Loxodromic elements in big mapping class groups via the Hooper–Thurston–Veech construction

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Let S be an infinite-type surface and $p \in S$. We show that the Thurston–Veech construction for pseudo-Anosov elements, adapted for infinite-type surfaces, produces infinitely many loxodromic elements for the action of $\mathrm{Mod}(S;p)$ on the loop graph $L(S;p)$ that do not leave any finite-type subsurface $S' \subset S$ invariant. Moreover, in the language of Bavard and Walker, Thurston–Veech's construction produces loxodromic elements of any weight. As a consequence of Bavard and Walker's work, any subgroup of $\mathrm{Mod}(S;p)$ containing two "Thurston–Veech loxodromics" of different weight has an infinite-dimensional space of non-trivial quasimorphisms.

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1 Introduction

Let S be an orientable infinite-type surface, $p \in S$ a marked point (through this text we think of p as either a marked point or a puncture) and $\mathrm{Mod}(S;p)$ the quotient of $\mathrm{Homeo}^+(S;p)$ by isotopies which fix p for all times. This group is related to the (big) mapping class group of S, $\mathrm{Mod}(S)$, which is the group defined as $\mathrm{Homeo}^+(S)$ modulo isotopy, via Birman's exact sequence:

$$1 \longrightarrow \pi_1(S,p) \xrightarrow{\textit{Push}} \mathrm{Mod}(S;p) \xrightarrow{\textit{Forget}} \mathrm{Mod}(S) \longrightarrow 1.$$

The group $\mathrm{Mod}(S;p)$ acts by isometries on the (Gromov-hyperbolic, infinite-diameter) loop graph $L(S;p)$. This is the graph whose vertices are isotopy classes of loops based at p and where adjacency is defined by disjointness (modulo isotopy). This action has been extensively studied by Bavard and Walker in [**?**] and [**?**]. One of the main objectives of this manuscript is the study of loxodromic elements for this action. For this, we use the description of the Gromov boundary of $L(S;p)$ in terms of *long rays* (see Section 2.3 for a precise definiton) made by Bavard and Walker in [*Ibid*]. Roughly speaking, each point of the boundary of $L(S;p)$ is identified with cliques in another graph (the completed ray graph, which contains $L(S;p)$ as a subgraph, see Section 2.3 for a precise definition) and the vertex set of each of these cliques is in bijection with a set of long rays. One of the main contributions of Bavard and Walker is to show that points in the boundary of $L(S;p)$ fixed by a loxodromic element f correspond to cliques with the same (finite) number of vertices. This numbe r is called *the weight* of f. For example, Bavard and Walker show that if f is supported on a compact subsurface $\Sigma \subset S$ containing p and its restriction to Σ defines a pseudo-Anosov class having a k-prong at p, then f is a loxodromic element of weight k and the vertices of the aforementioned cliques correspond to the separatrices of the stable and unstable measured foliations of f at p. On the other hand, up to date, the only known example of a loxodromic element which is not supported on a compact subsurface was defined by Bavard in [**?**]. In her example S is the sphere minus a Cantor set, $\mathrm{Mod}(S;p) = \mathrm{Mod}(S)$, h acts non-trivially on the space of ends of S, does not preserve any finite type subsurface and has weight 1.

In [**?**], the authors remark that *it would be interesting to construct examples of loxodromic elements of weight greater than 1 which do not preserve any finite type subsurface (up to isotopy).*

The purpose of this article is to show that such examples can be obtained by adapting the Thurston–Veech construction for pseudo-Anosov elements (see Thurston [**?**], Veech [**?**] or Farb and Margalit [**?**, Theorem 14.1]) to the context of infinite-type surfaces. The construction we present is based on ideas of Hooper [**?**], Thurston and Veech, hence we call it the Hooper–Thurston–Veech construction. Roughly speaking, we show that if one takes as input an appropiate pair of multicurves α, β whose union fills S, then the subgroup of $\mathrm{Mod}(S,p)$ generated by the (right) multitwists T_α and T_β contains infinitely many loxodromic elements. To state our first result properly, recall that given two multicurves $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ in S in minimal position, the *configuration graph* $\mathcal{G}(\alpha \cup \beta)$ of α and β is the graph whose vertex set is $I \cup J$, and there is an edge between $i \in I$ and $j \in J$ for every point of intersection between α_i and β_j.

Theorem 1.1 *Let S be an orientable infinite-type surface, $p \in S$ a marked point and $m \in \mathbb{N}$. Let $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ be multicurves in minimal position whose union fills S and such that:*

(1) *the configuration graph $\mathcal{G}(\alpha \cup \beta)$ is of finite valence, that is, the degree of the vertices of $\mathcal{G}(\alpha \cup \beta)$ is uniformly bounded,*

(2) *for some fixed $N \in \mathbb{N}$, every connected component D of $S \setminus \alpha \cup \beta$ is a polygon or a once-punctured polygon[1] with at most N sides and*

[1]The boundary of any connected component of $S \setminus \alpha \cup \beta$ is formed by arcs contained in the curves forming $\alpha \cup \beta$ and hence we can think of them as (topological) polygons.

(3) *the connected component of* $S \setminus \alpha \cup \beta$ *containing* p *is a* $2m$*-sided polygon.*

If T_α, $T_\beta \in \mathrm{Mod}(S;p)$ *are the (right) multitwists w.r.t.* α *and* β *respectively then any* $f \in \mathrm{Mod}(S;p)$ *in the positive semigroup generated by* T_α *and* T_β^{-1} *given by a word in which both generators appear is a loxodromic element of weight* m *for the action of* $\mathrm{Mod}(S;p)$ *on the loop graph* $L(S;p)$.

Remark 1.2 During the 2019 AIM-Workshop on Surfaces of Infinite Type we learned that Abbott, Miller and Patel also have a construction of loxodromic mapping classes whose support is an infinite type surface [**?**]. Their examples are obtained via a composition of handle shifts and are contained in the complement of the closure of the subgroup of $\mathrm{Mod}(S,p)$ defined by homeomorphisms with compact support. In contrast, the loxodromic elements given by Theorem 1.1 are limits of compactly supported mapping classes.

As we explain in Section 2.3, the weight of a loxodromic element $f \in \mathrm{Mod}(S;p)$ is defined by Bavard and Walker using a precise description of the (Gromov) boundary $\partial L(S;p)$ of the loop graph. For finite-type surfaces, if f is a pseudo-Anosov having a singularity at p, this number corresponds to the number of separatrices based at p of an f-invariant transverse measured foliation. This quantity is relevant because, as shown in [**?**] and using the language of Bestvina and Fujiwara [**?**], loxodromic elements with different weights are independent and anti-aligned. This has several consequences, for example the work of Bavard and Walker [**?**] immediately gives the following:

Corollary 1.3 *Let* f, g *be two loxodromic elements in* $\mathrm{Mod}(S;p)$ *as in Theorem 1.1 and suppose that their weights are* $m \neq m'$. *Then any subgroup of* $\mathrm{Mod}(S;p)$ *containing them has an infinite-dimensional space of non-trivial quasimorphisms.*

Recent work by Rasmussen [**?**] implies that the mapping classes given by Theorem 1.1 are not WWPD in the language of Bestvina, Bromberg and Fujiwara [**?**].

About the proof of Theorem 1.1. As in the case of Thurston's work, our proof relies on the existence of a flat surface structure M on S, having a conical singularity at p, for which the Dehn twists T_α and T_β are affine automorphisms. In the case of finite-type surfaces, this structure is unique (up to scaling) and its existence is guaranteed by the Perron-Frobenius theorem. For infinite-type surfaces the presence of such a flat surface structure is guaranteed once one can find a positive eigenfunction of the adjacency operator on the (infinite bipartite) configuration graph $\mathcal{G}(\alpha \cup \beta)$. The spectral theory of infinite graphs in this context secures the existence of uncountably many flat surface structures (which are not related by scaling) on which the Dehn twists T_α and T_β are affine automorphisms. The main difficulty we encounter is that the description of the Gromov boundary $\partial L(S;p)$ needed to certify that f is a loxodromic element depends on a hyperbolic structure on S which, a priori, is not quasi-isometric to any of the aforementioned flat surface structures. To overcome this difficulty we propose arguments which are mostly of topological nature.

We strongly believe that Theorem 1.1 does not describe all possible loxodromics living in the group generated by T_α and T_β.

Question 1.4 *Let α and β be as in Theorem 1.1 and consider the group generated by T_α and T_β. Is it true that an element of this group acts loxodromically on $L(S;p)$ if and only if it is not conjugated to a generator?*

Constructing explicit examples. We spend a considerable part of this text in the proof of the next result, which guarantees that Theorem 1.1 is not vacuous.

Theorem 1.5 *Let S be an infinite-type surface, $p \in S$ a marked point and $m \in \mathbb{N}$. Then there exist two multicurves α and β whose union fills S and such that:*

(1) *the configuration graph $\mathcal{G}(\alpha \cup \beta)$ is of finite valence,*
(2) *every connected component of $S \setminus \alpha \cup \beta$ which does not contain the point p is a polygon or a once-punctured polygon with at most $\max\{8, m\}$ sides, and*
(3) *p is contained in a connected component of $S \setminus \alpha \cup \beta$ whose boundary is a $2m$-sided polygon.*

A crucial part on the proof of this result is to find, for *any* infinite-type surface S, a simple way to portray S. We call this a topological normal form. Once this is achieved, we give a recipe to construct the curves α and β explicitly.

Phenomena unique to surfaces of infinite type. We find phenomena proper to big mapping class groups. Recall that the *Loch Ness monster surface* is the only surface (up to homeomorphism) which has infinite genus and only one end.

Corollary 1.6 *Let S be the Loch Ness monster and consider the action of $\mathrm{Mod}(S;p)$ on the loop graph. Then there exist a sequence of loxodromic elements (f_n) in $\mathrm{Mod}(S;p)$ which converge in the compact-open topology to a non-trivial elliptic element.*

Theorem 1.7 *There exists a family of translation surface structures $\{M_\lambda\}_{\lambda\in[2,+\infty]}$ on a Loch Ness monster S and $f \in \mathrm{Mod}(S)$ such that:*

- *For every $k \in \mathbb{N}$, f^k does not fix any isotopy class of essential simple closed curve in S.*
- *If $\tau_\lambda : \mathrm{Aff}(M_\lambda) \hookrightarrow \mathrm{Mod}(S)$ is the natural map sending an affine homeomorphism to is mapping class, then $D\tau_\lambda^{-1}(f) \in \mathrm{PSL}(2,\mathbb{R})$ is parabolic if $\lambda = 2$ and hyperbolic for every $\lambda > 2$.*

Recall that for finite-type surfaces S a class $f \in \mathrm{Mod}(S)$ such that for every $k \in \mathbb{N}$, f^k does not fix any isotopy class of essential simple closed curve in S is necessarily pseudo-Anosov. In particular, the derivative of any affine representative $\phi \in f$ is hyperbolic.

We want to stress that for many infinite-type surfaces $\mathrm{Mod}(S)$ does not admit an action on a metric space with unbounded orbits. For a more detailed discussion on this fact and the large scale geometry of big mapping class groups we refer the reader to Durham, Fanoni and Vlamis [**?**], Mann and Rafi [**?**], and references therein.

Outline. Section 2 is devoted to preliminaries about the loop graph, its boundary and infinite-type flat surfaces. In Section 3 we present the Hooper–Thurston–Veech construction. The construction here presented is a particular case of a more general construction built upon work of Hooper by Delecroix and the second author. An early version of this more general construction had mistakes that were pointed out by the first author. In Section 3 we also prove Theorem 1.7. Finally, Section 4 is devoted to the proof of Theorems 1.1, 1.5 and Corollary 1.6 (in this order).

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2 Preliminaries

2.1 Infinite-type surfaces

Any orientable (topological) surface S with empty boundary is determined up to homeomorphism by its genus (possibly infinite) and a pair of nested, totally disconnected, separable, metrizable topological spaces $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$ called the space of ends accumulated by genus and the space of ends of S, respectively. Any such pair of nested topological spaces occurs as the space of ends of some orientable surface, see Richards [**?**]. On the other hand, $S \cup \mathrm{Ends}(S)$ can be endowed with a natural topology which makes the correspoding space compact. This space is called the Freudenthal end-point compactification of S, see Raymond [**?**].

A surface S is of finite (topological) type if its fundamental group is finitely generated. In any other case we say that S is an *infinite-type* surface. All surfaces considered henceforth are orientable.

2.2 Multicurves

Let S be an infinite-type surface. If not explicitly stated otherwise, all curves are simple and closed in what follows. A collection of essential curves l in S is *locally finite* if for every $x \in S$ there exists a neighbourhood U of x which intersects finitely many elements of l. As surfaces are second-countable topological spaces, any locally finite collection of essential curves is countable.

A *multicurve* in S is a locally finite, pairwise disjoint, and pairwise non-isotopic collection of essential curves in S.

Let α be a multicurve in S. We say that α *bounds a subsurface* Σ of S, if the elements of α are exactly all the boundary curves of the closure of Σ in S. Also, we say that Σ is *induced* by α if there exists a subset $\alpha' \subset \alpha$ that bounds Σ and there are no elements of $\alpha \setminus \alpha'$ in the interior of Σ.

A multicurve α in S is of *finite type* if every connected component of $S \setminus \alpha$ is a finite-type surface.

Finite multicurves (that is, formed by a finite number of curves) are not necessarily of finite type. On the other hand, there are infinite multicurves which are not of finite type, *e.g.* the blue ("vertical") curves in the right-hand side of Figure 1.

Let α and β be two multicurves in S. We say that α and β are in *minimal position* if for every $\alpha_i \in \alpha$ and $\beta_j \in \beta$, $|\alpha_i \cap \beta_j|$ realizes the minimal number of (geometric) intersection points between representatives in the (free) isotopy classes of α_i and β_j. For every pair of multicurves one can find representatives in their isotopy classes which are in minimal position.

Let α and β be two multicurves in S in minimal position. We say that α and β *fill* S if every connected component of $S \setminus (\alpha \cup \beta)$ is either a disk or a once-punctured disk.

Remark 2.1 Let α and β be multicurves. Then:

(1) If α and β are of finite type and fill S, then every complementary connected component of $\alpha \cup \beta$ in S is a polygon with finitely many sides. The converse is not true, see the left-hand side of Figure 1.

(2) There are pair of multicurves α and β so that $S \setminus (\alpha \cup \beta)$ has a connected component that is a polygon with infinitely many sides, see the right-hand side of Figure 1.

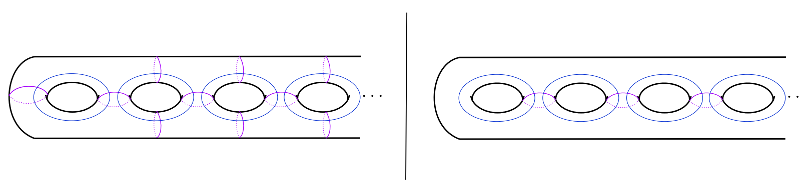


Figure 1:

2.3 The loop graph and its boundary

Bavard and Walker introduced in [?] and [?] the loop graph $L(S;p)$ and proved that it is hyperbolic graph on which $\mathrm{Mod}(S,p)$ acts by isometries. Moreover, they made a precise description of the Gromov boundary of $L(S;p)$ in terms of (hyperbolic) geodesics on the Poincaré disk. We recall the general aspects of their work in what follows. The exposition follows largely [?], [?] and Rasmussen [?].

The loop graph. Let S be an infinite type surface and $p \in S$. In what follows we think of p as a marked point in S. The isotopy class of a topological embedding $\gamma : (0,1) \hookrightarrow S$ is said to be a *loop* if it can be continuously extended to the end-point Freudenthal compactification $S \cup \mathrm{Ends}(S)$ of S with $\gamma(0) = \gamma(1) = p$. On the other hand, if the continuous extension of γ satifies that $\gamma(0) = p$ and $\gamma(1) \in \mathrm{Ends}(P)\backslash\{p\}$ we call it a short ray. The loop graph, denoted by $L(S;p)$, has as vertex set isotopy classes (relative to p) of loops and adjacency is defined by disjointness (modulo isotopy). This graph is hyperbolic w.r.t to the combinatorial distance, see Bavard and Walker [**?**].

The short-ray-and-loop-graph. The vertex set of this graph is the set of isotopy classes of simple short rays and loops based at p. Following [*Ibid.*], we denote it by $R_sL(S;p)$. This graph is quasi-isometric to the loop graph. We use it to show that the loxodromic elements given by Theorem 1.1 have unbouded orbits indeed.

Long rays and the completed ray graph. In order to describe the Gromov boundary of $L(S;p)$ we need to introduce the completed ray graph. From now on we fix a hyperbolic metric μ on S of the first kind[2] for which the marked point p is a cusp. Every short ray or loop has a unique geodesic representative in its isotopy class. We denote by $\pi : \hat{S} \to S$ the infinite cyclic cover of S defined by the (conjugacy class of) cyclic subgroup of $\pi_1(S,\cdot)$ generated by a simple loop around the cusp p and call it *the conical cover of S*. The surface $\hat{S}$ is conformally equivalent to a punctured disc and its unique cusp $\hat{p}$ projects to p. We denote by $\partial\hat{S}$ the Gromov boundary $\partial\hat{S}$ from which $\hat{p}$ has been removed. This cover is useful because for every geodesic representative of a short ray or loop in S there is a unique lift to $\hat{S}$ which is a geodesic with one end in $\hat{p}$ and the other in $\partial\hat{S}$.

A long ray on S is a *simple* bi-infinite geodesic of the form $\pi(\hat{\delta})$, where $\hat{\delta} \subset \hat{S}$ is a geodesic from $\hat{p}$ to $\partial\hat{S}$, which is not a short ray or a loop. By definition, each long ray limits to p at one end and does not limit to any point of Ends(S) on the other end. The vertices of the completed ray graph $\mathcal{R}(S;p)$ are isotopy classes of loops and short rays, and long rays. Two vertices are adjacent if their geodesic representatives in (S,μ) are disjoint. As before, we consider the combinatorial metric on $\mathcal{R}(S;p)$ defined by declaring that each edge has length 1.

Theorem 2.2 *[?] The completed ray graph $\mathcal{R}(S;p)$ is disconnected. There exists a component containing all loops and short rays, which is of infinite diameter and quasi-isometric to the loop graph. All other connected components are (possibly infinite) cliques and each of them is formed exclusively by long rays.*

The component of $\mathcal{R}(S;p)$ containing all loops and short rays is called the *main component* of the completed ray graph. Long rays not contained in the main component are called *high-filling* rays and each clique contains exclusively high-filling rays.

[2]That is, the Fuchsian group appearing in the uniformization $\mathbb{D} \to S$ has as limit set the whole boundary of the Poincaré disk.

The Gromov boundary of the loop graph. Let us denote by $\mathcal{H}(S,p)$ the set of all high-filling rays in $\mathcal{R}(S;p)$. Bavard and Walker endow $\mathcal{H}(S,p)$ with a topology. This topology is based on the notion of two rays k-beggining like each other, see Section 4.1 and Definition 5.2.4 in [**?**]. On the other hand, they define a $\mathrm{Mod}(S;p)$-action on $\mathcal{H}(S;p)$ by homeomorphisms. We sketch this action in what follows. First they show that endpoints of lifts of loops and short rays to the conical cover $\hat{S}$ are dense in $\partial\hat{S}$. Using this, and the fact that mapping classes in $\mathrm{Mod}(S;p)$ permute loops and short rays, they prove that any $\phi \in \mathrm{Mod}(S;p)$ lifts to a homeomorphism of $\hat{S}$ which admits a unique continuous extension to a homeomorphism of $\partial\hat{S}$. Finally, they show that this extension preserves the subset of $\partial\hat{S}$ formed by endpoints of high-filling rays, hence inducing the aforementioned action by homeomorphisms.

Theorem 2.3 *Let $\mathcal{E}(S;p) = \mathcal{H}(S;p)/\sim$, where $\sim$ identifies all high-filling rays in the same clique, and endow this set with the quotient topology. Then there exists a $\mathrm{Mod}(S;p)$-equivariant homeomorphism $F : \mathcal{E}(S;p) \to \partial L(S;p)$, where $\partial L(S;p)$ is the Gromov boundary of the loop graph.*

In consequence any loxodromic element $\phi \in \mathrm{Mod}(S;p)$ fixes two cliques of high-filling rays $C^{-}(\phi)$ and $C^{+}(\phi)$.

Theorem 2.4 ([**?**], Theorem 7.1.1) *The cliques $C^{-}(\phi)$ and $C^{+}(\phi)$ are finite and of the same cardinality.*

This allows us to define the *weight* of a loxodromic element ϕ as the cardinality of either $C^{-}(\phi)$ or $C^{+}(\phi)$. As said in the introduction, the importance of the weight of a loxodromic element is given by the following fact:

Lemma 2.5 ([**?**], Lemma 9.2.7) *Let $g, h \in \mathrm{Mod}(S;p)$ be two loxodromic elements with different weights. Then in the language of Bestvina and Fujiwara [**?**], g and h are independent and anti-aligned.*

2.4 Flat surfaces

In this section we recall only basic concepts about flat surfaces needed for the rest of the paper. It is important to remark that most of the flat surfaces considered in this text are of infinite type. For a detailed discussion on infinite-type flat surfaces we refer the reader to Delecroix, Hubert and Valdez [**?**].

We use x, y for the standard coordinates in $\mathbb{R}^2$, $z = x + iy$ the corresponding number in $\mathbb{C}$ and (r, θ) for polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (or $z = r\exp(i\theta)$). The Euclidean metric $dx^2 + dy^2$ can also be written as $(dr)^2 + (rd\theta)^2$.

Let S be an orientable surface and g be a metric defined on the complement of a discrete set $\Sigma \subset S$. A point $p \in S$ is called a *conical singularity of angle πn* for some $n \in \mathbb{N}$ if there exists an open neighbourhood U of p such that (U, g) is isometric to (V, g_n), where $V \subset \mathbb{C}^*$ is a (punctured) neighbourhood of the origin and $g_n = (dr)^2 + (nrd\theta)^2$. If $n = 2$ we call p a *regular point*. In general, regular points are not considered as singularities, though as we see in the proof of Theorem 1.1 sometimes it is convenient to think of them as marked points.

A *flat surface structure* on a topological surface S is a maximal atlas $\mathcal{T} = \{\phi_i : U_i \to \mathbb{C}\}$ where $(U_i)_{i\in\mathbb{N}}$ forms an open covering of S, each ϕ_i is a homeomorphism from U_i to $\phi(U_i)$ and for each i, j the transition map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a translation in $\mathbb{C}$ or a map of the form[3] $z \to -z + \lambda$ for some $\lambda \in \mathbb{C}$.

Definition 2.6 A *flat surface* is a pair $M = (S, \mathcal{T})$ made of a connected topological surface S and a flat surface structure $\mathcal{T}$ on $S \setminus \Sigma$, where:

(1) Σ is a discrete subset of S and

(2) every $z \in \Sigma$ is a conical singularity.

If the transition functions of $\mathcal{T}$ are all translations we call the pair $(S, \mathcal{T})$ a translation surface.

Remark 2.7 In the preceding definition S can be of infinite topological type and Σ can be infinite. All points in $M \setminus \Sigma$ are regular. Every flat surface carries a flat metric given by pulling back the Euclidean metric in $\mathbb{C}$. We denote by $\widehat{M}$ the corresponding metric completion and $\mathrm{Sing}(M) \subset \widehat{M}$ the set of non-regular points, which can be thought as singularities of the flat metric. We stress that the structure of M near a non-regular point is not well understood in full generality, see Bowman and Valdez [**?**], and Randecker [**?**].

Every flat surface M which is not a translation surface has a (ramified) double covering $\pi : \widetilde{M} \to M$ such that $\widetilde{M}$ is a translation surface whose atlas is obtained by pulling back via π the flat surface structure of M. This is called the orientation double covering. From the topological point of view $\pi : \widetilde{M} \to M$ is defined by the kernel of the *linear*

[3] These kind of maps are also called *half-translations* and for this reason flat surfaces containing half-translation are also known as half-translation surfaces.

holonomy of M. This is the group morphism $f : \pi_1(M \setminus \Sigma, x_0) \to \{+1, -1\}$ obtained by considering the linear part of transition maps along loops, where Σ is the set of conical singularities of M and $x_0 \in M$ is a basepoint. We use two facts about the double covering $\pi : \widetilde{M} \to M$ in the proof of Theorem 1.1 (we refer to the first Chapter of [**?**] for details):

(1) The preimage in $\widetilde{M}$ of the core curve γ of a cylinder C in M is formed by two closed curves of the same length. This follows from the fact that the image by the linear holomomy of γ is $f(\gamma) = +1$ and hence the action of γ on fibers of $\pi : \widetilde{M} \to M$ is trivial.

(2) If $z_0 \in M$ is a conical singularity of angle $n\pi$ then, if n is even, $\pi^{-1}(z_0)$ is formed by two conical singularities of total angle $n\pi$; whereas if n is odd, $\pi^{-1}(z_0)$ is a conical singularity of total angle $2n\pi$. Hence the branching points of the orientation double covering are the conical singularities in M of angle $n\pi$, with n odd.

On the other hand, flat surfaces can be defined using the language of complex geometry in terms of Riemann surfaces and quadratic differentials or by glueing (possibly infinite) families of polygons along their edges. A detailed discussion on these other definitions can be found in the first Chapter of [**?**].

Affine maps. A map $f \in Homeo(M)$ with $f(\Sigma) \subset \Sigma$ is called an *affine automorphisms* if the restriction $f : M \setminus \Sigma \to M \setminus \Sigma$ to flat charts is an $\mathbb{R}$-affine map. We denote by $\mathrm{Aff}(M)$ the group of affine homeomorphisms of M and by $\mathrm{Aff}^+(M)$ the subgroup of $\mathrm{Aff}(M)$ made of orientation preserving affine automorphisms (*i.e* their linear part has positive determinant). Remark that the derivative Df of an element $f \in \mathrm{Aff}(M)$ is an element of $\mathrm{GL}_2(\mathbb{R})/\pm Id$.

Translation flows. For each direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have a well-defined translation flow $F^t_{\mathbb{C},\theta} : \mathbb{C} \to \mathbb{C}$ given by $F^t_{\mathbb{C},\theta}(z) = z + te^{i\theta}$. This flow defines a constant vector field $X_{\mathbb{C},\theta}(z) := \frac{\partial F^t_{\mathbb{C},\theta}}{\partial t}|_{t=0}(z)$. Now let M be a translation surface and $X_{M,\theta}$ the vector field on $\widehat{M} \setminus \mathrm{Sing}(M)$ obtained by pulling back $X_{\mathbb{C},\theta}$ using the charts of the structure. For every $z \in M \setminus \mathrm{Sing}(M)$ let us denote by $\gamma_z : I \to M$, where $I \subset \mathbb{R}$ is an interval containing zero, the maximal integral curve of $X_{M,\theta}$ with initial condition $\gamma_z(0) = z$. We define $F^t_{M,\theta}(z) := \gamma_z(t)$ and call it the *translation flow* on M in direction θ. Let us remark that formally speaking $F^t_{M,\theta}$ is not a flow because the trajectory of the curve $\gamma_z(t)$ may reach a point of $\mathrm{Sing}(M)$ in finite time. A trajectory of the translation flow whose maximal domain of definition is a bounded interval (on both sides) is called a *saddle connection*. When there is no need to distinguish translation flows in different translation surfaces we abbreviate $F^t_{M,\theta}$ and $X_{M,\theta}$ by F^t_θ and X_θ respectively. If M is a flat surface but not a translation surface the pull back of the vector field $e^{i\theta}$ to M does not define a global vector field on M but it does define a *direction field*. In both cases integral curves define a foliation $\mathcal{F}_\theta$ on $M \setminus \mathrm{Sing}(M)$.

Definition 2.8 (Cylinders and strips) A *horizontal cylinder* $C_{c,I}$ is a translation surface of the form $([0,c] \times I)/\sim$, where $I \subset \mathbb{R}$ is open (but not necessarily bounded), connected, and where $(0,s)$ is identified with (c,s) for all $s \in I$. The numbers c and $h = |I|$ are called the *circumference* and *height* of the cylinder respectively. The *modulus* of $C_{c,I}$ is the number $\frac{h}{c}$.

A *horizontal strip* $C_{\infty,I}$ is a translation surface of the form $\mathbb{R} \times I$, where I is a bounded open interval. Analogously, the height of the horizontal strip is $h = |I|$.

An open subset of a translation surface M is called a *cylinder* (respectively a *strip*) in direction θ (or parallel to θ) if it is isomorphic, as a translation surface, to $e^{-i\theta}C_{c,I}$ (respect. to $e^{-i\theta}C_{\infty,I}$).

One can think of strips as cylinders of infinite circumference and finite height. For flat surfaces which are not translation surfaces the definition of cylinder still makes sense, though its direction is well defined only up to change of sign.

Definition 2.9 Let M be a flat surface and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ a fixed direction. A collection of maximal cylinders $\{C_i\}_{i\in I}$ parallel to θ such that $\cup_{i\in I} C_i$ is dense in M is called a *cylinder decomposition* in direction θ.

3 The Hooper–Thurston–Veech construction

The main result of this section is a generalization of the Thurston–Veech construction for infinite-type surfaces. The key ingredient for this generalization is the following:

Lemma 3.1 *Let M be a flat surface for which there is a cylinder decomposition in the horizontal direction. Suppose that every maximal cylinder in this decomposition has modulus equal to $\frac{1}{\lambda}$ for some $\lambda > 0$. Then there exists a unique affine automorphism ϕ_h which fixes the boundaries of the cylinders and whose derivative (in $\mathrm{PSL}(2,\mathbb{R})$) is given by the matrix $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, the automorphism ϕ_h acts as a Dehn twist along the core curve of each cylinder.*

In general, if M is a flat surface having a cylinder decomposition in direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ for which every cylinder has modulus equal to $\frac{1}{\lambda}$ for some $\lambda \in \mathbb{R}^*$, one can apply to M the rotation $R_\theta \in \mathrm{SO}(2,\mathbb{R})$ that takes θ to the horizontal direction and apply the preceding lemma. For example, if $\theta = \frac{\pi}{2}$, then there exists a unique affine automorphism φ_v which fixes the boundaries of the vertical cylinders, acting on each cylinder as a Dehn twist and with derivative (in $\mathrm{PSL}(2,\mathbb{R})$) given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -\lambda & 1 \end{smallmatrix}\right)$. In particular, if M is a flat surface having cylinder decompositions in the horizontal and vertical directions such that each cylinder involved has modulus $\frac{1}{\lambda}$, then $\mathrm{Aff}(M)$ has two affine multitwists ϕ_h and ϕ_v with $D\phi_h = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $D\phi_v = \left(\begin{smallmatrix} 1 & 0 \\ -\lambda & 1 \end{smallmatrix}\right)$ in $\mathrm{PSL}(2,\mathbb{R})$.

Let us recall now the Thurston–Veech construction (see Farb and Margalit [**?**, Theorem 14.1]):

Theorem 3.2 (Thurston–Veech construction) *Let $\alpha = \{\alpha_i\}_{i=1}^n$ and $\beta = \{\beta_j\}_{j=1}^m$ be two multicurves filling a finite type surface S. Then there exists $\lambda = \lambda(a,b) \in \mathbb{R}^*$ and a representation $\rho : \langle T_\alpha, T_\beta\rangle \to \mathrm{PSL}(2,\mathbb{R})$ given by:*

$$T_\alpha \to \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad T_\beta \to= \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}.$$

Moreover, an element $f \in \langle T_\alpha, T_\beta\rangle$ is periodic, reducible or pseudo-Anosov according to whether $\rho(f)$ is elliptic, parabolic or hyperbolic.

The proof of Theorem 3.2 uses Lemma 3.1. More precisely, one needs to find a flat surface structure on S which admits horizontal and vertical cylinder decompositions $\{H_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ such that each cylinder has modulus equal to $\frac{1}{\lambda}$ and for which α_i and β_j are the core curves of H_i and V_j for each $i = 1,\dots,n$ and $j = 1,\dots,m$, respectively. By the Perron–Frobenius theorem, such a flat structure always exists and is unique up to scaling.

Definition 3.3 Let $\alpha = \cup_{i\in I}\alpha_i$ and $\beta = \cup_{j\in J}\beta_j$ be two multicurves in a topological surface S (in minimal position, not necessarily filling). The *configuration graph* of the pair (α,β) is the bipartite graph $\mathcal{G}(\alpha\cup\beta)$ whose vertex set is $I \sqcup J$ and where there is an edge between two vertices $i \in I$ and $j \in J$ for every intersection point between α_i and β_j.

Cylinder decompositions, bipartite graphs and harmonic functions. Let M be a flat surface having horizontal and vertical cylinder decompositions $\mathcal{H} = \{H_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ such that each cylinder has modulus $\frac{1}{\lambda}$ for some $\lambda > 0$. For every $i \in I$ let α_i be the core curve of H_i and for every $j \in J$ let β_j be the core curve of V_j. Then

$\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ are multicurves whose union fills M. Let $\mathbf{h} : I \cup J \to \mathbb{R}_{>0}$ be the function which to an index associates the height of the corresponding cylinder. Then $A\mathbf{h} = \lambda\mathbf{h}$ where A is the adjacency operator of the graph $\mathcal{G}(\alpha \cup \beta)$, that is:

$$(A\mathbf{h})(v) := \sum_{w\sim v} \mathbf{h}(w) \tag{1}$$

where the sum above is taken over edges $\{v, w\}$ having v as one endpoint, that is, the summand $\mathbf{h}(w)$ appears as many times as there are edges between the vertices v and w.

Definition 3.4 Let $\mathcal{G} = (V, E)$ be a graph with vertices of finite degree and $A : V^{\mathbb{R}} \to \mathbb{R}$ as in (1). A function $\mathbf{h} : V \to \mathbb{R}$ that satisfies $A\mathbf{h} = \lambda\mathbf{h}$ is called a *λ-harmonic function* of $\mathcal{G}$.

In summary: the existence of a horizontal and a vertical cylinder decomposition where each cylinder has modulus $\frac{1}{\lambda}$ implies the existence of a *positive* λ-harmonic function of the configuration graph of the multicurves given by the core curves of the cylinders in the decomposition.

The idea to generalize Thurston–Veech's construction for infinite-type surfaces is to reverse this process: given a pair of multicurves α and β whose union fills S, every positive λ-harmonic function of $\mathcal{G}(\alpha \cup \beta)$ can be used to construct construct horizontal and vertical cylinder decompositions of S where all cylinders have the same modulus.

Theorem 3.5 (Hooper–Thurson–Veech construction) *Let S be an infinite-type surface. Suppose that there exist two multicurves $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ filling S such that:*

(1) *there is an uniform upper bound on the degree of the vertices of the configuration graph $\mathcal{G}(\alpha \cup \beta)$ and*

(2) *every component of the complement of $\alpha \cup \beta$ is a polygon with a finite number of sides, that is, each component of $S \setminus \alpha \cup \beta$ is a disc whose boundary consists of finitely many subarcs of curves in $\alpha \cup \beta$.*

Then there exists $\lambda_0 \geq 2$ such that for every $\lambda \geq \lambda_0$ there exists a positive λ-harmonic function $\mathbf{h}$ on $\mathcal{G}(\alpha \cup \beta)$ which defines a flat surface structure $M = M(\alpha, \beta, \mathbf{h})$ on S admiting horizontal and vertical cylinder decompositions $\mathcal{H} = \{H_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ where each cylinder has modulus $\frac{1}{\lambda}$. Moreover, for every $i \in I$ and $j \in J$ the core curves of H_i and V_j are α_i and β_j, respectively. In particular, we have (right) multitwists T_α and T_β in $\mathrm{Aff}(M)$ *which fix the boundary of each cylinder in $\mathcal{H}$ and $\mathcal{V}$,* *respectively. For each $\lambda \geq \lambda_0$, the derivatives of these multitwists define a representation $\rho : \langle T_\alpha, T_\beta \rangle \to \mathrm{PSL}(2,\mathbb{R})$ given by:*

$$T_\alpha \to \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad T_\beta \to \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}.$$

Remark 3.6 Theorem 3.5 is a particular case of a more general version of Hooper–Thurston–Veech's construction due to Delecroix and the second author whose final form was achieved after discussions with the first author, see Delecroix, Hubert and Valdez [**?**]. We do not need this more general version for the proof of our main results. On the other hand, the second assumption on the multicurves in Theorem 3.5 makes the proof simpler that in the general case and for this reason we decided to sketch it. Many of the key ideas in the proof of the result above and its general version appear already in the work of Hooper [**?**]. The main difference is that Hooper starts with a bipartite infinite graph with uniformly bounded valence and then, using a positive harmonic function h on that graph, constructs a translation surface. We take as input an infinite-type topological surface S and a pair of filling multicurves to construct a flat surface structure on S, which is a not *a priori* a translation surface structure.

Proof of Theorem 3.5. The union $\alpha \cup \beta$ of the multicurves α and β defines a graph embedded in S: the vertices are points in $\bigcup_{(i,j)\in I\times J} \alpha_i \cap \beta_j$ and edges are the segments forming the connected components of $\alpha \cup \beta \setminus \bigcup_{(i,j)\in I\times J} \alpha_i \cap \beta_j$. Abusing notation we write $\alpha \cup \beta$ to refer to this graph. It is important not to confuse the (geometric) graph $\alpha \cup \beta$ with the (abstract) configuration graph $\mathcal{G}(\alpha \cup \beta)$. To define the flat structure M on S we consider a dual graph $(\alpha \cup \beta)^*$ defined as follows. If S had no punctures then $(\alpha \cup \beta)^*$ is just the dual graph of $\alpha \cup \beta$. If S has punctures[4] we make the following convention to define the vertices of $(\alpha \cup \beta)^*$: for every connected component D of $S \setminus \alpha \cup \beta$ homeomorphic to a disc choose a unique point v_D inside the connected component. If D is a punctured disc, then choose v_D to be the puncture.

The points v_D chosen above are the vertices of $(\alpha \cup \beta)^*$. Vertices in this graph are joined by an edge in S if the closures of the corresponding connected components of $S \setminus \alpha \cup \beta$ share an edge of $\alpha \cup \beta$. Edges are chosen to be pairwise disjoint. Remark that $(\alpha \cup \beta)^*$ might have loops. See Figure 2.

Given that $\alpha \cup \beta$ fills, every connected component $S \setminus (\alpha \cup \beta)^*$ is a topological quadrilateral which contains a unique vertex of $\alpha \cup \beta$. Hence there is a well defined bijection between edges in the abstract graph $\mathcal{G}(\alpha \cup \beta)$ and the set of these quadrilaterals. This

[4] We think of punctures also as isolated ends or points at infinity.

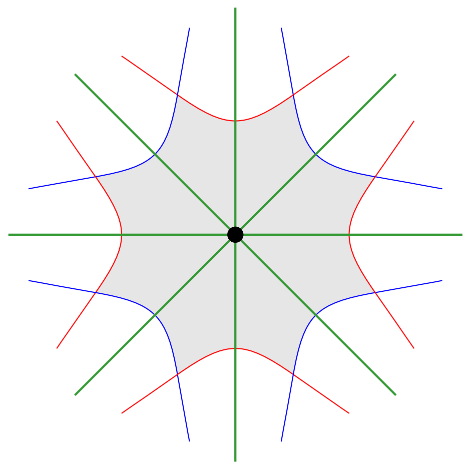


Figure 2: The graph $(\alpha \cup \beta)^*$.

way, for every edge $e \in E(\mathcal{G}(\alpha \cup \beta))$ we denote by R_e the closure in S of the corresponding topological quadrilateral with the convention to add to R_e vertices v_D corresponding to punctures in S.

Note that there are only two sides of R_e intersecting the multicurve α, which henceforth are called *vertical sides*. The other two sides are in consequence called *horizontal*, see Figure 3.

We now build a flat surface structure on S by declaring the topological quadrilaterals R_e of the dual graph $(\alpha \cup \beta)^*$ to be Euclidean rectangles. Given that there is a uniform upper bound on the degree of the vertices of the configuration graph $\mathcal{G}(\alpha \cup \beta)$ there exists $\lambda_0 \geq 2$ such that for every $\lambda \geq \lambda_0$ there exists a positive λ-harmonic function $\mathbf{h} : \mathcal{G}(\alpha \cup \beta) \to \mathbb{R}_{>0}$. For a more detailed discussion on λ-harmonic functions we recommend Appendix C in [**?**] and references therein. We use this function to define compatible heights of horizontal and vertical cylinders. More precisely, let us define the maps:

$$p_\alpha : E(\mathcal{G}(\alpha \cup \beta)) \to V(\mathcal{G}(\alpha \cup \beta)) \qquad \text{and} \qquad p_\beta : E(\mathcal{G}(\alpha \cup \beta)) \to V(\mathcal{G}(\alpha \cup \beta))$$

which to an edge e of the configuration graph $\mathcal{G}(\alpha \cup \beta)$ associate its endpoints $p_\alpha(e)$ in I and $p_\beta(e)$ in J. The desired flat structure is defined by declaring R_e to be the rectangle $[0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)]$, see Figure 3. For a formal description on how to identify R_e with $[0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)]$ we refer the reader to [**?**].

We denote the resulting flat surface $M(\alpha, \beta, \mathbf{h})$. Remark that by contruction a vertex v_D of the dual graph $(\alpha \cup \beta)^*$ of valence k defines a conical singularity of angle $\frac{\pi k}{2}$ in

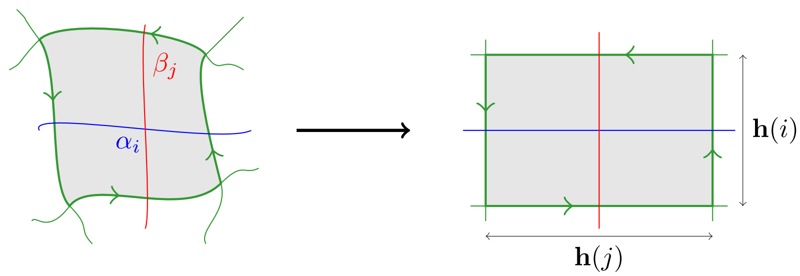


Figure 3: Transforming topological rectangles into Euclidean rectangles.

the metric completion of $M(\alpha,\beta,\mathbf{h})$. Given that k is always an even number we have that $M(\alpha,\beta,\mathbf{h})$ is a half-translation surface (*i.e.* given by a quadratic differential) when $k = 2(2n-1)$ for some $n \in \mathbb{Z}_{\geq 1}$.

Now, for every $i \in I$, the curve α_i is the core curve of the horizontal cylinder $H_i := \cup_{e\in p_\alpha^{-1}(i)} R_e$. Because $\mathbf{h}$ is λ-harmonic we have

$$\sum_{e\in p_\alpha^{-1}(i)} \mathbf{h}(p_\beta(e)) = \sum_{j\sim i} \mathbf{h}(j) = \lambda \mathbf{h}(i).$$

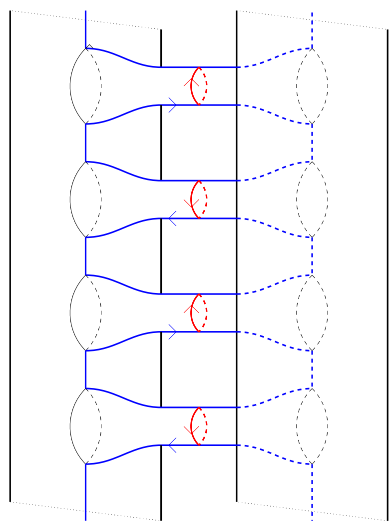
This equations say that the circumference $\sum_{e\in p_\alpha^{-1}(i)} \mathbf{h}(p_\beta(e))$ of H_i is λ times its height $\mathbf{h}(i)$, hence the modulus of H_i is equal to $\frac{1}{\lambda}$. The same computation with β_j shows that the vertical cylinders $V_j := \cup_{e\in p_\beta^{-1}(j)} R_e$ have core curve β_j and modulus $\frac{1}{\lambda}$. □

Remark 3.7 As said before, Hooper–Thurston–Veech construction can be applied to more general pairs of multicurves α,β. Consider for example the case in the Loch Ness monster depicted in Figure 4a: here the graph $\mathcal{G}(\alpha\cup\beta)$ has finite valence but there exist four connected components $\{C_i\}_{i=1}^4$ of $S\setminus\alpha\cup\beta$ which are infinite polygons, that is, whose boundary is formed by infinitely many segments belonging to curves in α and in β. In this situation the convention is to consider vertices in the dual graph $(\alpha\cup\beta)^*$ of infinite degree as points at infinity (that is, not in S). With this convention the Hooper–Thurston–Veech construction produces a translation surface structure on S, because each ∂C_i is connected. In Figure 5 we illustrate the case of a 2-harmonic function; the resulting flat surface is a translation surface known as the infinite staircase.

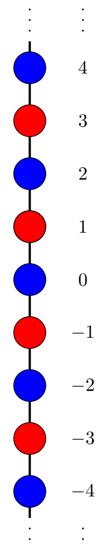
Proof of Theorem 1.7. Let $\lambda \geq 2$ and consider the subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$G_\lambda := \langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \rangle \tag{2}$$

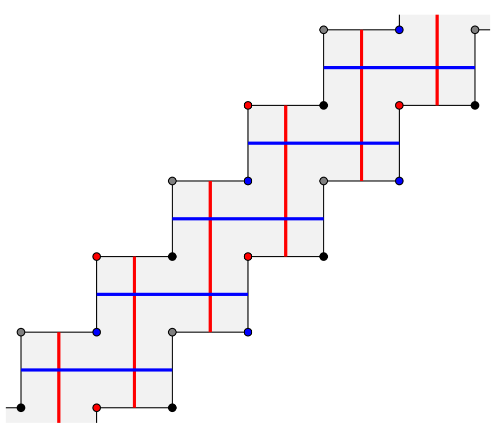
This group is free and its elements are matrices of the form $\begin{pmatrix} 1+k_{11}\lambda^2 & k_{12}\lambda \\ k_{21}\lambda & 1+k_{22}\lambda^2 \end{pmatrix}$, $k_{ij} \in \mathbb{Z}$, such that the determinant is 1 and $|\frac{1+k_{11}\lambda^2}{k_{12}\lambda}|$ does not belong to the interval (t^{-1}, t), where



Two oriented multicurves α (in blue) and β (in red) in the Loch Ness monster for which the Hooper–Thurston–Veech construction produces the infinite staircase.



The graph $\mathcal{G}(\alpha \cup \beta)$



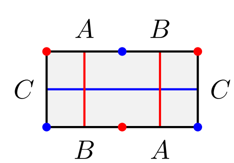


Figure 5: The infinite staircase (for $\lambda = 2$) as a Hooper–Thurson–Veech surface and a covering of a twice punctured torus.

$t = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$, see Brenner [**?**]. On the other hand, since

$$\left\{-\frac{\lambda}{2} < \Re(z) \leq \frac{\lambda}{2}\right\} \cap \left\{|z + \frac{1}{2\lambda}| > \frac{1}{2\lambda}\right\} \cap \left\{|z - \frac{1}{2\lambda}| \geq \frac{1}{2\lambda}\right\} \subset \mathbb{H}^2$$

is a fundamental domain in the hyperbolic plane for G_λ, this group has no elliptic elements. Moreover, if $\lambda > 2$ there are only two conjugacy classes of parabolics (corresponding to the generators of G_λ) and if $\lambda = 2$ then $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda^2 & \lambda \\ -\lambda & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda^2 & -\lambda \\ \lambda & 1 \end{pmatrix}$ determine, together with the generators of G_λ, the only 4 conjugacy classes of parabolics in G_λ. Remark that $\begin{pmatrix} 1-\lambda^2 & \lambda \\ -\lambda & 1 \end{pmatrix}$ and $\begin{pmatrix} 1-\lambda^2 & -\lambda \\ \lambda & 1 \end{pmatrix}$ are hyperbolic if $\lambda > 2$.

If α and β are the multicurves depicted in Figure 4a on the Loch Ness monster then $\mathcal{G}(\alpha \cup \beta)$ is the infinite bipartite graph in Figure 4b. Let us index the vertices of this graph by the integers as in the Figure so that $\mathbf{h}_2(n) = 1$, for all $n \in \mathbb{Z}$, is a positive 2-harmonic function on $\mathcal{G}(\alpha \cup \beta)$. If $\lambda > 2$ and $r_+ = \frac{\lambda + \sqrt{\lambda^2-4}}{2}$ the positive function $\mathbf{h}_\lambda(n) = r_+^n$, $n \in \mathbb{Z}$ is λ-harmonic on $\mathcal{G}(\alpha \cup \beta)$. The desired family of translation surfaces $\{M_\lambda\}_{\lambda \in [2,+\infty)}$, is obtained by applying Hooper–Thurston–Veech's construction to the multicurves α, β and the family of positive λ-harmonic functions $\{\mathbf{h}_\lambda\}_{\lambda \in [2,+\infty)}$. The desired class $f \in \mathrm{Mod}(S)$ is given by the product of (right) multitwists $T_\alpha T_\beta$.

We now show that no positive power of f fixes an isotopy class of a simple closed curve. Given that this is a purely topological property, we can work with a particular translation surface structure. Indeed, we pick $\lambda = 2$ and remark that the infinite staircase is a $\mathbb{Z}$-covering $\pi : M_\lambda \to T_\lambda$ of a two punctured torus T_λ. An instance of this covering is depicted in Figure 5. The product of multitwists f projects into a pseudo-Anosov class $g \in \mathrm{Mod}(T_\lambda)$. Pseudo-Anosov mapping classes act on the curve complex without periodic points, hence no positive power of f fixes an isotopy class of a simple closed curve. □

Remark 3.8 There are only 3 infinite graphs which admit 2-harmonic functions. These are depicted in Figure 6 together with their corresponding positive 2-harmonic functions (which are unique up to rescaling). None of them comes from a pair of multicurves satifying (2) in Theorem 3.5.

Renormalizable directions. The main results in Hooper's work [**?**] deal with the dynamical properties of the translation flow in *renormalizable directions*.

Definition 3.9 Consider the action of G_λ as defined in (2) by homographies on the real projective line $\mathbb{RP}^1$. We say that a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is λ-renormalizable if its projectivization lies in the limit set of G_λ and is not an eigendirection of any matrix conjugated in G_λ to a matrix of the form:

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

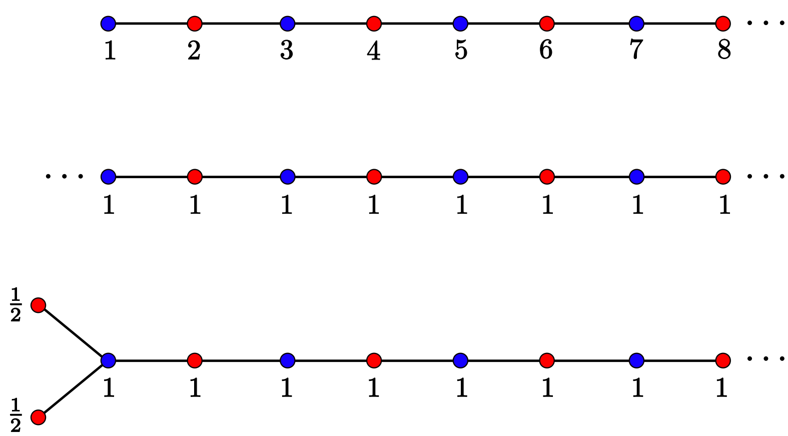


Figure 6: Graphs with 2-harmonic functions.

We use two of Hooper's results in the proof of Theorem 3.5. Recall that in Hooper's work one takes as input an infinite bipartite graph and a positive λ-harmonic function on this graph to produce a translation surface.

Theorem 3.10 (Theorem 6.2, [**?**]) *Let M be a translation surface obtained from an infinite bipartite graph as in [Ibid.] using a positive λ-harmonic function and let θ be a λ-renormalizable direction. Then the translation flow F_θ^t on M does not have saddle connections.*

Theorem 3.11 (Theorem 6.4, [**?**]) *Let M be a translation surface obtained from an infinite bipartite graph as in [Ibid.] using a positive λ-harmonic function and let θ be a λ-renormalizable direction. Then the translation flow F_θ^t is conservative, that is, given $A \subset M$ of positive measure and any $T > 0$, for Lebesgue almost every $x \in M$ there is a $t > T$ such that $F_\theta^t(x) \in A$*

4 Proof of results

4.1 Proof of Theorem 1.1

The proof is divided in three parts. In the first part we use the Hooper–Thurston–Veech construction (see Section 3) to find two transverse measured f-invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ on S for which p is a singular point and for which each foliation has m separatrices based at p. We prove that each separatrix based at p is dense in S. Then, we consider a hyperbolic metric on S of the first kind (allowing us to talk about the completed ray graph $\mathcal{R}(S;p)$). We stretch each separatrix of $\mathcal{F}^u$ and $\mathcal{F}^s$ based at p to a geodesic with respect to this metric. This defines two sets Γ^+ and Γ^- of geodesics, each having cardinality m. In the second part of the proof, we show that Γ^+ and Γ^- are the only cliques of high-filling rays fixed by f in the Gromov boundary of the loop graph. Finally, in the third part we discuss why f has an infinite diameter orbit in the loop graph.

Flat structures. We use the Hooper–Thurston–Veech construction (Section 3) for this part of the proof. Let α and β be two multicurves satisfying the hypothesis of Theorem 1.1. Fix $\mathbf{h} : \mathcal{G}(\alpha \cup \beta) \to \mathbb{R}_{>0}$ a positive λ-harmonic function on the configuration graph $\mathcal{G}(\alpha \cup \beta)$. Let $M = M(\alpha, \beta, \mathbf{h})$ be the flat structure on S given by the Hooper–Thurston–Veech construction and

$$\rho : \langle T_\alpha, T_\beta \rangle \to \mathrm{PSL}(2, \mathbb{R})$$

the corresponding representation. Here, we have chosen p as one of the vertices of the dual graph $(\alpha \cup \beta)^*$ (see the proof of Theorem 3.5) and therefore it makes sense to consider the classes that the affine multitwists T_α, T_β define in $\mathrm{Mod}(S;p)$. We abuse notation and denote also by T_α, T_β these classes.

From the proof of Theorem 1.7 one can deduce that $\rho(f)$ is a matrix of the form $\begin{pmatrix} 1+k_{11}\lambda^2 & k_{12}\lambda \\ k_{21}\lambda & 1+k_{22}\lambda^2 \end{pmatrix}$, where $k_{ij} \in \mathbb{Z}_{>0}$. In particular, $\rho(f)$ is always hyperbolic. Thus, the eigenspaces of the hyperbolic matrix $\rho(f)$ define two transverse (f-invariant) measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ (for unstable and stable, respectively) on M . Moreover, we have that $f \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \eta\mu_u)$ and $f \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \eta^{-1}\mu_s)$, where $\eta > 1$ is (up to sign) an eigenvalue of $\rho(f)$. For simplicity we abbreviate the notation for these foliations by $\mathcal{F}^u$ and $\mathcal{F}^s$.

The set $\mathfrak{V}$. Recall that $M = M(\alpha, \beta, \mathbf{h})$ is constructed by glueing a family of rectangles $\{R_e\}_{e \in E}$, where $E = E(\mathcal{G}(\alpha \cup \beta))$ is the set of edges of the configuration graph $\mathcal{G}(\alpha \cup \beta)$, along their edges using translations and half-translations. By the way the Hooper–Thurston–Veech construction is carried out, sometimes the corners of these rectangles are not part of the surface S: this is the case when there are connected components of $S \setminus \alpha \cup \beta$ which are punctured discs. However, every corner of a rectangle $\{R_e\}_{e \in E}$ belongs to the metric completion $\widehat{M}$ of M (w.r.t. the natural flat metric). We define $\mathfrak{V} \subset \widehat{M}$ to be the set of points that are corners of rectangles in $\{R_e\}_{e \in E}$ (after glueings). Remark that since all connected components of $S \setminus \alpha \cup \beta$ are (topological) polygons with an uniformly bounded number of sides, points in $\mathfrak{V}$ are regular points or conical singularities of $\widehat{M}$ whose total angle is uniformly bounded. Moreover the set $\mathrm{Fix}(f)$ of fixed points of the continuous extension of f to $\widehat{M}$ contains $\mathfrak{V}$. Indeed, if $\mathcal{H} = \{H_i\}$

and $\mathcal{V} = \{V_j\}$ denote the horizontal and vertical (maximal) cylinder decompositions of M, then $\mathfrak{V} = \cup(\partial H_i \cap \partial V_j)$, where the boundary of each cylinder is taken in the metric completion $\widehat{M}$. The claim follows from the fact that for every $i \in I$ and $j \in J$, T_α fixes ∂H_i and T_β fixes ∂V_j.

For each $q \in \mathfrak{V}$ we denote by $\mathrm{Sep}_q(*)$ the set of leaves of $* \in \{\mathcal{F}^u, \mathcal{F}^s\}$ based at q. We call such a leaf a *separatrix based at q*. Remark that if the total angle of the flat structure M around q is $k\pi$ then $|\mathrm{Sep}_q(\mathcal{F}^u)| = |\mathrm{Sep}_q(\mathcal{F}^s)| = k$. The following fact is essential for the second part of the proof.

Proposition 4.1 *Let $q \in \mathfrak{V}$. Then any separatrix in $\mathrm{Sep}_q(\mathcal{F}^u) \cup \mathrm{Sep}_q(\mathcal{F}^s)$ is dense in M.*

Proof We consider first the case when M is a translation surface. At the end of the proof we deal with the case when M is a half-translation surface.

We show that any separatrix in $\mathrm{Sep}_q(\mathcal{F}^u)$ is dense. The arguments for separatrices in $\mathrm{Sep}_q(\mathcal{F}^s)$ are analogous.

Claim: $\cup_{q\in\mathfrak{V}}\mathrm{Sep}_q(\mathcal{F}^u)$, the union of all separatrices of $\mathcal{F}^u$, is dense in M. To prove this claim we strongly use the work of Hooper [**?**]. In particular, we use the fact that leaves in $\mathcal{F}^u$ are parallel to a renormalizable direction, see Definition 3.9. We remark that Hooper only deals with the case when M is a translation surface. This is why when M is a half-translation surface we consider its orientation double cover. We proceed by contradiction by assuming that the complement of the closure of $\cup_{q\in\mathfrak{V}}\mathrm{Sep}_q(\mathcal{F}^u)$ in $\widehat{M}$ is non-empty. Let U be a connected component of this complement. Then U is $\mathcal{F}^u$-invariant. If U contains a closed leaf of $\mathcal{F}^u$ then it has to be a cylinder, but this cannot happen because there are no saddle connections parallel to renormalizable directions, see Theorem 6.2 in [**?**]. Then U contains an open transversal to $\mathcal{F}^u$ to which leaves never return. Indeed, let $T \subset U$ be an open segment, transversal to $\mathcal{F}^u$ and whose closure is also contained in U. The set of points in T that follow the same trajectory back T partition T into maximal subintervals. Given that there are no closed leaves in U, at least one of the extremities of each of these maximal subintervals must lie in a singular leaf, but this is impossible because the closure of T is contained in U which by definition contains no singular leaves. As a consequence, U contains an infinite strip, *i.e.* a set which (up to rotation) is isometric to $(a, b) \times \mathbb{R}$ for some $a < b$. This is impossible since the translation flow on M in a renormalizable direction is conservative, see Theorem 6.4 in [**?**]. The claim follows.

We strongly recommend that the reader uses Figure 7 as a guide for the next paragraphs.

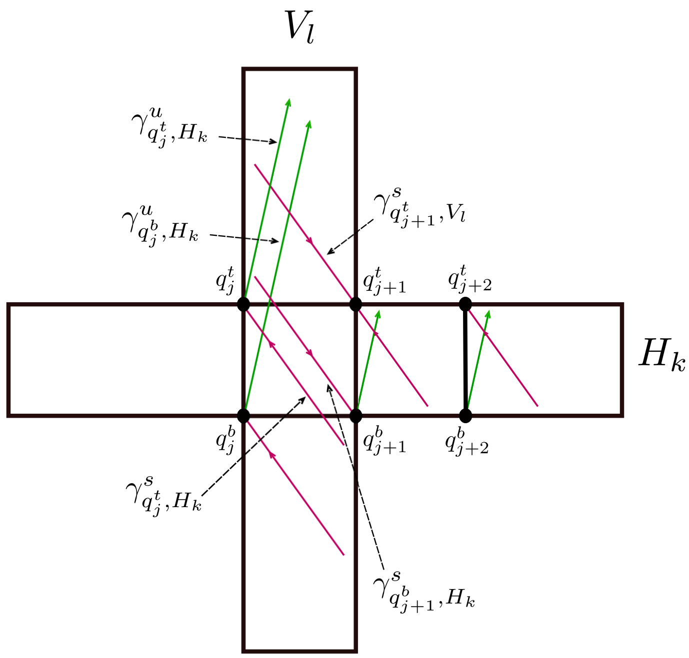


Figure 7:

Henceforth if γ is a separatrix of $\mathcal{F}^u$, we denote by $\gamma(t)$, $t > 0$ the parametrization for which $\lim_{t\to 0}\gamma(t) \in \mathfrak{V}$ and such that $|\gamma'| = 1$ (w.r.t. to the flat metric on M).

For each horizontal cylinder H_k in M and $\xi \in \mathfrak{V} \cap \partial H_k$ we denote by $\gamma^u_{\xi,H_k} \subset M$ (respectively γ^s_{ξ,H_k}) the unique separatrix of $\mathcal{F}^u$ (respect. of $\mathcal{F}^s$) based at ξ within H_k, that is, for which $\gamma^u_{\xi,H_k}(t) \in H_k$ for all t in a small neighbourhood of 0. For a vertical cylinder V_l, γ^u_{ξ,V_l} and γ^s_{ξ,V_l} are defined in a similar way. Let $\mathfrak{V}^b(H_k)$ and $\mathfrak{V}^t(H_k)$ denote the points in $\mathfrak{V} \cap H_k$ in the bottom and in the top[5] connected component of ∂H_k respectively; and for any vertical cylinder V_l let $\mathfrak{V}^e(V_l)$ and $\mathfrak{V}^w(V_l)$ denote the points in $\mathfrak{V} \cap V_l$ in the east and west connected component of ∂V_l respectively.

Without loss of generality we suppose that $q \in \mathfrak{V}^b(H_k) \cap \mathfrak{V}^e(V_l)$. We denote by $\omega(\gamma^u_{q,H_k})$ the ω-limit set of γ^u_{q,H_k}.

Claim: the union of all separatrices of $\mathcal{F}^u$ based at points in $\partial H_k \cup \partial V_l$ within H_k and

[5]We pull back the standard orientation of the Euclidean plane to M to make sense of the east-west and bottom-top sides of a cylinder.

V_l respectively

$$\left(\bigcup_{\xi\in\mathfrak{V}\cap\partial H_k}\gamma^u_{\xi,H_k}\right)\cup\left(\bigcup_{\xi\in\mathfrak{V}\cap\partial V_l}\gamma^u_{\xi,V_l}\right) \tag{3}$$

is contained in $\omega(\gamma^u_{q,H_k})$.

Proof of claim. Remark that since H_k is tiled by rectangles corresponding to points of intersection of the core curve α_k with curves in β, $|\mathfrak{V}^b(H_k)| = |\mathfrak{V}^t(H_k)|$ and for each $\xi \in \mathfrak{V}^b(H_k)$ there is exactly one point in $\mathfrak{V}^t(H_k)$ just above. Hence, using the east-west orientation of H_k, we can order the elements of $\mathfrak{V}^b(H_k)\cup\mathfrak{V}^t(H_k)$ cyclically: we write $\mathfrak{V}^b(H_k) = \{q^b_j\}_{j\in\mathbb{Z}/N\mathbb{Z}}$, $\mathfrak{V}^t(H_k) = \{q^t_j\}_{j\in\mathbb{Z}/N\mathbb{Z}}$ for some $N \geq 1$. The sets $\mathfrak{V}^e(V_l) = \{q^e_j\}_{j\in\mathbb{Z}/M\mathbb{Z}}$, $\mathfrak{V}^w(V_l) = \{q^w_j\}_{j\in\mathbb{Z}/M\mathbb{Z}}$, for some $M \geq 1$, are defined in a similar way.

We suppose that the labeling is such that above q^b_j lies q^t_j for all $j \in \mathbb{Z}/N\mathbb{Z}$, and that $q = q^b_0 = q^e_0$. Recall that $DT_\alpha = \left(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right)$, $DT^{-1}_\beta = \left(\begin{smallmatrix}1 & 0\\ \lambda & 1\end{smallmatrix}\right)$, hence $Df = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, with $a,b,c,d \in \mathbb{R}_{>0}$. In particular Df sends the positive quadrant $\mathbb{R}_{x\geq0,y\geq0}$ into itself. If we suppose, without loss of generality, that the unstable eigenspace of Df (without its zero) lies in the interior of $\mathbb{R}_{x\geq0,y\geq0}\cup\mathbb{R}_{x\leq0,y\leq0}$, then the stable eigenspace of Df (without its zero) has to lie in the interior of $\mathbb{R}_{x\geq0,y\leq0}\cup\mathbb{R}_{x\leq0,y\geq0}$. Hence, for every $j \in \mathbb{Z}/N\mathbb{Z}$ we have that $\gamma^u_{q^b_j,H_k}$ intersects γ^s_{ξ,H_k}, for $\xi \in \{q^t_j, q^b_{j+1}\}$. Moreover, we claim that $\gamma^u_{q^b_j,H_k}$ intersects also $\gamma^s_{q^t_{j+1},V_{l'}}$, where $V_{l'}$ is the vertical cylinder intersecting H_k and having $\{q^b_j, q^t_j, q^b_{j+1}, q^t_{j+1}\}$ in its boundary. Remark that in M these points need not to be all different from each other. For example $q^b_j = q^t_j$ and $q^b_{j+1} = q^t_{j+1}$ if the core curve of $V_{l'}$ only intersects the core curve of H_k. In any case the claims remain valid.

To see why this last claim is true remark that it is sufficient to show that $\gamma^u_{q^t_j,V_{l'}}$ intersects $\gamma^s_{q^t_{j+1},V_{l'}}$. Indeed, given that $\gamma^u_{q^b_j,H_k}$ intersects $\gamma^s_{q^t_j,H_k}$ there exist a divergent sequence of times (t_n) such that $\gamma^u_{q^b_j,H_k}(t_n) \in H_k\cap V_l\cap\gamma^s_{q^t_j,H_k}$ and $\gamma^u_{q^b_j,H_k}(t_n)$ converges to q^t_j. This implies that $\gamma^u_{q^b_j,V_{l'}}$ is arbitrarily close to $\gamma^u_{q^t_j,V_{l'}}$. To see that $\gamma^u_{q^t_j,V_{l'}}$ intersects $\gamma^s_{q^t_{j+1},V_{l'}}$ let $\rho : \widetilde{V_{l'}} \to V_{l'}$ be the universal covering. Without loss of generality, we can think of $\widetilde{V_{l'}}$ as the bi-infinite strip $B := (0,1)\times\mathbb{R}\subset\mathbb{R}\times\mathbb{R}$. Moreover, we can further suppose that:

- $(0,0)$ and $(1,0)$ are points in the fibers over q^t_j and q^t_{j+1} respectively, and
- $y = ax$, $a \in (0,\infty)$ and $y = b(x-1)$, $b \in (-\infty,0)$ are the equations of the lifts of $\gamma^u_{q^t_j,V_{l'}}$ and $\gamma^s_{q^t_{j+1},V_{l'}}$ to $(0,0)$ and $(1,0)$ respectively.

Then $\rho(\frac{b}{b-a},\frac{ab}{b-a})$ is the point of intersection of $\gamma^u_{q^t_j,V_{l'}}$ and $\gamma^s_{q^t_{j+1},V_{l'}}$.

By applying repeatedly f to all these points of intersection of separatrices we obtain that $\xi \in \omega(\gamma^u_{q^b_j}, H_k)$ for every $j \in \mathbb{Z}/N\mathbb{Z}$ and $\xi \in \{q^b_{j+1}, q^t_j, q^t_{j+1}\}$. This implies that $\gamma^u_{\xi,H_k} \subset \omega(\gamma^u_{q^b_j,H_k})$ for every $j \in \mathbb{Z}/N\mathbb{Z}$ and $\xi \in \{q^b_{j+1}, q^t_j, q^t_{j+1}\}$. In particular, we get that $\omega(\gamma^u_{q=q^b_0,H_k})$ contains γ^u_{ξ,H_k} for every $\xi \in \{q^b_1, q^t_1, q^t_0\}$. As a consequence, we have that $\omega(\gamma^u_{q,H_k})$ contains[6] $\omega(\gamma^u_{q^b_1,H_k})$, which in turn contains $\{\gamma^u_{q^b_2,H_k}, \gamma^u_{q^t_1,H_k}, \gamma^u_{q^t_2,H_k}\}$. Proceeding inductively we get that $\omega(\gamma^u_{q,H_k})$ contains

$$\bigcup_{\xi \in \mathfrak{V} \cap \partial H_k} \gamma^u_{\xi,H_k}.$$

The positivity of the matrix Df and the fact that its unstable eigenspace lies in $\mathbb{R}_{x\geq 0,y\geq 0} \cup \mathbb{R}_{x\leq 0,y\leq 0}$ also imply that for every $j \in \mathbb{Z}/M\mathbb{Z}$ the separatrix $\gamma^u_{q^e_j,V_l}$ intersects γ^s_{ξ,V_l} for $\xi \in \{q^w_j, q^e_{j+1}, q^w_{j+1}\}$. From here on, the logic to show that $\omega(\gamma^u_{q,H_k})$ contains $\bigcup_{\xi\in\mathfrak{V}\cap\partial V_l} \gamma^u_{\xi,V_l}$ is the same as the one presented in the preceding paragraph and the claim follows.

The arguments in the proof of the preceding claim are local so they can be used to show that:

- For every $j \in \mathbb{Z}/N\mathbb{Z}$, the limit set $\omega(\gamma^u_{q^b_j,H_k})$ contains all separatrices:

$$\left(\bigcup_{\xi \in \mathfrak{V} \cap \partial H_k} \gamma^u_{\xi,H_k}\right) \cup \left(\bigcup_{\xi \in \mathfrak{V} \cap \partial V_{l'}} \gamma^u_{\xi,V_{l'}}\right)$$

  where $V_{l'}$ is such that $q^b_j \in \partial H_k \cap \partial V_{l'}$.
- For every $j \in \mathbb{Z}/M\mathbb{Z}$, the limit set $\omega(\gamma^u_{q^e_j,V_l})$ contains all separatrices:

$$\left(\bigcup_{\xi \in \mathfrak{V} \cap \partial V_l} \gamma^u_{\xi,V_l}\right) \cup \left(\bigcup_{\xi \in \mathfrak{V} \cap \partial H_{k'}} \gamma^u_{\xi,H_{k'}}\right)$$

  where $H_{k'}$ is a horizontal cylinder such that $q^e_j \in \partial V_l \cap \partial H_{k'}$.

If we now denote by $\alpha_k \in \alpha$ the core curve of H_k then the preceding discussion can be summarized as follows: $\omega(\gamma^u_{q,H_k})$ contains all separatrices of $\mathcal{F}^u$ based at points in the boundary of cylinders (and starting within those cylinders) whose core curves belong to the link of α_k in the configuration graph $\mathcal{G}(\alpha \cup \beta)$. Moreover, if $\beta_l \in \mathrm{link}(\alpha_k)$ then $\omega(\gamma^u_{q,H_k})$ contains all separatrices of $\mathcal{F}^u$ based at points in the boundary of cylinders (and starting within those cylinders) whose core curves belong to $\mathrm{link}(\beta_l)$. This way we

[6] Here we are using the following general principle: if γ_1, γ_2 are trajectories of a vector field on a translation surface and γ_1 is contained in $\omega(\gamma_2)$, then $\omega(\gamma_1) \subset \omega(\gamma_2)$.

can extend the arguments above to the whole configuration graph $\mathcal{G}(\alpha \cup \beta)$ to conclude that $\omega(\gamma^u_{q,H_k})$ contains $\cup_{q\in\mathfrak{V}}\mathrm{Sep}_q(\mathcal{F}^u)$. Since the latter is dense in M we conclude that γ^u_{q,H_k} is dense in M.

We now suppose that M is a half-translation surface (*i.e.* given by a quadratic differential). Let $\pi : \widetilde{M} \to M$ be the orientation double cover of M.

We claim that for every horizontal cylinder H_i in M the lift $\pi^{-1}(H_i)$ is formed by two isometric copies $\widetilde{H}_{i_1}$, $\widetilde{H}_{i_2}$ of H_i and that these are maximal horizontal cylinders in $\widetilde{M}$ whose interiors are disjoint. From Remark 2.7 we know that each core curve of H_i lifts to a closed curve of the same length, hence $\pi^{-1}(H_i)$ is formed by two isometric copies of H_i. In what follows we argue why the core curves of these two isometric of H_i copies are not homotopic in $\widetilde{M}$. This implies that each of these isometric copies is maximal. Recall that if $p \in \mathfrak{V}$ is a conical singularity of angle $n\pi$, then $\pi^{-1}(p)$ is formed by two conical singularities of angle $n\pi$ if n is even, whereas if n is odd $\pi^{-1}(p)$ is a conical singularity of angle $2n\pi$. Given that the multicurves α and β are in minimal position, points in $\mathfrak{V} \cap \partial H_i$ which are conical singularities of angle π are actually punctures of S. This implies that the core curves of $\widetilde{H}_{i_1}$ and $\widetilde{H}_{i_2}$ cannot be homotopic. Points in $\mathfrak{V} \cap \partial H_i$ which are singularities of angle $n\pi$, $n > 1$, lift to conical singularities in the boudary of both $\widetilde{H}_{i_1}$ and $\widetilde{H}_{i_2}$. This allows us to conclude again that the core curves of $\widetilde{H}_{i_1}$ and $\widetilde{H}_{i_2}$ cannot be homotopic. Analogously, we have that for every vertical cylinder V_j in M the lift $\pi^{-1}(V_j)$ is formed by two disjoint isometric copies $\widetilde{V}_{j_1}$, $\widetilde{V}_{j_2}$ of V_j and these are maximal vertical cylinders in $\widetilde{M}$. The families $\widetilde{\mathcal{H}} = \{\widetilde{\mathcal{H}}_{i_1}, \widetilde{\mathcal{H}}_{i_2}\}$ and $\widetilde{\mathcal{V}} = \{\widetilde{\mathcal{V}}_{j_1}, \widetilde{\mathcal{V}}_{j_2}\}$ define horizontal and vertical (maximal) cylinder decompositions of $\widetilde{M}$ respectively.

Let $\tilde{\alpha}$, $\tilde{\beta}$ denote the lifts to the orientation double cover of α and β respectively. Given that the moduli of cylinders downstairs and upstairs is the same, we have a pair of affine multitwists $\widetilde{T_{\tilde{\alpha}}}, \widetilde{T_{\tilde{\beta}}} \in \mathrm{Aff}(\widetilde{M})$ with $DT_\alpha = D\widetilde{T_{\tilde{\alpha}}}$ and $DT_\beta = D\widetilde{T_{\tilde{\beta}}}$ in $\mathrm{PSL}(2,\mathbb{R})$. If we rewrite the word defining f replacing each appearence of T_α with $\widetilde{T_{\tilde{\alpha}}}$ and each appearence of T_β^{-1} with $\widetilde{T_{\tilde{\beta}}}^{-1}$ the result is an affine automorphism $\tilde{f}$ on $\widetilde{M}$ with $D\tilde{f} = Df$ in $\mathrm{PSL}(2,\mathbb{R})$. The eigendirections of $\tilde{f}$ define a pair of transverse $\tilde{f}$-invariant measured foliations $\widetilde{\mathcal{F}_u}$ and $\widetilde{\mathcal{F}_s}$. Moreover, we have that $\widetilde{\mathcal{F}_u} = \pi^{-1}(\mathcal{F}_u)$ and $\widetilde{\mathcal{F}_s} = \pi^{-1}(\mathcal{F}_s)$ (*i.e.* the projection π sends leaves to leaves). Let $\hat{\pi} : \widehat{\widetilde{M}} \to \widehat{M}$ be the continuous extension of the projection π to the metric completions of M and $\widetilde{M}$ and define $\tilde{\mathfrak{V}} := \hat{\pi}^{-1}(\mathfrak{V})$. Remark that $\tilde{\mathfrak{V}} = \cup(\partial\widetilde{H}_i \cap \partial\widetilde{V}_j)$, where the boundaries of the cylinders are taken in $\widehat{\widetilde{M}}$. As with M, for every $q \in \tilde{\mathfrak{V}}$ we define $\mathrm{Sep}_q(*)$ as the set of leaves of $* \in \{\widetilde{\mathcal{F}^u}, \widetilde{\mathcal{F}^s}\}$ based at q. In this context, the proof of Proposition 4.1 for translation surfaces then applies to $\widetilde{M}$ and we get the following:

Corollary 4.2 *Let $q \in \widetilde{\mathfrak{B}}$. Then any separatrix in* $\mathrm{Sep}_q(\widetilde{\mathcal{F}^u}) \cup \mathrm{Sep}_q(\widetilde{\mathcal{F}^s})$ *is dense in* $\widetilde{M}$.

If separatrices are dense upstairs they are dense downstairs. This ends the proof of Proposition 4.1. □

Let now $p \in S$ be the marked point and $\mathrm{Sep}_p(\mathcal{F}^u) = \{\gamma_1, \dots, \gamma_m\}$. We denote by S_μ a fixed complete hyperbolic structure on S (μ stands for the metric) of the first kind and define the completed ray graph $\mathcal{R}(S;p)$ with respect to μ. Remark that in S_μ the point p becomes a cusp, *i.e.* a point at infinity. In what follows we associate to each γ_i a simple geodesic in S_μ based at p. For elements in $\mathrm{Sep}_p(\mathcal{F}^s)$ the arguments are analogous. The ideas we present are largely inspired by the work of Levitt [**?**].

Henceforth $\pi : \mathbb{D} \to S_\mu$ denotes the universal cover, $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ the Fuchsian group for which $S_\mu = \mathbb{D}/\Gamma$, $\tilde{p} \in \partial\mathbb{D}$ a chosen point in the lift of the cusp p to $\partial\mathbb{D}$ and $\tilde{\gamma}_i = \tilde{\gamma}_i(\tilde{p})$ a lift of γ_i to $\mathbb{D}$ based at $\tilde{p}$.

Claim: $\tilde{\gamma}_i$ converges to two distinct points in $\partial\mathbb{D}$.

Recall the parametrization $\gamma_i(t)$ from the proof of Proposition 4.1 with $\gamma_i(0) = p$. This lifts to a parametrization $\tilde{\gamma}_i(t)$. We will show that $\tilde{\gamma}_i(t)$ converges to a point of $\partial\mathbb{D}$ as $t \to \infty$. Recall that in S, the point p is in a region bounded by a $2m$-sided polygon whose sides belong to closed curves $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ in $\alpha \cup \beta$. Each of these curves is transverse to the leaves of $\mathcal{F}^u$ and of $\mathcal{F}^s$.

Up to making an isotopy, we can suppose without loss of generality that the first element in $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ intersected by $\gamma_i(t)$ (for the natural parametrization used in the proof of Proposition 4.1) is α_j. Given that α_j is transverse to $\mathcal{F}^u$ and γ_i is dense in M we have that $\gamma_i \cap \alpha_j$ is dense in α_j. In consequence $\tilde{\gamma}_i$ intersects $\pi^{-1}(\alpha_j)$ infinitely often. Remark that $\tilde{\gamma}_i$ intersects a connected component of $\pi^{-1}(\alpha_j)$ at most once. Indeed, if this was not the case there would exist a disc D embedded in $\mathbb{D}$ whose boundary ∂D is formed by an arc in $\tilde{\gamma}_i$ and an arc contained in $\pi^{-1}(\alpha_j)$ transverse to $\widetilde{\mathcal{F}^u} := \pi^{-1}(\mathcal{F}^u)$. This is impossible because all singularities of $\widetilde{\mathcal{F}^u}$ are saddles (in particular only a finite number of separatrices can stem from each one of them) and there is a finite number of them inside D. Another way to see that the existence of such a disc D is impossible is by remarking that the winding number of $\widetilde{F^u}$ around ∂D should be $\frac{1}{2}$. Thus D must contain a 1-pronged singularity of $\mathcal{F}^u$. However, this is impossible as all 1-pronged singularities of $\mathcal{F}^u$ are punctures of S and these punctures lift to points of $\partial\mathbb{D}$. Then, all limit points of $\widetilde{\gamma_i}$ in $\mathbb{D} \cup \partial\mathbb{D}$ different from $\tilde{p}$ are in the intersection of an infinite family of nested domains $\mathbb{D} \cup \partial\mathbb{D}$ whose boundaries in $\mathbb{D}$ are components of $\pi^{-1}(\alpha_j)$.

Moreover, this intersection is a single point $q_i \in \partial\mathbb{D}$. If this intersection was not a single point then it would be a half disc (corresponding to the case that an infinite family of lifts of α_j limit to a single geodesic). However, this is ruled out by the fact that there is a lower bound on the distance between any pair of lifts of α_j. This finishes the proof of our claim above.

We define thus $\widetilde{\delta_i} = \widetilde{\delta_i}(\tilde{p})$ to be the geodesic in $\mathbb{D}$ whose endpoints are $\tilde{p}$ and q_i as above and $\delta_i := \pi(\widetilde{\delta_i})$. The geodesic δ_i is well defined: it does not depend on the lift of γ_i based at $\tilde{p}$ we have chosen and if we changed $\tilde{p}$ to some $\tilde{p}' = gp$ for some $g \in \Gamma$ then by continuity $q_i' = gq_i$. On the other hand δ_i is simple: if this was not the case two components of $\pi^{-1}(\delta_i)$ would intersect and this implies that two components of $\pi^{-1}(\gamma_i)$ intersect, which is imposible since $\widetilde{\mathcal{F}^u}$ is a foliation. Remark that if $\gamma_i \neq \gamma_j$ then δ_i and δ_j are disjoint. If this was not the case then there would be a geodesic in $\pi^{-1}(\delta_i)$ intersecting a geodesic in $\pi^{-1}(\delta_j)$, but this would imply that a connected component of $\pi^{-1}(\gamma_i)$ intersects a connected component of $\pi^{-1}(\gamma_j)$, which is impossible since $\widetilde{\mathcal{F}^u}$ is a foliation.

Hence we can associate to the set of separatrices $\{\gamma_1, \ldots, \gamma_m\}$ a set of pairwise distinct simple geodesics $\{\delta_1, \ldots, \delta_m\}$ based at p. Remark that by construction this set is f-invariant. In what follows we show that $\{\delta_1, \ldots, \delta_m\}$ is a clique of high-filling rays. By applying the same arguments to the separatrices of $\mathcal{F}^u$ based at p one obtains a different f-invariant clique of m high-filling rays. These correspond to the only two points in the Gromov boundary of the loop graph $L(S;p)$ fixed by f.

Let $\tilde{\delta}$ be a geodesic in $\mathbb{D}$ based at $\tilde{p}$ such that $\delta := \pi(\tilde{\delta})$ is a simple geodesic in S_μ which does not belong to $\{\delta_1, \ldots, \delta_m\}$. We denote by q the endpoint of $\tilde{\delta}$ which is different from $\tilde{p}$. Since every short ray or loop has a geodesic representative, it is sufficient to show that for every $i = 1, \ldots, m$ there exists a (geodesic) component of $\pi^{-1}(\delta_i)$ which intersects $\tilde{\delta}$. We recommend that the reader use Figure 8 as a guide for the next paragraph.

All components of $\pi^{-1}(\mathrm{Sep}_p(\mathcal{F}^u))$ with one endpoint in $\tilde{p}$ are of the form $\{g^k\tilde{\gamma}_1, \ldots g^k\tilde{\gamma}_m\}_{k\in\mathbb{Z}}$, with $g \in \Gamma$ parabolic fixing $\tilde{p}$. Hence, there exists a closed disc $D \subset \mathbb{D} \cup \partial\mathbb{D}$ whose boundary is formed by $\tilde{p} \cup \tilde{\gamma}_k \cup \tilde{\gamma}_l \cup A$, where A is a closed arc in $\partial\mathbb{D}$ containing q, $\tilde{\gamma}_k$, $\tilde{\gamma}_l$ are (lifts of) separatrices and there is no element of $\pi^{-1}(\mathrm{Sep}_p(\mathcal{F}^u))$ with one endpoint in $\tilde{p}$ in the interior of D. Remark that the endpoints of A are q_k and q_l (the endpoints of $\tilde{\gamma}_k$ and $\tilde{\gamma}_l$ respectively). Given that p is a saddle-type singularity of the foliation $\mathcal{F}^u$ there exists a neighbourhood of $\tilde{p}$ in D which contains a segment[7] Σ with one endpoint

[7] As a matter of fact this segment can be chosen to be contained in one of the (lifts of) the curves in $\alpha \cup \beta$ forming the boundary of the disc in $S \setminus \alpha \cup \beta$ containing p.

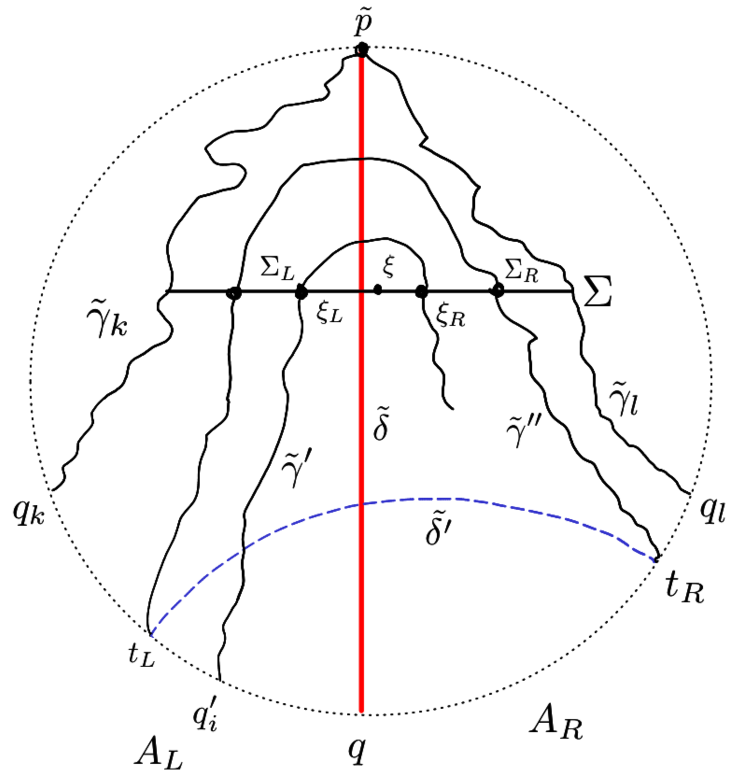


Figure 8:

in $\tilde{\gamma}_k$ and the other in $\tilde{\gamma}_l$, and which is transverse to $\widetilde{\mathcal{F}^u}$ except at one point ξ in its interior. Moreover, since p is an isolated singularity of $\mathcal{F}^u$, we can suppose that the closure of the connected component of $D \setminus \Sigma$ in $\mathbb{D}$ having $\tilde{p}$ in its boundary does not contain singular points of $\widetilde{\mathcal{F}^u}$ different from $\tilde{p}$. The point ξ divides Σ in two connected components Σ_L and Σ_R. On the other hand q divides A in two connected components A_L and A_R. Now let $i = 1, \ldots, m$ be fixed. Since γ_i is dense in M we have that $\pi^{-1}(\gamma_i)$ is dense in $\mathbb{D}$ and in particular $\pi^{-1}(\gamma_i) \cap \Sigma$ is dense in Σ. Hence we can pick a leaf $\tilde{\gamma}_i'$ in $\pi^{-1}(\gamma_i)$ passing through a point $\xi_L \in \Sigma_L$ and suppose without loss of generality that one of its endpoints q_i' is in A_L. Then $\tilde{\gamma}_i' \cap \Sigma = \{\xi_L, \xi_R\}$ with $\xi_R \in \Sigma_R$. Again, given that $\pi^{-1}(\gamma_i) \cap \Sigma$ is dense in Σ, we can find a leaf $\tilde{\gamma}_i'' \in \pi^{-1}(\gamma_i)$ which intersects Σ transversally at a point η_R between ξ_R and $\tilde{\gamma}_l$, and which has an endpoint t_R in A_R arbitrarily close to q_l. This is true because of the way q_l was found: there is a family of connected components of $\pi^{-1}(\alpha)$, for some closed curve α in M transverse to $\mathcal{F}^u$, bounding domains in $\mathbb{D} \cup \partial\mathbb{D}$ whose intersection is q_l. Now, by the way Σ was chosen we have that $\tilde{\gamma}_i'' \cap \Sigma = \{\eta_L, \eta_R\}$ with $\eta_L \in \Sigma_L$. This implies that $\tilde{\gamma}_i''$ has an endpoint t_L in A_L. Hence the geodesic $\tilde{\delta}'$ determined by the endpoints t_L and t_R intersects $\tilde{\delta}$ and $\delta_i = \pi(\tilde{\delta}')$ intersects $\delta = \pi(\tilde{\delta})$.

To finish we show that f has an infinite diameter orbit on the loop graph. Let $r \subset M$ be loop given by a topological embedding $r : (0, 1) \hookrightarrow S$. Without loss of generality we can suppose that:

(1) there exists $0 < \varepsilon < 1$ such that $r(0, \varepsilon) = I_0$ is an Euclidean segment in M.

(2) There exits a separatrix $\gamma_i \in \mathrm{Sep}_p(\mathcal{F}^u)$ of $\mathcal{F}^u$ based at p such that $I_0 \subset \gamma_i$ and (modulo taking an appropiate power) γ_i is invariant under f.

We now show that $(f^n(r))_{n\geq 0}$ is a sequence of loops which converges to the separatrix $\gamma_i \in \mathrm{Sep}_p(\mathcal{F}^u)$ and, in consequence, the clique of high-filling rays where this separatrix sits is attracting for the dynamics of f on the loop graph. Analogous arguments show that the clique of high-filling rays defined by $\mathrm{Sep}_p(\mathcal{F}^s)$ is repelling. The argument we provide is based on technical notions that we recall from Bavard and Walker [**?**] in the following paragraphs.

Equator and k-begin like. By Theorem 3.0.1 in [**?**], there is a geodesic fundamental polygon P for the hyperbolic surface $S_\mu = \mathbb{D}/\Gamma$ with no vertex in the interior of $\mathbb{D}$ and such that there is a parabolic element of Γ which fixes $\tilde{p}$. Moreover, Bavard and Walker show that there is a set of proper geodesic arcs on S_μ whose complementary components are two infinite-sided polygons. This set is called *an equator* on S and it is used by the same authors to topologize the Gromov boundary of the loop graph $L(S; p)$ by using the following combinatorial notion of convergence: two *oriented* loops (or rays) are said to $k - begin$ like each other if they cross the same initial k equatorial sides in the same direction. In simpler words: two loops (or rays) are close if they cross the same arcs of the equator in the same order for a long time.

Cover convergence. Let (x_n) be a sequence of rays or oriented loops and $\widetilde{x_n}$ a lift of x_n to $\mathbb{D}$ having $\tilde{p}$ as an extremity. The sequence (x_n) is said to *cover-converge* to a geodesic (ray or loop) l on S if the sequence of endpoints of the $\widetilde{x_n}$ which are different from $\tilde{p}$ converge to a point q such that the projection of the geodesic having as extremities $\tilde{p}$ and q is l. The following result relates the notions we have recalled.

Lemma 4.3 *Let (x_n) be a sequence of rays or loops and let l be a long ray. Then (x_n) cover-converges to l if and only if for all k there exists an N so that for all $n \geq N$, we have that x_n k-begins like l.*

This result is Lemma 5.2.5 in [**?**], which is based on Lemma 2.4.5 in [**?**], and the proof of the latter relies on the fact that the endpoint of a long ray in the conical cover is not on the boundary of the fundamental domain of S. We stress this property because it is also satisfied by any lift of the separatrix γ_i to the conical cover. Consequently, Lemma 4.3 remains valid if we replace (x_i) and l by $(f^n(r))_{n\geq 0}$ and γ_i respectively. We claim that for all k there exists an N such that for all $n \geq N$, we have that $f^n(r)$ k-begins like γ_i. As a result, we obtain that $(f^n(r))_{n\geq 0}$ cover converges to γ_i. That is, the endpoints of the lifts[8] of $(f^n(r))_{n\geq 0}$ converge to the endpoint of $\widetilde{\gamma_i}$ and hence we can conclude that $(f^n(r))_{n\geq 0}$ converges to γ_i.

The crucial remark to obtain the latest claim is that, since Df is a hyperbolic matrix and I_0 is parallel to one of its unstable eigendirections, the sequence of length of segments $(|f^n(I_0)|)_{n\geq 0}$ strictly increases with n and diverges. In consequence, $(f^n(I_0))_{n\geq 0}$ defines a saturation of the separatrix γ_i by Euclidean segments, that is, $f^n(I_0)$ is strictly contained in $f^{n+1}(I_0)$ and $\gamma_i = \cup_{n\geq 0} f^n(I_0)$. This implies that for all k there exists an N such that for all $n \geq N$, we have that $f^n(r)$ k-begins like γ_i. □

Remark 4.4 In the proof of Theorem 1.1 we made use of the fact that every short-ray or loop has a geodesic representative, but this is not necessary. As a matter of fact the following is true: if $\tilde{\delta}$ is any curve in $\mathbb{D}$ based at $\widetilde{p}$ whose extremities define two different points in $\partial\mathbb{D}$ and such that $\pi(\tilde{\delta}) = \delta$ is simple and does not belong to the set of separatrices $\{\gamma_1, \ldots, \gamma_m\}$, then for any $j = 1, \ldots, m$ the geodesic δ_j intersects δ.

On the other hand, in the proof of Theorem 1.1 the density of each separatrix of $\mathcal{F}^u$ or $\mathcal{F}^s$ on the *whole* surface S is not used. The proof remains valid if we only require separatrices of $\mathcal{F}^u$ and $\mathcal{F}^s$ to be dense on a subsurface $S' \subset S$ of finite type with enough topology, *e.g.* such that all curves defining the polygon on which p is contained are essential in S'. In particular we have the following:

Corollary 4.5 *Let $S' \subset S$ be an essential subsurface of finite topological type containing p and $h \in \mathrm{Mod}(S')$ a pseudo-Anosov element for which p is a k-prong singularity for some $k \in \mathbb{N}$. Let $\hat{h}$ be the extension (as the identity) of h to S. Then $\hat{h}$ is a loxodromic element of weight k. Moreover, by stretching (as described in the proof of Theorem 1.1) the separatrices of the invariant transverse measured foliations of h based at p define the cliques of high-filling rays fixed by $\hat{h}$.*

This result already appears in the work of Bavard and Walker [**?**], see Lemma 7.2.1 and Theorem 8.3.1.

[8] Here $\widetilde{\gamma_i}$ is a fixed lift of the separatrix and, if $f^n(r)(t)$ denotes the parametrization defined by considering the topological embedding $r : (0,1) \hookrightarrow S$ we fixed before, then we are considering the lifts $\widetilde{f^n(r)}$ to $\mathbb{D}$ with one extremity in $\tilde{p}$ such that $\widetilde{f^n(r)}(0,\epsilon)$ and $\widetilde{\gamma_i}$ intersect the same fundamental domain of S for sufficiently small ϵ.

4.2 Proof of Theorem 1.5

Unless otherwise stated, S denotes an infinite-type surface. In order to give the proof of Theorem 1.5, we develop the so-called *normal form* of the surface S. Very briefly, the normal form of the surface S is a surface homeomorphic to S that is constructed taking the boundary of a closed regular neighbourhood of a graph $T(S)$ embedded in the 3-dimensional hyperbolic space. Then, replacing S with its normal form and taking advantage of the combinatorics of the underlying graph $T(S)$ we show explicitly a pair of multicurves satisfying (1), (2), and (3).

4.2.1 Preliminaries

In this section, we present the *normal form* of S, that is, a model for S that is convenient for the proof of Theorem 1.5. We describe the general ideas behind this construction. Let $T2^{\mathbb{N}}$ be the rooted binary Cantor tree. We introduce the notion of *simple* subtree (Definition 4.6) of $T2^{\mathbb{N}}$ and we show that there exists a *simple* subtree $T_{\mathrm{Ends}(S)} \subseteq T2^{\mathbb{N}}$ such that its space of ends is homeomorphic to $\mathrm{Ends}(S)$ (Lemma 4.7). Here the notion of simple subtree is crucial to get a nice model for S. After this, we construct a subtree $\Gamma \subseteq T_{\mathrm{Ends}(S)}$ (non-empty if the genus of S is positive) such that the space of ends of Γ is homeomorphic to $\mathrm{Ends}_\infty(S)$. Modifying the vertices in Γ by simplicial triangles we obtain the graph $T(S)$; this operation is called *surgery*. Finally, we embed $T(S)$ in the 3-dimensional hyperbolic space and put S' equal to the boundary of a closed regular neighbourhood of $T(S)$. By the way $T(S)$ is constructed S and S' are homeomorphic. We call S' the *normal form* of S.

Normal forms for infinite-type surfaces. In what follows we detail how to construct, for any infinite type surface S, a graph $T(S) \subset \mathbb{H}^3$ having a regular neighbourhood whose boundary is homeomorphic to S (our choice of ambient space obeys illustrative purposes only). There are many ways to construct such graph. The one we present is intended to make the proof of Theorem 1.5 more transparent.

Let $2^{\mathbb{N}} := \prod_{j\in\mathbb{N}}\{0,1\}_j$ be the Cantor set. In general terms, the construction is as follows. We consider a rooted binary tree $T2^{\mathbb{N}}$, a homeomorphism $f : \prod_{j\in\mathbb{N}}\{0,1\} \to \mathrm{Ends}(T2^{\mathbb{N}})$ from the standard binary Cantor set to the space of ends of this tree, and we choose a topological embedding $i : \mathrm{Ends}(S) \hookrightarrow \prod_{j\in\mathbb{N}}\{0,1\}$. We show that there exists a subtree of $T2^{\mathbb{N}}$ whose space of ends is precisely $f \circ i(\mathrm{Ends}(S))$. For our purposes it is important that this subtree is *simple* (see Definition 4.6 below). Then, if S has genus, we perform a surgery on vertices of the aforementioned subtree of $T2^{\mathbb{N}}$ belonging to rays starting at the root and having one end on $f \circ i(\mathrm{Ends}_\infty(S))$.

The rooted binary tree. For every $n \in \mathbb{N}$ let $2^{(n)} := \prod_{i=1}^{n}\{0,1\}$ and $\pi_i : 2^{(n)} \to \{0,1\}$ the projection on to the i^{th} coordinate. The rooted binary tree is the graph $T2^{\mathbb{N}}$ whose vertex set $V(T2^{\mathbb{N}})$ is the union of the symbol $\mathfrak{r}$ (this will be the root of the tree) with the set $\{D : D \in 2^{(n)}$ for some $n \in \mathbb{N}\}$. The edges $E(T2^{\mathbb{N}})$ are $\{(\mathfrak{r},0),\ (\mathfrak{r},1)\}$ together with:

$$\{(D,D') : D \in 2^{(n)}\ D' \in 2^{(n+1)} \text{ for some } n \in \mathbb{N},\ \text{and } \pi_i(D) = \pi_i(D') \text{ for all } 1 \le i \le n\}$$

Henceforth $T2^{\mathbb{N}}$ is endowed with the combinatorial distance. For every $\hat{x} = (x_n) \in 2^{\mathbb{N}}$ we define $r(\hat{x}) := (\mathfrak{r}, x_1, x_2 \ldots, x_n, \ldots)$ to be the infinite geodesic ray in $T2^N$ starting from $\mathfrak{r}$ and ending in $\mathrm{Ends}(T2^{\mathbb{N}})$. Then, the map

$$f : \prod_{i \in \mathbb{N}}\{0,1\} \to \mathrm{Ends}(T2^{\mathbb{N}}) \tag{4}$$

which associates to each infinite sequence $\hat{x} = (x_n)_{n\in\mathbb{N}}$ the end $f(\hat{x})$ of $T2^{\mathbb{N}}$ defined by the infinite geodesic ray $r(\hat{x})$ is a homeomorphism.

Definition 4.6 Let v and v^* be two different vertices in a subtree $\mathcal{T}$ of $T2^{\mathbb{N}}$. If v is contained in the geodesic which connects v^* with $\mathfrak{r}$, then we say that v^* is a *descendant* of v. A connected rooted subtree of $\mathcal{T}$ without leaves is *simple* if all descendants of a vertex $v \neq \mathfrak{r}$ of degree two, also have degree two.

Lemma 4.7 *Let $F \subset \mathrm{Ends}(T2^{\mathbb{N}})$ be closed. Then there exists a* simple *subtree T of $T2^{\mathbb{N}}$ rooted at $\mathfrak{r}$ such that F is homeomorphic to* $\mathrm{Ends}(T)$.

We postpone the proof of this lemma to the end of the section.

Definition 4.8 Given a subset F of $\mathrm{Ends}(T2^{\mathbb{N}})$ we define $T_F := \bigcup_{\hat{x}\in f^{-1}(F)} r(\hat{x}) \subseteq T2^{\mathbb{N}}$ and call it the *tree induced by F*.

Surgery. Let T be a subtree of $T2^{\mathbb{N}}$ rooted at $\mathfrak{r}$ and having no leaves different from this vertex, if any. Let L be a subset of the vertex set of T. We denote by $\Gamma_{T,L}$ the graph obtained from T and L after performing the following operations on each vertex $v \in L$:

(1) If v has degree 3 with adjacent descendants v', v'' we delete first the edges $\{(v,v'),(v,v'',)\}$. Then we add to L two vertices v'_*, v''_* and the edges $\{(v,v'_*),$ $(v,v''_*),(v'_*,v'),(v''_*,v''),(v'_*,v''_*)\}$.

(2) If v has degree 2 and v' is its adjacent descendant, we delete first the edge (v,v'). Then we add to L two vertices v'_*, v''_* and the edges $\{(v,v'_*),(v,v''_*),(v'_*,v'),(v'_*,v''_*)\}$.

Definition 4.9 Let S be a surface of infinite type of genus $g \in \mathbb{N} \cup \{\infty\}$ and $\text{Ends}_\infty(S) \subset \text{Ends}(S) \subset 2^{\mathbb{N}}$ its space of ends accumulated by genus and space of ends, respectively. Abusing notation, we identify $\text{Ends}_\infty(S)$ and $\text{Ends}(S)$ with $f(\text{Ends}_\infty(S))$ and $f(\text{Ends}(S))$, respectively. We define the graph $T(S)$ according to the following cases. In all of them we suppose w.l.o.g that $T_{\text{Ends}(S)}$ is simple.

(1) If $g = 0$ let $T(S) := T_{\text{Ends}(S)}$,

(2) if $g \in \mathbb{Z}_{>0}$ let $T(S) := \Gamma_{T,L}$ where $T = T_{\text{Ends}(S)}$ and $L = a_1, a_2, \ldots, a_g$ and $(\mathfrak{r}, a_1, \ldots, a_g, \ldots) = r(\hat{x})$ for some $\hat{x} \in \text{Ends}(S) \subset 2^{\mathbb{N}}$, and

(3) if $g = \infty$ let $T(S) := \Gamma_{T,L}$ where $T = T_{\text{Ends}(S)}$ and L is the set of vertices of the subtree $T_{\text{Ends}_\infty(S)} \subset T_{\text{Ends}(S)}$.

By construction, there exists a geometric realization for $T(S)$ as a graph in the plane $\{(x, 0, z) \in \mathbb{H}^3 : z > 0\}$ in the 3-dimensional hyperbolic space (here we are using the upper half space model of $\mathbb{H}^3$), which we denote again by $T(S)$. Moreover there exists a closed regular neighbourhood $N(T(S))$ so that S is homeomorphic to $S' = \partial N(T(S))$, see Figure 9. Observe that $T(S)$ is a strong deformation retract of $N(T(S))$. We identify S with S', and we say that S is in *normal form*, and $T(S)$ is the *underlying* graph that induces S.

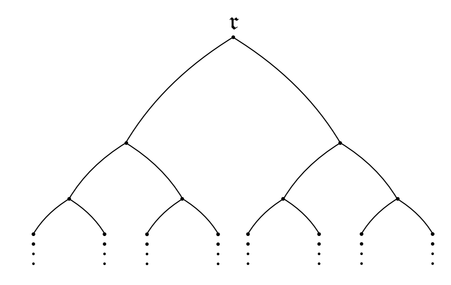


Figure 9: Embedding of the tree $T(S)$.

Remark 4.10 In [**?**], Walker and Bavard carry out a similar construction. For this they introduce the notion of *rooted core tree* T from which they construct a surface $\Sigma(T)$ homeomorphic to a given infinite type surface S, see Lemma 2.3.1 in [**?**]. The graph $T_{\text{Ends}(S)}$ (see Definition 4.9) is turned into a rooted core tree by declaring that the vertices of $T_{\text{Ends}_\infty(S)}$ are the marked vertices. The main difference with the work of Walker and Bavard is that the normal form we are proposing comes from a *simple* tree. This property is strongly used in the proof of Theorem 1.5.

Proof of Lemma 4.7. Let T_F be the subtree of $T2^{\mathbb{N}}$ induced by F. Let V' be the set of vertices of T_F of degree 2, different from $\mathfrak{r}$, having at least one descendant of degree 3. Then $V' = \sqcup_{i\in I} V'_i$, where:

(1) V'_i is a subset of the vertices of a ray $r(\hat{x})$ for some $\hat{x} \in f^{-1}(F)$,

(2) for every $i \in I$, one can label $V'_i = \{a_{i,1}, \ldots, a_{i,k_i}\}$ so that $a_{i,l+1}$ is a descendant of $a_{i,l}$ adjacent to $a_{i,l}$.

(3) for every $i \in I$, the vertex A_i of T_F adjacent to $a_{i,1}$ other than $a_{i,2}$ is either the root $\mathfrak{r}$ or a vertex of degree 3. Similarly, the vertex B_i of T_F adjacent to a_{i,k_i} other than a_{i,k_i-1} is of degree 3.

Replacing the finite simple path from A_i to B_i by an edge (A_i, B_i) does not modify the space of ends of T_F. By doing this for every $i \in I$ we obtain a simple tree as desired. □

Proof of Theorem 1.5. The proof is divided in two parts. First we show the existence of a pair of multicurves of finite type whose union fills S and which satisfy (1) and (2). In the second part we use these to construct the desired multicurves α and β satisfying (1), (2) and (3).

First part: Let S be an infinite-type surface in its normal form, and $T(S)$ the underlying graph that induces S. We are supposing that $T(S)$ is obtained after surgery from a simple tree as described above. The idea here is to construct two disjoint collections A (blue curves) and B (red curves) of pairwise disjoint curves in S such that after forgetting the non-essential curves in $A \cup B$, we get the pair of multicurves α and β which satisfy (1) and (2) as in Theorem 1.5. For this purpose we strongly use the combinatorics of $T(S)$.

Let $T_g(S)$ be the full subgraph of $T(S)$ generated by all the vertices which lie on a triangle in $T(S)$. Observe that, since $T(S)$ is constructed by performing a surgery on a simple tree, the graph $T_g(S)$ is connected.

Let $T'_g(S)$ be the subset obtained as the union of $T_g(S)$ with all the edges in $T(S)$ adjacent to $T_g(S)$. As $T_g(S)$ is connected, then $T'_g(S)$ is also connected. Let Δ be a triangle in $T_g(S)$, and Δ' be the disjoint union of Δ with all the edges in $T'_g(S)$ adjacent to Δ. We notice that Δ' is one of the following two possibilities: (1) the disjoint union of Δ with exactly three edges adjacent to it, or (2) the disjoint union of Δ with exactly two edges adjacent to it. For each case, we choose blue and red curves in S as indicated in Figure 10.

For each edge e in $T'_g(S)$ which connects two triangles in $T'_g(S)$, we choose a blue curve in S as indicated in Figure 11.

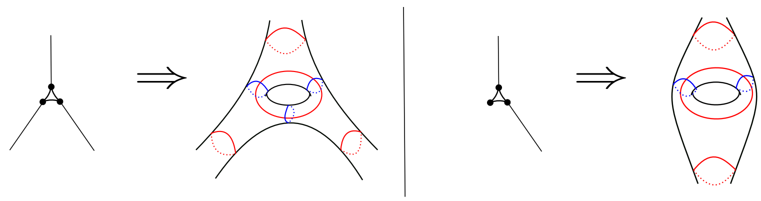


Figure 10: Blue and red curves associated to the neighbourhood Δ' of a triangle Δ.

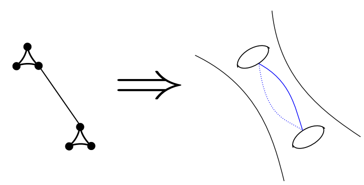


Figure 11: Blue curve associated to an edge which connects two triangles.

We consider the following cases.

$\underline{\mathrm{Ends}(S) = \mathrm{Ends}_\infty(S)}$. In this case $T_g(S) = T'_g(S) = T(S)$ and all curves in $A \cup B$ are essential simple close curves. Let $\alpha := A$ and $\beta := B$ up to isotopy. Given that every point in $T(S)$ is in a triangle or on an edge that connects two triangles we can deduce that $\alpha \cup \beta$ fills S. This implies the connectivity of the configuration graph $\mathcal{G}(\alpha \cup \beta)$. Condition (1) is satisfied because the geometric intersection of any curve in α with any other curve in β is at most 2, and any curve in α (respectively β) intersect at most 6 curves in β (respectively α). Finally, condition (2) is satisfied because we can verify that every complementary component of $S \setminus \alpha \cup \beta$ is a polygon with at most 6 sides. In Figure 12 we illustrate $T(S)$, the multicurves α and β in S, and the configuration graph $\mathcal{G}(\alpha \cup \beta)$ where S is the Jacob ladder surface, i.e., the infinite-type surface with exactly 2 ends each one accumulated by genus.

$\underline{\mathrm{Ends}(S) \neq \mathrm{Ends}_\infty(S)}$. Let C be a connected component of $T(S) - T'_g(S)$. Given that $T(S)$ is obtained from a simple tree, C is a tree with infinitely many vertices. If $\mathrm{Ends}_\infty(S) = \emptyset$ then $C = T(S)$ and we define $v := \mathfrak{r}$. In any other case, let v be the only vertex in C which is adjacent to an edge $e(v)$ in $T'_g(S)$. If v has degree one in C, then every vertex of C different from v has degree two because $T_{\mathrm{Ends}(S)}$ is a simple subtree of $T2^{\mathbb{N}}$. In this case, we have that the subsurface $S(C) \subset S$ induced by C is homeomorphic to a punctured disc. In particular, the red curve in S associated to the edge $e(v)$ chosen as depicted in Figure 10 is not essential in S.

Suppose now that v has degree two in C. We color with blue all the edges in C having vertices at combinatorial distances $2k$ and $2k + 1$ from v for every even $k \in \mathbb{Z}_{\geq 0}$. We

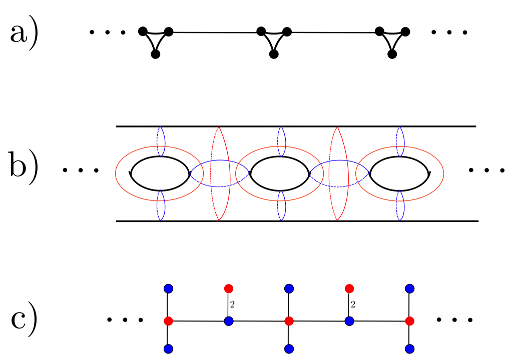


Figure 12: S is the Jacob ladder surface. a) Graph $T(S)$. b) The surface S with the multicurves α and β. c) Configuration graph $\mathcal{G}(\alpha \cup \beta)$; numbers indicate the multiplicity of the edges.

color all other edges in C in red, see the left-hand side in Figure 13. Let e and e' be two edges in C of the same color and suppose that they share a vertex v. Suppose that all vertices of $e \cup e'$ different from v have degree three. If e and e' are marked with blue color (respectively red color), we choose the red curve (respect. blue curve) in S as in the right-hand side of Figure 13.

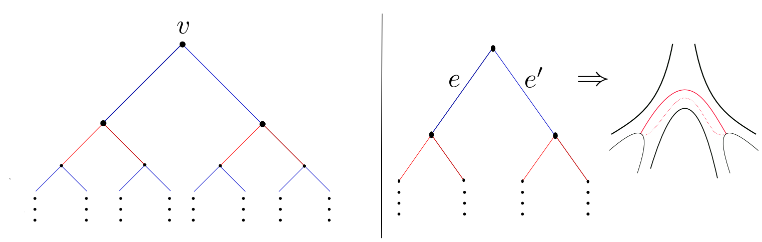


Figure 13: (Left) Edges of C colored with blue and red alternating in levels. (Right) The corresponding curve in S for the pair of edges e and e' marked with blue color.

For the edge $e(v) \in T'_g(S)$, we choose the blue curve in S as in the left-hand side of Figure 14. Finally, for each edge e of C, we take a curve in S with the same marked color of e as is shown in the right-hand side of Figure 14.

At this point we note that we must additionally treat the punctures on the surface because until now curves A and B do not necessarily fill S. For an example, in Figure 15 we illustrate the case of the flute surface together with blue and red curves built up to this point of the proof.

Let R be the full subgraph of $T(S)$ spanned by all the vertices of degree 3 in $T(S)$ together with the root vertex $\mathfrak{r}$, and define $T''_g(S)$ as the full subgraph of $T(S)$ generated

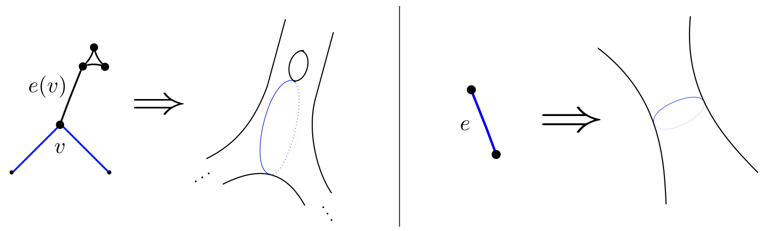


Figure 14: (Left) The corresponding curve for the e which connects F with $T'_g(S)$. (Right) Blue (red) curve associated to an edge e of F.

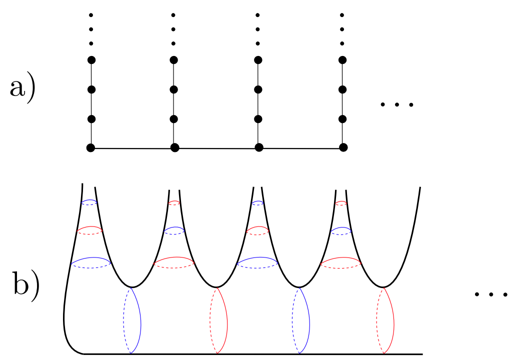


Figure 15: S is the flute surface. a) Graph $T(S)$. b) The surface S with the collection of curves A and B before treating the punctures.

by all the vertices in $T(S)$ at distance at most 1 from R. The graph $T''_g(S)$ is connected (again, because $T_{\text{Ends}(S)}$ is simple) and contains $T'_g(S)$. Suppose that $T(S) \setminus T''_g(S) \neq \emptyset$. Given that $T_{\text{Ends}(S)}$ is simple we have that every connected component of $T(S) \setminus T''_g(S)$ is a rooted tree of degree 2 (except at the root where it has degree 1); each corresponds to a puncture in the surface S. Let v_1 be a leaf in $T''_g(S)$, that is, a vertex of degree one. It exists because $T(S) \setminus T''_g(S)$ is not empty. If v_1 is adjacent to a vertex in a triangle at $T''_g(S)$, then the corresponding puncture is close to a genus in the surface. In this case, the connected component of $S \setminus A \cup B$ that contains the corresponding puncture is a punctured disc, so we are done. Suppose that v_1 is not adjacent to a vertex in a triangle. Let v be the vertex of $T''_g(S)$ adjacent to v_1, v_2 be the descendant of v at a distance 2 from v_1, and let e be the edge connecting v with v_2, see Figure 16. If e is marked with red color (respectively blue color), we choose the blue curve (respectively red curve) in S as is shown in Figure 16. This finishes the construction of the multicurves A and B and we define $\alpha = \{\alpha_i\}_{i \in I}$ and $\beta = \{\beta_j\}_{j \in J}$ as the set of essential curves in A and B, respectively. Figure 17 b) shows the collection of curves A and B in the flute surface after treating the punctures, compare with Figure 15.

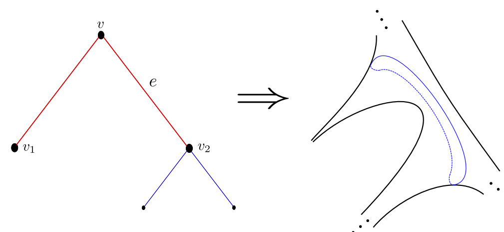


Figure 16: The corresponding blue curve in S for the red edge e.

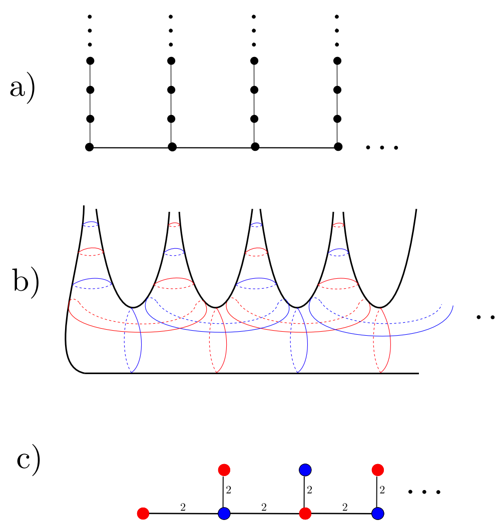


Figure 17: S is the flute surface. a) Graph $T(S)$. b) The surface S with the collection of curves A and B. c) Configuration graph $\mathcal{G}(\alpha \cup \beta)$; numbers indicate the multiplicity of the edges.

By construction, α and β are multicurves of finite type whose union fills S, and $\mathrm{i}(\alpha_{\mathrm{i}}, \beta_{\mathrm{j}}) \leq 2$ for every $i \in I$ and $j \in J$; hence (1) is satisfied. Upon investigation, one can observe that each connected component of $S \setminus \alpha \cup \beta$ is a disc or a punctured disc whose boundary is formed by 2, 4, 6 or 8 segments. This verifies condition (2), and we finish the proof of the first part.

Second part: In this part of the proof we construct a pair of multicurves satisfing (1), (2) and (3). Let $m \in \mathbb{N}$. Take a finite multicurve δ in S such that, if Q is the connected component of $S \smallsetminus \delta$ which contains p, we have that $Q \setminus p$ is homeomorphic to S_0^{m+3}, i.e., a genus zero surface with $m+3$ punctures. In $Q \setminus p$ we choose blue and red curves to form a chain as in Figure 18 and color them in blue and red so that no two curves of the same color intersect. We denote the blue and red curves in $Q \setminus p$ by α' and β' respectively. Remark that the connected component of $Q \smallsetminus (\alpha' \cup \beta')$ containing the point p is a $2m$-sided polygon.

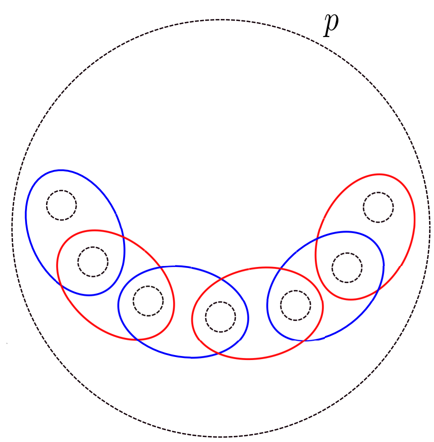


Figure 18: The multicurves α' (blue) and β' (red) in the surface $Q \setminus p$. The puncture p of $Q \setminus p$ is represented by the dotted outer circle.

The idea is to extend α' and β' to multicurves α and β, respectively, which satisfy all the desired properties. We consider two cases for the rest of the proof.

$\underline{m \text{ even}}$. Without loss of generality we suppose that all punctures in Q different from p are encircled by elements of α'. Now, let F be a connected component of $S \smallsetminus \alpha'$ not containing the point p. Then the closure $\overline{F}$ of F in S is a surface with $b > 0$ boundary components. Moreover, $\overline{F} \cap \beta'$ is a finite collection of disjoint essential arcs in $\overline{F}$ with end points in $\partial\overline{F}$, and the end points of an arc in $\overline{F} \cap \beta'$ are in a common connected component of $\partial\overline{F}$. We denote by θ_F the collection of arcs in $\overline{F} \cap \beta'$, and by δ_F the set of curves in δ contained in F.

Claim: There exists a pair of multicurves α''_F and β''_F whose union fills F, which satisfy (1) & (2) in Theorem 1.5 and such that $\theta_F \cap \beta''_F = \emptyset$.

Remark that if we define $\alpha := \alpha' \cup \left(\bigcup_{F \subset S \setminus \alpha'} \alpha''_F\right)$ and $\beta := \beta' \cup \left(\bigcup_{F \subset S \setminus \alpha'} \beta''_F\right)$, then α and β are the desired pair of multicurves.

We divided the proof of our claim in two cases: $b = 1$ and $b > 1$.

Case $b = 1$. If F is a finite-type surface it is not difficult to find the multicurves α''_F and β''_F. Suppose that F is of infinite-type. Notice that θ_F has at most two arcs. Take F in its normal form. We remark that the hole in F coming from the unique boundary component of $\overline{F}$ is a puncture in F; this puncture is called p'. Let α''_F and β''_F be the blue and red curves obtained from applying the first part of the proof of Theorem 1.5 to F. Up to taking a replacement of the arcs in θ_F, we can assume that θ_F does not intersect β''_F.

Case $b > 1$. Again, the case when F is a finite-type surface is left to the reader. If F is of infinite type, let γ be a separating curve in $\overline{F}$ which bounds a subsurface $W \subset \overline{F}$ of genus 0 with one puncture, b boundary components and such that $\partial\overline{W} = \partial\overline{F} \cup \gamma$ and write $\overline{F} \setminus \gamma = W \sqcup F_1$. Let θ_{F_1} be the set of arcs given by $\theta_F \cap \overline{F_1}$. Let η_1 and η_2 be two curves in F_1 (not necessary essential) such that γ, η_1 and η_2 bound a pair of pants P in F_1. If an element of θ_F intersects $\eta_1 \cup \eta_2$ then we replace it with one that doesn't and which is disjoint from all other arcs in θ_F. Up to making these replacements, we can assume that θ_F does not intersect $\eta_1 \cup \eta_2$, see Figure 19. Hence, $\theta_{F_1} \subseteq \overline{P}$. As $\overline{F_1}$ has one boundary component, by the case $b = 1$ above, there exists a pair of multicurves α''_{F_1} and β''_{F_1} which fills F_1 and such that $\theta_{F_1} \cap \beta''_{F_1} = \emptyset$. Define then $\alpha''_F := \{\gamma\} \cup \alpha''_{F_1}$ and $\beta''_F := \beta''_{F_1}$.

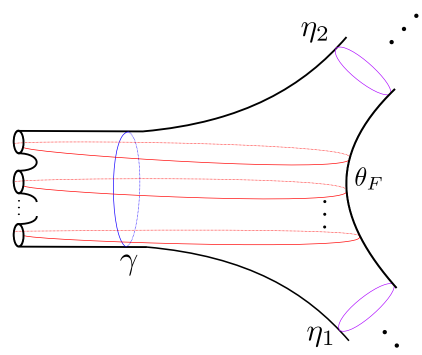


Figure 19: Case $b > 1$.

<u>m odd</u>. Without loss of generality we suppose that α' encircles all punctures in Q different from p except one. We add the curves α_1 and β_1 to α' and β' as depicted in Figure 20 respectively. Then we consider each connected component F of $S \setminus \alpha'$ and proceed as in the preceding case. □

Remark 4.11 If α and β are multicurves as constructed in the proof of Theorem 1.5, then $S \setminus \alpha \cup \beta$ is a family of polygons, each of which has either 2, 4, 6 or 8 sides. Hence, if $M = M(\alpha, \beta, \mathbf{h})$ is given by the Hooper–Thurston–Veech construction, then the set $\mathfrak{V}$ defined in the proof of Theorem 1.1 is formed by regular points and conical singularities of total angle π, 3π or 4π.

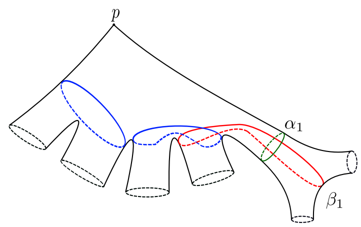


Figure 20: Case m odd.

4.3 Proof of Corollary 1.6

Our arguments use Figure 21. Let α and β be the multicurves in blue and red illustrated in the figure; the union of these fills the surface S in question (a Loch Ness monster). Let us write $\beta = \beta' \sqcup \{a_i\}_{i\in\mathbb{N}} \sqcup \{b_i\}_{i\in\mathbb{N}}$, and for each $n \in \mathbb{N}$ let $\beta_n := \beta' \sqcup \{a_i\}_{i\geq n} \sqcup \{b_i\}_{i\geq n}$. Theorem 1.1 implies that $f_n := T_\alpha \circ T_{\beta_n}^{-1}$ acts loxodromically on the loop graph $L(S;p)$ and hence it acts loxodromically on the main component of the completed ray graph $\mathcal{R}(S;p)$ (see Theorem 2.2). Remark that f_n converges to $f = T_\alpha \circ T_{\beta'}^{-1}$ in the compact-open topology. On the other hand, f fixes the short rays l and l' and hence it acts elliptically on both $\mathcal{R}(S;p)$ and $L(S;p)$. □

Remark 4.12 Using techniques similar to the ones presented in the proof of Theorem 1.5 one can construct explicit sequences (f_n) as above for any infinite-type surface S.

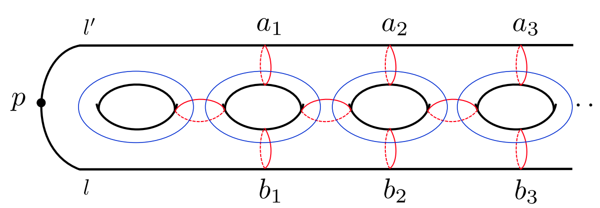


Figure 21:

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