Arch. Math. Logic (2016) 55:105–131
DOI 10.1007/s00153-015-0459-2

Mathematical Logic

Strong measure zero in separable metric spaces and Polish groups

Michael Hrušák[1] · **Wolfgang Wohofsky**[2] · **Ondřej Zindulka**[3]

Received: 23 January 2014 / Accepted: 26 January 2015 / Published online: 19 January 2016

Abstract The notion of strong measure zero is studied in the context of Polish groups and general separable metric spaces. An extension of a theorem of Galvin, Mycielski and Solovay is given, whereas the theorem is shown to fail for the Baer–Specker group $\mathbb{Z}^{\omega}$. The uniformity number of the ideal of strong measure zero subsets of a separable metric space is examined, providing solutions to several problems of Miller and Steprāns (Ann Pure Appl Logic 140(1–3):52–59, 2006).

Work on this project was partially conducted during the third author's stay at the Centro de Ciencias Matemáticas, Universidad Nacional Autonóma de México supported by CONACyT Grant No. 125108. The first author gratefully acknowledges support from PAPIIT Grant IN 108014 and CONACyT Grant 177758. The work of the second author was partially supported by the Czech Ministry of Education Grant 7AMB13AT011: Combinatorics and Forcing.

✉ Michael Hrušák
michael@matmor.unam.mx
http://www.matmor.unam.mx/~michael

Wolfgang Wohofsky
wolfgang.wohofsky@gmx.at
http://www.wohofsky.eu/math/

Ondřej Zindulka
zindulka@mat.fsv.cvut.cz
http://mat.fsv.cvut.cz/zindulka

1 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación Científica, Circuito exterior, Ciudad Universitaria, 04510 Mexico, D.F., Mexico

2 Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10/104, 1040 Vienna, Austria

3 Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

Keywords Strong measure zero · Separable metric space · Small ball property · Polish group · Elastic group · Baer–Specker group $\mathbb{Z}^\omega$ · Galvin–Mycielski–Solovay theorem · Meager · Uniformly meager · Translation · ω-Translatable · Uniformity number · Rothberger

Mathematics Subject Classification 03E17 · 22B05 · 22A10 · 54E53 · 54E52

1 Introduction

All spaces considered are separable and metrizable.

By the definition due to Borel [7], a metric space X has *strong measure zero* (**Smz**) if for any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive numbers there is a cover $\{U_n : n \in \omega\}$ of X such that diam $U_n \leqslant \varepsilon_n$ for all n.

In the same paper Borel conjectured that every **Smz** set of reals is countable. A consistent counterexample to his conjecture was given rather quickly by Sierpiński [32] who showed that the continuum hypothesis implies the existence of a counterexample by noting that every *Luzin set* (an uncountable set of reals every meager subset of which is countable) has strong measure zero.

As is well known now, the Borel Conjecture turned out to be independent of the usual axioms of set theory. Richard Laver in his ground-breaking work [22] proved that the Borel Conjecture is relatively consistent with ZFC. Not only did he answer a longstanding famous problem, but the method of the proof eventually led to the development of proper forcing and the consistency of strong forcing axioms [10,11, 26,35]. It was proved by Carlson [8] that the Borel Conjecture actually implies a formally stronger statement that all separable **Smz** metric spaces are countable.

Galvin, Mycielski, and Solovay [13] confirmed a conjecture of Prikry by proving that a set $A \subseteq \mathbb{R}$ is of strong measure zero if and only if $A + M \neq \mathbb{R}$ for every meager set $M \subseteq \mathbb{R}$. Recently Kysiak [21] and Fremlin [12] showed that an analogous theorem is true for all locally compact metrizable groups (see also the second author's PhD thesis [37]). Here we give a proof of Kysiak and Fremlin's result and consider the natural question as to how far the result can be extended.

On the other hand, we show that the theorem does not hold for all Polish groups. This, of course, depends on further set-theoretic axioms, as the result obviously holds for all Polish groups assuming, e.g., the Borel Conjecture. In particular, we show that the result consistently fails for the Baer–Specker group $\mathbb{Z}^\omega$.

Cardinal invariants associated with strong measure zero sets on $\mathbb{R}$, ω^ω, and 2^ω have been studied rather extensively in recent decades [2,15,38]. We present a study of the *uniformity* invariant of the σ-ideal $\mathsf{Smz}(X)$ of strong measure zero subsets of a general metric space X. With respect to strong measure zero, separable metric spaces seem to split into two disjoint classes, the dividing line being given by the so-called *small ball property* (**sbp**) introduced by Behrends and Kadets [3]. Via (fragments of) Galvin–Mycielski–Solovay type theorems this study connects to the investigation of the so-called *transitive coefficient* $\mathsf{cov}^*(\mathcal{M})$ in Polish groups [2,9,25]. In particular, we answer several questions concerning $\mathsf{cov}^*(\mathcal{M})$ posed by Miller and Steprāns in [25].

We include a section dealing with the small ball property in some detail. In particular, we characterize the **sbp** subsets of the Baire space, calculate the cardinal invariants of **sbp**, and show that a metrizable space has the Menger Property if and only if it has the small ball property in every compatible metric.

2 Notation and preliminary results

2.1 Set-theoretic notation

Our set-theoretic notation is mostly standard and follows e.g. [18,20]. In particular, the set of finite ordinals will be identified with the set of non-negative integers and denoted by ω. In the same vein, the non-negative integers themselves are identified with the set of smaller non-negative integers, in particular $2 = \{0, 1\}$.

2.2 Metric spaces

All spaces considered here will be separable and metrizable, hence second countable, typically endowed with an unspecified metric, often denoted d. We say that a metric d is *compatible* with the topology of a topological space X if it generates the topology of the space. We denote by $B(x, \varepsilon)$ the closed ball with radius ε centered at x, the corresponding open ball will be denoted by $B^\circ(x, \varepsilon)$.

The metrizable spaces we shall deal with are often of the type A^ω for some finite or countable set A—the Cantor cube 2^ω, naturally identified with the countable product of the two-element group, the *Baire space* ω^ω, and the *Baer–Specker group* $\mathbb{Z}^\omega$.

All of these may be endowed with the *metric of least difference* defined by $d(f, g) = 2^{-|f \wedge g|}$, where $f \wedge g = f \restriction n$ for $n = \min\{k : f(k) \neq g(k)\}$. The topology induced by this metric is the product topology of countably many copies of discrete A. The space is compact if and only if A is finite, otherwise it is nowhere locally compact. Basic clopen sets in the space A^ω can be conveniently represented by nodes of the tree $A^{<\omega}$: Given $s \in A^{<\omega}$, we let $\langle s\rangle = \{f \in A^\omega : s \subseteq f\}$. Also, closed sets in A^ω can be represented by subtrees of $A^{<\omega}$: Given a subtree T of $A^{<\omega}$, we let $[T] = \{f \in A^\omega : \forall n \in \omega\ f \restriction n \in T\}$ be the set of branches of T.

A metric space is *analytic* if it is a continuous image of ω^ω; it is *Borel* (*absolutely* G_δ, resp.) if it is Borel (G_δ, resp.) in its completion.

2.3 Separable metric groups

By a *Polish group* we understand a separable, completely metrizable topological group. A compatible metric d on a separable metrizable group $\mathbb{G}$ is *left-invariant* if $d(zx, zy) = d(x, y)$ for any $x, y, z \in \mathbb{G}$. Locally compact (equivalently, σ-compact) Polish groups admit a complete left-invariant compatible metric. On the other hand not all Polish groups do: A separable group $\mathbb{G}$ is a CLI *group* if it admits a complete left-invariant compatible metric. Note that CLI groups include all locally compact as well as all abelian Polish groups.

2.4 Cardinal invariants

Given a family $\mathcal{I}$ of subsets of a set X—usually an ideal—the following are the standard cardinal invariants associated with $\mathcal{I}$:

$$\begin{aligned}
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\},\\
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} \notin \mathcal{I}\},\\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} = X\},\\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\}.
\end{aligned}$$

We denote by $\mathcal{M}$, $\mathcal{N}$ the ideals of meager and Lebesgue null subsets of 2^ω, respectively. For $f, g \in \omega^\omega$, we say that $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$ (the order of *eventual dominance*). A family $F \subseteq \omega^\omega$ is *bounded* if there is an $h \in \omega^\omega$ such that $f \leqslant^* h$ for all $f \in F$; and F is *dominating* if for any $g \in \omega^\omega$ there is $f \in F$ such that $g \leqslant^* f$. The cardinal invariants related to eventual dominance are $\mathfrak{b}$ (the minimal cardinality of an unbounded family) and $\mathfrak{d}$ (the minimal cardinality of a dominating family).

We shall make use of two more, very similar, cardinal invariants. Following [25], we denote[1] by $\mathfrak{eq}$:

$$\mathfrak{eq} = \min\{|F| : F \subseteq \omega^\omega \text{ bounded}, \forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)\}.$$

It is a theorem of Bartoszyński and Judah [2, 2.4.1] that omitting "bounded" yields $\mathsf{cov}(\mathcal{M})$. We need a cardinal invariant similar to $\mathfrak{eq}$, only that partial functions take the place of functions (say that f is an *infinite partial function* if $f \in \omega^A$ for some $A \in [\omega]^\omega$):

$$\begin{aligned}
\mathfrak{ed} = \min\{|F| : F &\text{ is a bounded family of infinite partial functions},\\
&\forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \operatorname{dom} f\ f(n) \neq g(n)\}.
\end{aligned}$$

As shown in [17, Lemma 3.9], this cardinal invariant is actually equal to the uniformity number $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$ of the ideal $\mathcal{ED}_{\mathsf{fin}}$. We denote it $\mathfrak{ed}$ to avoid the lengthy $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$ and also to emphasize the similarity with $\mathfrak{eq}$. The provable inequalities between the listed cardinals are summarized in the following diagram (see [2,17] for proofs). As usual, the arrows in the diagram point from the smaller to the larger cardinal.

$$\begin{array}{ccccccccc}
\mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & & & \\
\uparrow & & \uparrow & & & & & & \\
\mathsf{add}(\mathcal{M}) & \longrightarrow & \mathsf{cov}(\mathcal{M}) & \longrightarrow & \mathfrak{ed} & \longrightarrow & \mathfrak{eq} & \longrightarrow & \mathsf{non}(\mathcal{N})
\end{array}$$

[1] The definition of $\mathfrak{eq}$ comes from Miller's [23] (see also [2, 2.7.14]) and the notation from [25].

Moreover, $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathfrak{eq}\} = \min\{\mathfrak{b}, \mathfrak{ed}\}$ and $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathfrak{ed}\}$, while $\mathsf{cov}(\mathcal{M}) < \min\{\mathfrak{d}, \mathfrak{eq}\}$ is consistent with ZFC by a theorem of Goldstern et al. [15].

2.5 Strong measure zero in metric spaces and groups

The notion of strong measure zero is in general neither a topological nor a metric property, it is a *uniform* property; in particular, a uniformly continuous image of a **Smz** set is **Smz**. Also, if X uniformly embeds into Y, then any set $A \subseteq X$ that is not **Smz** in X is not **Smz** in Y either. This has the following immediate corollary that will be relevant later on:

Lemma 2.1 (i) *If* $f : X \to Y$ *is uniformly continuous onto* Y, *then* $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathsf{non}(\mathbf{Smz}(Y))$.
(ii) *If* X *uniformly embeds into* Y, *then* $\mathsf{non}(\mathbf{Smz}(X)) \geqslant \mathsf{non}(\mathbf{Smz}(Y))$.

Another important observation deals with groups. As all left-invariant metrics on a separable metrizable group are uniformly equivalent, it does not matter which metric one chooses, so the notion becomes "topological": a subset S of a topological group $\mathbb{G}$ is *Rothberger bounded* if for every sequence $\langle U_n : n \in \omega\rangle$ of neighbourhoods of $1_{\mathbb{G}}$ there is a sequence $\langle g_n : n \in \omega\rangle$ of elements of the group $\mathbb{G}$ such that the family $\langle g_n \cdot U_n : n \in \omega\rangle$ covers S. It is easy to see [12] that a subset of a Polish group $\mathbb{G}$ is Rothberger bounded if and only if it is strong measure zero w.r.t. some (any) left-invariant metric on $\mathbb{G}$. Many of the results stated here could be phrased in the language of uniformities and/or in terms of the property of being Rothberger bounded (see [12] for such treatment). Extensions of Borel Conjecture to larger classes of non-separable groups were recently considered by Galvin and Scheepers [14].

Whenever $\mathbb{G}$ is a Polish group, $\mathbf{Smz}(\mathbb{G})$ denotes the strong measure zero sets with respect to any left-invariant metric (i.e., the Rothberger bounded sets as described above). In case of a metrizable space of the type A^ω, $\mathbf{Smz}(A^\omega)$ denotes the strong measure zero sets with respect to the least difference metric (which is left-invariant in case of the groups 2^ω and $\mathbb{Z}^\omega$, and therefore conforms with the above).

Cardinal invariants of strong measure zero were studied in quite some detail for two important, yet particular instances of separable metric spaces: the Cantor cube 2^ω and the Baire space ω^ω (both equipped with the metric of least difference) [2,15,38]. Here we shall concentrate on the uniformity number $\mathsf{non}(\mathbf{Smz}(X))$.

For a general separable metric space X there is a lower estimate of $\mathsf{non}(\mathbf{Smz}(X))$ found by Rothberger [31] in 1941 and an upper estimate found by Szpilrajn [33] in 1934. The uniformity invariant $\mathsf{non}(\mathbf{Smz}(X))$ for $X = 2^\omega$ and $X = \omega^\omega$ was calculated by Bartoszyński and Judah [2], and Fremlin and Miller [24], respectively.

Theorem 2.2 *Let* X *be a separable metric space.*

(i) [31] $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{non}(\mathbf{Smz}(X))$,
(ii) [33] *if* X *is not of universal measure zero,*[2] *then* $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathsf{non}(\mathcal{N})$,
(iii) [2,24] $\mathsf{non}(\mathbf{Smz}(\omega^\omega)) = \mathsf{cov}(\mathcal{M})$ *and* $\mathsf{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$.

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[2] Recall that a metric space X is of *universal measure zero* if there is no probability Borel measure on X vanishing on singletons.

3 The Galvin–Mycielski–Solovay theorem revisited

In this section we examine to what extent the Galvin–Mycielski–Solovay theorem can be extended to Polish groups other than $\mathbb{R}$. We denote by $\mathcal{M}(\mathbb{G})$ the ideal of meager subsets of $\mathbb{G}$. We begin by recalling an old result of Prikry:

Theorem 3.1 (Prikry [28]) *Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d and let $S \subseteq \mathbb{G}$ be such that $S \cdot M \neq \mathbb{G}$ for all $M \in \mathcal{M}(\mathbb{G})$. Then S has strong measure zero with respect to d (i.e., $S \in \mathbf{Smz}(\mathbb{G})$).*

Proof Let S be as above and let $\delta_n \searrow 0$ be given. Let $\{E_n : n \in \omega\}$ be a family of open sets such that diam $E_n < \delta_n$ and $E = \bigcup_n E_n$ is dense in $\mathbb{G}$. Then E^{-1} is also dense open in $\mathbb{G}$, hence, $M = \mathbb{G} \setminus E^{-1}$ is nowhere dense. By the assumption, there is $z \in \mathbb{G} \setminus S \cdot M$. Routine calculation yields $S \subseteq z \cdot E = \bigcup_n z \cdot E_n$. Since the metric is left-invariant, $\operatorname{diam}(z \cdot E_n) < \delta_n$ and thus S is covered by a sequence of open sets of diameters below δ_n. As this is true for every sequence $\delta_n \searrow 0$, we have that $S \in \mathbf{Smz}(\mathbb{G})$. □

Motivated by Prikry's result we introduce the following notation:

Definition 3.2 *Let $\mathbb{G} = (\mathbb{G}, \cdot)$ be a topological group. Let*

$$\mathbf{Pr}(\mathbb{G}) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}(\mathbb{G})\ A \cdot M \neq \mathbb{G}\}.$$

In other words, Prikry's result states that $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$ for any separable metric group. As mentioned in the introduction, Galvin, Mycielski, and Solovay answered Prikry's question by showing that the reverse inclusion holds for $\mathbb{R}$. The same was recently proved to hold for all locally compact groups by Kysiak [21] and Fremlin [12]. We shall present a proof of the theorem here. The main point is that Polish groups which are locally compact or carry an invariant metric admit a converse of Prikry's result with meager replaced by uniformly meager.

Definition 3.3 *A subset N of a separable metric space X is* uniformly nowhere dense *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x \in X$ there is a $y \in X$ such that $B(y, \delta) \subseteq B(x, \varepsilon) \setminus N$. A set M is* uniformly meager *if it can be written as a union of countably many uniformly nowhere dense sets.*

We shall denote the collection of all uniformly meager subsets of X by $\mathcal{UM}(X)$ (or just $\mathcal{UM}$).

Theorem 3.4 *Let $\mathbb{G}$ be a Polish group which is either locally compact, or equipped with a complete invariant metric d, and let $S \subseteq \mathbb{G}$ be* **Smz** *with respect to d (i.e., $S \in \mathbf{Smz}(\mathbb{G})$). Then $S \cdot M \neq \mathbb{G}$ for all $M \in \mathcal{UM}(\mathbb{G})$.*

Proof The proof of the case of a locally compact group is provided below in Sect. 6. It follows from Theorem 6.3 and the fact that in a locally compact group every meager set is uniformly meager (Proposition 3.6).

When the group admits a complete invariant metric d, then every uniformly nowhere dense set N in the group $\mathbb{G}$ satisfies the following condition:

$$\forall \varepsilon > 0\, \exists \delta > 0 \quad \forall x, y \in \mathbb{G}\, \exists z \in \mathbb{G} \quad B(z, \delta) \subseteq B(x, \varepsilon) \setminus (B(y, \delta) \cdot N).$$

To see this note that, as the metric is both left- and right-invariant, $B(y, \delta) \cdot N = B(1, \delta) \cdot y \cdot N$ and $y \cdot N$ is uniformly nowhere dense (witnessed by some δ which is independent of y). If δ is such a witness, then it is easy to see that $\frac{\delta}{2}$ works for the property above.

An immediate consequence is that given an increasing sequence $\langle N_n : n \in \omega \rangle$ of uniformly nowhere dense sets in a group equipped with a complete invariant metric d (an increasing sequence of compact (uniformly) nowhere dense, resp.) there is a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals, without loss of generality converging to 0, such that for every $n > 0$

$$\forall x, y \in \mathbb{G}\, \exists z \in \mathbb{G} \quad B(z, \varepsilon_n) \subseteq B(x, \varepsilon_{n-1}) \setminus (B(y, \varepsilon_n) \cdot N_n). \tag{1}$$

Now, fix a **Smz** set S and a uniformly meager set M written as the union of an increasing sequence $\langle N_n : n \in \omega \rangle$ of uniformly nowhere dense sets, (compact nowhere dense sets, resp.). Let $\langle \varepsilon_n : n \in \omega \rangle$ be the sequence given by (1). As S is **Smz**, there is a cover $\{U_n : n \in \omega\}$ of S such that $\operatorname{diam} U_n \leqslant \varepsilon_n$, for all $n \in \omega$, and such that each $s \in S$ is contained in infinitely many of the U_n's. Applying (1) recursively (note that for each $n \in \omega$ there is a $y \in \mathbb{G}$ such that $U_n \subseteq B(y, \varepsilon_n)$) one gets a sequence $\langle x_n : n \in \omega \rangle$ of elements of $\mathbb{G}$ such that for every $n \in \omega$

$$B(x_{n+1},\ \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus (U_{n+1} \cdot N_{n+1}).$$

The sequence $\langle x_n : n \in \omega \rangle$ is a Cauchy sequence, hence has a limit x which is the only point of $\bigcap_{n \in \omega} B(x_n, \varepsilon_n)$. Then $x \notin \bigcup_{n \in \omega} U_n \cdot N_n \supseteq S \cdot M$. To see the last inclusion note that, as the sequence $\langle N_n : n \in \omega \rangle$ is increasing and each element of S is contained in infinitely many of the U_n's, for every $(s, m) \in S \times M$ there is an $n \in \omega$ such that $s \in U_n$ and $m \in N_n$. □

An immediate corollary of the theorem and the following proposition is the Fremlin–Kysiak result.

Theorem 3.5 (Fremlin [12], Kysiak [21]) **Pr**$(\mathbb{G}) =$ **Smz**$(\mathbb{G})$ *for every locally compact group* $\mathbb{G}$.

On the other hand, Theorem 3.4 does not provide a proof of the Galvin–Mycielski–Solovay theorem for any group which is not locally compact, also by the following proposition.

Proposition 3.6 (i) $\mathcal{M}(X) = \mathcal{UM}(X)$ *for any locally compact metric space* X.

(ii) $\mathcal{M}(X) \neq \mathcal{UM}(X)$ *for every nowhere locally compact complete metric space* X.

(iii) *Hence, a Polish group* $\mathbb{G}$ *is locally compact if and only if* $\mathcal{M}(\mathbb{G}) = \mathcal{UM}(\mathbb{G})$.

Proof (i) It is easy to verify that it is sufficient to prove that every nowhere dense subset of a compact space is, in fact, uniformly nowhere dense. To that end fix a nowhere dense subset N of a compact space K and $\varepsilon > 0$. Let F be a finite subset of K such that $K = \bigcup_{x \in F} B(x, \frac{\varepsilon}{2})$. For every $x \in F$ let $y_x \in B(x, \frac{\varepsilon}{2})$ and $\delta_x > 0$ be such that $B(y_x, \delta_x) \subseteq B(x, \frac{\varepsilon}{2}) \setminus N$. Then $\delta = \min\{\delta_x : x \in F\}$ works as $B(x, \frac{\varepsilon}{2}) \subseteq B(z, \varepsilon)$ whenever $z \in B(x, \frac{\varepsilon}{2})$.

(ii) As the space is nowhere locally compact, i.e., nowhere totally bounded, for every U with non-empty interior there is an $\varepsilon^U > 0$ and a pairwise disjoint family $\{V_k^U : k \in \omega\}$ of open balls of radius ε^U contained in U. We want to construct a nowhere dense set N which is not uniformly meager. In order to do that we recursively construct a family $\{U_s : s \in \omega^{<\omega}\}$ of non-empty regular closed sets (i.e., for each s, the closure of the interior of U_s equals U_s) satisfying the following properties:

(1) $\operatorname{diam} U_s \leqslant 2^{-|s|}$ for every $s \in \omega^{<\omega}$,
(2) $\bigcup_{n \in \omega} U_{s^\frown n} \subseteq U_s$ for every $s \in \omega^{<\omega}$,
(3) $U_{s^\frown n} \cap U_{s^\frown m} = \emptyset$ for every $s \in \omega^{<\omega}$ and any two distinct $m, n \in \omega$,
(4) $\operatorname{int}(U_s \setminus \bigcup_{n \in \omega} U_{s^\frown n}) \neq \emptyset$ for every $s \in \omega^{<\omega}$,
(5) for all $s \in \omega^{<\omega}$ and $k \in \omega$ and $x \in V_k^{U_s}$ there is a $n \in \omega$ such that $U_{s^\frown n} \subseteq B(x, 2^{-k})$.

Such a family can be constructed by a routine recursion.

Having fixed a family as above let $N = \bigcap_{j \in \omega} \bigcup_{|s|=j} U_s$. We claim that this is the set we are looking for.

First, N is nowhere dense: a non-empty open set U is either disjoint from N, or contains U_s for some $s \in \omega^{<\omega}$. Then, however, $\emptyset \neq \operatorname{int}(U_s \setminus \bigcup_{n \in \omega} U_{s^\frown n}) \subseteq U \setminus N$ [see property (4) above].

Now we will prove that N is not uniformly meager in X. The set N is naturally homeomorphic to ω^ω [see properties (1)–(3) above], hence satisfies the Baire Category Theorem. Aiming toward a contradiction assume that $N \subseteq \bigcup_{l \in \omega} N_l$, where each N_l is a closed uniformly nowhere dense subset of X. By the Baire Category Theorem applied to N there is an $s \in \omega^{<\omega}$ and an $l \in \omega$ such that $U_s \cap N \subseteq N_l$, hence $U_s \cap N$ is uniformly nowhere dense. So, there is a $\delta > 0$ as in the definition of uniformly nowhere dense corresponding to ε^{U_s}. Consider $V_k^{U_s}$, for $2^{-k} < \delta$. Then, on the one hand there is an $x \in V_k^{U_s}$ such that $B(x, 2^{-k}) \subseteq V_k^{U_s} \setminus N$, and on the other hand, there is [see property (5) above] an $n \in \omega$ such that $\emptyset \neq N \cap U_{s^\frown n} \subseteq B(x, 2^{-k})$, which is a contradiction.

(iii) Follows directly from (i) and (ii). □

It is not clear to us whether Theorem 3.4 holds for all Polish groups. As in locally compact groups every meager set is uniformly meager, one has to wonder whether Prikry's result—Theorem 3.1—may hold with meager replaced by uniformly meager. It turns out that this is not the case, at least not in general. We give a (consistent) counterexample for the Baer–Specker group $\mathbb{Z}^\omega$.

Proposition 3.7 (CH) *There is a set $X \subseteq \mathbb{Z}^\omega$ such that $X + M \neq \mathbb{Z}^\omega$ for every $M \in \mathcal{UM}(\mathbb{Z}^\omega)$, and yet $X \notin$* **Smz**$(\mathbb{Z}^\omega)$.

Proof Suppose $\{M_\alpha : \alpha < \omega_1\}$ is a list of all uniformly meager F_σ subsets of $\mathbb{Z}^\omega$, and let $\{(s_n^\alpha)_{n\in\omega} : \alpha < \omega_1\}$ be a list of all sequences $(s_n)_{n\in\omega}$ satisfying $s_n \in \mathbb{Z}^{n+1}$ for all $n \in \omega$.

Claim 3.8 *Suppose $(s_n)_{n\in\omega}$ is a sequence satisfying $s_n \in \mathbb{Z}^{n+1}$ for all $n \in \omega$, and $(F_n)_{n\in\omega}$ are closed uniformly nowhere dense sets. Then there exists an $x \in \mathbb{Z}^\omega$ such that $x \notin \bigcup_{n\in\omega} F_n$ and $x \notin \bigcup_{n\in\omega}[s_n]$.*

Proof First note that whenever F is uniformly nowhere dense and $t \in \mathbb{Z}^{<\omega}$, there is an $l \in \omega$ such that for every $k \in \mathbb{Z}$ there is a $t' \supseteq t^\frown\langle k\rangle$ with $|t'| \leqslant l$ and $[t'] \cap F = \emptyset$.

Let $A = \mathbb{Z}^\omega \setminus \bigcup_{n\in\omega}[s_n]$. Since A is closed, we can fix a tree $T \subseteq \mathbb{Z}^{<\omega}$ such that the set $[T]$ of its branches is A. Note that T has the following property: whenever $t \in T$ and $l \in \omega$, there is a $k \in \mathbb{Z}$ (actually, this holds for all but at most l many k's) such that for all $t' \supseteq t^\frown\langle k\rangle$ with $|t'| \leqslant l$, we have $t' \in T$.

By the above, we can construct a sequence $t_0 \subsetneqq t_1 \subsetneqq \cdots$ such that $t_n \in T$ and $[t_n] \cap F_n = \emptyset$ for each $n \in \omega$. Let $x = \bigcup_{n<\omega} t_n$; then $x \notin \bigcup_{n\in\omega} F_n$, and $x \in [T] = \mathbb{Z}^\omega \setminus \bigcup_{n\in\omega}[s_n]$, as desired. □

We are now prepared to construct the set $X = \{x_\alpha : \alpha < \omega_1\}$ together with an auxiliary sequence $(z_\alpha)_{\alpha<\omega_1}$ as follows. At step α, we first pick

$$z_\alpha \notin \bigcup_{\beta<\alpha} (x_\beta + M_\alpha) \tag{2}$$

(this is possible since any countable collection of meager sets does not cover the whole group), and then pick

$$x_\alpha \notin \bigcup_{\beta\leqslant\alpha} (z_\beta - M_\beta) \tag{3}$$

such that

$$x_\alpha \notin \bigcup_{n\in\omega} \left[s_n^\alpha\right] \tag{4}$$

(this is possible by Lemma 3.8, just note that $z_\beta - M_\beta$ is uniformly meager for each $\beta \leqslant \alpha < \omega_1$, so there are countably many closed uniformly nowhere dense sets F_n with $\bigcup_{\beta\leqslant\alpha}(z_\beta - M_\beta) \subseteq \bigcup_{n\in\omega} F_n$).

It is easy to derive from (2) and (3) that the resulting set X satisfies $X + M \neq \mathbb{Z}^\omega$ for any $M \in \mathcal{UM}(\mathbb{Z}^\omega)$: since each uniformly meager set is contained in one of the M_α's, it suffices to show that $X + M_\alpha \neq \mathbb{Z}^\omega$ for any α; actually, $z_\alpha \notin X + M_\alpha$, where $z_\alpha \notin x_\beta + M_\alpha$ holds due to (2) (for $\beta < \alpha$) and (3) (for $\beta \geqslant \alpha$).

Moreover, $X \notin \mathbf{Smz}(\mathbb{Z}^\omega)$: if it were strong measure zero, there would be $(s_n)_{n\in\omega}$ with $s_n \in \mathbb{Z}^{n+1}$ for all $n \in \omega$ such that $X \subseteq \bigcup_{n\in\omega}[s_n]$; pick $\alpha < \omega_1$ such that $(s_n^\alpha)_{n\in\omega} = (s_n)_{n\in\omega}$ and recall that $x_\alpha \in X$ satisfies $x_\alpha \notin \bigcup_{n\in\omega}[s_n]$ by (4), a contradiction. □

Now we turn toward the main negative result. To that end we introduce three related properties. We shall call a Polish group $\mathbb{G}$

(1) **GMS** if it satisfies the Galvin–Mycielski–Solovay theorem absolutely, i.e., there is a ZFC proof of the fact.

(2) *strongly* GMS if for every nowhere dense $M \subseteq \mathbb{G}$ there is a $\langle \varepsilon_n : n \in \omega \rangle$ $\forall \langle U_n : n \in \omega \rangle$ such that diam $U_n < \varepsilon_n$ there is a $g \in \mathbb{G}$ such that $(g \cdot \bigcup_{n\in\omega} U_n) \cap M = \emptyset$.
(3) *weakly* GMS if for every closed nowhere dense $M \subseteq \mathbb{G}$ there is a $\langle \varepsilon_n : n \in \omega \rangle$ $\forall \langle U_n : n \in \omega \rangle$ such that diam $U_n < \varepsilon_n$ there is a $g \in \mathbb{G}$ such that $(g \cdot \bigcup_{n\in\omega} U_n) \cap M$ is not dense in M.

The property of being a GMS group is intentionally somewhat vague. Perhaps, a technically better property would be "$\mathbb{G}$ satisfies the Galvin–Mycielski–Solovay theorem [i.e., **Pr**$(\mathbb{G})$ = **Smz**$(\mathbb{G})$] after collapsing the continuum to ω_1 (with a σ-closed forcing)". The advantage would be that we could talk about the very same group $\mathbb{G}$, not having to worry about in which way $\mathbb{G}$ should be definable. We hope that the need for any such definition will soon be eliminated by a suitable theorem.

It is useful to explicitly state the negation of being weakly GMS:

A group $\mathbb{G}$ is *not weakly* GMS if there is a closed nowhere dense $M \subseteq \mathbb{G}$ such that $\forall \langle \varepsilon_n : n \in \omega \rangle$ $\exists \langle U_n : n \in \omega \rangle$ with diam $U_n < \varepsilon_n$ such that $\forall g \in \mathbb{G}$ $(g \cdot \bigcup_{n\in\omega} U_n) \cap M$ is comeager (or, equivalently, dense) in M.

We will show that, as the names suggest, strongly GMS $\Rightarrow$ GMS $\Rightarrow$ weakly GMS.

To prove the first implication, assume that a Polish group $\mathbb{G}$ is not GMS, that is, it is consistent that there are a **Smz** set S and a meager set M such that $S \cdot M = \mathbb{G}$. It follows from the proof of [25, Proposition 2,3] that there is a nowhere dense set N and a countable set C such that $M \subseteq C \cdot N$. Then of course $S \cdot C \cdot N = \mathbb{G}$, while $S \cdot C$ is **Smz**. Hence, M can actually be chosen nowhere dense.

Working in such model, given a sequence $\langle \varepsilon_n : n \in \omega \rangle$, let $\langle U_n : n \in \omega \rangle$ be such that diam $U_n < \varepsilon_n$ which covers S. Now, for any $g \in \mathbb{G}$ there are $u \in \bigcup_{n\in\omega} U_n$ and $m \in M^{-1}$ such that $g^{-1} = u \cdot m^{-1}$. Then $m = g \cdot u$, hence $(g \cdot \bigcup_{n\in\omega} U_n) \cap M^{-1} \neq \emptyset$.

The following proposition proves the second implication.

Proposition 3.9 (cov$(\mathcal{M})$=$\mathfrak{c}$) *If* $\mathbb{G}$ *is Polish and not weakly* GMS*, then*

$$\mathbf{Pr}(\mathbb{G}) \neq \mathbf{Smz}(\mathbb{G}).$$

Proof The proof proceeds by transfinite recursion. Enumerate the group $\mathbb{G}$ as $\{g_\alpha : \alpha < \mathfrak{c}\}$ and also enumerate all decreasing sequences $\langle \varepsilon_n : n \in \omega \rangle$ of real numbers converging to 0 as $\{\langle \varepsilon_n^\alpha : n \in \omega \rangle : \alpha < \mathfrak{c}\}$. Fix a closed nowhere dense set M which witnesses that $\mathbb{G}$ is not weakly GMS. For each $\alpha < \mathfrak{c}$ fix a corresponding sequence $\langle U_n^\alpha : n \in \omega \rangle$ such that diam $U_n^\alpha < \varepsilon_n^\alpha$ and such that for all $g \in \mathbb{G}$, $(g \cdot \bigcup_{n\in\omega} U_n^\alpha) \cap M$ is comeager in M. Let $U_\alpha = \bigcup_{n\in\omega} U_n^\alpha$.

As cov$(\mathcal{M}) = \mathfrak{c}$, the intersection of fewer than $\mathfrak{c}$ relatively dense open subsets of M is not empty. In particular, for every $\alpha < \mathfrak{c}$, there is an

$$m_\alpha \in M \cap \left(g_\alpha^{-1} \cdot \bigcap_{\beta \leqslant \alpha} U_\beta \right).$$

There is then an $x_\alpha \in \bigcap_{\beta \leqslant \alpha} U_\beta$ such that $m_\alpha = g_\alpha^{-1} \cdot x_\alpha$. That is

$$g_\alpha = x_\alpha \cdot m_\alpha^{-1}. \tag{5}$$

Let $X = \{x_\alpha : \alpha < \mathfrak{c}\}$. Then $\mathbb{G} = X \cdot M^{-1}$, so $X \notin \mathbf{Pr}(\mathbb{G})$. Let us argue that $X \in \mathbf{Smz}(\mathbb{G})$: Given a decreasing sequence $\langle \varepsilon_n : n \in \omega \rangle$ consider $\langle \varepsilon_{2n} : n \in \omega \rangle$. This sequence is listed as $\langle \varepsilon_n^\alpha : n \in \omega \rangle$ for some $\alpha < \mathfrak{c}$ and, by construction, every $x_\gamma \in U_\alpha$ for $\gamma \geqslant \alpha$. On the other hand, $X \setminus U_\alpha \subseteq \{x_\beta : \beta < \alpha\}$, hence has size less than $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. By Theorem 2.2(i), $X \setminus U_\alpha$ is a **Smz** set, hence can be covered by balls of diameters ε_{2n+1}, $n \in \omega$. We conclude that X has strong measure zero. □

In fact, the proof of the second implication (GMS $\Rightarrow$ weakly GMS) requires a little bit more of an argument: Assume that $\mathbb{G}$ is not weakly GMS in some model V of set theory. By σ-closed forcing collapse the continuum of V to ω_1 in a generic extension $V[G]$. Note that the property of being weakly GMS is projective, hence absolute for extension by σ-closed forcing. Hence, $\mathbb{G}$ is not weakly GMS in the model $V[G]$ either, while the continuum hypothesis (in particular, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$) holds in $V[G]$. By the claim, $V[G] \models$ "The Galvin–Mycielski–Solovay theorem fails for $\mathbb{G}$", hence $\mathbb{G}$ is not GMS.

Theorem 3.10 *The group $\mathbb{Z}^\omega$ is not weakly* GMS.

Proof Enumerate $\mathbb{Z}^{<\omega}$ as $\{t_n : n \in \omega\}$ and let $\{s_n : n \in \omega\} \subseteq \mathbb{Z}^{<\omega}$ be such that

(1) s_n end-extends t_n, and
(2) for every $m \in \omega$ there is at most one $n \in \omega$ such that $s_n \in \mathbb{Z}^m$.

Let

$$N = \mathbb{Z}^\omega \setminus \bigcup_{n\in\omega} \langle s_n \rangle.$$

The set N is then closed nowhere dense [by (1)]. Let $T = \{h \restriction n : h \in N\}$.

Claim 3.11 $\forall \varepsilon > 0\, \forall m \in \omega\, \exists M_\varepsilon^m \in \omega\, \forall x, y \in \mathbb{Z}^\omega$ $x \wedge y \in \mathbb{Z}^m \cap T$ *and* $|x(m) - y(m)| \geqslant M_\varepsilon^m$ *implies that* $N \cap (B(x,\varepsilon) \cup B(y,\varepsilon)) \neq \emptyset$.

Proof Define (for $s \in T$) $G(s)$ as the minimal $\delta > 0$ such that every ball with radius δ contained in the cone/ball $\langle s \rangle$ intersects N. Note that G is well-defined and, moreover, $\lim_{s\in T} G(s) = 0$ by (2), i.e., for every $\varepsilon > 0$ the set $R_\varepsilon = \{s \in T : G(s) > \varepsilon\}$ is finite. Note, by (2) again, that if $s \in T$ then $s^\frown n \in T$ for all but (possibly) one $n \in \mathbb{Z}$. Let M_ε^m be large enough so that for every $s \in T \cap \mathbb{Z}^m$ the set $\{n : s^\frown n \in R_\varepsilon\} \cup \{n : s \in T$ and $s^\frown n \in \mathbb{Z}^{m+1} \setminus T\}$ is contained in the interval $(-M_\varepsilon^m/2, M_\varepsilon^m/2)$.

If $x \wedge y \in \mathbb{Z}^m \cap T$ and $|x(m) - y(m)| \geqslant M_\varepsilon^m$, at most one of $x \restriction m+1$, $y \restriction m+1$ belongs to $R_\varepsilon \cup \{s_n : n \in \omega\}$, hence $N \cap B(x,\varepsilon) \neq \emptyset$ or $N \cap B(y,\varepsilon) \neq \emptyset$, and the claim is proved. □

Now, given a decreasing sequence $\langle \varepsilon_n : n \in \omega \rangle$ and $n \in \omega$, let

$$U_{2n} = B\left(t_n{}^\frown\bar{0}, \varepsilon_{2n}\right) \text{ and } U_{2n+1} = B\left(t_n{}^\frown M_{\varepsilon_{2n+1}}^{|t_n|}{}^\frown\bar{0}, \varepsilon_{2n+1}\right).$$

All that needs to be checked is that for every $h \in \mathbb{Z}^\omega$ and every $s \in T$

$$N \cap \langle s \rangle \cap \left(h + \bigcup_{n\in\omega} U_n\right) \neq \emptyset.$$

To see this fix h and $s \in T$ and let $t = s - h$. Then $t = t_n$ for some $n \in \omega$. Apply the above claim for $\varepsilon = \varepsilon_{2n+1}$ and $m = |t_n|$. Note that $x = h + t_n{}^\frown\bar{0}$ and $y = h + t_n{}^\frown M_\varepsilon^m{}^\frown\bar{0}$ satisfy $x \wedge y = s \in \mathbb{Z}^m \cap T$ and $|x(m) - y(m)| = M_\varepsilon^m$, hence $N \cap \langle s\rangle \cap (h + U_{2n}) \neq \emptyset$ or $N \cap \langle s\rangle \cap (h + U_{2n+1}) \neq \emptyset$. □

The properties strongly GMS and weakly GMS are related to recent work done by van Mill [36]. The result relevant for us is that an analytic group which is not Polish admits a meager set M and a countable set C such that for any $g \in \mathbb{G}$ $M \cap C \cdot g \neq \emptyset$. From van Mill's result one can deduce that an analytic group is either Polish or meager in itself, hence among analytic groups only the Polish ones have a chance to satisfy the Galvin–Mycielski–Solovay theorem.

We have seen that every locally compact group is GMS, hence, weakly GMS. Essentially the same proof gives that a compact group is strongly GMS.

Another class of related properties comes from the study of continuous actions of Polish groups [4,30,34]. A Polish group $\mathbb{G}$ is said to have the *Bergman* property if any left-invariant compatible metric on $\mathbb{G}$ is bounded. We suggest the following strong negation of the Bergman property: A group $\mathbb{G}$ is *elastic* if any open set has infinite diameter in some left-invariant compatible metric. That is, a group is non-elastic if it contains an open set which is bounded in all left-invariant compatible metrics. Trivially, each locally compact group is non-elastic, whereas each compact group is Bergman.

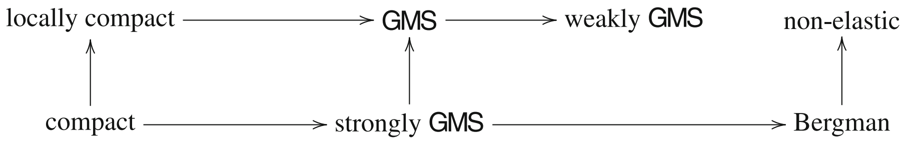
Proposition 3.12 *Every non-discrete strongly* GMS *group has the Bergman property.*

Proof Suppose $\mathbb{G}$ is not Bergman and fix an unbounded left-invariant metric d on $\mathbb{G}$. Let $\langle B(x_n, \varepsilon_n) : n \in \omega\rangle$ be a sequence of disjoint balls with diameters converging to 0 whose union is a dense subset of $\mathbb{G}$ and let $M = \{x_n : n \in \omega\}$ be the discrete set of centers of the balls. To see that $\mathbb{G}$ is not strongly GMS (witnessed by M, which is nowhere dense since the group is not discrete, i.e., has no isolated points), first note that

$$\forall\varepsilon > 0\, \exists D > 0 \quad \forall x, y \in \mathbb{G}\ d(x, y) \geqslant D \Rightarrow (B(x, \varepsilon) \cup B(y, \varepsilon)) \cap M \neq \emptyset.$$

To see this note that there are only finitely many n such that $\varepsilon_n > \varepsilon/3$. Let D be twice the diameter of $\bigcup\{B(x_n, \varepsilon_n) : \varepsilon_n > \varepsilon/3\}$. Such D obviously works. This finishes the proof as one does not even need the infinite sequence of ε's/balls, two suffice. □

So we have the following diagram of implications for non-discrete Polish groups (we omit the trivial implication locally compact ⇒ non-elastic):



It is not clear at the moment which implications can be reversed, other than, of course, no condition on the upper level implies any on the lower level, the real line being a counterexample. It is also not clear to us whether a (weakly) GMS group must be non-elastic.

For tangible examples, it would be interesting to know which properties are satisfied by the group $Sym(\omega)$ of permutations of ω. In particular, is $Sym(\omega)$ a GMS group? Of course the main remaining question is the following:

Question 3.13 Is there a Polish group which is (weakly) GMS but not locally compact?

We conjecture the answer to be negative.

Recall that Proposition 3.9 and Theorem 3.10 tell us the following: if $\mathbb{G}$ is a group which is not locally compact (e.g., the Baer–Specker group $\mathbb{Z}^\omega$), $X \in \mathbf{Smz}(\mathbb{G})$ (i.e., X strong measure zero with respect to any left-invariant metric) does not necessarily imply X being Prikry, i.e., $X \in \mathbf{Pr}(\mathbb{G})$.

We are going to prove that the above implication becomes true for all Polish groups if $X \in \mathbf{Smz}(\mathbb{G})$ is replaced by the following stronger property: A topological space X has the *Rothberger Property* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there are $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\}$ is a cover of X. Fremlin and Miller [24] proved that a metrizable space X has the Rothberger Property if and only if X has strong measure zero with respect to every compatible metric. In particular, every set $X \subseteq \mathbb{G}$ with the Rothberger Property is in $\mathbf{Smz}(\mathbb{G})$. The following theorem was suggested to the authors by the anonymous referee.

Theorem 3.14 *Suppose $\mathbb{G}$ is a non-discrete Polish group. If $X \subseteq \mathbb{G}$ has the Rothberger Property, then $X \in \mathbf{Pr}(\mathbb{G})$, i.e., $X \cdot M \neq \mathbb{G}$ for each $M \in \mathcal{M}(\mathbb{G})$.*

Proof Suppose $X \subseteq \mathbb{G}$ has the Rothberger Property. Given $M \in \mathcal{M}(\mathbb{G})$, fix a decreasing family of dense open sets $\{D_n : n \in \omega\}$ such that $\bigcap_{n\in\omega} D_n \subseteq \mathbb{G}\setminus M$.

Let ρ and d be, respectively, a complete and a right-invariant metric on $\mathbb{G}$.

Construct non-empty open sets $I_s, J_s \subseteq \mathbb{G}$ for $s \in \omega^{<\omega}$ such that

(1) $I_s \cdot J_s \subseteq D_{|s|}$,
(2) $I_s \subseteq \overline{I_s} \subseteq I_t$ whenever $s \supsetneq t$,
(3) diameter of I_s (with respect to ρ) is less than $\frac{1}{|s|+1}$,
(4) $\{J_{s^\frown n} : n \in \omega\}$ covers $\mathbb{G}$.

This can be done, e.g., as follows. Fix $s \in \omega^{<\omega}$ and suppose I_s and J_s are constructed. Put $m = |s| + 1$. First let I'_s be non-empty open such that $\overline{I'_s} \subseteq I_s$ and diameter of I'_s in ρ is below $\frac{1}{m+1}$. This will ensure (2) and (3). For $\varepsilon > 0$ let

$$A_\varepsilon = \{y \in D_m : d(y, \mathbb{G}\setminus D_m) > \varepsilon\}.$$

Since I'_s is open, the family $\mathcal{U} = \{x^{-1}A_\varepsilon : x \in I'_s, \varepsilon > 0\}$ covers $\mathbb{G}$. Since d is right-invariant, we also have $B_d(x, \varepsilon) \cdot x^{-1}A_\varepsilon \subseteq D_m$: indeed, if $z \in B_d(x, \varepsilon)$ and $y \in A_\varepsilon$, then $d(y, zx^{-1}y) = d(x, z) < \varepsilon$ and thus, for any $u \notin D_m$, $d(zx^{-1}y, u) \geqslant d(y, u) - d(y, zx^{-1}y) > \varepsilon - \varepsilon = 0$ and $zx^{-1}y \in D_m$ follows.

Since $\mathbb{G}$ is Lindelöf, $\mathcal{U}$ has a countable subcover $\{x_n^{-1}A_{\varepsilon_n} : n \in \omega\}$. Let $I_{s^\frown n} = B_d(x_n, \varepsilon_n) \cap I'_s$ and $J_{s^\frown n} = x_n^{-1}A_{\varepsilon_n}$.

According to a game characterization of Pawlikowski [27] (namely the fact that an uncountable set X has the Rothberger Property if and only if the respective game $G^{*\sigma}(X)$ is undetermined; see the main theorem of [27], and the subsequent discussion about equivalent games), there exists $f \in \omega^\omega$ such that $X \subseteq \bigcap_{m\in\omega}\bigcup_{n>m} J_{f\restriction n}$. Pick $g \in \bigcap_{n\in\omega} I_{f\restriction n}$; then we have

$$g \cdot X \subseteq g \cdot \bigcap_{m\in\omega}\bigcup_{n>m} J_{f\restriction n} \subseteq \bigcap_{n\in\omega} D_n \subseteq \mathbb{G}\setminus M.$$

So X has the following property: for each $M \in \mathcal{M}(\mathbb{G})$ there is a $g \in \mathbb{G}$ such that $g \cdot X \subseteq \mathbb{G}\setminus M$. A short computation shows that this is equivalent to $X \in \mathbf{Pr}(\mathbb{G})$. □

Denote by $\mathbf{Rbg}(X)$ the family of all Rothberger subsets of a metrizable space X. Combining the above theorem with Prikry's Theorem 3.1 we get the following corollary:

Corollary 3.15 *If $\mathbb{G}$ is a non-discrete Polish group equipped with a left-invariant metric, then* $\mathbf{Rbg}(\mathbb{G}) \subseteq \mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

By Theorem 3.10 and Proposition 3.9, the rightmost inclusion may be proper. Since there are (consistently) **Smz** sets on the line that are not Rothberger, the leftmost inclusion may be proper as well (see [24]). One just has to wonder if there is any topological or uniform combinatorial property between Rothberger (which is a topological property) and strong measure zero (which is a uniform property) that characterizes the Prikry sets.

4 The small ball property

The following notion introduced in [3] is closely related to strong measure zero, as we shall see in the next section.

Definition 4.1 *A metric space (X, d) has the* small ball property *(**sbp**) if for every $\varepsilon > 0$ there is a sequence of balls $B(x_n, \varepsilon_n)$ that covers X and such that $\varepsilon \geqslant \varepsilon_n$ for all n and $\varepsilon_n \to 0$.*

In [3], the notion is investigated within the framework of Banach spaces. Let us notice first that **Smz** obviously implies **sbp**, and second, that σ-totally bounded (hence in particular σ-compact) implies **sbp**.

We begin with establishing several equivalent definitions. Recall that a countable family $\mathcal{G}$ of subsets of X is termed a *λ-cover of $Y \subseteq X$* if every $y \in Y$ is contained in infinitely many $G \in \mathcal{G}$. We will also need a notion of Hausdorff measure. A right-continuous, nondecreasing function $g : (0, \infty) \to (0, \infty)$ with $\lim_{r\to 0} g(r) = 0$ is called a *gauge*. Given a gauge g and $Y \subseteq X$, the *g-dimensional Hausdorff measure of Y* is defined thus:

$$\mathcal{H}^g(Y) = \sup_{\delta>0} \inf \left\{ \sum_{E\in\mathcal{E}} g(\operatorname{diam} E) : Y \subseteq \bigcup \mathcal{E} \wedge \forall E \in \mathcal{E}\, (\operatorname{diam}(E) \leqslant \delta) \right\}.$$

The Hausdorff measure is an outer Borel measure. General reference: [29].

Lemma 4.2 *The following are equivalent.*

(i) *X has* **sbp***,*
(ii) *there is a family $\{E_n : n \in \omega\}$ that is a λ-cover of X and* $\operatorname{diam} E_n \to 0$*,*
(iii) *X admits a base $\{B_n : n \in \omega\}$ such that* $\operatorname{diam} B_n \to 0$*,*
(iv) *for every sequence $\langle \varepsilon_m : m \in \omega\rangle$ of positive reals there is a sequence $\langle F_m : m \in \omega\rangle$ of finite sets such that $\bigcup_{m\in\omega}\{B(x,\varepsilon_m) : x \in F_m\}$ covers X,*
(v) *there is a gauge g such that $\mathcal{H}^g(X) = 0$.*

Proof (i)$\Rightarrow$(ii) For each $k \in \omega$ let $\{B(x_i^k, \varepsilon_i^k) : i \in \omega\}$ be a cover of X such that $\lim_{i\to\infty} \varepsilon_i^k = 0$ and $\varepsilon_i^k < 2^{-k}$ for all $i \in \omega$. Then $\{B(x_i^k, \varepsilon_i^k) : i \in \omega, k \in \omega\}$ is the required family.

(ii)$\Rightarrow$(iii) For each $n \in \omega$ choose $x_n \in E_n$ and let $r_n = \operatorname{diam} E_n + 2^{-n}$ (the extra 2^{-n} is added to treat the case $\operatorname{diam} E_n = 0$). A straightforward argument proves that the family $\{B^\circ(x_n, r_n) : n \in \omega\}$ of open balls is the required base.

(iii)$\Rightarrow$(iv) We may clearly assume that the sequence $\langle \varepsilon_m : m \in \omega\rangle$ strictly decreases to zero. With no harm done suppose that the base given by (iii) consists of open balls $B^\circ(x_n, r_n)$ with $r_n < \varepsilon_0$ and $r_n \to 0$. Let $F_m = \{x_n : \varepsilon_{m+1} \leqslant r_n < \varepsilon_m\}$.

(iv)$\Rightarrow$(v) For $k, n \in \omega$ let $\varepsilon_n^k = 2^{-n-k}$. By (iv) there is, for each k, a sequence $\langle F_n^k : n \in \omega\rangle$ of finite sets such that $\{B(x, \varepsilon_n^k) : x \in F_n^k, n \in \omega\}$ covers X. Let g be a gauge satisfying, for all $j \in \omega$,

$$g(2^{1-j}) \leqslant \frac{2^{-j}}{\max\limits_{n+k=j} |F_n^k|}.$$

For each k consider the cover $\mathcal{B}_k = \{B(x, \varepsilon_n^k) : x \in F_n^k, n \in \omega\}$. It consists of sets of diameter at most 2^{1-k} and clearly

$$\sum_{B\in\mathcal{B}_k} g(\operatorname{diam} B) \leqslant \sum_{n\in\omega}\sum_{x\in F_n^k} g\left(2\varepsilon_n^k\right) = \sum_{n\in\omega} |F_n^k| g(2^{1-n-k}) \leqslant \sum_{n\in\omega} 2^{-n-k} = 2^{1-k}.$$

Since this holds for any $k \in \omega$, it follows that $\mathcal{H}^g(X) = 0$.

(v)$\Rightarrow$(i) Suppose $\mathcal{H}^g(X) = 0$. Then there is, for any $\varepsilon > 0$, a cover $\{E_n\}$ such that $\operatorname{diam} E_n \leqslant \varepsilon$ for all $n \in \omega$ and $\sum_{n\in\omega} g(\operatorname{diam} E_n) < 1$. In particular, $g(\operatorname{diam} E_n) \to 0$ and *a fortiori* $\operatorname{diam} E_n \to 0$, as required. □

It is worthwhile noticing that modifications of (ii)–(v) yield characterizations of **Smz**. For instance, X is **Smz** if and only if $\mathcal{H}^g(X) = 0$ for every gauge g [5,6].

Let X be a space that does not have **sbp** and denote **sbp**(X) the family of all **sbp** subsets of X. It is easy to see that **sbp**(X) is a σ-ideal. Later on it will turn useful to know its cardinal invariants. In order to calculate them we first describe a base of **sbp**(X). Denote d the metric of X. Fix a countable dense set $\{z_m : m \in \omega\} \subseteq X$. For $f \in \omega^\omega$ define

$$G_f = \left\{x \in X : \forall k \in \omega\ \exists n \geqslant k\ \exists m < f(n)\ d(x, z_m) < 2^{-n}\right\}.$$

Lemma 4.3 (i) $Y \in \mathbf{sbp}(X) \Leftrightarrow \exists f \in \omega^\omega\ Y \subseteq G_f$. *In other words, the family* $\{G_f : f \in \omega^\omega\}$ *is a base of* $\mathbf{sbp}(X)$.
(ii) $f \leqslant^* g \Rightarrow G_f \subseteq G_g$.

Proof (i) The family $\{B^\circ(z_m, 2^{-n}) : n \in \omega, m < f(n)\}$ is clearly a λ-cover of G_f with diameters going to zero. Therefore G_f has **sbp** by Lemma 4.2(ii).

On the other hand, if Y has **sbp**, then there is, again by Lemma 4.2(ii), a λ-cover $\{E_j : j \in \omega\}$ of Y such that $\operatorname{diam} E_j \to 0$. Since $\{z_m : m \in \omega\}$ is dense, there is, for every $j \in \omega$, a number m_j such that $\operatorname{diam}(\{z_{m_j}\} \cup E_j) \to 0$. Let $\varepsilon_j = \operatorname{diam}(\{z_{m_j}\} \cup E_j) + 2^{-j}$. For $n \in \omega$ set $A_n = \{j \in \omega : 2^{-n-1} \leqslant \varepsilon_j < 2^{-n}\}$ and $f(n) = \max_{j \in A_n} m_j + 1$. If $j \in A_n$, then $E_j \subseteq B^\circ(z_{m_j}, \varepsilon_j) \subseteq \bigcup \mathcal{G}^n_f$, where $\mathcal{G}^n_f = \{B^\circ(z_m, 2^{-n}) : m < f(n)\}$. Since $\{E_j : j \in \omega\}$ is a λ-cover of Y, it follows that $Y \subseteq G_f$.

(ii) If $x \in G_f$, and n_0 is such that $f(n) \leqslant g(n)$ for all $n \geqslant n_0$, then clearly $\forall k \geqslant n_0$ $\exists n \geqslant k\ \exists m < f(n) \leqslant g(n)\ d(x, z_m) < 2^{-n}$, i.e., $x \in G_g$. □

Consider the following mapping: for $x \in X$ define $\tilde{x} \in \omega^\omega$ by

$$\tilde{x}(n) = \min\left\{m \in \omega : d(x, z_m) < 2^{-n}\right\}.$$

Lemma 4.4 $x \notin G_f \Leftrightarrow f \leqslant^* \tilde{x}$.

Proof We have

$$\begin{aligned} x \notin G_f &\Leftrightarrow \exists k\ \forall n \geqslant k\ \forall m < f(n)\ d(x, z_m) \geqslant 2^{-n} \\ &\Leftrightarrow \exists k\ \forall n \geqslant k\ \forall m\ (d(x, z_m) < 2^{-n} \Rightarrow f(n) \leqslant m) \\ &\Leftrightarrow \exists k\ \forall n \geqslant k\ f(n) \leqslant \tilde{x}(n) \Leftrightarrow f \leqslant^* \tilde{x}. \end{aligned}$$ □

Corollary 4.5 $Y \in \mathbf{sbp}(X)$ *if and only if* $\tilde{Y} = \{\tilde{y} : y \in Y\}$ *is not dominating.*

Proof Clearly, $\tilde{Y}$ is not dominating iff $\exists f\ \forall y \in Y\ f \not\leqslant^* \tilde{y}$. The latter is by Lemma 4.4 equivalent to $\exists f\ \forall y \in Y\ y \in G_f$, i.e., $\exists f\ Y \subseteq G_f$, and Lemma 4.3(i) finishes the proof. □

It is worthwhile noticing that Y is σ-totally bounded if and only if $\tilde{Y}$ is eventually bounded in ω^ω.

Proposition 4.6 *If X does not have* **sbp**, *then*

(i) $\mathsf{non}(\mathbf{sbp}(X)) = \mathsf{cof}(\mathbf{sbp}(X)) = \mathfrak{d}$,
(ii) $\mathsf{add}(\mathbf{sbp}(X)) = \mathsf{cov}(\mathbf{sbp}(X)) = \mathfrak{b}$.

Proof $\mathfrak{d} \leqslant \mathsf{non}(\mathbf{sbp}(X))$: Let $Y \subseteq X$, $|Y| < \mathfrak{d}$. The family $\{\tilde{y} : y \in Y\}$ is not dominating, hence Y has **sbp** by Corollary 4.5.

$\mathsf{cof}(\mathbf{sbp}(X)) \leqslant \mathfrak{d}$: Let $D \subseteq \omega^\omega$ be dominating, $|D| = \mathfrak{d}$. The family $\{G_f : f \in D\}$ is cofinal in $\mathbf{sbp}(X)$ by Lemma 4.3.

$\mathfrak{b} \leqslant \mathsf{add}(\mathbf{sbp}(X))$: Let $\kappa < \mathfrak{b}$ and let $\{Y_\alpha : \alpha < \kappa\} \subseteq \mathbf{sbp}(X)$. By Lemma 4.3(i) there are f_α such that $Y_\alpha \subseteq G_{f_\alpha}$. The set $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$ is bounded, therefore there is f such that $f_\alpha \leqslant^* f$ for all $\alpha < \kappa$. By Lemma 4.3(ii) $\bigcup_{\alpha<\kappa} Y_\alpha \subseteq \bigcup_{\alpha<\kappa} G_{f_\alpha} \subseteq G_f$, and the latter set has **sbp** by Lemma 4.3(i).

$\mathsf{cov}(\mathbf{sbp}(X)) \leqslant \mathfrak{b}$: Let $B \subseteq \omega^\omega$ be unbounded, $|B| = \mathfrak{b}$. Then $X = \bigcup_{f\in B} G_f$; if not, fix $x \in X$ such that $\forall f \in B\ x \notin G_f$; it follows from Lemma 4.4 that $f \leqslant^* \tilde{x}$ for all $f \in B$, i.e., $\tilde{x}$ is an upper bound of B: a contradiction. □

Proposition 4.7 *For a subset $X \subseteq \omega^\omega$, the following are equivalent:*

(i) *X has* **sbp***,*
(ii) *no isometric copy of X in ω^ω is dominating,*
(iii) *no uniformly continuous image of X in ω^ω is dominating.*

Proof Let us prove first that if X has **sbp**, then it is not dominating. By Lemma 4.2(iii) there is a countable base consisting of cones, say $\{\langle s\rangle : s \in S\}$, such that $\{s \in S : |s| = n\}$ is finite for all n. Therefore there is $f \in \omega^\omega$ such that $s(n) < f(n)$ for all $s \in S$ such that $|s| = n + 1$. Since $\{\langle s\rangle : s \in S\}$ is a base for X, $\forall x \in X\ \exists^\infty s \in S\ s \subseteq x$ and therefore

$$\forall x \in X\ \exists^\infty n \in \omega\ x(n) \leqslant f(n). \tag{6}$$

Thus f witnesses that X is not dominating. (i)⇒(iii) now easily follows from the obvious fact that a uniformly continuous image of a **sbp** set is **sbp**.

(iii)⇒(ii) is trivial. In order to prove (ii)⇒(i) it is, by Corollary 4.5, enough to prove that the mapping $\tilde{}: \omega^\omega \to \omega^\omega$ is an isometry.

Recall that $x \wedge y$ denotes the common initial segment of $x, y \in \omega^\omega$ and that the metric on ω^ω is given by $d(x, y) = 2^{-|x\wedge y|}$. Let $n = |\tilde{x} \wedge \tilde{y}|$ and $j = \tilde{x}(n)$. Suppose without loss of generality that $j = \tilde{x}(n) < \tilde{y}(n)$. Then, by the definition of $\tilde{}$, $d(z_j, x) < 2^{-n}$, while $d(z_j, y) \geqslant 2^{-n}$. Hence $\forall i \leqslant n\ z_j(i) = x(i)$, while $\exists i \leqslant n\ z_j(i) \neq y(i)$. It follows that $\exists i \leqslant n\ x(i) \neq y(i)$, i.e., $|x \wedge y| \leqslant n$, and thus $d(x, y) \geqslant d(\tilde{x}, \tilde{y})$.

On the other hand, if $|\tilde{x} \wedge \tilde{y}| > |x \wedge y|$, then there would exist n such that $x(n) \neq y(n)$, but $\tilde{x}(n) = \tilde{y}(n)$. Letting $j = \tilde{x}(n)$ this would yield $z_j(n) = x(n)$ and $z_j(n) = y(n)$ and consequently $x(n) = y(n)$: a contradiction. Therefore $|\tilde{x} \wedge \tilde{y}| \leqslant |x \wedge y|$ and thus $d(x, y) \leqslant d(\tilde{x}, \tilde{y})$. □

Recall that a topological space X has the *Menger Property* if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of X.

As mentioned above, a metrizable space X has the Rothberger Property if and only if X is **Smz** with respect to every compatible metric. In view of Lemma 4.2(iv) it is no surprise that the Menger Property is characterized likewise by **sbp**. It was suggested to the authors by Marion Scheepers.

Theorem 4.8 *A metrizable space X has the Menger Property if and only if X has* **sbp** *with respect to every compatible metric.*

Proof The forward implication is trivial: If d is any compatible metric on X and $\langle \varepsilon_m : m \in \omega\rangle$ a sequence of positive reals, consider the sequence $\langle\{B(x, \varepsilon_m) : x \in X\} : m \in \omega\rangle$ of open covers. The Menger Property of X yields (iv) of Lemma 4.2.

The reverse implication is a bit harder. Let us first recall that a cover $\mathcal{V}$ *refines* a cover $\mathcal{U}$ if for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$; that for $x \in X$ we define $\mathrm{st}(x, \mathcal{V}) = \bigcup\{V \in \mathcal{V} : x \in V\}$ and that $\mathcal{V}$ *star-refines* $\mathcal{U}$ if $\{\mathrm{st}(x, \mathcal{V}) : x \in X\}$ refines $\mathcal{U}$.

Consider any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers. Since X is metrizable, every open cover has a star-refinement. Therefore there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of covers such that $\mathcal{V}_{n+1}$ star-refines both $\mathcal{V}_n$ and $\mathcal{U}_{n+1}$ for all n and moreover we may assume that $\lim_{n\to\infty} \sup\{\operatorname{diam} V : V \in \mathcal{V}_n\} = 0$ for some fixed compatible metric on X.

Therefore the number $n(x, y) = \max\{n \in \omega : \exists V \in \mathcal{V}_n\ \{x, y\} \subseteq V\}$ is well defined for distinct $x, y \in X$. Let $f(x, y) = 2^{-n(x,y)}$. By assumption $n(x, y) < \infty$. Since $\mathcal{V}_{n+1}$ star-refines $\mathcal{V}_n$ for all n, the function f satisfies the following condition:

$$\forall \varepsilon > 0\ (f(x, y) < \varepsilon \wedge f(y, z) < \varepsilon) \implies f(x, z) < 2\varepsilon.$$

Consequently, according to *Frink's lemma* (see [16, 2.6]), there is a compatible metric d on X such that $f(x, y) \leqslant d(x, y) \leqslant 4f(x, y)$. Therefore, if $x, y \in X$ and $d(x, y) \leqslant 2^{-n-1}$, then there is $V \in \mathcal{V}_{n+1}$ such that $\{x, y\} \subseteq V$; in other words, $B(x, 2^{-n}) \subseteq \mathrm{st}(x, \mathcal{V}_n)$. And since $\mathcal{V}_n$ star-refines $\mathcal{U}_n$, we have the following: For each $n \in \omega$, the cover $\{B(x, 2^{-n}) : x \in X\}$ refines $\mathcal{U}_n$. Since d has **sbp**, Lemma 4.2(iv) with $\varepsilon_n = 2^{-n}$ yields the required sequence of finite families. □

5 Uniformity of Smz in metric spaces and groups

In this section we take a closer look at the uniformity invariant of **Smz** in general separable metric spaces. The results of the previous section will be of use here. Recall that $\mathsf{non}(\mathbf{Smz}(\omega^\omega)) = \mathsf{cov}(\mathcal{M})$, while $\mathsf{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$ [see Theorem 2.2(iii)]. This shows that there are at least two distinct classes of metric spaces concerning their uniformity invariant. The following results at least partially confirm the suspicion that there may be exactly two classes of Polish (or even analytic) spaces as far as $\mathsf{non}(\mathbf{Smz}(X))$ is concerned, depending on whether the space in question has the small ball property:

Theorem 5.1 *Let X be a separable metric space. Then:*

(i) *If X is σ-totally bounded then $\mathfrak{eq} \leqslant \mathsf{non}(\mathbf{Smz}(X))$,*
(ii) *if X is* **sbp** *then $\mathfrak{ed} \leqslant \mathsf{non}(\mathbf{Smz}(X))$, and*
(iii) *if X is not* **sbp** *then $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathfrak{d}$.*

Proof (i) If X is compact, then there is a continuous mapping $f : 2^\omega \to X$ onto X, see e.g. [19, Theorem 4.18]. It is of course uniformly continuous, so $\mathsf{non}(\mathbf{Smz}(X)) \geqslant \mathfrak{eq}$ by Lemma 2.1(i) and Theorem 2.2(iii). If X is totally bounded, then it has a compact completion X^*, thus $\mathsf{non}(\mathbf{Smz}(X)) \geqslant \mathsf{non}(\mathbf{Smz}(X^*)) \geqslant \mathfrak{eq}$ by Lemma 2.1(ii). Since **Smz** is a σ-additive property, the same estimate holds for σ-totally bounded spaces.

(ii) Let $Y \subseteq X$, $|Y| < \mathfrak{ed}$. We will show that Y is **Smz**. Let $\delta_n \searrow 0$. Let $\mathcal{B} = \{B_j : j \in \omega\}$ be a base witnessing that X has the small ball property. For $n \in \omega$ let

$$F_n = \{B \in \mathcal{B} : \delta_{n+1} < \operatorname{diam} B \leqslant \delta_n\}.$$

The sets F_n are clearly finite, because diam $B_j \to 0$. To each $x \in X$ assign a partial function $\tilde{x}$ with range in $\prod_{n\in\omega} F_n$ as follows: given $n \in \omega$, if there is $B \in F_n$ such that $x \in B$, let $\tilde{x}(n) = B$; otherwise $\tilde{x}(n)$ is undefined. Since $\mathcal{B}$ is a base, $\tilde{x}(n)$ is defined for infinitely many n's. Therefore the set $\tilde{Y} = \{\tilde{y} : y \in Y\}$ is a bounded family of infinite partial functions, and surely $|\tilde{Y}| = |Y| < \mathfrak{eD}$. By the definition of $\mathfrak{eD}$ there is $f \in \mathcal{B}^\omega$ such that $\forall y \in Y\ \exists n_y \in \operatorname{dom} \tilde{y}$ with $\tilde{y}(n_y) = f(n_y)$. Let $I = \{n_y : y \in Y\}$. The family $\mathcal{F} = \{f(n) : n \in I\}$ is the required cover of Y: If $n = n_y \in I$, then $y \in \tilde{y}(n) = f(n)$, so $\mathcal{F}$ indeed covers Y, and also $f(n) = \tilde{y}(n) \in F_n$ and thus diam $f(n) \leqslant \delta_n$.

(iii) The left-hand inequality is Rothberger's estimate Theorem 2.2(i). The right-hand one follows at once from Proposition 4.6(i), since a **Smz** set is clearly **sbp**. □

Proposition 5.2 *Let X be an uncountable analytic separable metric space. Then* $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathfrak{eq}$.

Proof If X is analytic and uncountable, then it contains (by the Perfect Set Theorem) a (uniform) copy of the Cantor space and therefore $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathsf{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$ [cf. Lemma 2.1(ii) and Theorem 2.2(iii)]. □

Corollary 5.3 *Let X be an uncountable analytic separable metric space. Then:*

(i) *If X is* σ*-totally bounded then* $\mathsf{non}(\mathbf{Smz}(X)) = \mathfrak{eq}$,
(ii) *if X is* **sbp** *then* $\mathfrak{eD} \leqslant \mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathfrak{eq}$, *and*
(iii) *if X is not* **sbp** *then* $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{non}(\mathbf{Smz}(X)) \leqslant \min\{\mathfrak{eq}, \mathfrak{d}\}$.

Proof Follows immediately from Theorem 5.1 and Proposition 5.2. □

In particular, the above corollary applies to all uncountable Polish spaces. For **CLI** Polish groups we can do even better. First we prove the following Hurewicz-type result.

Lemma 5.4 *A Polish group equipped with a complete, left-invariant metric is either locally compact, or else contains a uniform copy of* ω^ω *and, in particular, is not* **sbp**.

Proof Let $\mathbb{G}$ be the group and d the metric. Suppose $\mathbb{G}$ is not locally compact. Since d is complete, no open set is totally bounded. Therefore for every $\varepsilon > 0$ exists $\delta > 0$ such that the ball $B(1, \varepsilon)$ contains an infinite set of points that are mutually at least δ apart. Use repeatedly this fact to construct, for each $n \in \omega$, $\varepsilon_n > 0$ and an infinite set $\{x_n^i : i \in \omega\} \subseteq B(1, \varepsilon_n)$ such that if $i \neq j$ then $d(x_n^i, x_n^j) > 5\varepsilon_{n+1}$. For $s \in \omega^n$ let $y_s = x_0^{s(0)} \cdot x_1^{s(1)} \cdot x_{n-2}^{s(n-2)} \cdot \dots \cdot x_{n-1}^{s(n-1)}$. The construction ensures that for any $f \in \omega^\omega$ the sequence $\langle y_{f\restriction n} : n \in \omega\rangle$ is Cauchy. Let z_f be its limit. It is easy to check that since d is left-invariant, the mapping $f \mapsto z_f$ is a uniform embedding of ω^ω into $\mathbb{G}$. □

Corollary 5.5 *Let* $\mathbb{G}$ *be a* **CLI** *Polish group.*

(i) *If* $\mathbb{G}$ *is locally compact, then* $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathfrak{eq}$,
(ii) *if* $\mathbb{G}$ *is not locally compact, then* $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}(\mathcal{M})$.

Proof (i) follows at once from Corollary 5.3(i), and (ii) from the above Lemma 5.4, Lemma 2.1 and Theorem 2.2. □

It is, of course, a natural question whether the result remains true also for groups which are not CLI.

5.1 Consistency results and questions

Our next result says that one cannot drop the assumption of not being of universal measure zero in Theorem 2.2(ii): $\mathsf{non}(\mathcal{N})$ is not an upper bound for $\mathsf{non}(\mathbf{Smz}(X))$ for all (non-$\mathbf{Smz}$) separable metric spaces X.

Theorem 5.6 *It is consistent with ZFC that there is a non-*$\mathbf{sbp}$ *set* $X \subseteq \omega^\omega$ *such that* $\mathfrak{d} = \mathsf{non}(\mathbf{Smz}(X)) > \mathsf{non}(\mathcal{N}) = \mathsf{cov}(\mathcal{M})$.

Proof Start with a ground model V with $V \vDash \mathfrak{b} = \mathfrak{d} = \mathsf{cov}(\mathcal{M}) = \omega_2$. By the assumption $\mathfrak{b} = \mathfrak{d} = \omega_2$, we can fix an ω_2-scale $X \in V$. Let $\mathbb{B}(\kappa)$ be the standard measure algebra for adding κ random reals. Let G be $\mathbb{B}(\kappa)$-generic over V for $\kappa \geqslant \omega_1$. Then, in $V[G]$, we have

- $\mathsf{cov}(\mathcal{M}) = \mathsf{non}(\mathcal{N}) = \omega_1$, because $\mathbb{B}(\kappa)$ adds a Sierpiński set of size ω_1,
- X is still an ω_2-scale (in particular, $\mathfrak{b} = \mathfrak{d} = \omega_2$), because $\mathbb{B}(\kappa)$ is ccc and ω^ω-bounding.

By Proposition 4.7(i), $V[G] \vDash X$ is not $\mathbf{sbp}$ and therefore also $V[G] \vDash X$ is not $\mathbf{Smz}$. It remains to prove that $\mathsf{non}(\mathbf{Smz}(X)) = \omega_2$ in $V[G]$, i.e., that every $Y \in [X]^{\leqslant\omega_1}$ is $\mathbf{Smz}$. Since $\mathbb{B}(\kappa)$ is *ccc*, there is $\overline{Y} \in V$ such that $|\overline{Y}| = \omega_1$ and $Y \subseteq \overline{Y} \subseteq X$. Since $V \vDash \mathsf{cov}(\mathcal{M}) > \omega_1$, Theorem 2.2(i) yields $V \vDash \overline{Y}$ is $\mathbf{Smz}$. It is easy to show that also $V[G] \vDash \overline{Y}$ is $\mathbf{Smz}$: Suppose $\langle \varepsilon_n : n \in \omega \rangle$ is a sequence of positive reals in $V[G]$. Since $\mathbb{B}(\kappa)$ is ω^ω-bounding, there is a ground-model sequence $\langle \overline{\varepsilon}_n : n \in \omega \rangle \in V$ of positive reals such that $\overline{\varepsilon}_n < \varepsilon_n$ for each n. As $\overline{Y}$ is $\mathbf{Smz}$ in V, there is a cover $\{U_n\}_{n \in \omega}$ of $\overline{Y}$ in V such that diam $U_n < \overline{\varepsilon}_n < \varepsilon_n$ for each n, which is enough. □

Question 5.7 Is $\mathsf{cof}(\mathcal{N})$ an upper bound for $\mathsf{non}(\mathbf{Smz}(X))$ for any non-$\mathbf{Smz}$ separable space X?

Theorem 5.6 also says the following: consistently, there is a non-$\mathbf{sbp}$ space with $\mathsf{non}(\mathbf{Smz}(X)) > \mathsf{cov}(\mathcal{M})$.

Question 5.8 Is $\mathsf{non}(\mathbf{Smz}(X)) = \mathsf{cov}(\mathcal{M})$ for all non-$\mathbf{sbp}$ analytic (Borel, absolutely $G_\delta, \ldots$) spaces X?

Consistently, there is also a $\mathbf{sbp}$ X such that $\mathsf{non}(\mathbf{Smz}(X)) < \mathfrak{eq}$:

Theorem 5.9 *It is consistent with ZFC that there is an* $\mathbf{sbp}$ *set* $X \subseteq \omega^\omega$ *such that* $\mathsf{non}(\mathbf{Smz}(X)) = \mathfrak{ed} < \mathfrak{eq}$.

Proof Goldstern et al. [15] have a model of $\mathfrak{d} = \mathfrak{eq} = \mathfrak{c} = \omega_2 + \mathsf{cov}(\mathcal{M}) = \omega_1$. Since $\min\{\mathfrak{ed}, \mathfrak{d}\} = \mathsf{cov}(\mathcal{M})$ (see [17, Prop. 3.6]), we have $\mathfrak{ed} = \omega_1$. Since $\mathsf{non}(\mathbf{Smz}(\omega^\omega)) = \mathsf{cov}(\mathcal{M}) = \omega_1$, there is a set $X \in [\omega^\omega]^{\omega_1}$ that is not $\mathbf{Smz}$. Thus clearly $\mathsf{non}(\mathbf{Smz}(X)) = \omega_1 = \mathfrak{ed}$. On the other hand, since $\mathfrak{d} > |X|$, it has the small ball property by Proposition 4.6(i). □

Question 5.10 Is $\mathsf{non}(\mathbf{Smz}(X)) = \mathfrak{eq}$ for all **sbp** analytic (Borel, absolutely $G_\delta, \ldots$) spaces X?

A related question was posed by Fremlin: Say that two spaces X and Y have the same **Smz**-*type* if there is a bijection $\varphi : X \to Y$ mapping **Smz** sets in X exactly onto **Smz** sets in Y. Fremlin [12] asked how many **Smz**-types of Polish spaces without isolated points are there.

6 Transitive covering in Polish groups

Bartoszyński and Judah [2, 2.7] calculated what they called "transitive covering for category": the minimal cardinality of a set $A \subseteq 2^\omega$ such that $A + M = 2^\omega$ for some meager set M. Miller and Steprāns [25] initiated the study of this cardinal invariant within the framework of a general Polish group. For such a group $\mathbb{G}$ the invariant (thereby denoted by $\mathsf{cov}^*_{\mathbb{G}}$) is defined as the minimal cardinality of a set $A \subseteq \mathbb{G}$ such that $A + M = \mathbb{G}$ for some meager set $M \subseteq \mathbb{G}$. It is clear that $\mathsf{cov}^*_{\mathbb{G}}$ is nothing but the uniformity number of $\mathbf{Pr}(\mathbb{G})$, i.e.,

$$\mathsf{cov}^*_{\mathbb{G}} = \mathsf{non}(\mathbf{Pr}(\mathbb{G})).$$

Miller and Steprāns [25] conclude their paper with the following questions regarding $\mathsf{cov}^*_{\mathbb{G}}$. Since this section is devoted to their work and questions, we will adhere to their notation $\mathsf{cov}^*_{\mathbb{G}}$ throughout.

(MS1) Is it consistent to have a compact Polish group $\mathbb{G}$ such that $\mathsf{cov}^*_{\mathbb{G}} > \mathfrak{eq}$?
(MS2) Is it true that $\mathsf{cov}^*_{\mathbb{G}} \geqslant \mathfrak{eq}$ for any infinite compact Polish group $\mathbb{G}$?
(MS3) Is it true that for every non-discrete Polish group $\mathbb{G}$ either $\mathsf{cov}^*_{\mathbb{G}} = \mathfrak{eq}$ or $\mathsf{cov}^*_{\mathbb{G}} = \mathsf{cov}(\mathcal{M})$?

We have enough to answer the first question already:

Corollary 6.1 *If* $\mathbb{G}$ *is a Polish group, then* $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{cov}^*_{\mathbb{G}} \leqslant \mathfrak{eq}$.

Proof If $A \notin \mathbf{Pr}(\mathbb{G})$, then $\mathbb{G}$ is covered by $|A|$-many translates of a meager set. Thus $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{cov}^*_{\mathbb{G}}$. On the other hand, Theorem 3.1 yields $\mathsf{cov}^*_{\mathbb{G}} \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G}))$. The right hand inequality thus follows from Proposition 5.2. □

We now treat the second question (MS2) by proving another extension of the Galvin–Mycielski–Solovay Theorem. The following lemma is set up in a generality that will prove useful later on.

Lemma 6.2 *Let* K *be a compact metric space and* X *a separable locally compact metric space. Let* $U \subseteq X$ *be an open set with compact closure* $C = \overline{U}$ *and* $P \subseteq X$ *be compact nowhere dense. Let* $\phi : K \times X \to X$ *be a continuous mapping such that for each* $y \in K$ *the image* $\phi(\{y\} \times P)$ *is nowhere dense. Then*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in C\ \forall y \in K\ \exists z \in C \quad B(z, \delta) \subseteq B(x, \varepsilon) \setminus \phi(B(y, \delta) \times P).$$

Proof Fix $\varepsilon > 0$. Since $\phi(\{y\} \times P)$ is nowhere dense for each y, the function $f : C \times K \to \mathbb{R}$ defined by

$$f(x, y) = \sup\{t : \exists z \in C \; B(z, t) \subseteq B^\circ(x, \varepsilon) \setminus \phi(\{y\} \times P)\}$$

is positive on $C \times K$. We claim that f actually attains its (positive) minimum. To see that, consider, for each $z \in C$, the functions

$$g_z(x) = \underline{d}(z, X \setminus B^\circ(x, \varepsilon)), \quad x \in C$$
$$h_z(y) = \underline{d}(z, \phi(\{y\} \times P)), \quad y \in K$$

and note that

$$f(x, y) = \sup_{z \in C} \min(g_z(x), h_z(y)). \tag{7}$$

Using compactness it is easy to show that, for each $z \in C$, the function h_z is lower semicontinuous and that while g_z does not have to, it has the following lower-semicontinuity property: if $x_n \to x$ and $g_z(x_n) \to 0$, then $g_z(x) = 0$.

Now suppose that there are $(x_n, y_n) \in C \times K$ such that $f(x_n, y_n) \to 0$. Since C, K are compact, passing to subsequences if needed we may assume $(x_n, y_n) \to (x, y) \in C \times K$. Use (7) and the semicontinuity properties of g_z and h_z to conclude that since $f(x_n, y_n) \to 0$, for any z either $g_z(x_n) \to 0$ and then $g_z(x) = 0$, or else $h_z(y_n) \to 0$ and then $h_z(y) = 0$. Use (7) again to conclude that $f(x, y) = 0$, the desired contradiction proving that there is $\eta > 0$ such that $f(x, y) > \eta$ for all x, y. It follows that

$$\forall x \in C \; \forall y \in K \; \exists z \in C \quad B(z, \eta) \subseteq B(x, \varepsilon) \;\wedge\; B(z, \eta) \cap \phi(\{y\} \times P) = \emptyset.$$

The latter of course yields $B(z, \frac{\eta}{2}) \cap B(\phi(\{y\} \times P), \frac{\eta}{2}) = \emptyset$. On the other hand, since ϕ uniformly continuous on $C \times K$, there is $\xi > 0$ such that

$$\forall y \in K \quad \phi(B(y, \xi) \times P) \subseteq B\left(\phi(\{y\} \times P), \tfrac{\eta}{2}\right).$$

It follows that $B(z, \frac{\eta}{2}) \cap \phi(B(y, \xi) \times P) = \emptyset$. Thus letting $\delta = \min\{\frac{\eta}{2}, \xi\}$ yields the lemma. □

Theorem 6.3 *Let Y be a σ-compact metric space and X a separable locally compact metric space. Let $\phi : Y \times X \to X$ be a continuous mapping such that for each $y \in Y$ and every compact nowhere dense set $P \subseteq X$ the image $\phi(\{y\} \times P)$ is nowhere dense. If $S \in$ **Smz**(Y) and $M \subseteq X$ is meager, then $\phi(S \times M) \neq X$.*

Proof Let K_n be an increasing sequence of compact sets with union Y and let P_n be an increasing sequence of compact nowhere dense sets with union M.

Choose $x_0 \in X$ and $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0)$ is compact. Let $C = B(x_0, \varepsilon_0)$. By the above lemma there is a sequence $\langle \varepsilon_n : n \in \omega \rangle \in (0, \infty)^\omega$ such that for every $n > 0$

$$\forall x \in C\ \forall y \in K_n\ \exists z \in C \quad B(z, \varepsilon_n) \subseteq B(x, \varepsilon_{n-1}) \setminus \phi((B(y, \varepsilon_n) \cap K_n) \times P_n). \quad (8)$$

We may clearly suppose that $\varepsilon_n \to 0$. Since S is **Smz**, there is a cover $\{E_n : n \in \omega\}$ of S such that diam $E_n < \varepsilon_n$ for all n. It is well-known that $\{E_n : n \in \omega\}$ can be chosen to be actually a λ-cover. Note that the family $\{E_n \cap K_n : n \in \omega\}$ is then a λ-cover as well and therefore we may actually suppose that $E_n \subseteq K_n$ for all $n \in \omega$.

For each n there is $y \in E_n$ such that $E_n \subseteq B(y, \varepsilon_n)$. Therefore, using repeatedly (8), there is a sequence $\langle x_n : n \in \omega \rangle$ in C (note that x_0 is already chosen) such that for all $n \in \omega$

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus \phi(E_{n+1} \times P_{n+1}).$$

Let x be the unique point in $\bigcap_{n \in \omega} B(x_n, \varepsilon_n)$ (there is one, since $B(x_0, \varepsilon_0)$ is compact and $\varepsilon_n \to 0$). Then x clearly belongs to none of the sets $\phi(E_n \times P_n)$ and consequently

$$x \notin \bigcup_{n \in \omega} \phi(E_n \times P_n) = \phi\left(\bigcup_{n \in \omega} E_n \times P_n\right).$$

Thus, to prove that x is not covered by $\phi(S \times M)$ it remains to show that $S \times M \subseteq \bigcup_{n \in \omega} E_n \times P_n$. Let $(s, m) \in S \times M$. There is k such that $m \in P_k$. Since there are infinitely many n such that $s \in E_n$, there is $n \geqslant k$ such that $s \in E_n$, and also $m \in P_n$, since $P_n \supseteq P_k$. The desired inclusion is proved. $\square$

Letting $X = Y = \mathbb{G}$ and $\phi(x, y) = xy$ immediately yields Theorem 3.5.

The answer to (MS2) now trivially follows from Corollary 5.3.

Corollary 6.4 *If $\mathbb{G}$ is a non-discrete, σ-compact Polish group, then* $\mathsf{cov}^*_{\mathbb{G}} = \mathfrak{eq}$.

6.1 Polish groups that are not σ- compact

If $\mathbb{G}$ is a non-σ-compact Polish group, then, as shown by Proposition 3.9 and Theorem 3.10, $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ may fail and Prikry's theorem only yields $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$. Together with Corollary 6.1 this yields $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{cov}^*_{\mathbb{G}} \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G}))$.

We proved earlier, in Lemma 5.4, that if $\mathbb{G}$ is a non-σ-compact CLI group, then it contains a uniform copy of ω^ω. If we combine this fact with Lemma 2.1(ii) and Theorem 2.2(iii), we get $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) \leqslant \mathsf{cov}(\mathcal{M})$. Thus we have, for non-$\sigma$-compact CLI groups, the following definite result that is also known to Dobrowolski and Marciszewski [9]:

Theorem 6.5 [9] *Let $\mathbb{G}$ be a* CLI *Polish group. If $\mathbb{G}$ is not locally compact, then* $\mathsf{cov}^*_{\mathbb{G}} = \mathsf{cov}(\mathcal{M})$.

Combine this theorem with Corollary 5.5 to get

Corollary 6.6 $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{non}(\mathbf{Pr}(\mathbb{G}))$ *for every* CLI *Polish group* $\mathbb{G}$.

It, however, remains a mystery if one can drop the CLI assumption.

Question 6.7 Is there a non-σ-compact Polish group $\mathbb{G}$ such that $\mathsf{cov}^*_{\mathbb{G}} > \mathsf{cov}(\mathcal{M})$?

Question 6.8 Is there consistently a Polish group $\mathbb{G}$ such that $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) \neq \mathsf{non}(\mathbf{Pr}(\mathbb{G}))$?

6.2 Actions

Miller and Steprāns [25] extend their definitions to actions of Polish groups on Polish spaces. Recall that an *action* of a Polish group $\mathbb{G}$ on a Polish space X is a continuous map $\phi : \mathbb{G} \times X \to X$,

$$\phi : (g, x) \mapsto gx$$

such that $1x = x$ for every $x \in X$ and $g(hx) = (gh)x$ for all $g, h \in \mathbb{G}$ and $x \in X$. (It is customary to write gx instead of $\phi(g, x)$. We, however, will resort to the latter whenever there is a danger of confusion of an action with a group operation.) It is well-known and easily seen that for each $g \in \mathbb{G}$ the map $x \mapsto gx$ is a homeomorphism of X whose inverse is the map $x \mapsto g^{-1}x$.

Definition 6.9 [25] *Let ϕ be an action of a Polish group $\mathbb{G}$ on a Polish metric space X. Denote by $\mathcal{M}(X)$ the ideal of meager subsets of X. Let*

$$\mathbf{Pr}(\phi) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}(X)\ \phi(A \times M) \neq X\}.$$

Note that when $\mathbb{G}$ acts on itself by left translation, $\mathbf{Pr}(\phi)$ is nothing but $\mathbf{Pr}(\mathbb{G})$.

Miller and Steprāns [25] do not provide any results on $\mathbf{Pr}(\phi)$ and its uniformity number. Instead they ask one more question.

(MS4) Let ϕ_n be the natural action of the isometry group on $\mathbb{R}^n$. Is it true that $\mathsf{non}(\mathbf{Pr}(\phi_m)) = \mathsf{non}(\mathbf{Pr}(\phi_n))$ for all m and n?

We will answer this question shortly. Since every action satisfies the hypotheses of Theorem 6.3, the following is straightforward.

Theorem 6.10 *If ϕ is an action of a σ-compact Polish group $\mathbb{G}$ on a Polish space, then $\mathbf{Smz}(\mathbb{G}) = \mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Pr}(\phi)$. Consequently $\mathsf{non}(\mathbf{Pr}(\phi)) \geqslant \mathfrak{eq}$.*

In general there is no upper estimate—consider an action of $\mathbb{G}$ on $\mathbb{G}$ defined by $\phi(x, y) = y$. However, for some actions one can get more. Given a σ-compact Polish group $\mathbb{G}$ we denote $H(\mathbb{G})$ the *homeomorphism group*, i.e., the group of all homeomorphisms of $\mathbb{G}$ onto itself, equipped with the compact-open topology.

Corollary 6.11 *Let $\mathbb{G}$ be a non-discrete σ-compact Polish group. Suppose $\mathbb{H} \subseteq H(\mathbb{G})$ is a σ-compact Polish group that includes all left translates. Let $\phi(h, g) = h(g)$ be the action of $\mathbb{H}$ on $\mathbb{G}$. Then $\mathsf{non}(\mathbf{Pr}(\phi)) = \mathfrak{eq}$.*

Proof Identify $\mathbb{G}$ with the subgroup of $\mathbb{H}$ consisting of left translates. It is clear that if $A \subseteq \mathbb{G}$ and $A \notin \mathbf{Pr}(\mathbb{G})$, then $A \notin \mathbf{Pr}(\phi)$. Therefore $\mathsf{non}(\mathbf{Pr}(\phi)) \leqslant \mathsf{non}(\mathbf{Pr}(\mathbb{G}))$. Since $\mathsf{non}(\mathbf{Pr}(\mathbb{G})) = \mathfrak{eq}$ by Corollary 6.4, we have $\mathsf{non}(\mathbf{Pr}(\phi)) \leqslant \mathfrak{eq}$. The opposite inequality comes from Theorem 6.10. □

This obviously answers question (MS4).

6.3 Translatability

The last result of this section deals with a notion introduced by Carlson [8, Section 5]: A collection $\mathcal{U}$ of subsets of $\mathbb{G}$ is *κ-translatable* if for any $M \in \mathcal{U}$, there is $M^* \in \mathcal{U}$ such that

$$\forall T \in [\mathbb{G}]^\kappa \ \exists g \in \mathbb{G} \quad M{\cdot}T \subseteq g{\cdot}M^*. \tag{9}$$

Carlson showed that the ideal of meager subsets of $\mathbb{R}$ is ω-translatable (see [8, Corollary 5.14]). Balka (see [1, Theorem 2.4]) recently showed that the ideal of meager sets is 2-translatable for all locally compact abelian Polish groups.

We present a generalization of his result which follows rather easily from Theorem 6.3 and its proof.

Theorem 6.12 *If $\mathbb{G}$ is a locally compact Polish group, then the ideal of meager sets in $\mathbb{G}$ is ω-translatable.*

Proof Let d be an invariant metric on $\mathbb{G}$. In Theorem 6.3, let $X = Y = \mathbb{G}$ and $\phi(x, y) = xy$. Let $M \subseteq \mathbb{G}$ be meager. The proof of Theorem 6.3 yields a sequence $\langle \varepsilon_n : n \in \omega\rangle$ such that for all $\langle U_n : n \in \omega\rangle$ with $\operatorname{diam} U_n < \varepsilon_n$ there is $g \in \mathbb{G}$ such that

$$\left(g \cdot \bigcap_{m\in\omega} \bigcup_{n\geqslant m} U_n\right) \cap M = \emptyset.$$

We have to find a meager set M^* such that (9) holds for $\kappa = \omega$. Fix $\langle A_k : k \in \omega\rangle$, a partition of ω into infinitely many infinite sets, and a sequence $\langle z_n : n \in \omega\rangle$ in $\mathbb{G}$ in such a way that $\{z_n : n \in A_k \wedge n \geqslant m\}$ is dense in $\mathbb{G}$ for any $k, m \in \omega$.

Let $U_n = B^\circ(z_n, \frac{\varepsilon_n}{2})$, and define

$$M^* = \mathbb{G}\setminus \bigcap_{k\in\omega} \bigcap_{m\in\omega} \bigcup_{n\in A_k\setminus m} U_n;$$

note that M^* is the complement of a countable intersection of open dense sets (by the choice of the z_n's) and hence meager. We have to show (9) for $\kappa = \omega$.

Fix $T = \{t_k : k \in \omega\} \subseteq \mathbb{G}$. Since the underlying metric d is right-invariant, $\operatorname{diam} U_n < \varepsilon_n$ yields $\operatorname{diam}(U_n \cdot t) < \varepsilon_n$ for any $t \in \mathbb{G}$. So we can pick $g \in \mathbb{G}$ such that

$$\left(g \cdot \bigcap_{m\in\omega} \bigcup_{n\geqslant m} U_n t_{k(n)}^{-1}\right) \cap M = \emptyset, \tag{10}$$

where $k(n)$ is the (unique) k such that $n \in A_k$. To show that $MT \subseteq gM^*$, assume towards a contradiction that $y \in MT$, but $y \notin gM^*$. Fix $k \in \omega$ with $y \in Mt_k$, i.e., $yt_k^{-1} \in M$. Since $y \notin gM^*$, we have (in particular) $y \in g \cdot \bigcap_{m\in\omega} \bigcup_{n\in A_k\setminus m} U_n$. Consequently, $yt_k^{-1} \in g \cdot \bigcap_{m\in\omega} \bigcup_{n\in A_k\setminus m} U_n t_{k(n)}^{-1}$, contradicting (10). □

Question 6.13 Is there a Polish group that is not locally compact such that its ideal of meager sets is ω-translatable (or at least 2-translatable)?

Acknowledgments The authors wish to thank Tomek Bartoszyński, Martin Goldstern, Osvaldo Guzmán, Marcin Kysiak, Marion Scheepers, Thilo Weinert, Piotr Zakrzewski, and Miroslav Zelený for stimulating conversations on the subject of the paper. In particular, they wish to thank Marion Scheepers for suggesting Theorem 4.8 and to Miroslav Zelený for pointing out the paper [3] to them. They also thank the anonymous referee for carefully reading the paper and for suggesting Theorem 3.14 as well as several improvements of the presentation of the results.

References

1. Balka, R.: Duality between measure and category in uncountable locally compact abelian Polish groups. Real Anal. Exchange **36**(2), 245–256 (2010/11)
2. Bartoszyński, T., Judah, H.: Set Theory: On the Structure of the Real Line. A K Peters Ltd., Wellesley (1995)
3. Behrends, E., Kadets, V.M.: Metric spaces with the small ball property. Stud. Math. **148**(3), 275–287 (2001)
4. Bergman, G.M.: Generating infinite symmetric groups (2008). http://arxiv.org/pdf/math/0401304.pdf
5. Besicovitch, A.S.: Concentrated and rarified sets of points. Acta Math. **62**(1), 289–300 (1933)
6. Besicovitch, A.S.: Correction. Acta Math. **62**(1), 317–318 (1933)
7. Borel, E.: Sur la classification des ensembles de mesure nulle. Bull. Soc. Math. Fr. **47**, 97–125 (1919)
8. Carlson, T.J.: Strong measure zero and strongly meager sets. Proc. Am. Math. Soc. **118**(2), 577–586 (1993)
9. Dobrowolski, T., Marciszewski, W.: Covering a Polish group by translates of a nowhere dense set. Topol. Appl. **155**(11), 1221–1226 (2008)
10. Foreman, M., Magidor, M., Shelah, S.: Martin's maximum, saturated ideals, and nonregular ultrafilters. I. Ann. Math. (2) **127**(1), 1–47 (1988)
11. Foreman, M., Magidor, M., Shelah, S.: Martin's maximum, saturated ideals and nonregular ultrafilters. II. Ann. Math. (2) **127**(3), 521–545 (1988)
12. Fremlin, D.H.: Measure Theory. Vol. 5, Set-Theoretic Measure Theory (2008). http://www.essex.ac.uk/maths/people/fremlin/mt5.2008/mt5.2008.tar.gz
13. Galvin, F., Mycielski, J., Solovay, R.M.: Strong measure zero sets. Not. Am. Math. Soc. **26**(3), A-280 (1979)
14. Galvin, F., Scheepers, M.: Borel's conjecture in topological groups. J. Symb. Logic **78**(1), 168–184 (2013)
15. Goldstern, M., Judah, H., Shelah, S.: Strong measure zero sets without Cohen reals. J. Symb. Logic **58**(4), 1323–1341 (1993)
16. Gruenhage, G.: Generalized Metric Spaces, Handbook of Set-Theoretic Topology. pp. 423–501. North-Holland, Amsterdam (1984)
17. Hrušák, M.: David Meza-Alcántara, and Hiroaki Minami, pair-splitting, pair-reaping and cardinal invariants of F_σ-ideals. J. Symb. Logic **75**(2), 661–677 (2010)
18. Jech, T.: Set Theory, Springer Monographs in Mathematics, The third millennium edition, revised and expanded. Springer, Berlin (2003)
19. Kechris, A.S.: Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156. Springer, New York (1995)
20. Kunen, K.: Set Theory, Studies in Logic and the Foundations of Mathematics, vol. 102. An introduction to independence proofs, Reprint of the 1980 original. North-Holland, Amsterdam (1983)
21. Kysiak, M.: On Erdős–Sierpiński duality between Lebesgue measure and Baire category (in Polish). Master's thesis (2000)

22. Laver, R.: On the consistency of Borel's conjecture. Acta Math. **137**(3–4), 151–169 (1976)
23. Miller, A.W.: Some properties of measure and category. Trans. Am. Math. Soc. **266**(1), 93–114 (1981)
24. Miller, A.W., Fremlin, D.H.: On some properties of Hurewicz, Menger, and Rothberger. Fund. Math. **129**(1), 17–33 (1988)
25. Miller, A.W., Steprāns, J.: The number of translates of a closed nowhere dense set required to cover a Polish group. Ann. Pure Appl. Logic **140**(1–3), 52–59 (2006)
26. Moore, J.T.: The proper forcing axiom. In: Proceedings of the International Congress of Mathematicians, vol. II, pp. 3–29. Hindustan Book Agency, New Delhi (2010)
27. Pawlikowski, J.: Undetermined sets of point-open games. Fund. Math. **144**(3), 279–285 (1994)
28. Prikry, K.: Unpublished result
29. Rogers, C.A.: Hausdorff Measures, Reprint of the 1970 original, With a foreword by K.J. Falconer. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1998)
30. Rosendal, C.: A topological version of the Bergman property. Forum Math. **21**(2), 299–332 (2009)
31. Rothberger, F.: Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C. Proc. Camb. Philos. Soc. **37**, 109–126 (1941)
32. Sierpiński, W.: Sur un ensemble non denombrable, dont toute image continue est de mesure nulle. Fund. Math. **11**, 302–304 (1928)
33. Szpilrajn, E.: Remarques sur les fonctions complètement additives d'ensemble et sur les ensembles jouissant de la propriété de Baire. Fund. Math. **22**(1), 303–311 (1934)
34. Thomas, S., Zapletal, J.: On the Steinhaus and Bergman properties for infinite products of finite groups. Conflu. Math. **4**(2), 1250002 (2012)
35. Todorčević, S.: Partition Problems in Topology. Contemporary Mathematics, vol. 84. American Mathematical Society, Providence (1989)
36. van Mill, J.: Analytic groups and pushing small sets apart. Trans. Am. Math. Soc. **361**(10), 5417–5434 (2009)
37. Wohofsky, W.: Special sets of real numbers and variants of the Borel Conjecture. Ph.D. thesis, Technische Universität Wien (2013)
38. Yorioka, T.: The cofinality of the strong measure zero ideal. J. Symb. Logic **67**(4), 1373–1384 (2002)