A COUNTABLE DENSE HOMOGENEOUS SET OF REALS OF SIZE $\aleph_1$

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Abstract. We prove there is a countable dense homogeneous subspace of $\mathbb{R}$ of size $\aleph_1$. The proof involves an absoluteness argument using an extension of $L_{\omega_1\omega}(Q)$ obtained by adding predicates for Borel sets.

A separable topological space X is *countable dense homogeneous (CDH)* if given any two countable dense subsets $D, D' \subseteq X$ there is a homeomorphism h of X such that $h[D] = D'$. The main purpose of this note is to show the following.

Theorem 1. *There is a countable dense homogeneous set of reals X of size $\aleph_1$. Moreover, X can be chosen to be a λ-set.*

Recall that a set of reals is a *λ-set* if all of its countable subsets are relatively G_δ, and therefore it cannot be completely metrizable. Theorem 1 and this remark solve problems 390 and 389 from [4]. Our construction necessarily uses the Axiom of Choice. In [6] it was shown that under sufficient large cardinal assumptions every CDH metric space in $L(\mathbb{R})$ is completely metrizable. Our proof of Theorem 1 uses Keisler's completeness theorem for logic $L_{\omega_1\omega}(Q)$ (see §2), and the secondary purpose of this note is stating a somewhat general method for proving absoluteness of the existence of an uncountable set of reals properties of which are described using Borel sets as parameters.

1. A meager countable dense homogeneous set

Recall that every compact zero-dimensional subset of $\mathbb{R}$ without isolated points is homeomorphic (even isomorphic as linearly ordered sets) to the Cantor set.

Lemma 1.1. *There is an uncountable F_σ set F containing the rationals $\mathbb{Q}$ and an F_σ equivalence relation $E \subseteq F \times F$ with all equivalence classes countable dense subsets of $\mathbb{R}$, such that for every dense $A \subseteq \mathbb{Q}$ there is a homeomorphism $h : F \longrightarrow F$ satisfying*

(1) *$h[\mathbb{Q}] = A$ and*
(2) *$h(x)\, E\, x$ for every $x \in F$.*

Proof. Let $F = \mathbb{Q} \cup D \cup \bigcup_{n\in\omega} F_n$, where $\mathbb{Q}$ and D are disjoint countable dense subsets of $\mathbb{R}$ and $\{F_n : n \in \omega\}$ is a family of pairwise disjoint copies of the Cantor set disjoint from both $\mathbb{Q}$ and D and such that every nonempty open set contains one of the F_ns. Denote by $\mathcal{C}$ the set of all relatively clopen subsets of all the Cantor sets

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F_n. For every pair $U, W \in \mathcal{C}$ fix $h_{U,W} \colon U \longrightarrow W$ an increasing homeomorphism. Let $\mathcal{F}$ be the (countable) family of all compositions of finitely many functions of the type $h_{U,W}$ and their inverses. Then define $x \, E \, y$ if and only if $x, y \in \mathbb{Q} \cup D$ or $y = h(x)$ for some $h \in \mathcal{F}$. The relation E is then obviously an equivalence relation with countable and dense equivalence classes and it is F_σ as it is a countable union of compact sets.

Let $A \subseteq \mathbb{Q}$ be dense. Enumerate $\mathcal{C}$ as $\{A_n : n \in \omega\}$, $\mathbb{Q}$ as $\{q_n : n \in \omega\}$, D as $\{d_n : n \in \omega\}$, $D \cup (\mathbb{Q} \setminus A)$ as $\{c_n : n \in \omega\}$ and A as $\{a_n : n \in \omega\}$. Using the back-and-forth argument of Cantor, construct the homeomorphism $h : F \longrightarrow F$ as an increasing union of strictly increasing partial homeomorphisms $h_n, n \in \omega$, so that, for every $n \in \omega$:

(1) h_n extends h_{n-1},
(2) $\mathrm{dom}(h_n)$ consists of a finite subset of $\mathbb{Q} \cup D$ and a finite union of elements of $\mathcal{C}$,
(3) $\mathrm{range}(h_n)$ consists of a finite subset of $\mathbb{Q} \cup D$ and a finite union of elements of $\mathcal{C}$,
(4) h_n restricted to $\mathrm{dom}(h_n) \setminus (\mathbb{Q} \cup D)$ is covered by finitely many elements of $\mathcal{F}$,
(5) $h_n(q) \in A$ for every $q \in \mathbb{Q} \cap \mathrm{dom}(h_n)$,
(6) $h_n(d) \in D \cup (\mathbb{Q} \setminus A)$ for every $d \in D \cap \mathrm{dom}(h_n)$,
(7) $\{q_m : m \leq n\} \cup \{d_m : m \leq n\} \cup \bigcup\{A_m : m \leq n\} \subseteq \mathrm{dom}(h_n)$,
(8) $\{a_m : m \leq n\} \cup \{c_m : m \leq n\} \cup \bigcup\{A_m : m \leq n\} \subseteq \mathrm{range}(h_n)$.

Then $h = \bigcup_{n \in \omega} h_n$ is the desired homeomorphism of F. □

Recall that if E is an equivalence relation then a set X is *E-saturated* if for all $x \, E \, y$ we have $x \in X$ if and only if $y \in X$.

Lemma 1.2. *Assume $\mathbb{Q}, D, F, E$ and $\mathcal{F}$ are as in Lemma 1.1 and its proof. If $X \subseteq F$ is an E-saturated set such that for every countable $B \subseteq X$ there is an E-saturated $A \subseteq X$ containing B and a homeomorphism $h \colon X \to X$ satisfying $h[A] = \mathbb{Q}$, then X is countable dense homogeneous.*

Proof. Fix a countable dense subset B of X. Let g be an autohomeomorphism of X such that $g^{-1}(\mathbb{Q})$ is an E-saturated set containing B. Then $A = g[B]$ is a dense subset of $\mathbb{Q}$. By Lemma 1.1 there is an autohomeomorphism h of F such that $h[\mathbb{Q}] = A$ and $h(x) \, E \, x$ for every $x \in F$. Therefore $h \restriction X$ is an autohomeomorphism of X. Then $H = h^{-1} \circ g$ is an autohomeomorphism of X such that $H[B] = \mathbb{Q}$ as required. □

2. Absoluteness

Recall that $L_{\omega_1\omega}(Q)$ is an extension of the first-order logic that allows countable disjunctions and has quantifier Qx, 'there exists uncountably many.' It is well-known that completeness of this logic is useful for proving that the existence of certain objects of size $\aleph_1$ is absolute between models of ZFC (see [7, 1, 3, 5, 9]).

Let $L^B_{\omega_1\omega}(Q)$ be the extension of $L_{\omega_1\omega}(Q)$ allowing countably many Borel predicates in the following sense. For some Borel sets $A_n \subseteq (\mathbb{N}^\mathbb{N})^{k_n}$ $(n \in \mathbb{N})$ and Borel functions $f_n \colon (\mathbb{N}^\mathbb{N})^{l_n} \to \mathbb{N}^\mathbb{N}$ $(n \in \mathbb{N})$, we have relation and function symbols $\mathbf{A}_n$ and $\mathbf{f}_n$ of matching arity, and for $b_n \in \mathbb{N}^\mathbb{N}$ $(n \in \mathbb{N})$ we have constant symbols $\mathbf{b}_n$ $(n \in \mathbb{N})$.

If ϕ is a sentence of $L^B_{\omega_1\omega}(Q)$, we say that a model $\mathfrak{X}$ of ϕ (with universe X) is *correct* if

1. each $\mathbf{A}_n$ is interpreted as $A_n \cap X^{k_n}$, each $\mathbf{f}_n$ is interpreted as $f_n \restriction X^{l_n}$, each $\mathbf{b}_n$ is interpreted as b_n, and
2. if A_n is countable then $A_n \subseteq X$.

A model of an $L_{\omega_1,\omega}(Q)$ sentence is *standard* if it interprets Qx as 'there exist uncountably many. Recall that a linear order is ω_1*-like* if it is uncountable yet each of its initial segments is countable.

Theorem 2. *An $L^B_{\omega_1\omega}(Q)$-sentence ϕ has a correct model if and only if it has a correct model in some forcing extension $V^{\mathbb{P}}$ of the universe V.*

Let us postpone the proof of Theorem 2 for a moment. Fix an $L^B_{\omega_1\omega}(Q)$-sentence ϕ. We shall define an $L_{\omega_1\omega}(Q)$ sentence ϕ^M as follows. (For simplicity we shall treat only the case when we have only one Borel set, $A \subseteq \mathbb{N}^{\mathbb{N}}$; a standard coding argument shows that the general case with infinitely many Borel sets, functions and constants is really not any more general.) First, the language of ϕ is expanded by adding new symbols $\mathbf{N}$, $\mathbf{M}$, $\{\mathbf{c}_n : n \in \mathbb{N}\}$, $\mathbf{B}$ and $\{\mathbf{N}_s : s \in \mathbb{N}^{<\mathbb{N}}\}$. Let ϕ_0 be the conjunction of sentences stating the following:

1. $(\forall x)\mathbf{N}(x) \Leftrightarrow \bigvee_{n\in\mathbb{N}} x = \mathbf{c}_n$,
2. $(\forall x)\mathbf{B}(x) \Leftrightarrow \bigvee_{s\in\mathbb{N}^{<\mathbb{N}}} x = \mathbf{N}_s$,
3. axioms of formal arithmetic for $\mathbf{c}_n$ ($n \in \mathbb{N}$),
4. first-order properties of basic open sets $[s] = \{x \in \mathbb{N}^{\mathbb{N}} : s \sqsubset x\}$ for $\mathbf{N}_s$ ($s \in \omega^{<\omega}$),
5. if $\mathbf{M}(x)$, then $x \in \mathbf{N}_s$ for exactly one s of length n for all n, and moreover $\{s : x \in \mathbf{N}_s\}$ forms a chain (all this can clearly be stated in $L_{\omega_1\omega}$).

Since A is a Borel set, we can fix arithmetic formulas $\psi_0(x,y)$ and $\psi_1(x,y)$ such that $x \in A \Leftrightarrow (\forall y)\psi_0(x,y) \Leftrightarrow (\exists y)\psi_1(x,y)$. Let ϕ_i ($i < 2$) be the translation of ψ_i into the language of $\mathbf{N}_s$ ($s \in \omega^{<\omega}$). Replace each occurrence of $\mathbf{A}(x)$ in ϕ by $\mathbf{M}(x) \wedge (\forall y)\phi_0(x,y)$, and let ϕ^M be the conjunction of thus modified ϕ, ϕ_0, and $(\forall x)((\exists y)\phi_0(x,y) \vee (\exists y)\neg\phi_1(x,y))$.

Lemma 2.1. *An $L^B_{\omega_1\omega}(Q)$ sentence ϕ has a correct model if and only if ϕ^M has a standard model.*

Proof. Assume ϕ has a correct model $\mathfrak{X} = (X, A, \dots)$. Extend its universe by adding all natural numbers, basic open subsets of $\mathbb{N}^{\mathbb{N}}$, and the set Y of 'witnesses' defined as follows. If $x \in X \cap A$, pick y_x such that $\phi_0(x, y_x)$ holds. If $x \in X \setminus A$, pick y_x such that $\neg\phi_1(x, y_x)$ holds. Let $Y = \{y_x : x \in X\}$. Finally interpret M as X. It is clear that thus obtained model is a standard model of ϕ^M.

Now assume ϕ^M has a standard model, $\mathfrak{Z} = (Z, A', \dots)$. Let $X = \{x \in Z : \mathfrak{Z} \models \mathbf{M}(x)\}$, and let $\mathfrak{X}$ be the reduction of $(X, A' \cap X, \dots)$ to the language of ϕ. We only need to check that $\mathbf{A}$ is interpreted as $A' \cap X$. Note that $\mathfrak{Z} \models \phi_i(x,y)$ iff $\phi_i(x,y)$ holds, for $i < 2$. For every $x \in X$ we either have $\mathfrak{Z} \models \phi_0(x,y)$ or $\mathfrak{Z} \models \neg\phi_1(x,y)$ for some y. If $\mathfrak{Z} \models \phi_0(x,y)$ for some y, then $\mathfrak{Z} \models \mathbf{A}(x)$ and $x \in A$. On the other hand, if $\mathfrak{Z} \models \phi_1(x,y)$ for some y, then $\mathfrak{Z} \models \neg\mathbf{A}(x)$ and $x \notin A$. □

Proof of Theorem 2. By Lemma 2.1 ϕ has a correct model if and only if ϕ^M has a standard model. By Keisler's completeness theorem for $L_{\omega_1\omega}(Q)$ ([8]), ϕ^M has a standard model if and only if it is not inconsistent in the proof system described in

[8]. However, if ϕ^M is inconsistent in V, then it would remain such in the extension. If ϕ^M has a model $\mathfrak{X}$ in V then $\mathfrak{X}$ is a *weak* model (see [8]) of ϕ^M in $V^{\mathbb{P}}$, and again by Keisler's theorem ϕ^M has a standard model in $V^{\mathbb{P}}$ as well. □

In the following lemma $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are unary relation symbols, $\mathbf{h}$ is a unary function symbol and $\mathbf{f}$ is a binary function symbol. We say that a property is *expressible* in $L^B_{\omega_1\omega}(Q)$ if there is a sentence of $L^B_{\omega_1\omega}(Q)$ such that in each of its correct models the interpretations A, B, C, D, f, h of these predicates satisfy the stated property.

Lemma 2.2. *The following properties are expressible in* $L^B_{\omega_1\omega}(Q)$.

(1) A *is countable.*
(2) *A binary relation* $<$ *is an* ω_1*-like linear order.*
(3) $h\colon A \to B$ *is a surjection.*
(4) $h\colon A \to B$ *is a continuous function.*
(5) $h\colon A \to B$ *is a homeomorphism.*
(6) $h\colon A \to B$ *and it satisfies* $h[C] = D$.
(7) $f(x, \cdot)\colon A \to B$ *is a homeomorphism for every* x.
(8) x *is in the closure of* A.
(9) A *is a dense subset of* B .
(10) A *is a relatively open subset of* B.
(11) A *is a relatively* G_δ *subset of* B.
(12) B *has a countable dense subset* K *that is relatively* G_δ *in* B.
(13) X *is* E*-saturated, for a given Borel equivalence relation* E *all of whose equivalence classes are countable.*

Proof. Items (3) and (6) are first-order definable, and (1) and (2) are straightforward to define using Qx.

For (4), (5) and (8) one only needs to observe that since we have a standard model of $L_{\omega_1\omega}(Q)$, quantifiers such as $(\forall\epsilon > 0)(\exists\delta > 0)$ are evaluated correctly. Item (7) is immediate from the preceding items, and (10) and (9) are immediate from (8). For (11), introduce new predicates $\mathbf{A}_n$ $(n \in \mathbb{N})$ and require that each A_n is a relatively open set of B and $A = \bigcap_n A_n$.

To see (12), add a predicate for A and then use (1), (11), (2) and (9).

Let E be as in (13). It is well-known that there are Borel functions f_n $(n \in \mathbb{N})$ such that $x\,E\,y$ if and only if $(\exists n)x = f_n(y)$, hence for (13) we only need to add names for f_n $(n \in \mathbb{N})$ to our language.

□

3. Proof of Theorem 1

Assume $\mathbb{Q}, D, F, E$ and $\mathcal{F} = \{g_n : n \in \mathbb{N}\}$ are as in Lemma 1.1 and its proof. By Lemma 1.2, an uncountable E-saturated $X \subseteq F$ with an ω_1-like ordering $<$ such that

(1) Each E-equivalence class is an interval in $<$,
(2) There is a function $H\colon X \times X \to X$ such that for every $x \in X$:
    (a) $H(x, \cdot)$ is an autohomeomorphism of X,
    (b) $H(x, y) \in \mathbb{Q}$ if and only if $y < x$

will be countable dense homogeneous. By Lemma 2.2, the existence of X and H can be expressed in $L^B_{\omega_1\omega}(Q)$, and by Theorem 2 it suffices to show that X exists in some forcing extension. In order to assure that X is uncountable, we will force with a ccc poset. In [2] it was proved that if $\{C_\alpha : \alpha < \omega_1\}$ and $\{D_\alpha : \alpha < \omega_1\}$ are two families of pairwise disjoint countable dense subsets of $\mathbb{R}$ then a ccc forcing adds a homeomorphism $h : \bigcup_{\alpha<\omega_1} C_\alpha \longrightarrow \bigcup_{\alpha<\omega_1} D_\alpha$ such that $h[C_\alpha] = D_\alpha$ for every $\alpha < \omega_1$. Therefore, if we pick any ω_1 sequence of equivalence classes so that the first one is $\mathbb{Q} \cup D$ and well-order their union X in type ω_1 then a standard ccc forcing such that MA holds in the extension adds H with the required properties.

Since $\mathbb{Q}$ is a relatively G_δ subset of F, it is a countable dense and relatively G_δ subset of X. By the countable dense homogeneity, X is a λ-set.

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