INVARIANCE PROPERTIES OF ALMOST DISJOINT FAMILIES

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Abstract. We answer a question of Gracia-Ferreira and Hrušák by constructing consistently a MAD family maximal in the Katětov order. We also answer several questions of Garcia-Ferreira.

1. Introduction

We consider two kinds of closely related mathematical structures in this paper: almost disjoint families and cofinitary groups.
An infinite family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is *almost disjoint* (AD) if the intersection of any two distinct elements of $\mathcal{A}$ is finite. It is *maximal almost disjoint* (MAD) if it is not properly included in any larger AD family or, equivalently, if given an infinite set $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$.
An attempt to classify MAD families via Katětov order was initiated by the second author in [6] and continued in [[3], [7], [9]]. This analysis is analogous to the study of ultrafilters via the Rudin-Keisler order. The following theorem can be considered the main result of the paper, it answers one of the basic question about this ordering.

Theorem 1.1. $(\mathfrak{t} = \mathfrak{c})$ *There exists a MAD family maximal in the Katětov order.*

It is worth mentioning that a MAD family maximal in the Katětov order is the analogue of a selective ultrafilter in this context. This will be explained in detail in Section 3.

Cofinitary groups are subgroups of the symmetric group on ω, and therefore they have a natural action on ω. The structure of (maximal) cofinitary groups has received a lot of attention (recently see e.g., [2], [8]). For a nice survey of algebraic aspects of cofinitary groups consult Cameron's [4].

Definition 1.2. (i) For any set A we denote by $Sym(A)$ the group of permutations from A onto A, with the group operation given by

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composition. We write Id_A for the identity or just Id in case A is clear from the context.

(ii) We say that a subgroup $G \leq Sym(A)$[1]is *cofinitary* if any $g \in G \setminus \{Id\}$ has finitely many fixed points. i.e., the set $Fix(g) = \{x \in A : g(x) = x\}$ is finite.

Some of the interest in cofinitary groups derives from the fact that they are groups in which the graphs of all members are *almost disjoint*.

We can associate to each AD family $\mathcal{A}$ the subgroup $Inv(\mathcal{A})$ of $Sym(\omega)$ which consists of the permutations that preserve $\mathcal{A}$, i.e., $f[A] \in \mathcal{A}$ for all $A \in \mathcal{A}$. Also, we shall consider its module finite version $Inv^*(\mathcal{A}) = \{f \in Sym(\omega) : \forall A \in \mathcal{A}\ \exists A' \in \mathcal{A}, A' =^* f[A]\}$. We consider $Sym(\omega)$ as a topological group with the subspace topology of the product ω^ω. $Sym(\omega)$ is a polish group since $Sym(\omega)$ is a G_δ subspace of ω^ω. Garcia-Ferreira in [5] asked several questions concerning the existence of invariant subgroups of $Sym(\omega)$ with certain topological properties. In Section 2 we answer these questions and in the process we also construct a cofinitary group with special topological properties which is of independent interest.

Theorem 1.3. *There exists a countable dense cofinitary group.*

For convenience of the reader we state the questions of [5].

Question 1.4. For any countable $F \subseteq Sym(\omega)$ is there a MAD family $\mathcal{A}$ so that $F \subseteq Inv(\mathcal{A})$?

Question 1.5. Is there a MAD family $\mathcal{A}$ so that $Inv(\mathcal{A})$ is a closed subspace?

Question 1.6. Is there a MAD family $\mathcal{A}$ such that $Inv(\mathcal{A})$ is a dense subspace?

We answer the first question in the negative and the other two questions in the affirmative.

2. Cofinitary groups

The following Proposition gives a negative answer to question 1.4.

Proposition 2.1. *There is a countable subset F of $Sym(\omega)$ such that $F \nsubseteq Inv(\mathcal{A})$ for any MAD family $\mathcal{A}$.*

Proof. We shall show that the set F consisting of functions which are almost equal to the identity is as required.

[1]Here $\leq$ denotes the subgroup relation.

For each MAD family $\mathcal{A}$, choose $A \in \mathcal{A}$, $n \in A$ and $m \in \omega \setminus A$. Define $f \in Sym(\omega)$ as follows:

$$f(k) = \begin{cases} n & \text{if } k = m \\ m & \text{if } k = n \\ k & \text{if } k \notin \{n, m\} \end{cases}$$

Then $f \in F$ but $f \notin Inv(\mathcal{A})$ since $f[A] = (A \cup \{m\}) \setminus \{n\} \notin \mathcal{A}$. □

We shall need the following simple facts.

Fact 2.2. *If $\mathcal{A}$ and $\mathcal{B}$ are MAD families such that, for any $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ so that $A =^* B$, then $Inv^*(\mathcal{A}) = Inv^*(\mathcal{B})$.*

Fact 2.3. *Let $\mathcal{A}$ be a MAD family. For any $g \in Inv^*(\mathcal{A})$ and $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| < |\mathcal{A}|$, there are $X, Y \in \mathcal{A} \setminus \mathcal{B}$ such that $Y =^* g[X]$.*

We are now in position to provide an answer to Question 1.5.

Proposition 2.4. *There is a MAD family $\mathcal{A}$ so that $Inv(\mathcal{A}) = \{Id\}$.*

Proof. Let $\mathcal{C}$ be a MAD family of cardinality $\mathfrak{c}$ and let $\{f_\alpha : \alpha < \kappa\}$ be an enumeration of the set $Inv^*(\mathcal{C} \setminus \{Id\})$. We will construct recursively a family $\{B^i_\beta : i < 2, \beta < \kappa\} \subseteq \mathcal{C}$ satisfying:

(1) $\{B^0_\alpha, B^1_\alpha\} \cap \{B^i_\beta : i < 2, \beta < \alpha\} = \emptyset$ for any $\alpha < \kappa$ and
(2) $B^1_\alpha =^* f_\alpha[B^0_\alpha]$ for any $\alpha < \kappa$.

Suppose that we have constructed $\mathcal{B} = \{B^i_\beta : i < 2, \beta < \alpha\}$ satisfying (1) and (2) for some α. Using Fact 2.3, we can find $A, B \in \mathcal{C} \setminus \mathcal{B}$ so that $B =^* f_\alpha[A]$, we set $B^0_\alpha = A$ and $B^1_\alpha = B$. This finishes the recursive construction.
For each $\alpha < \kappa$, we choose $n_\alpha, m_\alpha \in \omega$ such that $n_\alpha \neq m_\alpha$ and $f_\alpha(m_\alpha) = n_\alpha$. We now set $A^0_\alpha = B^0_\alpha \cup \{m_\alpha\}$ and $A^1_\alpha = B^1_\alpha \setminus \{n_\alpha\}$. Observe that $A^1_\alpha \neq f_\alpha[A^0_\alpha]$. We define

$$\mathcal{A} = (\mathcal{C} \setminus \{B^i_\alpha : i \in 2, \alpha < \kappa\}) \cup \{A^i_\alpha : i \in 2, \alpha < \kappa\}.$$

It is easy to see that $\mathcal{A}$ is a MAD family and moreover, by Fact 2.2, $Inv^*(\mathcal{A}) = Inv^*(\mathcal{C})$.
Suppose that there is $f_\alpha \in Inv(\mathcal{A}) \setminus \{Id\} \subseteq Inv^*(\mathcal{C}) \setminus \{Id\}$, then,

$$f_\alpha[A^0_\alpha] = f_\alpha[B^0_\alpha \cup \{m_\alpha\}] =^* B^1_\alpha =^* A^1_\alpha$$

and also $f_\alpha[A^0_\alpha] \neq A^1_\alpha$, which is a contradiction since both belong to the same MAD family $\mathcal{A}$. □

The following lemma give us a useful combinatorial characterization of cofinitary groups.

Lemma 2.5. *If $G < Sym(\omega)$ is a countable group, then the following are equivalent:*

(i) For any $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that the family $\{f[B] : f \in G\}$ is almost disjoint,
(ii) G is cofinitary.

Proof. Let us first show that (i) implies (ii). Suppose that this is not the case, then there is $f \in G \setminus \{Id\}$ so that $B \in [Fix(f)]^\omega$. It follows that $Id[B] \cap f[B] = B$, which is a contradiction.
For the reverse implication. Let $\{f_k : k \in \omega\}$ be an enumeration of G with $f_0 = Id$ and let $A \in [\omega]^\omega$ be given. We shall construct recursively a family $\mathcal{B} = \{B_n : n < \omega\}$ such that:

(1) $B_0 = A$,
(2) $B_{n+1} \subsetneq B_n$,
(3) $|B_{n+1}| = \omega$ and
(4) the family $\{f_i[B_n] : i \leq n\}$ is disjoint.

Suppose we have constructed $\{B_i : i \leq k\}$, since $f_{k+1} \in G \setminus \{Id\}$ has finitely many fixed points we can find $C_0 \in [B_k]^\omega$ such that $f_{k+1}[C_0] \cap C_0 = \emptyset$.

Moreover $f_j^{-1} \circ f_{k+1} \in G \setminus \{Id\}$ for $0 < j < k+1$, so there exists $C_j \in [C_{j-1}]^\omega$ such that $(f_j^{-1} \circ f_{k+1})[C_j] \cap C_j = \emptyset$.

As each f_j is a bijection, we can infer from the last equation that

$$f_{k+1}[C_1] \cap f_1[C_1] = f_{k+1}[C_2] \cap f_2[C_2] = ... = f_{k+1}[C_k] \cap f_k[C_k] = \emptyset \quad (^*)$$

Fix $b \in B_k$ and set $B_{k+1} = C_k \setminus \{b\}$. It should be clear that $B_{k+1} \in [B_k]^\omega$. We are left to show that the family $\{f_i[B_{k+1}] : i \leq k+1\}$ is disjoint. Let $i, j \leq k+1$, $i \neq j$ be given. If $i < k+1$ and $j < k+1$, then $f_i[B_{k+1}] \cap f_j[B_{k+1}] \subseteq f_i[B_k] \cap f_j[B_k] = \emptyset$. On the other hand, if we have $i = k+1$ and $j < k+1$, then, since $B_{k+1} \subseteq C_k \subseteq \cdots \subseteq C_0 \subseteq B_k$ and by $(^*)$ we have $f_i[B_{k+1}] \cap f_j[B_{k+1}] = f_{k+1}[B_{k+1}] \cap f_j[B_{k+1}] \subseteq f_{k+1}[C_j] \cap f_j[C_j] = \emptyset$. This finish the recursive construction.
Choose $b_0 \in B_0$ and for each $n > 0$ we choose $b_n \in B_n \setminus B_{n-1}$. Let $B = \{b_n : n \in \omega\}$. Note that $B \subseteq^* B_n$ for any $n \in \omega$ and moreover the family $\{f[B] : f \in G\} = \{f_i[B] : i \in \omega\}$ is almost disjoint. □

The following is the well-known result of Cayley that any group can be represented as a group of permutations.

Theorem 2.6 (Cayley)**.** *For any group G there is a subgroup $H < Sym(G)$ such that*

(i) $G \cong H$ and
(ii) $\forall \pi \in H \setminus \{Id\}, Fix(\pi) = \emptyset$.

Condition (ii) follows from Caley's proof since the left action does not have fixed points.

Definition 2.7. Let X and Y be given such that $X \subseteq Y$ and $G < Sym(X)$, $H < Sym(Y)$. We say H is *final extension* of G if there is an isomorphism $\psi : G \to H$ such that $\psi(g) \restriction X = g$ for any $g \in G$.

We are now in position to prove the main theorem of the section. For more on constructions of cofinitary groups see e.g. [**?**, K]

Theorem 2.8. *There is a countable dense cofinitary group $G < Sym(\omega)$.*

Proof. Choose an enumeration $\{\pi_i : i \in \omega\}$ of $\bigcup_{i\in\omega\setminus\{0\}} Sym(i)$ with $\pi_0 \in Sym(1)$. We will construct recursively a family of groups $\{G^i_n : n \leq i < \omega\}$ and at the same time a strictly increasing sequence of natural numbers $\{n_i : i \in \omega\}$ such that $n_0 = 1$, $G^0_0 = \{Id\}$ and

(1) $\forall n \leq i < \omega \quad G^i_n < Sym(n_i)$,
(2) $\forall n \leq j < i < \omega \;\; G^i_n$ is a final extension of G^j_n,
(3) $\forall n < \omega \;\exists\, g \in G^n_n$ such that $\pi_n \subseteq g$ and
(4) $\forall j \leq i < \omega \forall f \in G^j_i$, $Fix(f) \subseteq n_j$.

Suppose that $\{G^i_n : n \leq i \leq k\}$ and $\{n_i \in \omega : i \leq k\}$ have been already constructed for some k.

Let t be minimal so that $n_k + t|G^k_k| \geq dom(\pi_{k+1})$ and let $n_{k+1} = n_k + t|G^k_k|$.

Claim: There is $G^{k+1}_k < Sym(n_{k+1})$ which is a final extension of G^k_k such that $\forall f \in (G^{k+1}_k \setminus \{Id\}), \quad Fix(f) \subseteq n_k$.

Proof of Claim: Apply Cayley's Theorem successively t times starting with $H_0 = G^k_k$ to obtain a sequence H_i $(i < t)$ so that $H_{i+1} < Sym(H_i)$ and $H_i \cong H_0$ for all $i < t$. Let ϕ_i denote the isomorphism between H_0 and H_i given by composition of Cayley's ones.
Let $X = n_k \cup \bigcup_{i<t} H_i$. Observe that $|X| = n_{k+1}$. For each $h \in H_0$, we define a permutation $\phi_h : X \to X$ given by $\phi_h(x) = \phi_i(h)(x)$ where i is the unique integer so that $x \in H_{i-1}$. Fix a bijection $\psi : X \to n_{k+1}$ and define $G^{k+1}_k = \{\psi \circ \phi_h \circ \psi^{-1} : h \in H_0\}$. It is easy to prove, by using the fact that Cayley representation does not have fixed points, that G^{k+1}_k is as required.

Let F be an isomorphism witnessing that G^{k+1}_k is a final extension of G^k_k. We know that $G^k_0 \leq G^k_1 \leq \cdots \leq G^k_{k-1} \leq G^k_k$. For each $j < k$, set $G^{k+1}_j = F[G^k_j]$, since F is an isomorphism, $G^{k+1}_0 \leq G^{k+1}_1 \leq \cdots \leq G^{k+1}_{k-1}$ and moreover, G^{k+1}_j is a final extension of G^k_j for each $j < k$.
In order to define G^{k+1}_{k+1}, consider the function $\overline{\pi} : n_{k+1} \to n_{k+1}$ defined as

$$\overline{\pi}(x) = \begin{cases} \pi_{k+1}(x) & \text{if } x \in dom(\pi_{k+1}) \\ x & \text{otherwise.} \end{cases}$$

Now we set

$$G^{k+1}_{k+1} \text{ to be the subgroup generated by } G^{k+1}_k \text{ and } \overline{\pi}.$$

It is clear, due to the construction, that n_{k+1}, $G^{k+1}_0, ..., G^{k+1}_{k+1}$ satisfy conditions (1)-(4),
It follows from condition (2) that for fix i the sequence G^j_i $(i \leq j)$ is a chain of a final extensions. Thus, there exists a group $G^\omega_i < Sym(\omega)$ which is a final extension of G^j_i for all $j \geq i$ (the group is constructed by gluing together the all the groups in the obvious way). We now define $G = \bigcup_{i\in\omega} G^\omega_i$. Note that G is a subgroup since for each i, $G^i_m \leq G^i_n$ whenever $m \leq n \leq i$. Therefore $G^\omega_m \leq G^\omega_n$ whenever $m \leq n$. It is easy to see that G is the desired group. □

We are ready to provide an answer to Question 1.6.

Theorem 2.9. *There is a MAD family $\mathcal{A}$ such that $Inv(\mathcal{A})$ is dense in $Sym(\omega)$.*

Proof. Let $G < Sym(\omega)$ be like in Theorem 2.8 and let

$$\Sigma = \{\mathcal{A} : \mathcal{A} \text{ is an AD family and } A \in \mathcal{A} \text{ iff } \{f[A] : f \in G\} \subseteq \mathcal{A}\}.$$

Note that by Lemma 2.5 $\Sigma \neq \emptyset$. Also $(\Sigma, \subseteq)$ is a partial order in which every chain has an upper bound. By an application of Zorn's Lemma there is $\mathcal{A}_0$ maximal in $(\Sigma, \subseteq)$. Note that $\mathcal{A}_0$ is dense since $G \subseteq Inv(\mathcal{A}_0)$. So it suffices to show that $\mathcal{A}_0$ is a MAD family. Suppose this is not the case, then there is $X \in [\omega]^\omega$ almost disjoint from $\mathcal{A}_0$. We infer from lemma 2.5 that there exists an infinite subset $Y \subseteq X$ so that $\{f[Y] : f \in G\}$ is almost disjoint. It follows that $\mathcal{B} = \mathcal{A}_0 \cup \{f[Y] : f \in G\}$ is almost disjoint and $\mathcal{B} \in \Sigma$ which contradicts the maximality of $\mathcal{A}_0$. □

3. A Katětov maximal MAD family

If $\mathcal{A}$ is a MAD family then $\mathcal{J}(\mathcal{A})$ denotes the ideal of all subsets of ω which can be almost covered by finitely many elements of $\mathcal{A}$, $\mathcal{J}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{J}(\mathcal{A})$ denotes the family of sets of positive measure. We also need the set $J^{++}(\mathcal{A})$ consisting of all $X \in P(\omega)$ so that there exists $\langle A_n : n \in \omega \rangle \subseteq \mathcal{A}$ such that $|X \cap A_n| = \omega$ for all $n \in \omega$. Note that for any MAD family $\mathcal{A}$, $J^+(\mathcal{A}) = J^{++}(\mathcal{A})$. In the case $\mathcal{A}$ is just an AD family the set $J^{++}(\mathcal{A})$ consist of the sets that remain positive for any AD family extending $\mathcal{A}$.
Recall the definition of Katětov order.

Definition 3.1. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. We say that $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}(I) \in \mathcal{J}$ for all $I \in \mathcal{I}$. If $\mathcal{A}$ and $\mathcal{B}$ are MAD families then we write $\mathcal{A} \leq_K \mathcal{B}$ for $\mathcal{J}(\mathcal{A}) \leq_K \mathcal{J}(\mathcal{B})$.

We refer to $\leq_K$ as the Katětov ordering.

For $h \in \omega^\omega$, a function $\phi : \omega \to [\omega]^{<\omega}$ with $|\phi(n)| \leq h(n)$ for all n is called an *h-slalom*. A function $\pi : [\omega]^{<\omega} \to \omega$ is said to be a *predictor*. If $h : \omega^{<\omega} \to \omega$, a function $\pi : \omega^{<\omega} \to [\omega]^{<\omega}$ with $|\pi(s)| \leq h(s)$ for all s is called an *h-slalom predictor*.
The following theorem give us a several characterizations of $non(\mathcal{M})$ in terms of families of functions.

Theorem 3.2. *The following are equivalent for any cardinal κ.*

(i) $non(\mathcal{M}) > \kappa$,
(ii) for all $\mathcal{F} \subseteq \omega^\omega$ *of size* $\leq \kappa$ *there is* $g \in \omega^\omega$ *such that for all* $f \in \mathcal{F},\ f(n) \neq g(n)$ *holds for almost all* n,
(iii) for all families Π *of predictors of size* $\leq \kappa$ *there is* $g \in \omega^\omega$ *such that for all* $\pi \in \Pi,\ g(n) \neq \pi(g \restriction n)$ *holds for almost all* n,
(iv) any of (ii) through (iii) with the additional stipulation that g *be injective.*
(v) any of (ii) through (iii) with the additional assumptions that the families consists of partial functions. Moreover, for every $X \in [\omega]^\omega$ *we can find* g *so that the range of* g *is contained in* X.

Proof. (i) to (iii) is the well-known Bartoszynski-Miller characterization of $non(\mathcal{M})$ (see [1]). Details for showing that (iv) is equivalent to (ii) can be found in [2]. Since (v) is a strengthening of the preceding ones, it suffices to prove that (ii) implies (v). Let $\mathcal{F}$ be a family of $\leq \kappa$ partial functions by extending every function arbitrarily we may assume that the domain of each function is all ω. Now, let $\mathcal{F}' = \{f \restriction_{f^{-1}(X)} : f \in \mathcal{F}\}$ applying (iii) to the space X^ω and the family $\mathcal{F}'$ we obtain the desired conclusion. □

In order to prove Theorem 1.1 we shall need a slight generalization of the concept of cofinitary group.

Definition 3.3. Let G be a subset of injective partial functions from ω into ω closed under compositions and inverses. We say that G is a *partial cofinitary semigroup* if for every $f \in G$ either f is a partial identity or f has finitely many fix points.

The following lemma will play a key role in the construction of a MAD family maximal in the Katětov order.

Lemma 3.4. *Let* G *be a partial cofinitary semigroup of cardinality* $< non(\mathcal{M})$ *and* $X \in [\omega]^\omega$ *then there exists* $f : \omega \to X$ *such that* $G * f$ *is a partial cofinitary semigroup.*

Proof. Define an operation $F : \omega^{\leq\omega} \to \omega^\omega$ recursively as follows: let $n \in \omega, f \in \omega^{\leq\omega}$ and assume $F(f)(k)$ and $F(f)^{-1}(k)$have been defined for $k < n$. If $F(f)^{-1}(k) = n$ for some $k < n$, then clearly $F(f)(n) = k$. If not, then let

$F(f)(n) = f(2n)$. If $F(f)(k) = n$ for some $k < n$, then clearly $F(f)^{-1}(n) = k$. If $n \in X$, then let $F(f)^{-1}(n) = f(2n+1)$. If $n \notin X$, then $F(f)^{-1}$ is not defined at n.

If H is a partial cofinitary semigroup, a *word* $w(x)$ in variable x from H is an expression of the form

$$g_0 \cdot x^{m_0} \cdot ... \cdot g_{l-1} \cdot x^{m_{l-1}} \cdot g_l$$

such that $g_i \in H,\ g_i \neq Id$ for $1 \leq i \leq l-1$, and $m_i \in \mathbb{Z} \setminus \{0\}$ for all i. The *length* of such a $w(x)$ is $lg(w(x)) = |\{i \leq l : g_i \neq Id\} + \sum_{i<l} |m_i|$. For a word $w(x)$, an injective finite partial function (not necessarily in $\omega^{<\omega}$), we form the (possible empty) injective partial function $w(t)$ in the usual manner. Also, if g is an injective partial function, we define $w(g)$ as usual. Given a word $w(x)$, define a predictor $\pi_{w(x)}(s)$ by $w(F(s))(n)$ where $2n+e = |s|$ ($e \in \{0,1\}$) for $s \in S$ (S denotes the set of injective finite functions from ω into ω).

Now let H be a partial cofinitary semigroup of size $< non(\mathcal{M})$. We have to show that H is not maximal. By the injective version of (v) in Theorem 3.3, there is $f : \omega \to X$ injective such that for all $\pi_{w(x)}$ with $w(x)$ being a word from H, $\pi_{w(x)}(f \restriction n) \neq f(n)$ holds for almost all n. We claim that $G = H * F(f)$ is a partial cofinitary semigroup. Since all elements of G are of the form $w(F(f))$, where $w(x)$ is a word from H, it suffices to show that that for all such words $w(x) \neq Id$. This is done by induction on $lg(w(x))$.

Basic Step. $lg(w(x)) = 1$. Then either $w(x) = g_0$ for $g_0 \in H \setminus \{Id\}$ in which case there is nothing to prove, or $w(x) = x$ or $w(x) = x^{-1}$. Since $\pi_1(f \restriction n) \neq f(n)$ for almost all n (where π_1 is the predictor associated with the word representing the identity), it follows that $F(f)(k) = f(2k) \neq k$ for almost all k.

Induction Step. Assume $w(x) = g_0 \cdot x^{m_0} \cdot ... \cdot g_{l-1} \cdot x^{m_{l-1}} \cdot g_l$ is a word of length at least two and the claim has been proved for all shorter words. For $k < \sum_{i<l} |m_i|$ we define the *chopped word* $w_k(x)$ and the *inverse chopped word* $w_k^{-1}(x)$ basically by removing the occurrence of x, as follows. First let $j < k$ be such that $\sum_{i<j} |m_i| \leq k < \sum_{i<j+1} |m_i|$ and assume $k = \sum_{i<j} |m_i| + k'$ with $0 \leq k' < |m_j|$. Then $w_k(x)$ is the reduced word obtained from the word

$$x^{sgn(m_j)(|m_j|-k'-1)} \cdot g_{j+1} \cdot x^{m_{j+1}} \cdot ... \cdot x^{m_{i-1}} \cdot g_l \cdot g_0 \cdot x^{m_0} \cdot ... \cdot g_j \cdot x^{sgn(m_j)k'},$$

and w_k^{-1} is simply its inverse.

Now let n^* be large enough so that for all $n \geq n^*$ the following hold:

(i) the values

$$n, (F(f)^{sgn(m_{l-1})} \cdot g_l)(n),$$

$$(F(f)^{sgn(m_{l-2}) \cdot 2} \cdot g_l)(n), ..., (F(f)^{m_{l-1}} \cdot g_l)(n), ...,$$

$$(F(f)^{m_0 - sgn(m_0)} \cdot g_1 \cdot ... \cdot g_{l-1} \cdot F(f)^{m_{l-1}} \cdot g_l)(n),$$

and in case $g_l \neq Id$ also $g_l(n)$, and in case $g_0 \neq Id$ also

$$(F(f)^{m_0} \cdot g_1 \cdot \ldots \cdot g_{l-1} \cdot F(f)^{m_{l-1}} \cdot g_l)(n),$$

are all distinct as well as
(ii) for each $k < \sum_{i<l} |m_i|$ with $k = \sum_{i<j} |m_i| + k'$, if

$$n' = (F(f)^{-sgn(m_j) \cdot k'} \cdot g_j^{-1} \cdot \ldots \cdot F(f)^{-m_0} \cdot g_0^{-1})(n),$$

then $f(2n') \neq \pi_{w_k^{-1}(x)}(f \restriction 2n')$.
By induction hypothesis, and since there are only finitely many k and for each k only finitely many n' for which (ii) can fail, it is clear that there is such an n^*. We claim that $w(f)(n) \neq n$ for each $n \geq n^*$.
Assume this were not the case and fix $n \geq n^*$ with $w(F(f))(n) = n$. For each $k < \sum_{i<l} |m_i|$ with $k = \sum_{i<j} |m_i| + k'$, let

$$n_k = \min\{(f^{sgn(m_j)(|m_j|-k'-1)} \cdot \ldots. \cdot f^{m_{l-1}} \cdot g_l)(n),$$
$$(f^{sgn(m_j)(|m_j|-k')} \cdot \ldots \cdot f^{m_{l-1}} \cdot g_l)(n)\}.$$

Now note that by (i), there can be at most two values k_0 and k_1 for k such that n_k is maximal; and if there are two they must be adjacent; i.e., $k_1 = k_0 + 1$ without loss. Let $j < l$ be such that this (these) maximal value(s) n_k occur(s) at $k = \sum_{i<j} |m_i| + k'$ for some k'. We need to consider four cases.
Case 1. $m_j > 0$, and either there are $k_1 = k_0 + 1$ such that $n_{k_0} = n_{k_1}$ is maximal in which case we let $k = k_1$, or there is a unique k such that n_k is maximal and one has $n_k = (f^{sgn(m_j)(|m_j|-k')} \cdot \ldots. \cdot f^{m_{l-1}} \cdot g_l)(n)$. Note that in the former case n_k must necessarily have the value $(f^{sgn(m_j)(|m_j|-k')} \cdot \ldots \cdot f^{m_{l-1}} \cdot g_l)(n)$. Also note that since we assume $w(f)(n) = n$ we additionally have $n_k = (f^{-sgn(m_j)k'} \cdot \ldots \cdot f^{-m_0} \cdot g_0^{-1})(n)$. Now,

$$\pi_{w_k(x)}(f \restriction_{n_k+1}) = w_k(f \restriction_{n_k+1})(n_k)$$

because the right-hand side is indeed defined by maximality of n_k. $w(f)(n) = n$ clearly entails

$$w_k(f \restriction_{n+1})(n_k) = f^{-1}(n_k).$$

However, by (ii), we get

$$\pi_{w_k(x)}(f \restriction_{n_k+1}) \neq f(n_k),$$

a contradiction.
Case 2. $m_j < 0$, and either there are $k_1 = k_0 + 1$ such that $n_{k_0} = n_{k_1}$ is maximal in which case we let $k = k_0$, or there is a unique k such that n_k is maximal and one has $n_k = (f^{sgn(m_j)(|m_j|-k'-1)} \cdot \ldots. \cdot f^{m_{l-1}} \cdot g_l)(n)$. In this case use $\pi_{w_k^{-1}(x)}(f \restriction_{n_k+1})$ to derive a contradiction.
Case 3. $m_j > 0$ and there is a unique k such that n_k is maximal and one has $n_k = (f^{sgn(m_j)(|m_j|-k'-1)} \cdot \ldots \cdot f^{m_{l-1}} \cdot g_l)(n)$. Use $\pi_{w_k^{-1}(x)}(f \restriction_{n_k})$.
Case 4. $m_j < 0$ and there is a unique k such that n_k is maximal and one

has $n_k = (f^{sgn(m_j)(|m_j|-k')} \cdot ... \cdot f^{m_{l-1}} \cdot g_l)(n)$. Use $\pi_{w_k(x)}(f \restriction_{n_k+1})$. These contradictions complete the proof of the theorem. □

We recall the following definitions from [6].

Definition 3.5. We say that a MAD family $\mathcal{A}$ is K-*uniform* if $\mathcal{A} \leq_K \mathcal{A} \restriction X$ for every $X \in J^+(\mathcal{A})$.

Definition 3.6. We say that a MAD family $\mathcal{A}$ is *tight* (*weakly tight*) if for every $\langle X_n : n \in \omega \rangle \subseteq J^+(\mathcal{A})$ there is $A \in \mathcal{A}$ so that $\forall n\ (\exists^\infty n),\ |A \cap X_n| = \omega$.

The following proposition from [6] shows that (weakly) tight MAD families are almost maximal in the Katětov order.

Proposition 3.7. *Let $\mathcal{A}$ be a weakly tight MAD family and let $\mathcal{B}$ be a MAD family. If $\mathcal{A} \leq_K \mathcal{B}$ then there exists an $X \in J^+(\mathcal{A})$ such that $\mathcal{B} \leq_K \restriction X$.*

Recently Raghavan and Steprans [11], using a novel technique of Shelah, showed that assuming $\mathfrak{s} \leq \mathfrak{s}$ there is a weakly tight MAD family.
We are now in position to prove the main theorem of the paper.

Theorem 3.8. *Assuming $\mathfrak{t} = \mathfrak{c}$. There exists a MAD family maximal in the Katětov order.*

Proof. By propotion 3.7, it suffices to construct a tight K-uniform MAD family. In order to do this, enumerate $([\omega]^\omega)^\omega$ as $\{\vec{X}_\alpha : \alpha < \mathfrak{c}\}$ in such a way that each sequence appears cofinally many times. We shall construct recursively an increasing sequence $\mathcal{A}_\alpha,\ \alpha < \mathfrak{c}$ of almost disjoint families and a sequence $\{_\alpha\ \alpha < \mathfrak{c}$ of injective partial functions from ω into ω so that $\mathcal{A}_0$ is a partition of ω into infinitely many infinite pieces and $f_0 = Id$ for every $\alpha < \mathfrak{c}$:

1. $|\mathcal{A}_\alpha| < \mathfrak{c}$,
2. the set $\mathcal{F}_\alpha$ consisting of elements of the form $w(f_{\xi_1}, ..., f_{\xi_n})$ is a partial cofinitary semigroup where $w(x_1, .., x_n)$ is a reduced word in n variables and $\xi_1, ..., \xi_n < \alpha$,
3. $\mathcal{F}_\alpha$ is a strictly increasing sequence of partial cofinitary semigroups of cardinality $< \mathfrak{c}$,
4. $\mathcal{F}_\alpha$ respects $\mathcal{A}_\alpha$, i.e., $f^{-1}(A) \in \mathcal{A}_\alpha$ for all $A \in \mathcal{A}_\alpha$ and all $f \in \mathcal{F}_\alpha$,
5. if $\vec{X}_\alpha \subseteq \mathcal{J}(\mathcal{A}_\alpha)^{++}$ then there exists $A \in \mathcal{A}_{\alpha+1}$ such that $A \cap \vec{X}_\alpha(n)$ is infinite for all $n \in \omega$,
6. if $\vec{X}_\alpha(0) \in \mathcal{J}(\mathcal{A}_\alpha)^{++}$ then there exists $f : \omega \to \vec{X}_\alpha(0)$ with $f \in \mathcal{F}_{\alpha+1}$.

For α limit let $\mathcal{F}_\alpha = \bigcup\{\mathcal{F}_\beta : \beta < \alpha\}$ and $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$.
For $\alpha = \beta + 1$ consider $\mathcal{A}_\beta$ and $\mathcal{F}_\beta$. If $\vec{X}_\alpha(0) \in \mathcal{J}(\mathcal{A}_\alpha)^{++}$ then, using Lemma 3.3, we can find a bijection $f : \omega \to X$ between ω and a subset X almost disjoint from every element of $\mathcal{A}_\beta$ so that $\mathcal{F}_\beta * f$ is a partial cofinitary

semigroup, we set $f_\alpha = f$. It is easy to verify that (1), (2) and (4) holds.
In order to construct $\mathcal{A}_\alpha$, enumerate $\mathcal{F}_\alpha$ as $\{f_\gamma : \gamma < \kappa\}$, and assume that $\vec{X}_\alpha \subseteq \mathcal{J}(\mathcal{A}_\alpha)^{++}$. We may assume that $\vec{X}$ is a partition of ω. For each n, recursively choose a $\subseteq^*$-decreasing sequence T^n_γ ($\gamma < \kappa$) of infinite subsets of $\vec{X}_\alpha(n)$ so that:

(i) $T^n_0 \subseteq \vec{X}_\alpha(n)$ is almost disjoint from all elements of $\mathcal{A}_\alpha$,
(ii) for $\gamma < \kappa$, $f^{-1}_\gamma(T^n_\alpha)$ is almost disjoint from every element of $\mathcal{A}_\alpha$,
(iii) for every $\xi, \eta \leq \gamma < \kappa$,and for every $n, m < \omega$ $f^{-1}_\xi(T^m_\gamma) \cap f^{-1}_\eta(T^n_\gamma)$ is finite.

Note that (ii) follows directly from (i) and the fact that $\mathcal{F}_\alpha$ respects $\mathcal{A}_\beta$.
Assume that T^n_ξ, $\xi < \gamma$ has been successfully constructed. Choose $S^n \in [\vec{X}_\alpha(n)]^\omega$ such that $S^n \subseteq^* T^n_\xi$ for $\xi < \gamma$. Since $\mathcal{F}_\alpha$ is a partial cofinitary semigroup there exists $S^n_0 \in [S^n]^\omega$ so that $f^{-1}_\alpha(S^n_0)$ is almost disjoint from $\mathcal{A}_\beta$. Note that if T^n_α is a subset of S^n_0 then (i) and (ii) are satisfied. In order to find T^n_α so that (iii) holds enumerate all pairs ξ, η, $\xi, \eta \leq \alpha$ as $\{(\xi_\zeta, \eta_\zeta) : \zeta < \lambda$. Note that $\lambda < \mathfrak{t}$. Construct another decreasing sequence $\{S^n_\zeta : \zeta < \lambda\}$ (S^n_0 has already been chosen) so that for all $n.m < \omega$

$$f^{-1}_{\xi_\zeta}(S^n_{\zeta+1}) \cap f^{-1}_{\eta_\zeta}(S^m_{\zeta+1}) =^* \emptyset.$$

Now that is easy to do as $\mathcal{F}_\alpha$ is a partial cofinitary semigroup we can always find an infinite subset of S^n_ζ and S^m_ζ so that their pre images are almost disjoint. Finally choose $T^n_\alpha \in [S^n_0]$ so that $T^n_\alpha \subseteq^* S^n_\zeta$ for all $\zeta < \lambda$. This finishes the construction.
Let $\{T^n_\gamma : \gamma < \kappa, n < \omega\}$ be the sequence satisfying the above requirements (i)-(iii). As $\kappa < \mathfrak{t}$ we can find a pseudo-intersection T^n of the family $\{T^n_\gamma : \gamma < \kappa\}$ for all $n \in \omega$.
Let $T = \bigcup T_n$. Fix an enumeration $\{f_\gamma : \gamma < \kappa\}$ of $\mathcal{F}_{\alpha+1}$ and let $\{(\gamma_\xi, \delta+\xi) : \xi < \kappa\}$ be an enumeration of all ordered pairs $(\gamma, \delta) \in \kappa \times \kappa$. For each $\xi < \kappa$ and $n < \omega$, let f^n_ξ be the function from ω into ω defined as follows:

$$f^n_\xi(k) = \max(f_{\gamma_\xi}([T^n] \cap f_{\delta_\xi}[T^k].$$

Since $\kappa < \mathfrak{b}$ we can find $h : \omega \to \omega$ so that $f^n_\xi \leq^* h$ for all $\xi < \kappa$ and all $n < \omega$. Let $A = \bigcup_{n\in\omega}(T^n \setminus h(n))$. Set

$$\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{w(f_{\beta_1}, ..., f_{\beta_n})[A] : w(x_1, ..., x_n) \text{ is a reduced word in n variables}$$
$$\text{and } f_{\beta_1}, ..., f_{\beta_n} \in \{f_g : \gamma \leq \alpha+1\}\}.$$

It is easy to see that $\mathcal{A}_{\alpha+1}$ is an AD family and satisfies the required properties. This finishes the proof of the Theorem. □

We will finish with some open questions.

Question 3.9. Does there exists a MAD family maximal in the Katětov order which is weakly tight but not tight?

Question 3.10. Is every MAD family maximal in the order of Katětov weakly tight?

Question 3.11. Is it consistent with ZFC that there are no Katětov maximal MAD families?

References

[1] T. Bartoszynski and H. Judah, *Set Theory, On the Structure of the Real Line,* Peters, Wellesley, MA, 1995.

[2] J. Brendle, O. Spinas, Y. Zhang, *Uniformity of the Meager ideal and Maximal Cofinitary Groups,* Journal of Algebra 232, 209-225 (2000).

[3] J. Brendle and S. Yatabe, *Forcing indestructibility of MAD families*, Ann. Pure Appl. Logic, 132 (2-3): 271-312, 2005.

[4] P. Cameron, *Cofinitary permutation groups*, Bull. London Math. Soc., 28 (2): 113-140, 1996.

[5] S. Garcia-Ferreira, *Continuos functions between Isbell-Mrówka spaces*, Comment. Math. Univ. Carolinae 21 (1980), 742-769.

[6] S. Garcia-Ferreira and M. Hrušak, *Ordering MAD families a la Katětov*, J. Symbolic Logic, Vol. 68, No. 4, (2003), pp. 1337-1353.

[7] S. Garcia-Ferreira and P. Szeptycki, *MAD families and P-points*, Comment. Math. Univ. Carolin., 48 (4): 699-705, 2007.

[8] M. Hrušák, J. Steprans and Y. Zhang, *Cofinitary Groups, almost Disjoint and Dominating Families*, J. Symbolic Logic, Vol. 66, No. 3, (2001), pp. 1259-1276.

[9] M. Hrušák and J. Zapletal, *Forcing with quotients*, Arch. Math. Logic, 47 (7-8): 719-739, 2008.

[10] B. Kastermans, J. Steprans and Y. Zhang, *Analytic and coanalytic families of almost disjoint functions*, J. Symb. Logic, 73 (4): 1158-1172, 2008.

[11] D. Raghavan and J. Steprans, *On weakly tight families*, pre-print 2012.

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