NEARLY COUNTABLE DENSE HOMOGENEOUS SPACES

MICHAEL HRUŠÁK AND JAN VAN MILL

Abstract. We study separable metric spaces with few types of countable dense sets. We present a structure theorem for locally compact spaces having precisely n types of countable dense sets: such a space contains a subset S of size at most $n-1$ such that S is invariant under all homeomorphisms of X and $X \setminus S$ is countable dense homogeneous. We prove that every Borel space having fewer than $\mathfrak{c}$ types of countable dense sets is Polish. The natural question of whether every Polish space has either countably many or $\mathfrak{c}$ many types of countable dense sets, is shown to be closely related to the Topological Vaught Conjecture.

1. Introduction

All spaces under discussion are separable and metrizable. A metric on a space X is *admissible* if it generates the topology on X. A space is *Polish* if it has an admissible *complete* metric.

Recall that a space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$. This is a classical notion that can be traced back to the works of Cantor, Brouwer, Fréchet, and others. Examples of CDH-spaces are the Euclidean spaces, the Hilbert cube and the Cantor set. In fact, every strongly locally homogeneous Polish space is CDH, as was shown by Bessaga and Pełczyński [4]. Recall that a space X is *strongly locally homogeneous* if it has a basis $\mathscr{B}$ of open sets such that for every $U \in \mathscr{B}$ and every $x, y \in U$ there is a autohomeomorphism h of X such that $h(x) = y$ and $h \restriction X \setminus U = id$.

In this paper we consider the number of types of countable dense sets that a given space has. As usual, we let $\mathfrak{c}$ denote the cardinality of the continuum. Given a space X and a cardinal number $1 \leq \kappa \leq \mathfrak{c}$ we say that a space X has κ *types of countable dense sets* provided that κ is the least cardinal for which there is a collection $\mathscr{A}$ of countable dense subsetes of X such that $|\mathscr{A}| = \kappa$ while for any given countable dense set B of X there exist $A \in \mathscr{A}$ and a homeomorphism $f\colon X \to X$ such that $f(A) = B$.

We prove that a Borel space that has fewer than continuum many types of countable dense sets is Polish. This improves the result of Hrušák and Zamora Avilés [13] that a Borel

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CDH-space is Polish. It is a natural question whether a Polish space having uncountably many types of countable dense sets, has in fact $\mathfrak{c}$ many such types. We show that the question is strongly related to the Topological Vaught Conjecture.

The topological sum of n copies of $[0,1)$ has $n+1$-many types of countable dense sets, while the topological sum of countably many copies of $[0,1)$ has countably many types of countable dense sets, i.e., simple examples of locally compact spaces having at most countably many types of countable dense sets can be constructed by adding a finite number of points to a CDH Polish space (e.g., finite graphs). Perhaps the main result of the present paper presenting a structure theorem on locally compact spaces X having at most countably many types of countable dense sets shows that these are the only examples.

Theorem 1.1. *Let X be a locally compact space having at most countably many types of countable dense sets. Then X contains a closed and scattered subset S of finite Cantor-Bendixson rank which is closed under all homeomorphisms of X and has the property that $X \setminus S$ is* CDH. *If X has at most n types of countable dense sets for some $n \in \mathbb{N}$, then $|S| \leq n-1$.*

The pseudoarc P is an example of a homogeneous continuum that has $\mathfrak{c}$ types of countable dense sets, the maximum number possible. To see this, let A and B be disjoint composants of P. Moreover, let D and E be countable dense sets of A and B, respectively. By Lemma 4.3 below, there is a collection $\mathscr{F}$ consisting of $\mathfrak{c}$ pairwise nonhomeomorphic subsets of E, none of which is homeomorphic to $\mathbb{Q}$. Since every homeomorphism of P permutes its composants, it is easy to see that for distinct $F, F' \in \mathscr{F}$ we have that $D \cup F$ and $D \cup F'$ are of different type.

The proofs of our results depend heavily on the Effros Theorem [8] about actions of Polish groups on Polish spaces as well as on Ungar's analysis of various homogeneity notions in [22, 23].

2. Ungar's Theorem revisited

Recall that a space X is *n-homogeneous* provided that for all subsets F and G of X of size n there is a homeomorphism f of X such that $f(F) = G$. In addition, X is *strongly n-homogeneous* provided that for all n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points of X there is a homeomorphism f of X such that $f(x_i) = y_i$ for all $i \leq n$.

The natural question whether these homogeneity notions are actually equivalent to countable dense homogeneity for certain classes of spaces was addressed by Ungar in [22, 23]. He showed that for highly connected locally compact spaces this is indeed the case[1]:

Theorem 2.1 (Ungar [22, 23])**.** *Let X be a locally compact space such that no finite set separates X. Then the following statements are equivalent:*

(1) *X is* CDH.
(2) *X is n-homogeneous for every n.*

[1]See van Mill [20] for an example of a Polish space that is strongly n-homogeneous for all n, but not CDH.

(3) *X is strongly n-homogeneous for every n.*

The proof of this very elegant result is based on the Effros Theorem [8] on actions of Polish groups on Polish spaces (see below). For a generalization of the implication (1) $\Rightarrow$ (3) in Theorem 2.1 see [20, Theorem 1.2]. The principal aim of this section is to prove an extension of this result without the connectivity assumptions.

First we recall relevant definitions and facts concerning actions of Polish groups.

An *action* of a topological group G on a space X is a continuous map

$$(g, x) \mapsto gx \colon G \times X \to X$$

such that $ex = x$ for every x in X and $g(hx) = (gh)x$ for g and h in G and x in X (here e denotes the neutral element of G). It is easily seen that for each g in G the map $x \mapsto gx$ is a homeomorphism of X whose inverse is the map $x \mapsto g^{-1}x$.

If x belongs to X and U is a subset of G then $Ux = \{gx : g \in U\}$. The action of G on X is *transitive* if $Gx = X$ for every x in X. It is *micro-transitive* if for every x in X and every neighborhood U of e in G the set Ux is a neighborhood of x in X.

Let G be a topological group acting on a space X. For every $F \subseteq X$, we put

$$G_F = \{g \in G : (\forall\, x \in F)(gx = x)\}, \text{and } G^F = \{g \in G : gF = F\}.$$

If F is a singleton, say $F = \{x\}$, then we denote G_F by G_x; it is the *stabilizer* of x by G. Observe that both G_F and G^F act on $X \setminus F$ and that G_F is a normal subgroup of G^F. Moreover, clearly, if F is finite, then G_F has finite index in G^F.

We say that a group G *makes* X CDH if G acts on X in such a way that for any two countable dense subsets $D, E \subseteq X$ there is a $g \in G$ such that $gD = E$. The following result appears in [20, Proposition 3.1].

Proposition 2.2. *Let X be a space, and let G be a group that makes X* CDH*. If $F \subseteq X$ is finite, and $D, E \subseteq X \setminus F$ are countable and dense in X, then there is an element $g \in G_F$ such that $gD \subseteq E$.*

A space X is *Baire* if the complement of every first category subset of X is dense in X. A space is *analytic* if it is a continuous image of a Polish space. It is well known that an absolute Borel set is analytic and that a Borel subspace of an analytic space is analytic.The following interesting fact, was proved in a more general context by Levi [16] (see also [17, §A13]):

Theorem 2.3. *Every analytic Baire space has a dense Polish subspace.*

As already mentioned, our main tool is the so-called *Effros Theorem* from [8], also known as the 'Open Mapping Principle'. It was generalized in [19], as follows:

Theorem 2.4 (Open Mapping Principle). *Suppose that an analytic group G acts transitively on a space X. If X is of second category, then G acts micro-transitively on X.*

This extremely useful result was first proved in its original form by Effros [8] using a Borel selection argument. Simpler proofs were found independently by Ancel [1], Hohti [11], and

Toruńczyk (unpublished). The proofs of Ancel and Toruńczyk are based on an ingenious technique of Homma [12], while Hohti uses an open mapping theorem due to Dektjarev [7].

The Open Mapping Principle implies the classical Open Mapping Theorem of functional analysis (for separable Banach spaces). Indeed, let B and E be separable Banach spaces, and let $h\colon B \to E$ be a continuous linear surjection. We think of B as a topological group, and define an action of B on E by $(x, y) \mapsto h(x) + y$. This action is transitive, since if y and y' in E and x in B are such that $h(x) = y' - y$, then $(x, y) \mapsto y'$. By Theorem 2.4, the map $B \to E$ defined by $x \mapsto h(x) + 0$ is open.

More important for our considerations is the fact that the Open Mapping Principle also implies that for every homogeneous compactum (X, ϱ) and every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x and y in X satisfy $\varrho(x, y) < \delta$, then there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and such that f does not move any point more than ε far from itself. (This goes half way towards explaining the word micro-transitive.) This interesting and surprising fact, first discovered by Ungar [23], was used with great success by continuum theorists in their study of homogeneous continua. See Ancel [1] and Charatonik and Maćkowiak [6] for details and further references.

Lemma 2.5. *Let G be an analytic group acting on a space Y. Suppose that Gy is of second category in Y for every $y \in Y$. Then for every $y \in Y$, Gy is clopen in Y and the map $G \to Y$ defined by $g \mapsto gy$ is open.*

Proof. Since G is analytic, it follows that Gy is analytic, hence it contains a dense Polish subspace by Theorem 2.3. Moreover, $\overline{Gy}$ has nonempty interior, say U. It then clearly follows that GU is a dense open subset of $\overline{Gy}$. Assume that for some $y \in Y$ we have that there exists $z \in \overline{Gy} \setminus Gy$. Then by what we just proved, the interior V of $\overline{Gz}$ is dense in $\overline{Gz} \subseteq \overline{Gy}$. However, this is impossible since then $Gz \cap V$ and $Gy \cap V$ contain disjoint dense Polish subspaces of V. This implies that Gy is closed in Y for every $y \in Y$, and hence that Gy is a clopen subset of Y since it has, as we just observed, nonempty interior in Y. Observe that the evaluation of G at a given point $y \in Y$ is open, is a direct consequence of Theorem 2.4. □

Now we are ready to state and prove the following generalization of Ungar's Theorem.

Theorem 2.6. *Let G be a Polish group acting on a Baire space X. Then the following statements are equivalent:*

(1) *G makes X* CDH*,*
(2) *for every finite subset F of X and $y \in X \setminus F$, $G^F y$ is of second category in X,*
(3) *for every finite subset F of X and $y \in X \setminus F$, $G_F y$ is of second category in X.*

Moreover, X is Polish.

Proof. We first prove that (2) $\Leftrightarrow$ (3). Indeed, pick a finite $F \subseteq X$, and let $y \in X \setminus F$. Since there is a finite subset A of G such that

$$G^F y = (AG_F)y = \bigcup_{a \in A} a(G_F y),$$

it follows that $G^F y$ is a second category subset of X if and only if $G_F y$ is.

Hence it suffices to prove that (1) $\Leftrightarrow$ (3). For (1) $\Rightarrow$ (3), pick an arbitrary finite subset $F \subseteq X$ and $y \in X \setminus F$. Striving for a contradiction, assume that $G_F y$ is a first category subset of X. Since X is Baire, there is a countable dense subset D of X such that $D \subseteq X \setminus (F \cup G_F y)$. By Proposition 2.2, we may pick $g \in G_F$ such that $g(D \cup \{y\}) \subseteq D$. However, this contradicts the fact that $gy \in G_F y \subseteq X \setminus D$. For (3) $\Rightarrow$ (1), first observe that by Lemma 2.5 we have that $G_F y$ is clopen in $X \setminus F$ for all finite subsets $F \subseteq X$ and $y \in X \setminus F$. Let D and E be countable dense subsets of X. Enumerate D as $\{d_n : n \in \mathbb{N}\}$ and E as $\{e_n : n \in \mathbb{N}\}$, respectively. By the above, Gd_0 is clopen, so we may pick $g_0 \in G$ such that $gd_0 \in E$. Let $m = \min\{n \in \mathbb{N} : e_n \neq gd_0\}$. Consider the set $F = \{gd_0\}$ and the point $e_m \in X \setminus F$. The set $G_F e_m$ is clopen in $X \setminus F$. Pick a very small symmetric open neighborhood V of the identity of the neutral element e of G_F. Then Ve_m is open in $X \setminus F$. It therefore intersects D, say in the point d. Let $h \in V$ be such that $hd = e_m$. Hence h fixes gd_0 and moves d to e_m by a small move. Continuing in this way by the standard back and forth method will produce precisely such as in the proof of Theorem 3.3 in Ungar [23] a sequence of elements of G that converges to an element of the Polish group G that takes D onto E.

That X is Polish follows from the following observations. If $x \in X$, then Gx is clopen in X. Hence by Theorem 2.4, the evaluation mapping $g \mapsto gx$ is an open surjection. Hence Gx is Polish by Hausdorff's Theorem in [10] that an open image of a Polish space is Polish, cf. [1]. Hence X is Polish since $\{Gx : x \in X\}$ is a clopen partition of X by Polish subspaces. $\square$

The promised strengthening of Ungar's Theorem 2.1 (1) $\Leftrightarrow$ (3) follows by an application of Lemma 2.5 and Theorem 2.6:

Corollary 2.7. *Let X be a locally compact space. Then the following statements are equivalent:*

1. *X is* CDH*,*
2. *for every finite subset F of X there is a partition $\mathscr{U}$ of $X \setminus F$ into relatively clopen sets such that for every $U \in \mathscr{U}$ and every $x, y \in U$ there is a homeomorphism h of X such that $h(x) = y$ while $h(F) = F$,*
3. *for every finite subset F of X there is a partition $\mathscr{U}$ of $X \setminus F$ into relatively clopen sets such that for every $U \in \mathscr{U}$ and every $x, y \in U$ there is a homeomorphism h of X such that $h(x) = y$ while $h \restriction F = id$.*

That Theorem 2.1 (2) $\Leftrightarrow$ (3) holds requires a different application of the Effros Theorem, see Ungar [22] for details.

3. Spaces with few types of countable dense sets

In this section we study the structure of spaces having fewer than $\mathfrak{c}$ types of countable dense sets. We are interseted mostly in locally compact spaces, but since our proofs only require the existence of suitable actions by Polish groups, we formulate our results first in the language of G-spaces.

Let X be a *G-space*, for some topological group G, i.e., a space with a fixed action of G on X. By the *G-type* of a countable dense set $D \subseteq X$ we mean the collection $\{gD : g \in G\}$. We are interested in the structure of G-spaces having κ G-types of countable dense sets, where κ is a cardinal number below $\mathfrak{c}$. Different groups may yield different cardinal numbers, of course. If X is a crowded space, then the trivial group $G = \{e\}$ has $\mathfrak{c}$ G-types of countable dense sets. More importantly, if X is locally compact, then the number of $\mathscr{H}(X)$-types of countable dense sets is equal to the number of types of countable dense sets. Here $\mathscr{H}(X)$ denotes the group of autohomeomorphisms of X. It is well-known, and easy to prove, that for a locally compact space, $\mathscr{H}(X)$ can be endowed with a Polish group topology such that the natural action

$$\mathscr{H}(X) \times X \to X, \quad (g, x) \mapsto g(x) \quad (g \in \mathscr{H}(X), x \in X),$$

is continuous. For details, see Kechris [15].

Here is our first structure theorem.

Theorem 3.1. *Let G be a Polish group, and let X be a Baire G-space. Assume that X has fewer than $\mathfrak{c}$ G-types of countable dense sets. Then*

$$S = \big\{x \in X : Gx \textit{ is of first category in } X\big\}$$

is a closed and scattered (hence countable) subspace of X. Moreover, S has finite Cantor-Bendixson rank, X is Polish, S is invariant under he action of G and, assuming $X \setminus S$ is connected, G makes $X \setminus S$ homogeneous.

Proof. First note that if $x \in S$ then $Gx \subseteq S$. Striving for a contradiction, assume first that S is not scattered. Then it contains a copy Q of the space of rational numbers $\mathbb{Q}$. Put $T = \bigcup_{x \in Q} Gx$. Then T is clearly of first category, hence $Y = X \setminus T$ is dense and Baire. By Lemma 4.3 below there is a family $\mathscr{A}$ consisting of $\mathfrak{c}$ countable and pairwise nonhomeomorphic subsets of T. Let $D \subseteq Y$ be any countable dense set. Then the collection

$$\{D \cup A : A \in \mathscr{A}\}$$

is clearly a collection of $\mathfrak{c}$ countable dense subsets of X pairwise non-equivalent under the action of G. This contradicts our assumptions. Hence S is scattered, and so countable.

Now, given $x \in S$, the orbit Gx is scattered, hence discrete. We claim that there does not exist an infinite collection of first category orbits. To this end, assume that there are $\{x_n : n \in \mathbb{N}\}$ in X such that Gx_n is first category for every n, and $Gx_n \cap Gx_m = \emptyset$ if $n \neq m$. Let $D \subseteq X \setminus \bigcup_{n \in \mathbb{N}} Gx_n$ be a countable dense set. For every $A \in \mathscr{P}(\mathbb{N})$, put

$$D(A) = D \cup \bigcup_{n \in A} Gx_n.$$

It is clear that the collection $\{D(A) : A \in \mathscr{P}(\mathbb{N})\}$ is a collection of $\mathfrak{c}$ pairwise non-G-equivalent countable dense sets, which is a contradiction. Hence S has finite scattered rank.

Next we claim that S is closed. Put $Z = X \setminus S$. Assume that there exists an element $z \in \overline{S} \setminus S$. Then Gz is clopen in Z by Lemma 2.5. Let U be an open subset of X such that $U \cap Z = Gz$. Observe that $U \cap S$ is scattered and S does not contain an isolated point (by definition). Hence $U \cap S$ is nowhere dense in U. This implies that there is an element $z' \in U \setminus \overline{S}$. Pick $g \in G$ such that $gz' = z$. Since $z \in \overline{S}$ and $gS = S$, we get that $z' = gz \in \overline{S}$. This is a contradiction.

The fact that X is Polish follows from the following observation: If $z \in Z$, then Gz is clopen in Z. Hence by Theorem 2.4, the evaluation mapping $g \mapsto gz$ is an open surjection. Hence Gz is Polish by Hausdorff's Theorem in [10] that an open image of a Polish space is Polish, cf. [1]. Hence Z is Polish since $\{Gz : z \in Z\}$ is a clopen partition of Z by Polish subspaces. As a consequence, X is Polish since S is closed in X.

That G makes Z homogeneous provided it is connected, is clear from the above. ☐

Corollary 3.2. *Let X be a locally compact space with fewer than $\mathfrak{c}$ types of countable dense sets. Then there is a closed and scattered subset S of X of finite Cantor-Bendixson rank which is invariant under all homeomorphisms of X. Moreover, $X \setminus S$ is homogeneous provided it is connected.*

In the remainder of this section we will present the proof of Theorem 1.1. Hence we show that the Structure Theorem 3.1 can be improved if we additionally assume that the number of types of countable dense subsets of X is countable. The question whether the number of types of countable dense sets of a Polish space can be uncountable, but less than $\mathfrak{c}$, is an open problem.

Again, we formulate and prove our results in terms of group actions.

Theorem 3.3. *Let G be a Polish group, and let X be a Baire G-space with at most countably many G-types of countable dense sets. Then*

$$S = \{x \in X : Gx \text{ is of first category in } X\}$$

is closed, scattered of finite Cantor-Bendixson rank, invariant under the action of G. X is Polish and G makes $X \setminus S$ CDH.

Moreover, $|S| \leq n{-}1$ if X has at most n G-types of countable dense sets.

Note that the bound $|S| \leq n{-}1$ is not necessarily optimal, as there are examples of locally compact spaces with n-types of countable dense sets for which $|S| < n{-}1$, e.g. if X is the subspace of the plane consisting of the union of the following three objects: The circle centered at $(0, 0)$ of diameter 1, the line segment joining the points $(1, 0)$, $(2, 0)$ and the geometric interior of the line segment joining the points $(2, 1)$, $(2, -1)$. Then X has 4 types of countable dense sets, yet $S = \{(1, 0), (2, 0)\}$ has size 2.

In order to prove the theorem, let X, G and S be as in the theorem, and let $Y = X \setminus S$. Suppose that X has at most n G-types of countable dense sets. We will prove that $|S| \leq n{-}1$. Assume that $|S| \geq n$, and let A be a subset of S of size n. Observe that $B = GA$ is G-invariant and of first category. Let D be a countable dense subset of $X \setminus B$. For every $E \subseteq B$, put $D_E = D \cup E$. Then D_E is a countable dense subset of X for every $E \subseteq B$, and clearly $gD_E \neq D_{E'}$ for E and E' contained in B of different cardinality. Simply observe that B is G-invariant, and hence $|gD_E \cap B| = |E|$ for every $E \subseteq B$. From this it follows that X has more than n G-types of countable dense sets, which is a contradiction.

Hence by Theorem 3.1 it suffices to prove that if X has countably many G-types of countable dense sets, then $Y = X \setminus S$ is CDH. In order to demonstrate this, we prove several preliminary results. First note that:

Claim 1. Y is a topological sum of a discrete space and a crowded[2] space.

To see this, denote by $Is(X)$ the set of isolated points of X, and note that the set $C = \overline{Is(X)} \setminus Is(X)$ of accumulation points $Is(X)$ is contained in S (C is closed nowhere dense and $Gx \subseteq C$ for every $x \in C$).

So we can and will assume that the set Y **is crowded**.

For a space X and $n \geq 1$, let

$$F^n(X) = \{x \in X^n : x_i = x_j \text{ iff } i = j\}.$$

This space is sometimes called the *n-th configuration space of X*. It is an open subset of X^n. Hence if X is Polish, so is every $F^n(X)$.

The following lemma, crucial here, was proved by Ungar [23] for locally compact spaces. We present it with a simpler proof that also works for Polish spaces.

Lemma 3.4. *Let X be a Polish space. Let T be a first category subset of $F^n(X)$. Then there is a countable dense $D \subseteq X$ such that $F^n(D) \cap T = \emptyset$.*

Proof. On X we choose some admissible complete metric. Let $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open base for X such that $U_i \neq \emptyset$ for every i, and without loss of generality write T as $\bigcup_{i \in \mathbb{N}} T_i$, where each T_i is closed and nowhere dense in $F^n(X)$. If

$$(U_1 \times \cdots \times U_n) \cap F^n(X) = \emptyset,$$

then we shrink every U_i to a nonempty open set V_i^1 such that $V_i^1 \subseteq \overline{V_i^1} \subseteq U_i$ and $\operatorname{diam} V_i^1 < 2^{-1}$; we put $V_m^1 = X$ for every $m > n$. If

$$(U_1 \times \cdots \times U_n) \cap F^n(X) \neq \emptyset$$

then it contains a point not in T_1. Hence we may assume that there are open subsets V_i^1 for $1 \leq i \leq n$ such that

(1) $V_i^1 \cap V_j^1 \neq \emptyset$ iff $i = j$,
(2) $V_i^1 \subseteq \overline{V_i^1} \subseteq U_i$,
(3) $\operatorname{diam} V_i^1 < 2^{-1}$,
(4) for every permutation $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$,
$$(V_{\pi(1)}^1 \times \cdots \times V_{\pi(n)}^1) \cap T_1 = \emptyset.$$

Put $V_m^1 = X$ for every $m > n$. Now we continue in this way. In each step we do nothing but a basic shrinking in case we run into a set having empty intersection with $F^n(X)$. The conclusion is that we can find open subsets V_i^2 for $i \leq n{+}1$ such that

(5) $V_i^2 \subseteq \overline{V_i^2} \subseteq V_i^1$ if $i \leq n$,
(6) $V_{n+1}^2 \subseteq \overline{V_{n+1}^2} \subseteq U_{n+1}$,

[2]A space is *crowded* if it has no isolated points.

(7) $\operatorname{diam} V_i^2 < 2^{-2}$,
(8) for every permutation $\pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ and every $k \le n$
$$(V^2_{\pi(1)} \times \cdots \times V^2_{\pi(k-1)} \times V^2_{n+1} \times V^2_{\pi(k+1)} \times \cdots \times V^2_{\pi(n)}) \cap T_2 = \emptyset.$$

Put $V^1_m = X$ for every $m \ge n+2$. Continue in this way recursively.

At the end of the construction, for every $k \in \mathbb{N}$, let q_k be the unique point in the intersection $\bigcap_{i\in\mathbb{N}} V^i_k$. Then $D = \{q_k : k \in \mathbb{N}\}$ is clearly as required. □

The following result appears in van Mill [18] as Theorem 2.12. For the sake of completeness, we present its standard proof. Recall that given a cover $\mathscr{U}$ of a space X, a set $A \subseteq X$ and a function $f\colon A \to X$ we say that *f is limited by $\mathscr{U}$* if for every $x \in A$ there is a $U \in \mathscr{U}$ containing both x and $f(x)$.

Lemma 3.5. *Let G be an analytic group acting on a space Y such that Gy is a second category subset of Y for every $y \in Y$. Then for every open cover $\mathscr{U}$ of Y and every compact subset $K \subseteq Y$ there is an open cover $\mathscr{V}$ of Y with the following property: for all $V \in \mathscr{V}$ and $x, y \in V$ there exists $g \in G$ such that $gx = y$ and $g{\restriction}K$ is limited by $\mathscr{U}$.*

Proof. By continuity of the action, for $x \in K$, we may pick an open neighborhood V_x of the neutral element $e \in G$ such that $V_x^2 x$ is contained in an element of $\mathscr{U}$. Since every set of the form $V_x x$ is open by Lemma 2.5, there is a finite $F \subseteq K$ such that
$$K \subseteq \bigcup_{x\in F} V_x x.$$
Let $V = \bigcap_{x\in F} V_x$, and let W be a symmetric open neighborhood of e such that $W^2 \subseteq V$. Put $\mathscr{V} = \{Wy : y \in Y\}$. Then $\mathscr{V}$ is an open cover by Lemma 2.5. We claim that $\mathscr{V}$ is as required. To this end, pick arbitrary $z, p, q \in Y$ such that $p, q \in Wz$. There are $h, g \in W$ such that $hz = p$ and $gz = q$. Then $\xi = gh^{-1} \in V$ and $\xi p = q$. So it suffices to prove that for $y \in K$ there exists $U \in \mathscr{U}$ containing both y and ξy. Pick $x \in F$ such that $y \in V_x x \subseteq V_x^2 x$. There is an element $h \in V_x$ such that $hx = y$. Since $\xi y = (\xi h)x \in V_x^2 x$ and $V_x^2 x$ is contained in an element of $\mathscr{U}$, this completes the proof. □

Fix $m \ge 1$. We let G act on $F^m(Y)$ as follows:
$$(g, (y_1, \dots, y_m)) \mapsto (gy_1, \dots, gy_m), \qquad g \in G, (y_1, \dots, y_m) \in F^m(Y).$$

Lemma 3.6. *For every $(y_1, \dots, y_m) \in F^m(Y)$ there exists a Cantor set K in G such that the collection*
$$\big\{\{gy_1, \dots, gy_m\} : g \in K\big\}$$
is pairwise disjoint.

Proof. Let ϱ be an admissible metric on X.

The construction of K is, of course, similar to the standard construction of a Cantor set in an uncountable analytic space. We only need to ensure that for all distinct g and h in K we have
$$\{gy_1, \dots, gy_m\} \cap \{hy_1, \dots, hy_m\} = \emptyset.$$
Pick an arbitrary $(y_1, \dots, y_m) \in F^m(X)$. We need the following:

Claim 1. For every neighborhood W of the neutral element e of G, there exist $a \in W$ such that

$$\{ay_1, \ldots, ay_m\} \cap \{y_1, \ldots, y_m\} = \emptyset.$$

Pick an open neighborhood V of e such that $V^m \subseteq W$, and put $F = \{y_1, \ldots, y_m\}$. Since F is finite, there exists by Lemma 2.5 an element $a_1 \in V$ such that $a_1 y_1 \notin F$. Let $\varepsilon_1 = \varrho(a_1 y_1, F)$ and put $\delta_1 = \varepsilon_1/m$. By Lemma 3.5 there exists an $a_2 \in G$ such that $(a_2 a_1) y_2 \notin F$ and $\varrho((a_2 a_1) y_1, a_1 y_1) < \delta_1$. By continuing in this way, it is clear that in m steps we have succeeded in making $\{ay_1, \ldots, ay_m\}$ and $\{y_1, \ldots, y_m\}$ disjoint, where $a = a_m \cdots a_1$.

Now, by using Claim 1 at each step of the standard construction of a Cantor set as a nested intersection of finite unions of disjoint balls of smaller and smaller diameter in the complete space G, we can construct K. □

In what follows, ω_1 denotes the first uncountable cardinal and, at the same time, the set of all countable ordinal numbers. It is a result in Baumgartner [2, Corollary 2.4], attributed to Weiss, that there is a family $\mathscr{A} = \{A_\alpha : \alpha \in \omega_1\}$ of countable ordinal numbers such that

(1) if $\alpha < \beta \in \omega_1$, then A_β does not embed in A_α,
(2) for every $\alpha \in \omega_1$, if $\mathscr{S}$ is a finite partition of A_α, then for some $S \in \mathscr{S}$ we have that S and A_α are homeomorphic.

Proposition 3.7. *If $(y_1, \ldots, y_m) \in F^m(Y)$, then $G(y_1, \ldots, y_m)$ is of second category in $F^m(Y)$.*

Proof. Striving for a contradiction, assume that for some $(y_1, \ldots, y_m) \in F^m(Y)$ we have that $T = G(y_1, \ldots, y_m)$ is of first category in $F^m(Y)$. By Lemma 3.4, there is a countable dense subset D of Y such that $F^m(D) \cap T = \emptyset$. Let $K \subseteq T$ be the Cantor set given by Lemma 3.6. We may assume that the ordinal numbers from the family $\mathscr{A}$ above are all contained in K. Observe that for every $i \leq m$ we have that the function $K \to Y$ defined by $g \mapsto gy_i$, is an embedding. For every $\alpha \in \omega_1$, put

$$D_\alpha = D \cup \bigcup_{g \in A_\alpha} \{gy_1, \ldots, gy_m\}.$$

Then for every $\alpha \in \omega_1$, D_α is a countable dense subset of X. We claim D_α and D_β are not G-equivalent for distinct α and β. To this end, pick $\alpha < \beta \in \omega_1$, and assume that there is an element $h \in G$ such that $hD_\beta = D_\alpha$. Since $F^m(D) \cap T = \emptyset$, for every $g \in A_\beta$ there exists $i(g) \leq m$ such that $hgy_{i(g)} \notin D$. By (2) above, we may assume that there are $B \subseteq A_\beta$ and $i \leq m$ such that B is homeomorphic to A_β while moreover $hgy_i \notin D$ for every $g \in B$. Again by (2) there exist $B' \subseteq B$ and $j \leq m$ such that B' and A_β are homeomorphic and $hgy_i \in A_\alpha y_j$ for every $g \in B'$. However, this shows that A_β can be embedded in A_α, which is a contradiction.

Hence we have created uncountably many G-types of countable dense sets in X, which violates our assumptions. □

Hence we conclude that every G-orbit of $F^m(Y)$ is clopen by Lemma 2.5. From this we conclude by Theorem 2.6 that G makes Y CDH, and this is what we had to prove. This completes the proof of Theorem 3.3, and hence also its corollary Theorem 1.1.

Corollary 3.8. *Let X be a homogeneous locally compact space. If X is not* CDH*, then X has uncountably many types of countable dense sets.*

Question 3.9. Let S be the set in Theorem 3.1. Is it true that $X \setminus S$ is CDH?

Question 3.10. Let X be a homogeneous locally compact space which is not CDH. Does X have $\mathfrak{c}$ types of countable dense sets?

4. Spaces with many types of countable dense sets

The aim of this section is to prove that every Borel space which is not completely metrizable has $\mathfrak{c}$ many types of countable dense sets (Corollary 4.6). This generalizes the main result in Hrušák and Zamora Avilés [13]. Before we prove the theorem we recall several known facts from descriptive set-theory (for more see e.g. Kechris [15]).

It is a well-known theorem of Souslin's [15, 14.13] that every uncountable analytic space contains a homeomorphic copy of a Cantor set. The following results and its corollary are due to Hurewicz [14] (see also [17, pages 78 and 79]).

Theorem 4.1. *If a space X is not a Baire space, then X contains a closed subspace homeomorphic to $\mathbb{Q}$.*

A space X is *completely Baire* if all of its closed subspaces are Baire.

Corollary 4.2. *A space X is completely Baire if and only if X does not contain a closed copy of $\mathbb{Q}$.*

The following lemma can be found in Brian, van Mill and Suabedissen [5]. For the sake of completeness, we include its simple proof.

Lemma 4.3. *The number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is $\mathfrak{c}$.*

Proof. Every countable subset of $\mathbb{R}$ can be embedded in $\mathbb{Q}$, so the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at most $|\mathscr{P}(\mathbb{Q})| = \mathfrak{c}$.

Let $X \subseteq \mathbb{R}$. Let P be the largest crowded subset of X and let $S = X \setminus P$ be the scattered part of X. We define the *scattered signature* $H(X)$ of X as follows: $H(X)$ is a set of ordinals, and $\alpha \in H(X)$ if and only if there is some $p \in P$ such that p has Cantor-Bendixson rank α in $S \cup \{p\}$.

Let $A = \{\alpha_n : n \in \mathbb{N}\}$ be a countable subset of ω_1. We show that there is a countable subset of $\mathbb{R}$ with scattered signature A. On the interval $[n+1/4, n+1/2]$, embed $\omega^{\alpha_n}+1$, making sure that the point ω^{α_n} maps to the point $n+1/2$. Include the points $\mathbb{Q} \cap [n+1/2, n+3/4]$ and call the resulting set X. It is a routine exercise to show that $H(X) = A$.

As there are $\mathfrak{c}$-many countable subsets of ω_1, this proves that the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at least $\mathfrak{c}$. $\square$

Fitzpatrick and Zhou [9] proved the following useful lemma.

Lemma 4.4. *A crowded space X is meager in itself if and only if there is a countable dense $D \subseteq X$ which is G_δ in X.*

The main theorem of this section is:

Theorem 4.5. *Let X be an analytic space with fewer than $\mathfrak{c}$ types of countable dense sets. Then X is completely Baire.*

Proof. We can write X as $A \cup B$, where A is open and scattered, and B is crowded. Observe that A is countable and invariant under the homeomorphisms of X. Hence B is also an analytic space with fewer than $\mathfrak{c}$ types of countable dense sets. Moreover, B is completely Baire if and only if X is completely Baire. It therefore suffices to consider the case that $A = \emptyset$, hence that X is crowded.

Claim 2. X is nowhere countable.

Assume not, that is

$$V = \bigcup\{U : U \text{ is a countable open subset of } X\}$$

is not empty. Then V is itself a countable open subset of X. Since V is crowded, $V \approx \mathbb{Q}$. By Lemma 4.3, we may pick a family $\mathscr{E}$ consisting of $\mathfrak{c}$ pairwise nonhomeomorphic nowhere dense subsets of V. Let D be a countable dense subset of $X \setminus V$, and for $E \in \mathscr{E}$, put $D(E) = (V \setminus E) \cup D$. Since V is invariant under the homeomorphisms of X, clearly $D(E)$ and $D(E')$ are not equivalent for $E \neq E'$, which proves that X has $\mathfrak{c}$ types of countable dense sets. This is a contradiction.

Claim 3. X is Baire.

Suppose that it is not the case. Then there is a nonempty open subset U of X which is meager in itself. Let V be a nonempty open subset of U such that $\overline{V} \subseteq U$ while moreover $U \setminus \overline{V} \neq \emptyset$. By Claim 1, V is uncountable, hence, being analytic, contains a Cantor set K by Souslin's Theorem. We may assume without loss of generality that K is nowhere dense in X. Hence we may pick nonempty disjoint open sets W and W' in $V \setminus K$ such that $K = \overline{W} \cap \overline{W'}$. Observe that this implies that the interior of the closure of W does not intersect K. Hence we may assume without loss of generality that W is regular open.

Since W is meager in itself, we may pick a countable dense subset $D_0 \subseteq W$ which is a G_δ-subset of W (Lemma 4.4). Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis for $X \setminus \overline{W}$. By Claim 1 and Souslin's Theorem, we may pick for every n a Cantor set F_n in U_n. Choose for every $n \in \mathbb{N}$ a countable dense subset $C_n \subseteq F_n$, and put $D_1 = \bigcup_{n \in \mathbb{N}} C_n$.

Note that $D_1 \cap O$ is not a G_δ-subset of O for every nonempty open subset O of $X \setminus \overline{W}$. For if this were true, we could pick $n \in \mathbb{N}$ such that $F_n \subseteq O$ which would imply that $D_1 \cap F_n$ would be G_δ in F_n. However, $D_1 \cap F_n$ contains the dense set C_n. Hence $D_1 \cap F_n$ is a countable dense subset of F_n which is G_δ in F_n. This shows by Lemma 4.4 that F_n is meager in itself, which contradicts the fact that F_n is compact.

By Lemma 4.3, we may pick a family $\mathscr{F}$ consisting of $\mathfrak{c}$ pairwise disjoint nonhomeomorphic countable subsets of K. For every $F \in \mathscr{F}$ put $D(F) = D_0 \cup F \cup D_1$. Then $D(F)$ is dense in X for every $F \in \mathscr{F}$, and we claim that $D(F)$ and $D(F')$ are not equivalent if $F \neq F'$. To this end, pick distinct $F, F' \in \mathscr{F}$, and let $f\colon X \to X$ be a homeomorphism such that $f(D(F)) = D(F')$. Since $f(D_0)$ is G_δ in $f(W)$, by the above we obtain that $f(W) \subseteq \overline{W}$. For if $A = f(W) \setminus \overline{W} \neq \emptyset$, it would follow that

$$f(D_0) = f(W \cap D(F)) = f(W) \cap f(D(F)) = f(W) \cap D(F'),$$

and hence

$$f(D_0) \cap A = f(D_0) \setminus \overline{W} = (f(W) \cap D(F')) \setminus \overline{W} = (f(W) \setminus \overline{W}) \cap D_1 = D_1 \cap A.$$

However, this is a contradiction since $f(D_0) \cap A$ is a G_δ-subset of A while $D_1 \cap A$ is not. Since W is regularly open, this consequently implies that $f(W) \subseteq W$. A similar analysis with f replaced by f^{-1} gives us that $f^{-1}(W) \subseteq W$. Hence we conclude that $f(W) = W$, and so $f(\overline{W} \setminus W) = \overline{W} \setminus W$. From this we conclude that $f(F) = F'$, which is a contradiction.

Hence again we conclude that X has $\mathfrak{c}$ types of countable dense sets, which contradicts our assumptions.

We are now ready to show that X is completely Baire. By Claim 2 and Theorem 2.3 there is a Polish $G \subseteq X$ which is dense in X. Let D_0 be any countable dense subset of G (and consequently also a dense subset of X). Note that D_0 has the property that if $E \subseteq D_0$ is crowded, then E is not a G_δ-subset in $\overline{E}$. For if E were G_δ in $\overline{E}$, then E would be G_δ in $\overline{E} \cap G$, but $\overline{E} \cap G$ is G_δ in G, hence is Polish. This contradicts Lemma 4.4; simply observe that $\overline{E} \cap G$ is crowded since E is.

Let Q be a closed homeomorphic copy of $\mathbb{Q}$ in X. We will derive a contradiction, which means that we will be done by Corollary 4.2. We may assume without loss of generality that $Q \cap G = \emptyset$. Again by Lemma 4.3 we may fix a collection $\mathscr{A}$ of $\mathfrak{c}$ pairwise nonhomeomorphic nowhere dense subsets of Q. For every $A \in \mathscr{A}$, put $D(A) = (Q \setminus A) \cup D_0$. We claim that the countable dense subsets $D(A)$ and $D(A')$ are of different type if $A \neq A'$. To this end, let $f\colon X \to X$ be a homeomorphism such that $f(D(A)) = D(A')$. Assume that there exists $x \in Q \setminus A$ such that $f(x) \notin Q$. Since Q is closed in X, there is a neighborhood U of x in $Q \setminus A$ such that $f(U) \subseteq D_0$. However, $Q \setminus A \approx \mathbb{Q}$, hence U is crowded. Hence U is G_δ in $\overline{U}$, but by the above, $f(U)$ is not G_δ in $\overline{f(U)}$. This is a contradiction. From this we conclude that $f(Q \setminus A) = Q \setminus A'$, and hence, $f(A) = A'$. This again contradicts our assumptions. $\square$

By a result of Hurewicz every co-analytic completely Baire space is Polish (see [15, 21.21]). The following generalizes the main result in Hrušák and Zamora Avilés [13].

Corollary 4.6. *If X is Borel and has fewer than $\mathfrak{c}$ types of countable dense sets then X is Polish.*

In other words:

Corollary 4.7. *If X is Borel and not an absolute G_δ set then X has $\mathfrak{c}$ types of countable dense sets.*

5. ω_1-many types and Vaught's Conjecture

The *Vaught's Conjecture* is a conjecture in model theory posed by Vaught in 1961 [24]. It states that any first-order complete theory in a countable language has either at most countably many or $\mathfrak{c}$ many non-isomorphic countable models. Morley [21] showed that number of countable models is at most ω_1 or $\mathfrak{c}$. Morley's proof led to the formulation of the, so called, *Topological Vaught's Conjecture* - the statement that whenever a Polish group acts continuously on a Polish space, there are either countably or $\mathfrak{c}$ many orbits. The topological Vaught's conjecture is, in fact, a stronger statement than the Vaught's conjecture,

The results contained in this paper suggest the following natural question:

Question 5.1. Is there a Polish space X with ω_1 types of countable dense sets?

We will not answer the question here. We will show that it has a close connection to the Topological Vaught Conjecture.

Let S_∞ denote the group of all permutations of $\mathbb{N}$ with the topology of pointwise convergence. Then S_∞ is a Polish group. It admits a standard action on every infinite product $X^{\mathbb{N}}$, as follows:

$$\big(\pi, (x_1, x_2, \dots)\big) \mapsto (x_{\pi(1)}, x_{\pi(2)}, \dots),$$

where $\pi \in S_\infty$ and $(x_1, x_2, \dots) \in X^{\mathbb{N}}$. Consider the Cantor set[3] $2^{\mathbb{N}}$ and the standard action of S_∞ on it. This action has countably many orbits. If we let G denote the subgroup of S_∞ consisting of the neutral element only, then its natural action on $2^{\mathbb{N}}$ has $\mathfrak{c}$ orbits. For an arbitrary closed subgroup G of S_∞, it is unknown whether the number of the orbits of its natural action on $2^{\mathbb{N}}$ is countable or $\mathfrak{c}$. This is a special case of the Topological Vaught Conjecture, and we refer to Becker and Kechris [3] for more information on this.

The connection between the number of types of countable dense sets and the Topological Vaught Conjecture is established by the following two results.

Theorem 5.2. *Let G be a closed subgroup of S_∞, and let κ be the number of orbits for the canonical action $G \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Then there is an action of a Polish group H on $X = \mathbb{N} \times [0,1)$ such that X has κ H-types of countable dense sets.*

Proof. Let G act on X in the following natural way:

$$(g, (n, t)) \mapsto (g(n), t) \qquad (g \in G, n \in \mathbb{N}, t \in [0,1)).$$

Put

$$F = \{f \in \mathscr{H}(X) : (\forall\, n \in \mathbb{N})(f(n,0) = (n,0))\}.$$

Then F is a closed subgroup of $\mathscr{H}(X)$ and hence is Polish. Moreover, for any two countable dense subsets D and E of $\mathbb{N} \times (0,1)$ there exists $f \in F$ such that $f(D) = E$. Observe that every $g \in G$ commutes with every $f \in F$. This means that the Polish group $H = F \times G$ acts on X as follows:

$$((f,g),x) \mapsto (f \circ g)(x) \qquad (f \in F, g \in G, x \in X).$$

[3]Following set theoretic notation we identify $2 = \{0,1\}$.

A typical countable dense subset of X has the form $D \cup A$, where D is a countable dense subset of $\mathbb{N} \times (0,1)$, and $A \subseteq \mathbb{N} \times \{0\}$. By identifying $\mathscr{P}(\mathbb{N} \times \{0\})$ and $2^{\mathbb{N}}$ in the standard way, it is clear that we get what we want. $\square$

Hence, in particular, an action with ω_1 orbits would produce a space with ω_1 H-types of countable dense sets. We now aim at proving the converse.

Let X be a Polish space. Put

$$\mathrm{CD}(X) = \{x \in X^{\mathbb{N}} : \{x_1, x_2, \dots\} \text{ is dense in } X\}.$$

We think of $\mathrm{CD}(X)$ as the *space of countable dense subsets* of X.

Lemma 5.3. *If X is Polish, then so is* $\mathrm{CD}(X)$*.*

Proof. Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open base for X. Now simply observe that $\mathrm{CD}(X)$ is equal to

$$X^{\mathbb{N}} \setminus \bigcup_{n \in \mathbb{N}} (X \setminus U_n)^{\mathbb{N}}$$

and hence is a G_δ-subset of $X^{\mathbb{N}}$. Hence since $X^{\mathbb{N}}$ is Polish, so is $\mathrm{CD}(X)$. $\square$

This leads us to the following result:

Theorem 5.4. *Let G be a Polish group for which there is a Polish G-space X with κ G-types of countable dense sets. Then there is an action of a Polish group H on a Polish space Y having exactly κ orbits.*

Proof. Consider the standard action of S_∞ on $X^{\mathbb{N}}$. It is clear that $\mathrm{CD}(X)$ is invariant under the action of S_∞. We let G act on $\mathrm{CD}(X)$ as follows:

$$\big(g, (x_1, x_2, \dots)\big) \mapsto (gx_1, gx_2, \dots).$$

Observe that for $\pi \in S_\infty$, $g \in G$ and $x \in \mathrm{CD}(X)$ we have that $\pi g x = g \pi x$. Hence we can let the Polish group $H = S_\infty \times G$ act on the Polish space $\mathrm{CD}(X)$ (Lemma 5.3) as follows:

$$\big((\pi, g), (x_1, x_2, \dots)\big) \mapsto (gx_{\pi(1)}, gx_{\pi(2)}, \dots).$$

It is left as an exercise to the reader to show that H has exactly κ orbits. $\square$

Corollary 5.5. *If there is a locally compact space X with κ types of countable dense sets, then there is are Polish group G and a Polish G-space Y with κ orbits.*

This suggests the following problem, equivalent to the problem whether 'the Topological Vaught Conjecture' is true for locally compact spaces.

Question 5.6. Let X be a locally compact space. Does X have either at most ω or exactly $\mathfrak{c}$ types of countable dense sets?

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Centro de Ciencas Matemáticas, UNAM, A.P. 61-3, Xangari, Morelia, Michoacán, 58089, México

E-mail address: michael@matmor.unam.mx

Faculty of Sciences, Department of Mathematics, VU University Amsterdam, De Boelelaan 1081^a, 1081 HV Amsterdam, The Netherlands

E-mail address: j.van.mill@vu.nl