by the notion of *rigid expansion* in the curve complex, in the sense of [1], [2] and particularly the graph theoretic reformulation of [16] with the methods of Stochastic Algebraic Topology.

One of the motivations for rigid expansions is that the proces of rigid expansion allows to create rigid supergraphs out of rigid subgraphs. Moreover, with this method one can exhaust the curve graph, which leads to several results concerning simplicial rigidity of the curve graph. This method was originally introduced by Aramayona-Leininger [2], and in the specific situation that we study here by the third author in [17] and [16]. We review this concept in Section 2.

In [17], the third author proved that the curve graph of a closed surface of genus at least three can be exhausted via rigid expansions by a finite subgraph consisting of a closed chain of length $2g + 2$, and a system of external vertices. In Section 2 we prove we only need the closed chain and two external vertices.

After translating the previous situation to purely combinatorial terms, in this work we study how common finding this subgraphs is by defining a random variable on the multiparametric model for random simplicial complexes of Costa and Farber.

We formulate the problem in the expectation of a random variable which counts *closed chains of length 2g+2 with a system of two "alternating" external vertices inside a 2g+4 simplex.*

The outline of the main result in this note is the following. Take a random simplicial complex $Y \subset \Omega_{\mathfrak{P}}^r$ with at most $n = 2^g > 0$ vertices and vector of probabilities $\mathfrak{P} = (p_0, p_1, \ldots, p_n)$ in the multiparametric Model, with $p_i = n^{-\alpha_i}$, where α_i is a function of n, such that α_i has a limit as n tends to infinity. See Section 3 for more details.

Assume that the simplicial complex is nonempty, connected and Gromov hyperbolic. Recall that as a consequence of Theorem 5 in [9], this is granted whenever the following set of inequalities hold:

Condition 1.1. [Conditions to model the curve complex]

- For the simplicial complex to be hyperbolic: $\alpha_0 + 3\alpha_1 + 2\alpha_2 > 1$ with $\alpha_2 > 0$, and with $0 < \alpha_0 + \alpha_1 < 1$ to be connected. (Theorem 5 in page 449 of [9]).

The following condition is the fundamental threshold for the main statements of this article to hold or not asymptotically almost surely :

Condition 1.2. Assume that the parameters for the random simplicial complex satisfy the hypothesis of critical dimension in the sense of Costa and Farber [10].

The condition of critical dimension was originally defined in the context of phase transitions in homology of random simplicial complexes.

The reason behind the use of this condition is that the argument that we present consists of analyzing the expectation of a random variable related to a geometric condition inside a complete $4g + 2$ graph which is a subcomplex of the random simplicial complex, and the conditions 1.1 pose restrictions on the first three parameters, the critical dimension gives control of the remaining parameters in estimates for the expectation and variance of a random variable CH.

Precisely, we denote by CH the discrete random variable counting closed chains of length $2g+2$ with an alternating pair of external vertices inside a $2g+4$ complete subgraph. See section 4 for precise definitions.

Theorem 1.3. *Assume Y is a $(3g − 3)$-dimensional random simplicial complex and satisfies Conditions 1.1 and 1.2, as well as the technical condition 3.4.*
   Then the discrete random variable CH tends to infinity as g tends to infinity.

This result can be interpreted as a statement giving probabilistic evidence for the Costa-Farber multiparametric model to be a geometric/simplicial limit for the curve complexes of closed surfaces when the genus tends to infinity.

In this sense, for studying several assymptotic properties of a geometric and large-scale nature, the Costa-Farber model with specific parameters is better suited than the curve complex of an infinite-genus surface, since in the latter case the curve complex has finite diameter.

Other developments in this direction, which complement the view on this problems include the result by Bering and Gaster that the curve complex of an infinite genus surface contains the random graph (which is the limit of an Erdös-Rényi Graph) in [5], as well as the definition by Farber, Mead and Strauss of the Radó simplicial complex, including the recent observation by Farber [13], that the Radó complex is highly symmetric, in the sense that local injective automorphisms extend to global simplicial automorphisms.

2. The curve complex and rigid expansions.

Let S be an orientable surface. A *curve* is a topological embedding of the unit circle into S. We often abuse notation and call "curve" the embedding, its image on S or its isotopy class. The context makes clear which use we mean.

A curve is *essential* if it is neither null-homotopic nor homotopic to the boundary curve of a neighbourhood of a puncture.

The *(geometric) intersection number* of two (isotopy classes of) curves α and β is defined as follows:

$$i(\alpha, \beta) := \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

The *curve complex of S*, denoted by $\mathcal{C}(S)$ is the abstract simplicial complex whose vertices are the isotopy classes of essential curves on S, and the set $\{\alpha_0, \ldots, \alpha_k\}$ is a k-simplex if for all $i, j \leq k$ we have that $i(\alpha_i, \alpha_j) = 0$. Note that $\mathcal{C}(S)$ is a flag complex. The 1-skeleton of $\mathcal{C}(S)$ is called the *curve graph of S*.

A result due to Ivanov [21], Korkmaz [24] and Luo [25], asserts that, excluding finitely-many well-understood cases, the action of the extended mapping class group is effective, and the automorphisms of the curve complexes are all geometric, i.e. they are induced by a homeomorphism. This means that the group $Aut(C(S))$ of simplicial automorphisms of $C(S)$ is isomorphic to the extended mapping class group MCG$^*(S)$.

In [1] and [2], the concept of a subgraph of the curve graph being rigid was introduced. Later, in [16], this definition was generalised to the context of simplicial

graphs. The following definition is an obvious generalisation to the context of abstract simplicial complexes.

Definition 2.1. Let Γ be an abstract simplicial complex and $Y < \Gamma$ be a vertex-induced subcomplex. We say Y is *rigid* if every locally injective map $Y \to \Gamma$ can be extended to an automorphism of Γ.

Recall that a map $f : Y \to \Gamma$ is *locally injective* if $f|_{star(v)}$ is injective for all $v \in V(Y)$, where Γ is a simplicial complex, Y is a vertex-induced subcomplex, and $star(v)$ is the subcomplex induced by all the simplices containing v as a vertex.

In general, if $Y < X < \Gamma$ are vertex-induced subcomplexes and Y is rigid, there is no reason why X should also be rigid. In [1], a method for enlarging subgraphs was developed to solve this issue, and in [16] it was also generalised to simplicial graphs. Here we present the obvious analogue to abstract simplicial complexes:

Definition 2.2. Let Γ be an abstract simplicial complex and Y be a vertex-induced subcomplex of Γ. Denoting by $V(\Gamma)$ the set of vertices of Γ, and by $adj(v)$ the set of vertices of Γ that span a 1-simplex with v, we have the following definitions:

(i) Let $A \subset V(\Gamma)$ and $v \in V(\Gamma)$. We say v is *uniquely determined by* A, denoted by $a = \langle A \rangle$ if they satisfy that

$$\{v\} = \bigcap_{w \in A} adj(w).$$

(ii) The *zeroth rigid expansion of* Y, denoted by Y^0 is defined as Y.

(iii) If α is a countable ordinal that is the successor of β, then the α-*th rigid expansion of* Y, denoted by Y^α is the vertex-induced subcomplex whose vertices are

$$V(Y^\alpha) := V(Y^\beta) \cup \{v \in V(\Gamma) : \exists A \subset V(Y^\beta) \text{ with } v = \langle A \rangle\}.$$

(iv) If α is a countable limit ordinal, then the α-*th rigid expansion of* Y, denoted by Y^α is the vertex-induced subcomplex whose vertices are

$$V(Y^\alpha) := \bigcup_{\beta < \alpha} V(Y^\beta).$$

Using the same arguments as in Section 1 of [16] and the same proof as Proposition 3.5 in [2] we can easily verify that rigid expansions satisfy the following result.

Proposition 2.3. *Let Γ be an abstract simplicial complex, $Y < \Gamma$ be a rigid subcomplex, and α be a countable ordinal. Then Y^α is a rigid subcomplex of Γ.*

Now, let $Y < \Gamma$ be a vertex-induced subcomplex, we say Y is a *seed subcomplex* if there exists a countable ordinal α such that $Y^\alpha = \Gamma$. In [17] the third author proved the following result concerning the curve graph of a closed surface.

Theorem 2.4 (3.1 in [16])**.** *Let S be an orientable closed surface of genus at least 3. Then there exists a finite seed subcomplex $Y < \mathcal{C}(S)$.*

To construct the subcomplex Y exhibited in [17] we need the following definitions.

Let $k \in \mathbb{Z}^+$ and $C = \{\gamma_0, \ldots, \gamma_k\}$ be an ordered set of $k + 1$ curves in S. We call C a *chain* of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \le i \le k - 1$, and γ_i is disjoint from γ_j for $|i - j| > 1$. On the other hand, C is called a *closed chain* of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \le i \le k$ modulo $k + 1$, and γ_i is disjoint from γ_j for $|i - j| > 1$ (modulo $k + 1$); a closed chain has maximal length if it has length $2g + 2$. A *subchain* is an ordered subset of either a chain or a closed chain which is itself a chain, and its length is its cardinality.

Recalling that $k \geq 1$, note that if C is a chain (or a subchain) of odd length, a closed regular neighbourhood $N(C)$ has two boundary components; we call these curves the bounding pair associated to C.

Let $C = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ be the closed chain in S depicted in Figure 1. Observe it is a closed chain of maximal length, and given any other maximal closed chain C' there exists an element of $\mathrm{MCG}^*(S)$ that maps C' to C (see [12]). Then, we define the set $B(C)$ as the union of the bounding pairs associated to the subchains of odd length of C. The seed subcomplex Y of Theorem 2.4 is $C \cup B$.
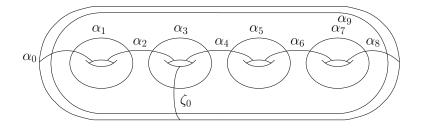

FIGURE 1. The set $C = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ and an example of a curve ζ_0.

An immediate result coming from the definition of rigid expansions is that if $Y < X < \Gamma$ are vertex-induced subcomplexes, then for every countable ordinal α, we have that $Y^\alpha < X^\alpha$. For the curve complex of an orientable closed surface S of genus at least 3, this implies that if Z is a vertex-induced subcomplex of $\mathcal{C}(S)$ such that there exists a countable ordinal α that satisfies $C \cup B < Z^\alpha$, then Z is a seed subcomplex of $\mathcal{C}(S)$. As such, we have the following result.

Lemma 2.5. *Let S be an orientable closed surface S of genus at least 3, $C = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ be a closed chain of maximal length, ζ_0 be a curve on S disjoint and different from all α_i with i even, ζ_1 be a curve on S disjoint and different from all α_i with i odd. Then the subcomplex of $\mathcal{C}(S)$ induced by the vertices $C \cup \{\zeta_0, \zeta_1\}$ is a seed subcomplex.*

Proof. Given C as in the hypotheses, we define $C_0 = \{\alpha_i \in C : i \text{ is even}\}$ and $C_1 = C - C_0$. Let $\Sigma_0^\pm$ be the two connected components of $S - C_0$, and $\Sigma_1^\pm$ be the two connected components of $S - C_1$. Since for $i = 1, 2$, ζ_i is disjoint and different from every element in C_i, we have that ζ_i must be contained in either Σ_i^+ or Σ_i^-; without loss of generality we can assume it is contained in Σ_i^+.

Now, if $C' = \{\alpha_j, \ldots, \alpha_{j+2k}\}$ is a subchain of C of odd length with j of the same parity as i, and we denote by β^+ and β^- the curves that constitute the bounding pair associated to C', then by the same reasoning as above we have that (up to relabeling) $\beta^\pm$ is contained in $\Sigma_i^\pm$.

Thus, there exists a subchain C' of C such that β^+ intersects ζ_i. Then we have the following:

$$\beta^- = \langle \{\zeta_i\} \cup C - \{\alpha_{j-1}, \alpha_{j+2k+1}\} \rangle \in Z^1,$$
$$\beta^+ = \langle \{\beta^-\} \cup C - \{\alpha_{j-1}, \alpha_{j+2k+1}\} \rangle \in Z^2.$$

Repeat the process for every β^+ that intersects ζ_i, and then substitute ζ_i by all the $\beta^\pm$ obtained this way. Since for every β^+ and $(\beta^+)'$, contained in Σ_i^+ either they intersect or there exists a $(\beta^+)''$ that intersects both, we have that all curves in B that are disjoint and different from C_i are contained in Z^5. Thus B is contained in Z^5 which implies that $C \cup B$ is contained in Z^5, and by the argument above we have that Z is a seed subcomplex. $\square$

2.1. **Translation to combinatorial terms.** Now that we have proved that $C \cup \{\zeta_0, \zeta_1\}$ is a seed subcomplex of $\mathcal{C}(S)$, we dedicate this subsection to describe it in combinatorial terms. To do this, we need to describe geometric intersection 0 and 1 in combinatorial terms.

For the rest of this subsection, let Γ be an abstract simplicial complex and S be a closed surface of genus $g \geq 3$.

We say two vertices $v, w \in V(\Gamma)$ have *intersection* 0 if $\{x, y\}$ is a simplex in Γ.

To encode intersection 1 combinatorially, we need to encode several other concepts first.

Let $v, w \in V(\Gamma)$ be two different vertices, and σ be a top-dimensional simplex in Γ with $v \in \sigma$. We say *v can be exchanged with w with respect to σ* if $(\sigma \backslash \{v\}) \cup \{w\}$ is a top-dimensional simplex in Γ. In the context of $\mathcal{C}(S)$, this means we have a pants decomposition and substitute one of its curves with another with the resulting set being still a pants decomposition. In particular, this means that the two curves are contained in a subsurface of complexity one (i.e. in a one-holed torus or a four-holed sphere) whose boundary curves are elements of the pants decomposition.

Let P be a pants decomposition of S, and $\gamma_1, \gamma_2 \in P$ be two different curves. We say *γ_1 and γ_2 are adjacent with respect to P* if there exists a closed subsurface Σ whose interior is isomorphic to a thrice-punctured sphere, such that Σ has γ_1 and γ_2 as two of its boundary curves. The *adjacency graph of P*, denoted $\mathcal{A}(P)$, is the simplicial graph whose vertex set is P, and two vertices span an edge if the curves they represent are adjacent with respect to P. This graph was first introduced by Behrstock and Margalit in [4] and has been extensively used in works related to the combinatorial rigidity of the curve graph. Note that in the proof of Lemma 7 in [27], Shackleton implicitly gives a characterisation of being adjacent: γ_1 and γ_2 are adjacent with respect to P if and only if there exists curves δ_1 and δ_2 such that $i(\delta_1, \delta_2) \neq 0$ and for $i = 1, 2$ we have that $(P \backslash \{\gamma_i\}) \cup \{\delta_i\}$ is a pants decomposition. Based on this characterisation we give the following definitions.

Let $v_1, v_2 \in V(\Gamma)$ be two different vertices, and σ be a top-dimensional simplex in Γ with $v_1, v_2 \in \sigma$. We say *v_1 and v_2 are adjacent inside σ* if there exist $w_1, w_2 \in V(\Gamma)$ such that $\{w_1, w_2\}$ is not a simplex in Γ, and for $i = 1, 2$ we have that v_i can be exchanged with w_i with respect to σ.

Following the same analogy as above, given σ a top-dimensional simplex in Γ the *adjacency graph of σ*, denoted $\mathcal{A}(\sigma)$ is the simplicial graph whose vertex set is σ, and two vertices span an edge if they are adjacent inside σ.

Recall that a curve α in S is called *separating* if $S \backslash \alpha$ is not connected, and is called *non-separating* otherwise.

Now, let P be a pants decomposition, and $\gamma_1, \gamma_2 \in P$. It is clear that, since S is closed, a vertex in $\mathcal{A}(P)$ is a cut vertex if and only if it correspond to a separating curve of S. Thus, since S is a closed surface with genus at least three we have the following: if γ_1 is a leaf in $\mathcal{A}(P)$ and γ_2 is the unique vertex in $\mathcal{A}(P)$ adjacent to γ_1 with respect to P, then γ_2 is a separating curve that bounds a one-holed torus and γ_1 is a non-separating curve contained in said torus.

Analogously, let σ be a top-dimensional simplex, and $v_1, v_2 \in \sigma$. We say *v_2 separates a torus containing v_1 with respect to σ* if v_1 is a leaf in $\mathcal{A}(\sigma)$ and v_2 is the unique vertex in $\mathcal{A}(\sigma)$ adjacent to v_1 inside σ.

Then, to encode geometric intersection 1, we use Ivanov's characterisation of it (Lemma 1 in [21]) and use the terminology defined above.

Let v_1 and v_2 be two different vertices of Γ. We say they have *intersection* 1 if there exists different vertices v_3, v_4 and v_5, and a top-dimensional simplex σ such that:

    (i) $\{v_i, v_j\}$ is a simplex if and only if $\{i, j\}$ is an edge in Figure 2.

(ii) $v_1, v_4 \in \sigma$.

(iii) v_4 separates a torus containing v_1 with respect to σ.

(iv) v_1 can be exchanged with v_2 with respect to σ.
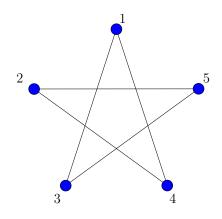

FIGURE 2. The graph used for the definition of intersection 1.

Thus, we can define a closed chain as follows: the set $\{v_0, \ldots, v_k\}$ is a *closed chain of length* $k+1$ if v_i and v_j have intersection 0 whenever $|i - j| > 1$, and if v_i and v_j have intersection 1 whenever $|i - j| = 1$.

Finally, to encode the set used in Lemma 2.5 we define it as $Y = C \cup \{w_0, w_1\}$, where $C = \{v_0, \ldots, v_{2g+1}\}$ is a closed chain, and w_0 and w_1 are two vertices such that $\{w_i, v_j\}$ is a simplex if and only if i and j have the same parity.

3. THE MULTIPARAMETRIC MODEL FOR RANDOM SIMPLICIAL COMPLEXES

In this section we recall the definition and basic results of the multiparametric model for random simplicial complexes due to Costa and Farber [8], [9], [10].

Let g be a natural number, $n = 2^g$ and Δ_n be the n-dimensional simplex.

We want to consider r-dimensional simplicial complexes, where $r = 3g - 3 \leq n$.

Given a simplicial subcomplex $Y \subset \Delta_n$, we denote by $f_i(Y)$ the number of i-dimensional simplices.

Also, given σ a simplex in Δ_n, recall that σ is an *external face* of Y if $\partial \sigma \subset Y$ but σ is not completely contained in Y. We denote by $E(Y)$ the set of exterior faces of Y and by $F(Y)$ the set of faces of Y.

We recall the definition of the multiparametric model for random simplicial complexes of Costa and Farber.

Definition 3.1. Let r be a natural number between 0 and n. Let $\mathfrak{P} = (p_0, p_1, \ldots, p_n)$ be an $n + 1$-tuple of probabilities, that is, real numbers in $[0, 1]$. The space $\Omega_n^{r, \mathfrak{P}}$ is the measure space where the objects are r-dimensional simplicial subcomplexes of the n-dimensional simplex Δ_n, and where the probability function is given by

$$\mathbb{P}_{\mathfrak{P}}(Y) = \prod_{\sigma \in F(Y)} p_\sigma \prod_{\sigma \in E(Y)} q_\sigma.$$

Here, $p_\sigma = p_i$ if σ is an i-dimensional face, $q_\sigma = q_i$, and $q_i = 1 - p_i$.

The following result is proved in [8] as Lemma 2.3.

Theorem 3.2. *Let $A \subset B \subset \Delta_n$ be simplicial subcomplexes such that the boundary of any external face of B of dimension $\leq r$ is contained in A. Then,*

$$\mathbb{P}_{\mathfrak{P}}\{A \subset Y \subset B\} = \prod_{\sigma \in F(A)} p_\sigma \prod_{\sigma \in E(B)} q_\sigma.$$

The last result characterizes the probability measure, in the sense that any other probability measure with the property that if $\mathbf{P}$ is some probability measure for which, given an arbitrary subcomplex $A \subset Y \subset \Delta_n$, the equality

$$\mathbf{P}(A \subset \Delta_n) = p_i^{f_i(A)}$$

implies that $\mathbf{P}$ is the measure $\mathbb{P}$.

The notion of critical dimension was introduced in [9]. The idea is that the probability vector $\mathfrak{P}$ of the random simplicial complex lies between affine subspaces $\mathfrak{D}_i$, for $i \in \{0, n\}$ where the homology in degree i has rank significatively bigger than in any other degree.

Consider the linear functions $\psi_k : \mathbb{R}^{r+1} \to \mathbb{R}$ defined by

$$\psi_k(\alpha) = \psi_k(\alpha_0, \dots, \alpha_r) = \sum_{i=0}^{r} \binom{k}{i} \alpha_i,$$

with the conventions $\binom{k}{i} = 0$ for $i < k$ and $\binom{0}{0} = 1$. Since $\binom{k}{i} < \binom{k+1}{i}$, one has

$$\psi_0(\alpha) \le \psi_1(\alpha) \le \dots \le \psi_r(\alpha).$$

Definition 3.3. The domain $\mathfrak{D}_k$ is the affine subspace defined by the inequalities

$$\{\alpha \in \mathbb{R}^{n+1} \mid \psi_k(\alpha) < 1 < \psi_{k+1}(\alpha)\}.$$

Now, the probabilistic model for the curve complex of the orientable genus g closed surface is the multiparametric random simplicial complex, in the sense of Costa and Farber with the following specific parameters:

Let $r = 3g - 3$, consider the Costa and Farber multiparametric model for random r-dimensional simplicial complexes, and assume that the probability vector $\mathfrak{P}$ is of the form $\mathfrak{P} = (n^{-\alpha_0}, n^{-\alpha_1}, n^{-\alpha_2}, \dots, n^{-\alpha_n})$ for α satisfying Conditions 1.1 and 1.2, which we state additionally here for the sake of completness:

(i) For the simplicial complex to be hyperbolic, connected and non-empty: $\alpha_0 + 3\alpha_1 + 2\alpha_2 < 1$ with $\alpha_0 + \alpha_1 < 1$. (Theorem 5 in page 449 of [9]).
(ii) For the simplicial complex to have critical dimension $k = 4g + 2$ the parameters α_* satisfy the condition of belonging to the domain $\mathfrak{D}_k$.

We state now the technical condition.

Condition 3.4. Assume $\alpha_1 < \frac{1}{g^2}$, $\alpha_0 < \frac{g^2 - 1}{g^2}$, and $\alpha_2 > \frac{1 - 2g^2}{g^2}$.

Condition 3.4 is a specific realization of hyperbolicity, non emptyness and connectedness of a random simplicial complex for parameters of fixed decrease order in funcion of g which appear in our probabilistic estimates.

4. Geometric estimates.

We now give estimates for the geometric condition of a closed chain length $2g + 2$ and a pair of external alternating vertices as in the situation described in Lemma 2.5.

Let Δ be the $(2g + 3)$-simplex with vertex set $\{v_0, \dots, v_{2g+1}\} \cup \{w_0, w_1\}$. We define A as the subgraph of Δ that has the same vertex set as Δ, and whose edges are defined as follows:

(i) $\{v_i, v_j\}$ is an edge in A if and only if $|i - j| > 1$ modulo $2g + 2$.
(ii) $\{w_i, v_j\}$ is an edge in A if and only if i and j have the same parity.
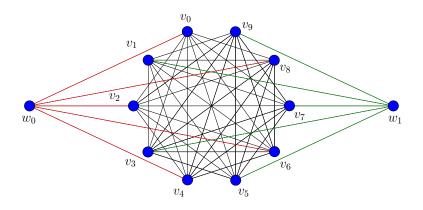
FIGURE 3. The subgraph A for the case $g = 4$.

See Figure 3 for an example with $g = 4$.

As it can be deduced from Subsection 2.1, any seed graph given by Lemma 2.5 has A as a subgraph of its subjacent graph, but not every subgraph isomorphic to A comes from a seed graph given by Lemma 2.5.

Now, if Y is the subjacent subgraph of a seed graph given by Lemma 2.5, it may happen that Y had more edges than A. Since the vertices $v_0, \ldots, v_{2g+1}$ represent a closed chain in the surface, if Y has more edges than A, they have to be edges of the form $\{w_0, v_{2k+1}\}$ for some k, $\{w_1, v_{2k}\}$ for some k, or $\{w_0, w_1\}$. Since we can create examples of Y with edges of these forms for each value of k from $k = 0$ to $k = g$, we have to consider the possibility that our random complex may have any of these edges too.

For this reason, we define B to be the flag complex induced by the subgraph of Δ with vertex set $\{v_0, \ldots, v_{2g+1}\} \cup \{w_0, w_1\}$, and whose edges are the following:

   (i) $\{v_i, v_j\}$ is an edge in B if and only if $|i - j| > 1$ modulo $2g + 2$.
   (ii) $\{w_i, v_j\}$ is an edge in B for all j.
   (iii) $\{w_0, w_1\}$.

Thus, if Y is the subjacent subgraph of a seed graph given by Lemma 2.5, we have that $A \subset Y \subset B$. Now, we compute the probability of this event happening.

Lemma 4.1. *Let $Y \in \Omega_n^{r,\mathfrak{P}}$ be a random simplicial complex in the multiparametric model. Then, the probability of the event described by A being contained in Y and Y being contained in B is given by*

$$p_0^{2g+4} \cdot p_1^{2g^2+3g+1}(1 - p_0)^0(1 - p_1)^{2g+2}.$$

Proof. Recall from Theorem 3.2, that the probability of a subcomplex Y to appear in the sequence $A \subset Y \subset B$ is given by

$$\mathbb{P}_{\mathfrak{P}}\{A \subset Y \subset B\} = \prod_{\sigma \in F(A)} p_\sigma \prod_{\sigma \in E(B)} q_\sigma,$$

Where $F(A)$ denote the faces of A and $E(B)$ denote the external faces of B.

This is the expression

$$\prod_{i=0}^{r} p_i^{f_i(A)} \cdot \prod_{i=0}^{r} q_i^{e_i(B)},$$

where r denotes the dimension of B. In our case, $r = g + 2$, $f_0 = 2g + 4$, $e_0 = 0$, and we use a counting argument to determine f_1 for the diagram A, and e_1 for the flag complex B. Also we argue why $e_i = 0$ for $i \geq 2$.

Notice that in A there are $2g^2 + g - 1$ edges spanned between the vertices $v_0, \ldots, v_{2g+1}$ (these are the diagonals of a $(2g + 2)$-gon). Also, there are $g + 1$ edges joining w_0 to the vertices v_i with even i, and $g + 1$ joining w_1 to the vertices v_i with odd i. Thus $f_1(A) = (2g^2 + g - 1) + (g + 1) + (g + 1) = 2g^2 + 3g + 1$.

For computing $e_1(B)$ notice that the exterior edges of B are exactly those which are in Δ but not B. By definition of B, those edges are exactly the sides of the $(2g + 2)$-gon with vertices $v_0, \ldots, v_{2g+1}$. This implies $e_1 = 2g + 2$.

Since B is a flag complex, if there is a complete subgraph with at least 3 vertices, then the corresponding simplex is contained in B. As such, B does not have any exterior faces of dimension greater or equal to 2. Hence, $e_i = 0$ for all $i \geq 2$.

Substituting in the formula $\prod_{i=0}^{r} p_i^{f_i} \cdot \prod_{i=0}^{r} q_i^{e_i(B)}$ gives the desired expression. $\qquad \square$

5. Proof of Theorem 1

We define now the random variable which is the main object of study.

Definition 5.1. Let CH be the discrete random variable defined on the probability space $\Omega_n^{r, \mathfrak{P}}$ which is one in the case of an appearance of a diagram A inside a $(2g+4)$-complete graph as subsimplicial complex.

We now give the proof of Theorem 1.3.

Proof. The proof consists of two arguments

- Prove using the estimates of Section 4 that the expectation of CH tends to infinity as g tends to infinity.
- Prove using a second moment argument that the random variable is asymptotically almost surely positive.

Recall from [9], Theorem 1 in page 444, that the expectation of the random variable which counts embeddings of a complete graph in $4g + 2$ vertices is:

$$\binom{n}{4g + 2}(4g + 2)! \prod_{i=0}^{3g-3} p_i^{\binom{4g+1}{i}}.$$

We claim that for $g > 1$, $n = 2^g$ and the parameters $p_i = \frac{1}{n^{\alpha_i}}$ satisfying the critical dimension condition, 1.2, the logarithm in base $n = 2^g$, this expression is greater than 6.

The given expression is greater than

$$\frac{2^g}{4g + 2}^{4g+2}(4 + 2)! 2^{\psi_k(\alpha)} > \frac{1}{2}(4g + 1!)(2^{4g^2 + 2g}).$$

which has logarithm base 2^g greater than 7 if $g \geq 2$.

Thus, we have the following estimate for the expectation of the random variable CH

$$(5.2) \quad \mathbb{E}[CH(Y)] \approx \binom{n}{4g + 2}(4g + 2)!$$
$$\prod_{i=0}^{3g-3} p_i^{\binom{4g+1}{i}} p_0^{2g+4} \cdot p_1^{2g^2+3g+1}(1 - p_0)^0(1 - p_1)^{2g+2}..$$

which tends to ∞.

Let us verify this fact.

Consider $\log_n \mathbb{E}[CH(Y)]$. First notice that due to the fact that the critical dimension equals $4g + 2$, Approximation 5.2 becomes then

$$(5.3) \quad \log_n(\mathbb{E}[CH(Y)]) \geq \log_n \left(\frac{n}{4g+2}^{4g+2} \right)$$

$$- 1 - \alpha_0(4g+2) - \alpha_1(2g^2 + 3g + 1) + \log_n \left(1 - \frac{1}{n^{\alpha_1}}^{(2g+2)} \right)$$

Under the hypothesis of critical dimension, we have verified that the first summand is at least 7. The technical condition 3.4 permits to bound this expression from below by

$$7 - 5 - \frac{2}{g} - 2 + \frac{3}{g} + \frac{1}{g^2} + \log_n \left(1 - \frac{1}{n^{\alpha_1}}^{(2g+2)} \right).$$

We handle now the last summand. Recall that from the Taylor expansion around zero, $\log(1 - z) \approx \sum_{i=0}^{\infty} -\frac{z^i}{i}$, and thus the last summand has the assymptotic expansion

$$(2g + 2) \left[\sum_{i=0}^{\infty} 2^{-g\alpha_1 i} \right].$$

From the technical condition 3.4, we have that $\alpha_1 < \frac{1}{g^2}$, and thus the summand $(4g + 5)[(\sum_{i=0}^{\infty} 2^{-g\alpha_1 i})]$ Is bounded below by the espression

$$(2g + 2) \sum_{i=1}^{\infty} 2^{-g},$$

which diverges to ∞.

We now finish the proof recalling that the variance of the random variable CH is the same as the variance of the random variable which counts $(4g+1)$- dimensional faces in a random simplicial complex, denoted by f_{4g+1} in [10]. It is proved there that there exists a constant C such that

$$\frac{\mathbb{V}(f_{4g+1})}{\mathbb{E}(f_{4g+1})^2} < Cn^{-\delta_{4g+1}(\alpha_*)/2},$$

where $\delta_{4g+1}(\alpha_*)$ is the minimum between $\tau_0(\alpha_*)$ and $\tau_{4g+1}(\alpha_*)$. It follows by the work of Costa and Farber there that it tends to 0 as g tends to infinity under condition 1.2. $\qquad \square$

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