n-Luzin gaps

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Abstract

We introduce the notion of an n-Luzin gap, which is a natural generalization of a Luzin gap. We prove that under Martin's Axiom, every AD family $\mathcal{A}$ of size less than $\mathfrak{c}$ contains an n-Luzin gap or the corresponding Mrówka-Isbell space $\Psi(\mathcal{A})$ is normal.

0 Introduction

An infinite family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *almost disjoint (AD)* if the intersection of any two distinct elements of $\mathcal{A}$ is finite. It is *maximal almost disjoint (MAD)* if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$. Given an almost disjoint family $\mathcal{A}$ and two subfamilies $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$ we say that a set $X \subseteq \omega$ *separates* $\mathcal{B}$ and $\mathcal{C}$ if $A \subseteq^* X$ for every $A \in \mathcal{B}$ and $A \cap X =^* \varnothing$ for every $A \in \mathcal{C}$.

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One of the first constructions of almost disjoint families with special properties is the construction of Luzin [12] of an uncountable almost disjoint family $\mathcal{A}$ such that no two uncountable subfamilies of $\mathcal{A}$ can be separated. The ingenious property used in the proof deserves a name:

Definition 0.1 *An almost disjoint family $\mathcal{A}$ is* Luzin *if it can be enumerated as $\{A_\alpha : \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1\ \forall n \in \omega\ \{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}$ is finite.*

Abraham and Shelah [1] called (and so do we) an almost disjoint family $\mathcal{A}$ *inseparable* if no two uncountable subfamilies can be separated. It is easy to see that $\mathcal{A}$ is inseparable if and only if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite. The point of Luzin's proof was that, Luzin families are inseparable. Abraham and Shelah proved that (1) assuming CH, there is an inseparable AD family which contains no Luzin subfamily, while (2) under $\mathsf{MA} + \neg\mathsf{CH}$ every inseparable AD family is a countable union of Luzin subfamilies.

Roitman and Soukup in [14] introduced the notion of an anti-Luzin family: An AD family $\mathcal{A}$ is an *anti-Luzin* family if for every $\mathcal{B} \in [\mathcal{A}]^{\aleph_1}$ there are $\mathcal{C}, \mathcal{D} \in [\mathcal{B}]^{\aleph_1}$ which can be separated (or equivalently, $\mathcal{A}$ does not contain uncountable inseparable families) and proved that assuming $\mathsf{MA} + \neg\mathsf{CH}$, every AD family is either anti-Luzin or contains an uncountable Luzin subfamily, and assuming $\upharpoonleft$[1], there is an uncountable almost disjoint family which contains no uncountable anti-Luzin and no uncountable Luzin subfamilies.

More recently, Dow [6] showed that PFA implies that every MAD family contains an uncountable Luzin subfamily. Dow and Shelah in [7] showed that Martin's Axiom does not suffice by showing that it is relatively consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that there is a maximal almost disjoint family which is ω_1*-separated*, i.e. any disjoint pair of $\leq \omega_1$-sized subfamilies are separated.

To every almost disjoint family one can naturally associate the so called *Mrówka-Isbell space*:

Definition 0.2 *Given an AD family $\mathcal{A}$, define a space $\Psi(\mathcal{A})$ as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.*

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a separable, scattered, zero-dimensional, first countable, locally compact Moore space [13]. Normality of Ψ-spaces is characterized using separation as follows:

[1] Recall that $\upharpoonleft$ is the following weakening of CH: There is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ of size $\aleph_1$ such that every uncountable subset of ω_1 contains an element of $\mathcal{S}$.

Proposition 0.3 ([17]) *$\Psi(\mathcal{A})$ is normal if and only if $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated for every $\mathcal{B} \subseteq \mathcal{A}$.*

Abusing notation we will call $\mathcal{A}$ *normal* if the space $\Psi(\mathcal{A})$ is normal. A natural choice would be to call $\mathcal{A}$ *completely separated*, but unfortunately a very similar term is already in use [16, 8, 4].

Up until recently Luzin families were often referred to as Luzin gaps. However, that name has recently [18, 9] been used to describe a weaker notion.

Definition 0.4 ([18]) *A pair $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$, $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ of subfamilies of $[\omega]^\omega$ is called a* Luzin gap *if there is an $m \in \omega$ such that*

1. *$A_\alpha \cap B_\alpha \subseteq m$ for all $\alpha < \omega_1$, and*
2. *$A_\alpha \cap B_\beta$ is finite yet $(A_\alpha \cap B_\beta) \cup (A_\beta \cap B_\alpha) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.*

Every Luzin family $\mathcal{A}$ contains many Luzin gaps: given a pair $\{A_\alpha : \alpha < \omega_1\}$, $\{B_\alpha : \alpha < \omega_1\}$ of subfamilies of $\mathcal{A}$, there is an uncountable $X \subseteq \omega_1$ such that $\{A_\alpha : \alpha \in X\}$, $\{B_\alpha : \alpha \in X\}$ forms a Luzin gap. The basic property of a Luzin gap is that the two families $\mathcal{A}$ and $\mathcal{B}$ can not be separated, and the property of being a Luzin gap is indestructible by forcing preserving ω_1 (see [18, 9] or section 1). Hence, the space $\Psi(\mathcal{A})$ can not be normal (in any forcing extension preserving ω_1) for any AD family $\mathcal{A}$ containing a Luzin gap. The notion of a Luzin gap suggest the following generalization,

Definition 0.5 *Let $n \in \omega$ and $\mathcal{B}_i = \{\mathcal{B}^i_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of an AD family $\mathcal{A}$ for $i < n$. We call $\langle \mathcal{B}_i | i < n \rangle$ an n-Luzin* gap *if there is $m \in \omega$ such that*

1. *$B^i_\alpha \cap B^j_\alpha \subseteq m$ for all $i \neq j$, $\alpha < \omega_1$ and*
2. *$\bigcup_{i \neq j} \left(B^i_\alpha \cap B^j_\beta \right) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.*

We say that $\mathcal{A}$ contains an n-Luzin gap if there is an n-Luzin gap $\langle \mathcal{B}_i | i < n \rangle$ where each $\mathcal{B}_i$ is a subfamily of $\mathcal{A}$. We will see that any family contanining an n-Luzin gap is not normal, and our main theorem states that the converse is also true assuming Martin's Axiom:

Theorem 0.6 *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Assuming PFA the theorem can be strenghten. We also show that the result does not follow from MA(σ-centered), as

Theorem 0.7 *It is consistent with* MA(*σ-centered*) *that there is an inseparable AD family of size ω_1 which does not contain Luzin gaps for every $n \in \omega$.*

The situation is reminiscent of ω_1-trees and Hausdorff gaps, an inseparable family that does not contain n-Luzin gaps for any $n \in \omega$ being the equivalent of a Suslin tree or a ccc destructible gap. A Suslin tree can be destroyed by two different means: (1) one can force with the tree an add an uncountable branch and (2) one can specialize the tree by a ccc forcing making it a union of countably many antichains. Similar situation occurs with ccc destructible Hausdorff gaps ([**?**] see [15]) a destructible Hausdorff gaps can be either (1) filled or (2) frozen, both by ccc forcing. Here, an inseparable family with no n-Luzin gaps can be either (1) forced normal or (2) frozen by forcing it to contain a Luzin gap, both by a ccc forcing.

An early (probably the first) example of a Ψ-space appears in [2]: A topology of the real line is refined by declaring all rational points isolated. To each irrational point a convergent sequence is chosen and the cofinite subsets of the given convergent sequence are declared basic open neighborhoods of the irrational number.

We call an almost disjoint family $\mathcal{A}$ $\mathbb{R}$-*embeddable* (see [10]) if there is an injection $e : \omega \to \mathbb{Q}$ such that for every $A \in \mathcal{A}$ there is an $r_A \in \mathbb{R}$ such that $e[A]$ converges to r_A and, moreover, $r_A \neq r_B$ whenever $A \neq B$. Evidently, this is equivalent that there is an injective and continous $f : \Psi(\mathcal{A}) \longrightarrow \mathbb{R}$ such that $f(n) \in \mathbb{Q}$ for every $n \in \omega$. Using Tietze's theorem, it is easy to show that every normal family is $\mathbb{R}$-embedabble.

The notion of $\mathbb{R}$-embeddability together with a strengthening of the notion of an anti-Luzin family are the main tools here.

Definition 0.8 *An almost disjoint family $\mathcal{A}$ is* partially separated *if given a pair $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$, $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ of pairwise disjoint subfamilies of $\mathcal{A}$ there is an uncountable $X \subseteq \omega_1$ such that the families $\{B_\alpha : \alpha \in X\}$, $\{C_\alpha : \alpha \in X\}$ are separated.*

We call an AD family $\mathcal{A}$ *potentially* $\mathcal{P}$ (for a property $\mathcal{P}$) if there is a ccc forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathcal{A}$ *has* $\mathcal{P}$". Similarly, we say that $\mathcal{A}$ is *indestructibly* $\mathcal{P}$, if $\mathcal{A}$ has property $\mathcal{P}$ in all ccc forcing extensions. We show that

Theorem 0.9 *The following are equivalent for an AD family* $\mathcal{A}$*:*

1. $\mathcal{A}$ *is does not contain* n*-Luzin gaps for any* $n \in \omega$*,*
2. $\mathcal{A}$ *is potentially normal,*
3. $\mathcal{A}$ *is potentially* $\mathbb{R}$*-embeddable,*
4. $\mathcal{A}$ *is potentially partially separated.*

Dow and Shelah's [7] result mentioned above shows that it is consistent with MA that there is a MAD family which is potentially normal, while assuming PFA ([6]) all MAD families contain Luzin families, hence, also Luzin gaps. It is worth mentioning that Aviles and Todorcevic studied gaps of higher dimensions in [3].

1 Forcing an AD to be normal

In the following, $\mathcal{A}$ will always be an AD family. Given $\mathcal{B}, \mathcal{C}$ disjoint subsets of $\mathcal{A}$, we will define a forcing that adds a set separating $\mathcal{B}$ from $\mathcal{C}$. Let $\mathbb{S}_{\mathcal{BC}}$ be the set of all $(s, \mathcal{F}, \mathcal{G})$ such that,

1. $s \in {}^{<\omega}2$, $\mathcal{F} \in [\mathcal{B}]^{<\omega}$, $\mathcal{G} \in [\mathcal{C}]^{<\omega}$.
2. If $B \in \mathcal{F}$ and $C \in \mathcal{G}$ then $B \cap C \subseteq |s|$.

We say $(s, \mathcal{F}, \mathcal{G}) \leq (s', \mathcal{F}', \mathcal{G}')$ if and only if,

1. $s' \subseteq s$, $\mathcal{F}' \subseteq \mathcal{F}$, $\mathcal{G}' \subseteq \mathcal{G}$.
2. If $i \in dom(s) \setminus dom(s')$ then,
   a) If $i \in \bigcup \mathcal{F}'$ then $s(i) = 1$.
   b) If $i \in \bigcup \mathcal{G}'$ then $s(i) = 0$.

It is easy to prove that for all $n \in \omega$, $B \in \mathcal{B}$ and $C \in \mathcal{C}$ the sets $\{(s, \mathcal{F}, \mathcal{G}) \mid |s| \geq n\}$, $\{(s, \mathcal{F}, \mathcal{G}) \mid B \in \mathcal{F}\}$ and $\{(s, \mathcal{F}, \mathcal{G}) \mid C \in \mathcal{G}\}$ are dense, so $\mathbb{S}_{\mathcal{BC}}$ adds a set separating $\mathcal{B}$ from $\mathcal{C}$.

Lemma 1.1 *If $\mathcal{A}$ is partially separated, then $\mathbb{S}_{\mathcal{BC}}$ is ccc.*

Proof. Let $\{p_\alpha \mid \alpha \in \omega_1\}$ be a set of conditions, and write $p_\alpha = (s_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$. With out lose of generality, we may assume that there are $n, m \in \omega$ such that $|\mathcal{F}_\alpha| = n$ and $|\mathcal{G}_\alpha| = m$ for every $\alpha \in \omega_1$. Let us enumerate $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha(i) \mid i < n\}$ and $\mathcal{G}_\alpha = \{\mathcal{G}_\alpha(i) \mid i < m\}$.

Let $\mathcal{B}_0 = \{\mathcal{F}_\alpha(0) \mid \alpha \in \omega_1\}$ and $\mathcal{C}_0 = \{\mathcal{G}_\alpha(0) \mid \alpha \in \omega_1\}$, since $\mathcal{A}$ is partially separated, there are $Z_0 \in [\omega_1]^{\omega_1}$ and k_0 such that $\mathcal{F}_\alpha(0) \cap \mathcal{G}_\beta(0) \subseteq k_0$ for every $\alpha, \beta \in Z_0$. Now let $\mathcal{B}_1 = \{\mathcal{F}_\alpha(0) \mid \alpha \in Z_0\}$, $C_1 = \{\mathcal{G}_\alpha(1) \mid \alpha \in Z_0\}$ and find $Z_1 \in [Z_0]^{\omega_1}$, $k_1 \in \omega$ such that $\mathcal{F}_\alpha(0) \cap \mathcal{G}_\beta(1) \subseteq k_1$ for every $\alpha, \beta \in Z_1$. Repeating this process (mn times) we conclude there is $Z \in [\omega_1]^{\omega_1}$ and k such that $B_\alpha(i) \cap C_\beta(j) \subseteq k$ for every $\alpha, \beta \in Z$ and $i < n$, $j < m$.

For every $\alpha \in Z$, take s'_α such that $(s'_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha) \leq (s_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$ and $k < |s'_\alpha|$. Naturally, there are $s \in {}^{<\omega}2$ and $\alpha, \beta \in Z$ with the property that $s = s_\alpha = s_\beta$. We claim that $(s, \mathcal{F}_\alpha, \mathcal{G}_\alpha)$ and $(s, \mathcal{F}_\beta, \mathcal{G}_\beta)$ are compatible (and then, so are p_α and p_β). To prove this, we only need to note that $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta, \mathcal{G}_\alpha \cup \mathcal{G}_\beta)$ is a condition, but this is trivial since $k < |s'_\alpha|$. ■

Now, we will prove that $\mathbb{R}$-embedabbility implies partial separability.

Proposition 1.2 *If $\mathcal{A}$ is $\mathbb{R}$-embeddable, then it is partially separated.*

Proof. Let $h : \Psi(\mathcal{A}) \longrightarrow \mathbb{R}$ witness that $\mathcal{A}$ is $\mathbb{R}$-embeddable and take $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ disjoint subsets of $\mathcal{A}$. Fix D a countable base for $\mathbb{R}$ and for every $\alpha \in \omega_1$, find disjoint $U_\alpha, V_\alpha \in D$ such that $h(B_\alpha) \in U_\alpha$ and $h(C_\alpha) \in V_\alpha$. We may also choose $m_\alpha \in \omega$ such that $h[B_\alpha \setminus m_\alpha] \subseteq U_\alpha$ and $h[C_\alpha \setminus m_\alpha] \subseteq V_\alpha$. Now, let $X \in [\omega_1]^{\omega_1}$ be such that there are $U, V \in D$ and m with the property that $U_\alpha = U$, $V_\alpha = V$ and $m_\alpha = m$ for all $\alpha \in X$. It is clear that if $\alpha, \beta \in X$ then $B_\alpha \cap C_\alpha \subseteq m$. ■

We may conclude even more from the above, note that being $\mathbb{R}$-embeddable is an indestructible property, so an $\mathbb{R}$-embeddable family is actually indestructibly partially separated. In this way, we may conclude,

Corollary 1.3 *The following are equivalent,*

1. *$\mathcal{A}$ is potentially $\mathbb{R}$-embeddable,*
2. *$\mathcal{A}$ is potentially indestructibly partially separated ,*
3. *$\mathcal{A}$ is potentially normal.*

Proof. We already note that 1 implies 2 and it is clear that 3 implies 1. Let us prove that 2 implies 3, let $\mathbb{P}$ be a ccc forcing such that $1_{\mathbb{P}}$ forces that $\mathcal{A}$ is indestructibly partially separated . In this way, the forcings $\mathbb{S}_{\mathcal{BC}}$ will always be ccc (under any extension) so we may iterate them and get a model where $\mathcal{A}$ is normal. ■

As a consequence, under Martin's Axiom, the small almost disjoint families that can become normal, are precisely those that are already normal.

Corollary 1.4 *Assume* MA. *Let $\mathcal{A}$ be an AD with $|\mathcal{A}| < \mathfrak{c}$, then $\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is normal.*

Proof. Let $\mathcal{A}$ be potentially normal and of size less than $\mathfrak{c}$. We must prove that every $\mathcal{B}, \mathcal{C}$ disjoint subsets of $\mathcal{A}$ can be separated. Since we are assuming MA, it is enough to show that the forcing $\mathbb{S}_{\mathcal{BC}}$ is ccc (because we only need $|\mathcal{B}| + |\mathcal{C}| + \omega$ dense sets to do the job). Now, let $\mathbb{P}$ be a ccc forcing such that $\mathcal{A}$ is partially separated in $V[G]$ for every generic filter $G \subseteq \mathbb{P}$. Note that $\mathbb{S}_{\mathcal{BC}}$ is the same as $\mathbb{S}_{\mathcal{BC}}^{V[G]}$ and since $\mathcal{A}$ is partially separated, then it is ccc in $V[G]$. This implies that it $\mathbb{S}_{\mathcal{BC}}$ is ccc in V (since any uncountable antichain in V would still be an uncountable antichain in $V[G]$). ■

Assuming MA, we may get another equivalence of potentially normal,

Corollary 1.5 *Assume* MA. *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is indestructibly partially separated .*

Proof. Let $\mathcal{A}$ be potentially normal, $\mathbb{P}$ a ccc forcing and $G \subseteq \mathbb{P}$ a generic filter. We must prove that $\mathcal{A}$ is partially separated in $V[G]$. For this, it is enough to see that every subfamily of $\mathcal{A}$ of size ω_1 is partially separated. In this way, in $V[G]$ choose $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ and since $\mathbb{P}$ is ccc, then there is $\mathcal{A}'' \in V$ a subset of $\mathcal{A}$ of size ω_1 such that $\mathcal{A}' \subseteq \mathcal{A}''$. Since MA is true in V, then $\mathcal{A}''$ is $\mathbb{R}$ embedabble, so it is partially separated in $V[G]$ and also $\mathcal{A}'$. ■

We remark that the previous corollary can not be proved in ZFC, as we will see in section 3.

2 n-Luzin gaps

We start by proving some elementary facts about n-Luzin gaps.

Lemma 2.1 *If $\langle \mathcal{B}_i | i < n \rangle$ is an n-Luzin (with $\mathcal{B}_i = \{\mathcal{B}^i_\alpha \mid \alpha \in \omega_1\}$) then, for every $X \in [\omega_1]^{\omega_1}$ and $k \in \omega$, there are $\alpha, \beta \in X$ such that $\bigcup\limits_{i \neq j} \left(B^i_\alpha \cap B^j_\beta\right) \nsubseteq k$.*

Proof. Let $m \in \omega$ testify that $\langle \mathcal{B}_i \mid i < n \rangle$ is n-Luzin. With out losing generality, we may assume $\kappa > m$. First, we find $Y \in [X]^{\omega_1}$ such that if $\alpha, \beta \in Y$ and $i < n$, then $B^i_\alpha \cap k = B^i_\beta \cap k$. Take $\alpha, \beta \in Y$ distinct, we know there are $i \neq j$ such that $B^i_\alpha \cap B^j_\beta \nsubseteq m$, but since $B^i_\alpha \cap k = B^i_\beta \cap k$ and $B^j_\beta \cap B^i_\beta \subseteq m \subseteq k$, then B^i_α and B^j_β must intersect above k. ■

With the aid of this lemma, we can prove,

Lemma 2.2 *If $\mathcal{A}$ is partially separated, then it does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. Let $\mathcal{A}$ be partially separated and take $\{B^n_\alpha \mid n \in \omega\}$ such that $B^i_\alpha \cap B^j_\alpha \subseteq m$ when $i \neq j$. Since $\mathcal{A}$ is partially separated, we may find $X \in [\omega_1]^{\omega_1}$ and $k \in \omega$ such that $B^i_\alpha \cap B^j_\alpha \subseteq k$ for all $\alpha, \beta \in X$. In this way, $\mathcal{A}$ can not contain n-Luzin gaps by the previous lemma. ■

Since normal families are partially separated, we immediately conclude:

Corollary 2.3 *If $\mathcal{A}$ contains an n-Luzin gap, then it is not normal.*

Using this, we will be able to give a nice combinatorial reformulation of potential normality of AD families. First, we will introduce a forcing that makes $\mathcal{A}$ an $\mathbb{R}$-embeddable family. Instead of trying to embedd $\Psi(\mathcal{A})$ into $\mathbb{R}$, we will try to do it in the Cantor space ${}^{\omega}2$, identifying the rational numbers with the eventually 0 functions. It is easy to see that this is enough. Let $\mathcal{R}(\mathcal{A})$ be the set of all $(s, \mathcal{F})$ such that,

1. $s \in {}^{<\omega}\mathbb{Q}$ is injective and $\mathcal{F} \in [\mathcal{A}]^{<\omega}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \subseteq |s|$.

And $(s, \mathcal{F}) \leq (s', \mathcal{F}')$ if,

1. $s' \subseteq s$, $\mathcal{F}' \subseteq \mathcal{F}$.

2. If $i \in dom(s) \setminus dom(s')$ and there is $A \in \mathcal{F}'$ such that $i \in A$ and $j = max\{A \cap dom(s')\}$ then $\Delta(s(i), s'(j)) \geq |s'|$ (where $\Delta(x,y)$ is the first n such that $x(n) \neq y(n)$).

Note that the A as is unique since $(s', \mathcal{F}')$ is a condition.

Lemma 2.4 *If $\mathcal{R}(\mathcal{A})$ is ccc, then $\mathcal{A}$ is potentially $\mathbb{R}$-embeddable.*

Proof. Given $n \in \omega$, it is easy to prove that the set $D_n = \{(s, \mathcal{F}) \mid n < |s|\}$ is dense (this is due to the fact that if $A, B \in \mathcal{F}$ then $A \cap B \subseteq \mathcal{F}$, so we may extend the condition $(s, \mathcal{F})$ without changing $\mathcal{F}$). Also, if $A \in \mathcal{A}$ then the set $E_A = \{(s, \mathcal{F}) \mid A \in \mathcal{F}\}$ is dense. Given $(s, \mathcal{F})$ we first find $m \in \omega$ such that $X \cap Y \subseteq m$ for every $X \neq Y \in \mathcal{F} \cup \{A\}$ and then we extend $(s, \mathcal{F})$ to a condition $(s', \mathcal{F})$ such that $m < |s'|$. In this way, $(s', \mathcal{F} \cup \{A\})$ is smaller than $(s, \mathcal{F})$.

Fix G a generic filter for $\mathcal{R}(\mathcal{A})$, we will prove that $\mathcal{A}$ is $\mathbb{R}$-embeddable in $V[G]$. Let $e = \bigcup\limits_{(s,\mathcal{F}) \in G} s$ since the D_n are dense, then e is a function from ω to $\mathbb{R}$. We will show that if $A \in \mathcal{A}$ then $e[\mathcal{A}]$ is a convergent sequence. For this, just note that if $A \in \mathcal{F}$ and $A \cap dom(s) \neq \emptyset$ then $(s, \mathcal{F}) \Vdash$ "if $x, y \in A \setminus dom(s)$, then $\dot{e}(x) \upharpoonright |s| = \dot{e}(y) \upharpoonright |s|$".

Let us call $r_A \in {}^{\omega}2$ the limit of $e[\mathcal{A}]$. It remains to see that $r_A \neq r_B$ whenever $A \neq B$. Let D_{AB} be the set of the $(s, \mathcal{F})$ that force r_A to be different from r_B. It is enough to show that this set is dense. Take $(s, \mathcal{F})$ a condition, without loosing generality, we may assume $A, B \in \mathcal{F}$ and $A \cap dom(s)$, $B \cap dom(s)$ are not empty. Now, it is easy to extend this condition in such a way that r_A and r_B belong to different clopen sets. ■

We are finally ready to prove one of the main results.

Theorem 2.5 *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. If $\mathcal{A}$ contains an n-Luzin gap, then $\mathcal{A}$ still contains it in any forcing extension that preserves ω_1. Since this families are not normal, we may conclude that $\mathcal{A}$ can not be potentially normal. Now, we only need to prove that if $\mathcal{A}$ does not contain n-Luzin gaps then it is potentially normal, or equivalently that it is potentially $\mathbb{R}$-embeddable. For this, we just need to see that $\mathcal{R}(\mathcal{A})$ is ccc.

Assume this is not the case, then there is a set $\{(s_\alpha, \mathcal{F}_\alpha) \mid \alpha \in \omega_1\}$ of pairwise incompatible conditions. We may assume there is $s \in {}^{<\omega}\mathbb{R}$ such that $s_\alpha = s$ for all $\alpha \in \omega_1$ and $\{\mathcal{F}_\alpha \mid \alpha \in \omega_1\}$ forms a Δ system with root R. Note that since $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are incompatible so are $(s, \mathcal{F}_\alpha \setminus R)$ and $(s, \mathcal{F}_\beta \setminus R)$. In this way, we may assume that R is the empty set and all $\mathcal{F}_\alpha$ are of the same size, say n. We may also assume that if $i < n$ then $\mathcal{F}_\alpha(i) \cap m = \mathcal{F}_\beta(i) \cap m$ for all $\alpha, \beta \in \omega_1$.

Enumerate $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha(i) \mid i < n\}$ and let $\mathcal{B}_i = \{\mathcal{F}_\alpha(i) \mid \alpha \in \omega_1\}$. Note that, since each $(s, \mathcal{F}_\alpha)$ is a condition, then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\alpha(j) \subseteq m$. Since $\mathcal{A}$ does not contain n-Luzin gaps, there are $\alpha \neq \beta$ such that if $i \neq j$ then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$. We claim that $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are compatible, which will be a contradiction. Note that $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ may fail to be a condition, since there could be $A, B \in \mathcal{F}_\alpha \cup \mathcal{F}_\beta$ such that $A \cap B \nsubseteq |s| = m$. But in this case, A must be of the form $\mathcal{F}_\alpha(i)$ and B must be $\mathcal{F}_\beta(i)$ (because $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are conditions and $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$ when $i \neq j$). However, since $\mathcal{F}_\alpha(i)$ and $\mathcal{F}_\beta(i)$ agree up to m, it is easy to extend $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ to a condition. ■

Evidently, we may conclude,

Corollary 2.6 *If $\mathcal{A}$ is partially separated then it is potentially $\mathbb{R}$-embeddable.*

We may also prove the promised result,

Theorem 2.7 *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. The forward implication is clear, for the converse, just recall that under MA normality and potential normality are equivalent for families of size less than $\mathfrak{c}$. ■

We will show that, under the Proper Forcing Axiom, we may "remove the n" from the previous result. Assume $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ are disjoint subfamilies of $\mathcal{A}$ and let $X = \{(B_\alpha, C_\alpha) \mid \alpha \in \omega_1\}$. For every $m \in \omega$ we define the "coloring",

$$c_m : [X]^2 \longrightarrow 2$$

$$c_m\left(\left(B_\alpha, C_\alpha\right), \left(B_\beta, C_\beta\right)\right) = \begin{cases} 0 & if \quad \left(B_\alpha \cap C_\beta\right) \cup \left(B_\beta \cap C_\alpha\right) \nsubseteq m \\ 1 & if \quad \left(B_\alpha \cap C_\beta\right) \cup \left(B_\beta \cap C_\alpha\right) \subseteq m \end{cases}$$

We may see X as a subset of the polish space ${}^\omega 2 \times^\omega 2$, so it carries a natural topology. In this way, note that $c^{-1}(\{0\}) \subseteq X^2$ is an open set. Let us recall *Todorcevic's Axiom* (see [19]),

TA) If X is a separable metric space and $c : [X]^2 \longrightarrow 2$ is such that $c^{-1}(\{0\})$ is open, then one of the following holds,

$\star$) There is $M \in [X]^{\omega_1}$ that is monochromatic of color 0 (i.e. c restricted to $[M]^2$ is the constant 0).

$\star\star$) X may be cover by ω monochromatic sets of color 1.

For us, it will be enough to observe that TA implies that (given X is uncountable) there is always an uncountable monochromatic set in one color. Note that our X is indeed a separable metric space, since polish spaces are hereditarily separable. The following result may be seen as a consequence of theorem 13.5 in [19], but we prove it for the sake of completeness.

Proposition 2.8 ([19]) *If* TA *is true, then every almost disjoint family is partially separated or contains a Luzin gap.*

Proof. Assume $\mathcal{A}$ is not partially separated, so there are two disjoint subfamilies $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ of $\mathcal{A}$ such that for every $Y \in [\omega_1]^{\omega_1}$ and $n \in \omega$, there are $\alpha, \beta \in Y$ with the property that $B_\alpha \cap C_\beta \nsubseteq n$. We may assume there is $m \in \omega$ such that $B_\alpha \cap C_\alpha \subseteq m$ for all $\alpha \in \omega_1$.

Let X, and c_m be defined as above. In this way, the previous remark tells us that there are no uncountable 1 monochromatic sets, so TA implies the existence of an uncountable 0 monochromatic set Y. Clearly $\{B_\alpha \mid \alpha \in Y\}$, $\{C_\alpha \mid \alpha \in Y\}$ is a Luzin gap. ■

The above result can not be proven in ZFC, since it is consistent with $\mathsf{MA}(\sigma - centered)$ that there are 3-Luzin gaps that do not contain Luzin gaps. It is well known that PFA implies both TA and MA, so we may conclude the following,

Corollary 2.9 *Assume* PFA*. Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain Luzin gaps.*

However, we do not know if we really need PFA (TA) for the strenghten result, so the following question is unanswered,

Questions 2.10 *Does the previous corollary holds assuming* MA*?*

3 Schizophrenic AD Families

Recall that $\mathcal{A}$ is inseparable if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite or equivalently, for every $m \in \omega$ there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $B \cap C \nsubseteq m$. Clearly, every uncountable subfamily of an inseparable family is inseparable and $\mathcal{A}$ is inseparable if and only if all of its subfamilies of size ω_1 are inseparable.

Let us introduce a forcing aiming to add a Luzin family. Assume $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ and for every $p \in [\omega_1]^{<\omega}$ let m_p be the smallest integer such that $A_\alpha \cap A_\beta \subseteq m_p$ for all $\alpha, \beta \in p$ distinct. We define the poset $\mathcal{SR}(\mathcal{A}) = [\omega_1]^{<\omega}$ (see [14]) and we say $p \leq q$ if and only if,

1. $q \subseteq p$,
2. If $\alpha \in p \setminus q$ and there is $\beta \in q$ with $\alpha < \beta$, then $A_\beta \cap A_\alpha \nsubseteq m_q$.

Lemma 3.1 ([14]) *If $\mathcal{SR}(\mathcal{A})$ is ccc, then $\mathcal{A}$ potentially contains a Luzin family.*

Proof. For every $\alpha \in \omega_1$ define $D_\alpha = \{p \mid p \nsubseteq \alpha\}$. It is easy to see that this set is dense, since if $p \subseteq \alpha$ then $p \cup \{\alpha\} \leq p$. Let G be a generic filter and in $V[G]$ define $\mathfrak{B} = \{A_\alpha \mid \alpha \in \bigcup G\}$, then $\mathfrak{B}$ is uncountable (since the forcing is ccc) and it is easy to see that it is indeed a Luzin family. ■

With the aid of the previous result, we may obtain the following characterization due to Roitman and Soukup in [14].

Proposition 3.2 ([14]) *$\mathcal{A}$ is inseparable if and only if every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family.*

Proof. First, assume every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family. Let $\mathcal{B}, \mathcal{C}$ be uncountable subfamilies of $\mathcal{A}$ and define $\mathcal{A}' = \mathcal{B} \cup \mathcal{C}$. We know there is $\mathbb{P}$ a ccc forcing such that $1_{\mathbb{P}}$ forces that $\mathcal{A}'$ contains a Luzin family. Aiming for a contradiction, assume there is $m \in \omega$ such that $B \cap C \subseteq m$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Let $G \subseteq \mathbb{P}$ be a generic filter and in $V[G]$ find $\mathcal{D} = \{X_\alpha \mid \alpha \in \omega_1\} \subseteq \mathcal{A}'$ be a Luzin family. Clearly, there is $\alpha \in \omega_1$ such that $X_\alpha \in \mathcal{B}$ and $\{X_\xi \mid \xi < \alpha\} \cap \mathcal{C}$ is infinite, but then the set $\{\xi < \alpha \mid X_\alpha \cap X_\xi \subseteq m\}$ is infinite, which contradicts that $\mathcal{D}$ is a Luzin family.

For the other implication, it is enough to prove that if $\mathcal{A}$ is inseparable of size ω_1, then $\mathcal{SR}(\mathcal{A})$ is ccc. We will proceed by contradiction, suppose $\{p_\alpha \mid \alpha \in \omega_1\}$ is an antichain, we may assume it forms a Δ system with root r, every $p_\alpha \setminus r$ has size n and and there is $m \in \omega$ such that $m_{p_\alpha} = m$ for all $\alpha \in \omega_1$. Furthermore, thinning our family, we may assume that for all α, every member of r is below every member of $p_\alpha \setminus r$ and if $\alpha < \beta$, then every member of $p_\alpha \setminus r$ is below every member of $p_\beta \setminus r$. Write $p_\alpha \setminus r = \{p_\alpha(i) \mid i < n\}$ and we may suppose there is $k > m$ such that $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$ for all $\alpha \in \omega_1$. Thinning our family again, we may assume $p_\alpha(i) \cap k = p_\beta(i) \cap k$ for all $\alpha, \beta \in \omega_1$.

We will now see that there are $X_0, Y_0 \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X_0$ and $\beta \in Y_0$ then $p_\alpha(0) \cap p_\beta(1) \nsubseteq m$. Suppose this is false, then for every $x > m$, at least one of the sets $B_x = \{\alpha \mid x \in p_\alpha(0)\}$, $C_x = \{\alpha \mid x \in p_\alpha(1)\}$ is countable (and they are disjoint, since x is bigger than m). Let $\mathcal{B}$ be the set of all the $p_\alpha(0)$ such that $\alpha \notin \bigcup\limits_{|B_x| \leq \omega} B_x$ and $\mathcal{C}$ be the set of all $p_\alpha(1)$ such that $\alpha \notin \bigcup\limits_{|C_x| \leq \omega} C_x$. In this way, $\mathcal{B}$ and $\mathcal{C}$ are two uncountable subfamilies of $\mathcal{A}$. However, if $B \in \mathcal{B}$ and $C \in \mathcal{C}$ then $B \cap C \subseteq m$, which contradicts that $\mathcal{A}$ was inseparable.

Repeating this process several times, we find there are $X, Y \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X$ and $\beta \in Y$ then $p_\alpha(i) \cap p_\beta(j) \nsubseteq m$ when $i \neq j$. However, we already knew that $p_\alpha(i) \cap p_\beta(i) \nsubseteq m$, since $p_\alpha(i) \cap k = p_\beta(i) \cap k$ and $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$. This implies that $p_\alpha \cup p_\beta$ is a common extension of p_α and p_β, which is a contradiction. ■

Now, we will introduce an interesting class of families,

Definition 3.3 *We say that an AD family $\mathcal{A}$ is* schizophrenic *if it is inseparable and potentially normal.*

Schizophrenic families are rather peculiar, since there is a ccc forcing that makes them normal, and another one that freezes them by adding them a Luzin gap, so it become not indestructibly not normal! In this way we have the following result,

Corollary 3.4 *If $\mathcal{A}$ is schizophrenic, then $\mathcal{R}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are two ccc forcings such that $\mathcal{R}(\mathcal{A}) \times \mathcal{SR}(\mathcal{A})$ is not ccc.*

In this way, MA implies that there are no schizophrenic families (another way to prove this, is to remember that MA implies that potentially normal entails indestructibly partially separated , and partially separated families does not contain Luzin gaps). In the next section, we will see that the existence of schizophrenic families is consistent with ZFC.

While there are no a schizophrenic families under MA, we will prove that they may exist under $\mathsf{MA}(\sigma - centered)$. We will denote the Cohen forcing $\mathbb{C} = {}^{<\omega}2$.

Lemma 3.5 *If $\dot{A}$ is a $\mathbb{C}$ name for an uncountable subset of ordinals, then there is $s \in \mathbb{C}$ and $X \in V$ uncountable such that $s \Vdash$ "$X \subseteq \dot{A}$". In other words, any new uncountable set of ordinals contains an old uncountable set of ordinals.*

Proof. For every $s \in \mathbb{C}$, let $A_s = \{a \mid s \Vdash \text{“}a \in A\text{”}\}$. Clearly, if $G \subseteq \mathbb{C}$ is generic, then $A = \bigcup_{s \in G} A_s$ and since A is uncountable, then one of the A_s must be uncountable. ■

Now we will prove,

Theorem 3.6 *The existence of an schizophrenic family is consistent with $\mathsf{MA}(\sigma - centered)$.*

Proof. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ be an inseparable family (take a Luzin family, for example) and let $D \subseteq \omega$ be a Cohen real over V. In $V[D]$ define $\mathcal{A} \restriction \mathcal{D}$ to be the set of all $A_\alpha \cap D$ with $\alpha \in \omega_1$. We will show that this is an schizophrenic family (note first that $\mathcal{A} \restriction \mathcal{D}$ is still an almost disjoint family).

Let us see that it is inseparable. In $V[D]$, let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be uncountable subfamilies of $\mathcal{A}$. In this way, we may define $h : \omega_1 \longrightarrow \omega_1 \times \omega_1$ in such a way that $B_\alpha = A_{h(\alpha)_0} \cap D$ and $C_\alpha = A_{h(\alpha)_1} \cap D$ where $h(\alpha) = (h(\alpha)_0, h(\alpha)_1)$. By the previous lemma, there is $s \in \mathbb{C}$ and $X \in [\omega_1]^{\omega_1}$ (in V) such that s knows $h \restriction X$. We will find an extension of s that forces what we need.

Fix $m \in \omega$, we need to show that there are $\alpha, \beta \in \omega_1$ such that $B_\alpha \cap C_\beta = \left(A_{h(\alpha)_0} \cap A_{h(\beta)_1}\right) \cap D$ is not contained in m. Let $\mathcal{B}' = \left\{A_{h(\alpha)_0} \mid \alpha \in X\right\}$ and $\mathcal{C}' = \left\{A_{h(\alpha)_1} \mid \alpha \in X\right\}$ since $\mathcal{A}$ is inseparable, there are $\alpha, \beta \in X$ and $k > m, |s|$ such that $k \in A_{h(\alpha)_0} \cap A_{h(\beta)_1}$. If s' is any extension of s such that $s'(k) = 1$, then $s' \Vdash$ "$k \in B_\alpha \cap C_\beta$" and we are done.

Now, we will prove that it is potentially normal, or equivalently, that there are no n-Luzin gaps for any $n \in \omega$. Let $n \in \omega$ and assume for every $i < n$ we have $\langle B^i_\alpha \mid \alpha \in \omega_1 \rangle$ subfamilies of $\mathcal{A} \restriction \mathcal{D}$ such that there is $m \in \omega$ with the property that $B^i_\alpha \cap B^j_\alpha \subseteq m$ whenever $i \neq j$. As before, define a function $h : \omega_1 \longrightarrow \omega_1^n$ such that $B^i_\alpha = A_{h(\alpha)_i} \cap D$ (with the same notation as before). Find $s \in \mathbb{C}$ that forces all of this, and an uncountable $X \in V$ such that s knows $h \restriction X$. Let $l = |s|$ and we may assume $m < l$.

Find $\alpha, \beta \in X$ distinct such that $A_{h(\alpha)_i} \cap l = A_{h(\beta)_i} \cap l$ for all $i < n$. Note that if $i \neq j$ then $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \cap l = A_{h(\alpha)_i} \cap A_{h(\alpha)_j} \cap l \subseteq m$. Let $r > l$ such that $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \subseteq r$ when $i \neq j$. Choose s' any extension of s such that $r < |s'|$ and if $x \in dom(s') \setminus dom(s)$ then $s'(x) = 0$. In this way, s' forces $B^i_\alpha \cap B^j_\beta \subseteq m$ for all $i \neq j$, so it is not an n-Luzin gap.

To finish the proof, assume MA holds in V, then $\mathsf{MA}(\sigma - centered)$ is still true after we add a Cohen real (this is a theorem of Roitman, see [5] theorem 3.3.8). ■

Using the same ideas as above, we will (consistently) construct a 3-Luzin gap that does not contain Luzin gaps. Recall that a family $\mathcal{D} \subseteq \omega$ is *independent* if for any distinct $A_0, ..., A_n, B_0, ..., B_m \in \mathcal{D}$ the set $A_0 \cap ... \cap A_n \cap$

$(\omega \setminus B_0) \cap \ldots \cap (\omega \setminus B_m)$ is infinite. We say that $\mathcal{D}$ *separates points* if for every distinct $n, m \in \omega$, there is $D \in \mathcal{D}$ such that $\{n, m\} \cap D$ has size 1.

Given $\mathcal{D}$ an independent family that separates points, we define the topological space $(\omega, \tau_{\mathcal{D}})$ which has $\mathcal{D} \cup \{\omega - D \mid D \in \mathcal{D}\}$ as a subbase.

Lemma 3.7 *(ω, τ_D) is homeomorphic to the rationals with the usual topology.*

Proof. This space is countable, first countable, zero dimentional without isolated points, and this characterizes $\mathbb{Q}$ (This an old result of Sierpiński, see [11]). ■

To construct our 3-Luzin gap, we will first construct (in ZFC) a special type of a Luzin gap, which is interesting on its own,

Lemma 3.8 *There is a Luzin gap $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ such that $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable.*

Proof. Let $\mathcal{D} = \{D_n \mid n \in \omega\}$ and $\mathcal{E} = \{E_n \mid n \in \omega\}$ be disjoint families such that both separate points and $\mathcal{D} \cup \mathcal{E}$ is an independent family. As was remarked above, $(\omega, \tau_{\mathcal{D}})$ and $(\omega, \tau_{\mathcal{E}})$ are both homeomorphic to the rationals, and every open set of one topology is dense in the other. Identifying ω with $\mathbb{Q}$, we may view $\mathbb{R}$ as the metric completion of $(\omega, \tau_{\mathcal{D}})$ and $(\omega, \tau_{\mathcal{E}})$. Pick $\{r_\alpha \mid \alpha \in \omega_1\}$ a set of distinct irrationals, we will recursively build $\mathcal{B}$ and $\mathcal{C}$ such that,

1. In $(\omega, \tau_{\mathcal{D}})$, B_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_{\mathcal{E}})$.
2. In $(\omega, \tau_{\mathcal{E}})$, C_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_{\mathcal{D}})$.
3. $B_\alpha \cap C_\alpha = \varnothing$ while $B_\alpha \cap C_\beta$, $B_\beta \cap C_\alpha$ are non empty finite sets for every $\beta < \alpha$.

It is clear that if the recursion could be carried out, we would have constructed the desired family. Assume B_ξ, C_ξ had been constructed for every $\xi < \alpha$, let's find B_α and C_α. Let $\{U_n \mid n \in \omega\}$ be a local base for r_α with $U_0 \supseteq U_1 \supseteq U_2 \ldots$ in $(\omega, \tau_{\mathcal{D}})$ and $\{V_n \mid n \in \omega\}$ a base in $(\omega, \tau_{\mathcal{E}})$. Enumerate $\alpha = \{\xi_n \mid n \in \omega\}$ and we recursively build $B_\alpha = \{x_n \mid n \in \omega\} \cup \{y_n \mid n \in \omega\}$ such that:

1. $x_n, y_n \in U_n$,

2. $x_n \in V_n \setminus \bigcup_{m<n} C_{\xi_m}$,

3. $y_n \in C_{\xi_n} \setminus \bigcup_{m<n} C_{\xi_m}$.

It is easy to do that, since each U_n is dense in $(\omega, \tau_{\mathcal{E}})$ and all the V_n and C_{ξ_n} are dense in $(\omega, \tau_{\mathcal{D}})$. C_α is build in the same way, just taking care to be disjoint with B_α. ■

Mimicking the construction of the schizophrenic family, we will finally show:

Proposition 3.9 *The existence of a 3-Luzin gap without Luzin subgaps is consistent with* $\mathsf{MA}(\sigma - centered)$.

Proof. Like before, it is enough to see that there is such family after adding a Cohen real. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a Luzin gap with $B_a \cap C_\alpha = \varnothing$ such that both $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable. Assume D is a Cohen real. In $V[D]$, define $\mathcal{B}_1 = \{B_\alpha \cap D \mid \alpha \in \omega_1\}$, $\mathcal{B}_2 = \{B_\alpha \setminus D \mid \alpha \in \omega_1\}$, we will prove that $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{C} \rangle$ is the family we are looking for. It is easy to see that it is indeed a 3-Luzin gap, so it remains to show that it has no Luzin gaps.

In $V[D]$, let $m \in \omega$ and $\mathcal{X} = \{X_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{Y} = \{Y_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of $\mathcal{A}$ such that $X_\alpha \cap Y_\alpha \subseteq m$. We may assume $\mathcal{X}$ is a subset of $\mathcal{B}_1, \mathcal{B}_2$ or $\mathcal{C}$ (similarly for $\mathcal{Y}$). However, since $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable and every member of $\mathcal{B}_1$ is disjoint from every member of $\mathcal{B}_2$, then we only need to consider the case where $\mathcal{X}$ is a subset of $\mathcal{B}_1$ or $\mathcal{B}_2$ and $\mathcal{Y}$ is a subset of $\mathcal{C}$. For concreteness, we will assume $\mathcal{X} \subseteq \mathcal{B}_1$, while the other case is similar. Find a function $h : \omega_1 \longrightarrow \omega_1 \times \omega_1$ such that $X_\alpha = B_{h(\alpha)_0} \cap D$ and $Y_\alpha = C_{h(\alpha)_1}$. We know there is an uncountable $W \in V$ and $s \in \mathcal{U}2102$ such that s knows $h \restriction W$, we may assume $m < |s| = l$. Let $\alpha, \beta \in W$ dictinct such that $B_{h(\alpha)_0} \cap l = B_{h(\beta)_0} \cap l$ and $C_{h(\alpha)_1} l = C_{h(\beta)_1} l$. Let $r > l$ such that $B_{h(\alpha)_0} \cap C_{h(\beta)_1}, B_{h(\beta)_0} \cap C_{h(\alpha)_1} \subseteq r$ and choose s' any extension of s such that $r < |s'|$ and if $x \in dom(s') - dom(s)$ then $s'(x) = 0$. In this way, s' forces $X_\alpha \cap Y_\beta$, $X_\beta \cap Y_\alpha \subseteq m$ so $(\mathcal{X}, \mathcal{Y})$ is not a Luzin gap. ■

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