KATĚTOV ORDER ON BOREL IDEALS

MICHAEL HRUŠÁK

Abstract. We study the Katětov order on Borel ideals. We prove two structural theorems (dichotomies), one for Borel ideals, the other for analytic P-ideals. We isolate nine important Borel ideals and study the Katětov order among them. We also present a list of fundamental open problems concerning the Katětov order on Borel ideals.

Introduction

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω we shall say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$ ($\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$. This order, called *Katětov order* was introduced by M. Katětov [21] in 1968 to study convergence in topological spaces. This study was continued by M. Daguenet in [8] but otherwise the Katětov order has remained mostly unstudied for more than 30 years. It has been used implicitly by J. Baumgartner in [3] to classify ultrafilters on ω. According to Baumgartner, given an ideal $\mathcal{I}$ on ω, a free ultrafilter $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for any $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ such that $f[U] \in I$. This is, of course, equivalent to saying the the dual ideal $\mathcal{U}^*$ is not above $\mathcal{I}$ in the Katětov order. Many, if not most, commonly used properties of ultrafilters can be characterized in this way [12, 4, 10, 14].

The Katětov order was used by Solecki in [31] characterized ideals satisfying *Fatou's lemma* (equivalently, ideals satisfying the *Fubini property*) as exactly those ideals no positive restriction of which is above a certain critical ideal $\mathcal{S}$ in the Katětov order. This F_σ ideal figures also in one of our dichotomies.

Our interest in the Katětov order stems from the study of destructibility of ideals by forcing [25, 11, 6, 16, 5, 15, 12, 17]. Downward cones of definable ideals in the Katětov order are of interest here.

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These considerations showed that for many combinatorial and forcing properties of filters and ideals on countable sets there are "critical" ideals with respect to the property, which are definable, in most cases even Borel of a low Borel complexity., i.e for a given property $\mathcal{P}$ there is a (Borel) ideal $\mathcal{I}_\mathcal{P}$ such that an ideal $\mathcal{J}$ has $\mathcal{P}$ if and only if $\mathcal{J} \not\leq_K \mathcal{I}_\mathcal{P}$, or such that an ideal $\mathcal{J}$ has $\mathcal{P}$ if and only if for any $X \in \mathcal{J}^+$ $\mathcal{I}_\mathcal{P} \not\leq_K \mathcal{J} \restriction X$. As we saw, both of these patterns appear naturaly. Therefore, understanding of the *Katětov order on Borel ideals* is crucial for a possible classification of non-definable objects such as ultrafilters [3, 12, 4, 10, 14] and maximal almost disjoint families [11, 6, 13, 12, 1], as both upward and downward cones of these ideals in the Katětov order naturally stratify and classify these non-definable objects. For further information consult [12].

This paper is devoted to basic structural analysis of the Katětov order on Borel ideals. We present two dichotomies for Borel ideals and analytic P-ideals, respectively. We also isolate nine important Borel ideals and study the Katětov order among them. We list several fundamental open problems concerning the order.

1. Preliminaries

1.1. **Filters and ideals.** A family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of a given set X is an *ideal* on X if it is closed under taking subsets and finite unions of its elements. We always assume that the ideal is *proper*, i.e., does not contain X, and contains all finite subsets of X. Dual is the notion of a *filter* on X, i.e. a family of subsets of X closed under taking finite intersections and supersets. Given an ideal $\mathcal{I}$ on X we denote by $\mathcal{I}^*$ the *dual filter*, consisting of complements of the sets in $\mathcal{I}$. Similarly if $\mathcal{F}$ is a filter on X, $\mathcal{F}^*$ denotes the dual ideal. We say an ideal $\mathcal{I}$ on X is *tall* if for each $Y \in [X]^\omega$ there exists $I \in \mathcal{I}$ such that $I \cap Y$ is infinite. Given an ideal $\mathcal{I}$ on a set X, we denote by $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets, i.e. subsets of X which are not in $\mathcal{I}$. If $\mathcal{I}$ is an ideal on X and $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on Y.

We will consider mostly ideals and filters on countable sets. In that case, we treat them as ideals or filters on ω. The set $\mathcal{P}(\omega)$ is equipped with the natural topology inherited form 2^ω with the product topology via characteristic functions. We say that an ideal or filter $\mathcal{X}$ is a Borel (analytic) ideal (resp. filter) on ω if $\mathcal{X}$ is Borel (analytic) in this topology. We let fin denote the ideal of finite subsets of ω.

1.2. **Basics of the Katětov order.** We will consider also the following variant of the Katětov order - the *Katětov-Blass order*: Given ideals $\mathcal{I}$ and $\mathcal{J}$ on ω, we say that $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$. We will say $\mathcal{I}$ and $\mathcal{J}$ are *Katětov-equivalent* ($\mathcal{I} \simeq_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$, and analogously for the Katětov-Blass order.

The basic properties of Katětov order are listed here. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

(1) $\mathcal{I} \simeq_K \mathsf{fin}$ if and only if $\mathcal{I}$ is not tall.
(2) If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$.
(3) If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_K \mathcal{I} \restriction X$.
(4) $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{I}, \mathcal{J}$.
(5) $\mathcal{I}, \mathcal{J} \leq_K \mathcal{I} \times \mathcal{J}$.

Here $\mathcal{I} \oplus \mathcal{J}$ denotes the disjoint sum of $\mathcal{I}$ and $\mathcal{J}$, and

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega : \{n : (A)_n \notin \mathcal{J}\} \in \mathcal{I}\}$$

denotes the *Fubini product* of the ideals. It is easy to see that both the disjoint sum and the Fubini product of Borel ideals are Borel ideals. Hence, the Katětov order on Borel ideals is both upward and downward directed. However, it seems to be an open problem whether the Katětov-Blass order on Borel ideals is upward directed.

The following theorem of D. Meza [28] shows that there is enough structure to be studied here.

Theorem 1.1 (D. Meza [28]). *There is an order embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into Borel ideals ordered by the Katětov order.*

An easy but useful consequence of the Shoenfield's absoluteness Theorem is that the Katětov order among Borel ideals is absolute.

An ideal $\mathcal{I}$ is said to be *K-uniform* if $\mathcal{I} \restriction X \leq_K \mathcal{I}$ (equivalently, $\mathcal{I} \restriction X \simeq_K \mathcal{I}$) for every $\mathcal{I}$-positive set X.

2. Some Borel ideals

In this section we present several Borel ideals, together with the properties they are critical for (see [12] for more information) and specify the relations they have in the Katětov order.

• The *nowhere dense ideal* nwd is the ideal on the set of rational numbers $\mathbb{Q}$ consisting of nowhere dense subsets of $\mathbb{Q}$. The ideal nwd is an $F_{\sigma\delta}$ ideal. An ideal $\mathcal{I}$ on ω is Cohen-indestructible[1] if and only if $\mathcal{I} \not\leq_K \mathsf{nwd}$.

• The *eventually different ideal* is defined by

$$\mathcal{ED} = \{A \subset \omega \times \omega : (\exists m, n \in \omega)(\forall k > n)\,(|\{l : \langle k, l\rangle \in A\}| \leq m)\}.$$

[1]Given a forcing notion $\mathbb{P}$, a tall ideal $\mathcal{I}$ is $\mathbb{P}$-*indestructible* if it remains tall after forcing with $\mathbb{P}$.

The ideal $\mathcal{ED}$ is critical for *selectivity* of ideals: Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{ED} \leq_K \mathcal{I}$ if and only if there is a partition of ω into sets in $\mathcal{I}$ such that every selector is in $\mathcal{I}$.

• We also consider the ideal $\mathcal{ED}_{fin} = \mathcal{ED} \restriction \triangle$, where $\triangle = \{\langle m, n\rangle : n \leq m\}$. It is critical among *Q-ideals*, in much the same way [18]: Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$ if and only if there is a partition of ω into finite sets such that every selector is in $\mathcal{I}$. Moreover, $\mathcal{ED}_{fin}$ is the KB-least ω-hitting[2] ideal among definable ideals.

• The ideal $\mathsf{fin} \times \mathsf{fin}$ is an $F_{\sigma\delta\sigma}$ ideal. It is critical with respect to the following P-like property: Given an ideal $\mathcal{I}$ on ω, $\mathcal{I} \geq_K \mathsf{fin} \times \mathsf{fin}$ if and only if there is a partition $\{Q_n : n < \omega\}$ of ω into sets in $\mathcal{I}$ such that every $A \subseteq \omega$ satisfying $|A \cap Q_n| < \aleph_0$ is in $\mathcal{I}$. □

• An ideal closely related to $\mathsf{fin} \times \mathsf{fin}$ is the ideal conv, defined as the ideal on $\mathbb{Q} \cap [0, 1]$ generated by sequences in $\mathbb{Q} \cap [0, 1]$ convergent in $[0, 1]$. The ideal conv is an $F_{\sigma\delta\sigma}$ ideal. Every conv-positive set contains a positive subset X such that $\mathsf{conv} \restriction X$ is naturally isomorphic to the ideal $\mathsf{fin} \times \mathsf{fin}$.

For an ideal $\mathcal{I}$ on ω, $\mathcal{I} \geq_K \mathsf{conv}$ if and only if there is a countable family $\mathcal{X} \subseteq [\omega]^\omega$ such that for every $Y \in \mathcal{I}^+$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.

Also, in [19], it is shown that, if $\mathcal{I}$ is an ideal on ω such that the quotient $\mathcal{P}(\omega)/\mathcal{I}$ is a proper forcing adding a new real, then there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_K \mathsf{conv}$.

• We denote by $\mathcal{R}$ the ideal on ω generated by the homogeneous sets (cliques and free sets) in Rado's *random graph*. The ideal $\mathcal{R}$ is a tall F_σ ideal such that given an ideal $\mathcal{I}$ on ω,

$$\omega \longrightarrow (\mathcal{I}^+)^2_2 \quad \text{if and only if} \quad \mathcal{I} \not\geq_K \mathcal{R}.$$ [3]

• The *Solecki ideal* $\mathcal{S}$ [31] is the ideal on the countable set

$$\Omega = \{A \in Clop(2^\omega) : \lambda(A) = \frac{1}{2}\},$$

generated by the sets of the form $I_x = \{A \in \Omega : x \in A\}$, $x \in 2^\omega$. Here λ denotes the standard Haar measure on 2^ω.

[2]Recall that an ideal $\mathcal{I}$ on ω is *ω-hitting* if for any countable family of infinite subsets of ω there is an element of $\mathcal{I}$ having infinite intersection with all of them.

[3]We write $\omega \longrightarrow (\mathcal{I}^+)^2_2$ to mean that for every $\varphi : [\omega]^2 \to 2$ there is an $\mathcal{I}$-positive φ-homogeneous set. Similarly, $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$ denotes that for every $\mathcal{I}$-positive set X and every coloring $\varphi : [X]^2 \to 2$ there is an $\mathcal{I}$-positive φ-homogeneous subset Y of X.

The ideal $\mathcal{S}$ is a tall F_σ ideal critical for the Fubini property[4] [31]: An ideal $\mathcal{I}$ fails to satisfy the Fubini property if and only if there is an $\mathcal{I}$-positive set X such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.

An ideal $\mathcal{I}$ on ω is a P-*ideal* if for any sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that $X_n \subseteq^* X$ for all $n \in \omega$, i.e. $X \setminus X_n$ is finite for all $n \in \omega$. The most common examples of analytic P-ideals are the summable ideal and the density zero ideal.

• The *summable ideal* is the ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}.$$

It is a tall F_σ P-ideal.

• The ideal $\mathcal{Z}$ of subsets of ω of *asymptotic density zero* is the ideal

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \tfrac{|A \cap n|}{n} = 0 \right\}.$$

Equivalently, $A \in \mathcal{Z}$ if and only if

$$\lim_{n\to\infty} \frac{|A \cap [2^n, 2^{n+1})|}{2^n} = 0.$$

The ideal $\mathcal{Z}$ is a tall $F_{\sigma\delta}$ P-ideal. We shall need the following fact

Proposition 2.1. *The ideal $\mathcal{Z}$ is K-uniform.*

Proof. Given $X \in \mathcal{Z}^+$ there is an $N > 0$ and a strictly increasing $h : \omega \to \omega$ such that for every $k \in \omega$

$$|X \cap [2^{h(k)}, 2^{h(k)+1})| \geq 2^{h(k)-N}.$$

Fix, for every $k \in \omega$, a set $F_k \subseteq X \cap [2^{h(k)}, 2^{h(k)+1})$ of size $2^{h(k)-N}$, and let $f : \omega \to \omega$ be such that for all $m, k \in \omega$ such that $h(k) \leq m < h(k+1)$ and every $l \in F_k$

$$f[[2^m, 2^{m+1})] = F_k \text{ and } |f^{-1}(l) \cap [2^m, 2^{m+1})| = 2^{m-h(k)+N}.$$

It is easy to see that if $Z \subseteq [2^{h(k)}, 2^{h(k)+1})$ has size at most $\varepsilon \cdot 2^{h(k)}$, i.e.

$$\frac{|Z|}{2^{h(k)}} \leq \varepsilon$$

then $|f^{-1}[Z]| \leq \varepsilon \cdot 2^{m+N}$, that is

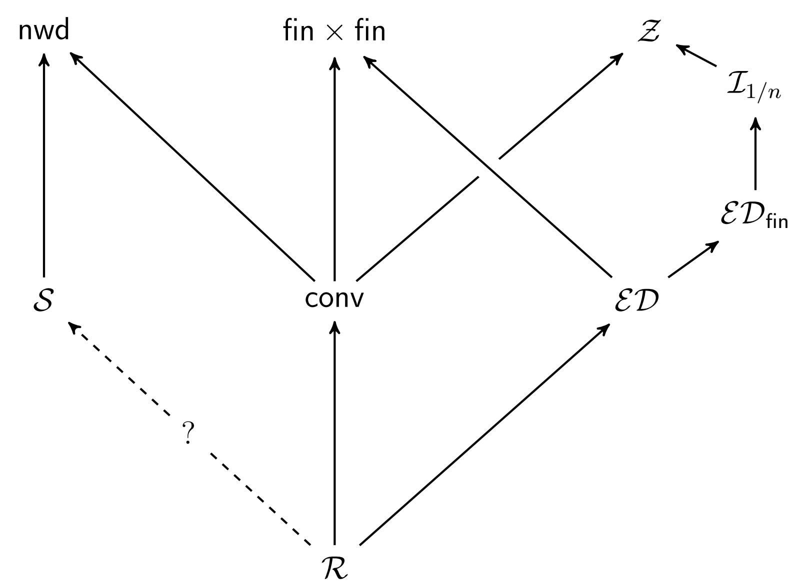
$$\frac{|f^{-1}[Z]|}{2^m} \leq \varepsilon \cdot 2^N.$$

From this it immediately follows that f witnesses $\mathcal{Z} \restriction X \leq_K \mathcal{Z}$. □

[4]An ideal $\mathcal{I}$ satisfies the *Fubini property* if for any Borel subset A of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda^*(A_n) > \varepsilon\} \in \mathcal{I}^+$ implies $\lambda^*(\{x \in 2^\omega : A^x \in \mathcal{I}^+\}) \geq \varepsilon$, where λ^* denotes the outer Lebesgue (Haar) measure on 2^ω.

We will be discussing analytic P-ideals in general and the asymptotic density zero ideal $\mathcal{Z}$ in particular in section 4.

The following diagram shows the complete picture (and one open problem) of these nine ideals in the Katětov order (a similar diagram was presented by Brendle and Flašková in [4]):



We will briefly sketch the reasons why the ideals are related.

- $\mathcal{R} \leq_K \mathcal{ED}, \mathsf{conv}$. For both conv and $\mathcal{ED}$ it is easy to define colorings of pairs for which all homogeneous sets are in the respective ideal. For conv enumerate $\mathbb{Q} \cap [0,1]$ as $\{q_n : n \in \omega\}$ and let $\varphi(\{q_m, q_n\}) = 0$ if $m < n \Leftrightarrow q_m < q_n$. For $\mathcal{ED}$ let $\varphi((m,n),(k,l)) = 0$ if and only if $m = k$.
- $\mathsf{conv} \leq_K \mathsf{fin} \times \mathsf{fin}$ and $\mathcal{ED} \leq_K \mathcal{ED}_{\mathsf{fin}}$. Follows from the fact that that the larger ideal is a restriction of the smaller to a positive set.
- $\mathsf{conv} \leq_K \mathsf{nwd}$, $\mathcal{ED} \leq_K \mathsf{fin} \times \mathsf{fin}$ and $\mathcal{I}_{\frac{1}{n}} \leq_k \mathcal{Z}$. Trivially, as the smaller ideal is contained in the larger one (i.e. the identity is the witnessing function).
- $\mathcal{ED}_{\mathsf{fin}} \leq_K \mathcal{I}_{\frac{1}{n}}$. Define $f : \omega \to \triangle$ by $f(2^m + k) = (2^m, k)$ for all $m, k \in \omega$ such that $k < 2^m$. It is easy to check that this works.
- $\mathsf{conv} \leq_K \mathcal{Z}$. Given $m, k \in \omega$ let $X^k_m = \{l \cdot 2^m + k : l \in \omega\}$. It is easy to verify that for any $\mathcal{Z}$-positive set Y there are $m, k \in \omega$

such that the sets $X^k_m \cap Y$ and $Y \setminus X^k_m$ are both infinite. By the characterization of $\mathsf{conv} \leq_K \mathcal{I}$, the result follows.

- $\mathcal{S} \leq_K \mathsf{nwd}$. Let T be the tree consisting of all finite decreasing sequences of clopen subsets of 2^ω of measure $> 1/2$. The ideal nwd is isomorphic to the ideal

$$\mathsf{nwd}' = \{A \subseteq T : \forall s \in T\ \exists t \in T\ s \subseteq t\ \&\ A \cap \{r \in T : t \subseteq r\} = \emptyset\}.$$

  Let $f : T \to \Omega$ be such that $f(s) \subseteq \bigcap_{i<|s|} s(i)$. It is easy to verify that $f^{-1}[I_x] \in \mathsf{nwd}'$ for every $x \in 2^\omega$, and the result follows.
- It is an open problem whether $\mathcal{R} \leq_K \mathcal{S}$.

Let us trurn to the negative results now.

- None of $\mathcal{S}, \mathcal{R}, \mathcal{ED}, \mathcal{ED}_{\mathsf{fin}}$ and $\mathcal{I}_{\frac{1}{n}}$ is Katětov above conv (and hence also not above $\mathsf{nwd}, \mathsf{fin} \times \mathsf{fin}$ and $\mathcal{Z}$). To see this, note that by the characterization of $\mathsf{conv} \leq_K \mathcal{I}$, no F_σ ideal is Katětov above conv. As all of $\mathcal{S}, \mathcal{R}, \mathcal{ED}, \mathcal{ED}_{\mathsf{fin}}$ and $\mathcal{I}_{\frac{1}{n}}$ are F_σ, the result follows.
- None of the ideals in the diagram is above $\mathsf{fin} \times \mathsf{fin}$. This can be easily proved directly, but it can also be proved combining the results of Solecki [31], and Recław-Laczkovich [24] to get that no $F_{\sigma\delta}$ ideal is Katětov above $\mathsf{fin} \times \mathsf{fin}$. Recall that both nwd and $\mathcal{Z}$ are $F_{\sigma\delta}$.
- $\mathcal{ED} \not\leq_K \mathsf{nwd}$. This follows directly form the observation that if one partitions $\mathbb{Q}$ into nowhere dense sets, there is a dense selector.
- Neither $\mathsf{fin} \times \mathsf{fin}$ nor $\mathcal{Z}$ are above $\mathcal{S}$. This follows from results of V. Kanovei and M. Reeken [20] that both $\mathsf{fin} \times \mathsf{fin}$ and $\mathcal{Z}$ satisfy the Fubini property, and a result of Solecki [31] that any filter Katětov above $\mathcal{S}$ fail to have the Fubini property[5]. In particular, none of the ideals in the diagram is above nwd (this follows from the statement for all the ideals other than $\mathcal{S}$, which has been taken care of above).
- $\mathcal{ED}_{\mathsf{fin}} \not\leq_K \mathsf{fin} \times \mathsf{fin}$. This can be proved directly form the definitions, but we give a short proof involving forcing. Note that if $\mathcal{I} \leq_K \mathcal{J}$ and a forcing $\mathbb{P}$ destroys $\mathcal{J}$ then it also destroys $\mathcal{I}$. Also note that any forcing adding a dominating real destroys $\mathsf{fin} \times \mathsf{fin}$. On the other hand, the ideal $\mathcal{ED}_{\mathsf{fin}}$ is ω-hitting, and there are forcing notions (e.g. Laver, Hechler) which add dominating

[5]Recall that according to Kanovei and Reeken [20] a Borel ideal $\mathcal{I}$ satisfies the *Fubini property* if for any Borel subset A of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda((A)_n) > \varepsilon\} \in \mathcal{I}^+$ implies $\lambda(\{x \in 2^\omega : (A)^x \in \mathcal{I}^+\}) \geq \varepsilon)$.

reals while preserving ω-hitting families. In particular, these forcing notions do not destroy $\mathcal{ED}_{\mathsf{fin}}$, hence $\mathcal{ED}_{\mathsf{fin}} \not\leq_K \mathsf{fin} \times \mathsf{fin}$.

From this (and previous observations) it immediately follows that none of the ideals is above $\mathcal{Z}$ and also that $\mathcal{ED}_{\mathsf{fin}} \not\leq_K \mathcal{ED}$.

- $\mathcal{I}_{\frac{1}{n}} \not\leq_K \mathcal{ED}_{\mathsf{fin}}$. To prove this let $f : \triangle \to \omega$ be given. We can assume that $f^{-1}(n) \in \mathcal{ED}_{\mathsf{fin}}$, otherwise a singleton shows that f is not a witness to $\mathcal{I}_{\frac{1}{n}} \leq_K \mathcal{ED}_{\mathsf{fin}}$. Now, for each $k \in \omega$ choose $n_k \in \omega$ and $F_k \subseteq \{n_k\} \times n_k + 1$ of size k such that $\min f[F_k] \geq k \cdot 2^k$. To do this is easy. Then let $X = \bigcup_{k\in\omega} F_k$ and note that $X \in \mathcal{ED}^+_{\mathsf{fin}}$, while $f[X] \in \mathcal{I}_{\frac{1}{n}}$. Hence f is not a witness to $\mathcal{I}_{\frac{1}{n}} \leq_K \mathcal{ED}_{\mathsf{fin}}$.

With a little bit of patience one can directly deduce from these results, that there are no arrows missing in the diagram (with the posible exception of $\mathcal{R} \leq_K \mathcal{S}$).

Let us also remark, that as far as the ideals involved are concerned, there is no difference between Katětov order and the Katětov-Blass order, in fact, all the witnessing functions can be chosen to be one-to-one.

3. The Category dichotomy

In this section we will prove the following structural theorem for Borel ideals announced in [12].

Theorem 3.1 (Category Dichotomy)**.** *Let $\mathcal{I}$ be a Borel ideal. Then either $\mathcal{I} \leq_K \mathsf{nwd}$ or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$.*

Proof. The proof uses the following game $G(\mathcal{I})$ (introduced by Laflamme in [26]) associated to an ideal $\mathcal{I}$: At stage k of the game Player I chooses an element I_k of $\mathcal{I}$ and Player II responds by choosing an $n_k \in \omega \setminus I_k$.

I	$I_0 \in \mathcal{I}$		$I_1 \in \mathcal{I}$		$\dots$	
II		$n_0 \notin I_0$		$n_1 \notin I_1$		$\dots$

Player I wins if $\{n_k : k < \omega\} \in \mathcal{I}$, otherwise Player II wins.

By Borel determinacy (see [22]), it is sufficient to prove:

(1) if for every $\mathcal{I}$-positive set X Player II has a winning strategy in the game $G(\mathcal{I} \restriction X)$ then $\mathcal{I} \leq_K \mathsf{nwd}$, and
(2) if there is an $\mathcal{I}$-positive set Y such that Player I has a winning strategy for $G(\mathcal{I} \restriction Y)$ then there is an $\mathcal{I}$-positive set $X \subseteq Y$ such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$.

Let us deal with case (1) first.

Claim 3.2. *The following are equivalent:*

(1) *Player II has a winning strategy in the game* $G(\mathcal{I})$,
(2) *there is an* $\mathcal{I}^+$*-branching tree* $S \subseteq \omega^{<\omega}$ *such that* $rng(x) \in \mathcal{I}^+$ *for every* $x \in [S]$, *and*
(3) *there is a pairwise disjoint family* $\{X_n : n < \omega\} \subseteq \mathcal{I}^+$ *such that for every* $I \in \mathcal{I}$ *there is* $n < \omega$ *such that* $X_n \cap I = \emptyset$.

Proof of the claim. To see that (1) implies (2), let τ be a winning strategy for Player II. Construct the tree S recursively, for each $s \in S$ simultaneously fixing a sequence $\langle I_j : j < |s|\rangle$ so that:

(1) the sequence $\langle I_0, s(0), I_1, s(1), \ldots, I_{|s|-1}, s(|s|-1)\rangle$ is a partial legal play of the game for every $s \in S$, in which the Player II plays according to τ, i.e. $s(j) = \tau(\langle I_0, s(0), I_1, s(1), \ldots, I_j\rangle)$ for all $j < |s|$, and
(2) $s^\frown\tau(\langle I_0, s(0), I_1, s(1), \ldots, I_{|s|-1}, s(|s|-1), I\rangle) \in S$ for every $s \in S$ and $I \in \mathcal{I}$.

The tree S is then $\mathcal{I}^+$-branching by (2) and $rng(x) \in \mathcal{I}^+$ for every $x \in [S]$, as x is a run of the game played according to τ, hence winning for Player II.

Ad (2) implies (3): Having fixed a tree as in (2), enumerate S as $\langle s_n : n < \omega\rangle$ and let $Y_n = succ_S(s_n)$. Let $\{X_n : n \in \omega\}$ be a disjoint refinement of $\{Y_n : n \in \omega\}$, consisting of $\mathcal{I}^+$-positive sets (such a refinement exist as the ideal $\mathcal{I}$ is Borel, hence hereditarily meager, by the Talagrand–Jalali-Naini theorem [2]). Then every set which intersects all of the X_n's is positive, as it contains the range of a branch of S.

For the same reason (3) provides a winning strategy for Player II in the game $G(\mathcal{I})$: She shall pick $n_k \in X_k \setminus I_k$. □

Using the claim we can prove (1): Suppose that Player II has a winning strategy in $G(\mathcal{I} \restriction X)$ for every $\mathcal{I}$-positive set X. We shall show that $\mathcal{I} \leq_K \mathsf{nwd}$.

Using the claim repeatedly one can construct sets $\{X_s : s \in \omega^{<\omega}\}$ such that

(1) $X_\emptyset = \omega$,
(2) $\{X_{s^\frown n} : n < \omega\}$ is a partition of X_s into $\mathcal{I}$-positive sets,
(3) for any $s \in \omega^{<\omega}$ and $I \in \mathcal{I}$ there is $n < \omega$ such that $I \cap X_{s^\frown n} = \emptyset$,
(4) for every pair of integers $n \neq m \in \omega$ there is an $s \in \omega^{<\omega}$ such that $|X_s \cap \{m, n\}| = 1$.

Let τ be the topology on ω generated by $\{X_s : s \in \omega^{<\omega}\}$. The space $\langle\omega, \tau\rangle$ is then a countable Hausdorff second countable and zero-dimensional topological space without isolated points, so by Sierpiński's theorem (see [22]), it is homeomorphic to $\mathbb{Q}$. Every $I \in \mathcal{I}$ is nowhere dense in τ since for any basic open set X_s there is $n < \omega$ such that $I \cap X_{s^\frown n} = \emptyset$. Hence, any homeomorphism φ between $\langle \omega, \tau \rangle$ and $\mathbb{Q}$ is the Katětov function requested.

Before we turn to case (2) we again first give a combinatorial reformulation of the existence of a winning strategy for Player I.

Claim 3.3. *Player I has a winning strategy in $G(\mathcal{I})$ if and only if there is an $\mathcal{I}^*$-branching tree $T \subseteq \omega^{<\omega}$ such that $rng(x) \in \mathcal{I}$ for every $x \in [T]$.*

Proof of the claim. If σ is a winning strategy for Player I, let T be the tree of all possible responses by Player II to Player I following σ. The tree is obviously as required.

On the other hand, given an $\mathcal{I}^*$-branching tree $T \subseteq \omega^{<\omega}$ such that $rng(x) \in \mathcal{I}$ for every $x \in [T]$, we can define a strategy σ as follows: Let $\sigma(\emptyset) = \omega \setminus \{n < \omega : \langle n \rangle \in T\}$, and if $k < \omega$ and a sequence $\langle I_0, n_0, \ldots, n_{k-1}, I_k \rangle$ is played following σ then for all $l \notin I_k$ put $\sigma(\langle I_0, n_0, \ldots, I_k, l \rangle) = \omega \setminus succ_T(\langle n_0, \ldots, n_{k-1}, l \rangle)$.

It is then clear that if $\langle I_0, n_0, I_1, n_1, \ldots \rangle$ follows σ then $\langle n_k : k < \omega \rangle$ follows a branch of T, so it is in $\mathcal{I}$. □

To finish the proof we will show that if Y is an $\mathcal{I}$-positive set such that Player I has a winning strategy for $G(\mathcal{I} \restriction Y)$ then there is an $\mathcal{I}$-positive set $X \subseteq Y$ such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$.

By the last claim, there is an $(\mathcal{I} \restriction Y)^*$-branching tree T with all branches enumerating elements of $\mathcal{I}$. Let $\langle s_n : n < \omega \rangle$ be an enumeration of T and let $Y_n = succ_T(s_n)$, for each $n \in \omega$.

Case 1. *The family $\{Y_n : n \in \omega\}$ does not have an $\mathcal{I}$-positive pseudointersection.* Let $I_0 = Y \setminus Y_0$ and $I_{n+1} = (\bigcap_{k \leq n} Y_k) \setminus Y_{n+1}$ for all $n < \omega$, and let $X = Y$. Note that $\{I_n : n < \omega\}$ is a partition of X into elements of $\mathcal{I}$, and any $I \subseteq X$ such that $|I \cap I_n| < \omega$ for all n is in $\mathcal{I}$ since it is a pseudointersection of the family $\langle Y_n : n < \omega \rangle$. So, in Case 1, we proved that $\mathcal{I} \restriction X \geq_K \mathsf{fin} \times \mathsf{fin} \geq_K \mathcal{ED}$.

Case 2. *The family $\{Y_n : n < \omega\}$ does have an $\mathcal{I}$-positive pseudointersection.* Let X be such a pseudointersection and define a strictly increasing function $g : \omega \to X$ such that for any $t \in T$, if $rng(t) \subseteq X \cap g(n)$ then $X \setminus g(n+1) \subseteq succ_T(t)$. To do this let $g(0) = \min X$ and for every $n < \omega$ let $W_n = \bigcap \{succ_T(t) : rng(t) \subseteq X \cap g(n)\}$. Clearly $W_n \in \mathcal{I}^*$ and $X \subseteq^* W_n$. $g(n+1) = \min\{k > g(n) : X \setminus k \subseteq W_n\}$.

Let $A = \bigcup_{n<\omega} [g(2n), g(2n+1))$ and $B = \bigcup_{n<\omega} [g(2n+1), g(2n+2))$. Note that if S is a selector of the partition$\{[g(n), g(n+1)) : n < \omega\}$ of X, and we split $S = (S \cap A) \cup (S \cap B)$, then $S \cap A$ and $S \cap B$ are in $\mathcal{I}$ since the enumerating function of each of them forms a branch through

T. Since $\mathcal{I} \geq_K \mathcal{ED}_{fin}$ is equivalent to the existence of a partition $\{I_n : n < \omega\}$ into finite sets such that every selector is in $\mathcal{I}$, we have proved the theorem. □

Note that the proof of the theorem actually produces a trichotomy: *For every Borel ideal $\mathcal{I}$ either $\mathcal{I} \leq_K \mathsf{nwd}$ or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_K \mathsf{fin} \times \mathsf{fin}$ or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_{KB} \mathcal{ED}_{fin}$.*

Seemingly the proof also gives a stronger result in that the ideal nwd can be replaced by the ideal of *porous* set defined as follows: Identify $\mathbb{Q}$ a countable dense subset of ω^ω, say with the set of all functions which are eventually 0, let $\langle s \rangle = \{f \in \omega^\omega : s \subseteq f\}$ be the cone determined by $s \in \omega^{<\omega}$, and define

$$\mathsf{por} = \{A \subseteq \mathbb{Q} : \ \forall s \in \omega^{<\omega} \ \exists n \in \omega \ A \cap \langle s^\frown n \rangle = \emptyset\}.$$

However, as $\mathsf{por} \simeq_K \mathsf{nwd}$ the "stronger" result is equivalent to the theorem stated.

Let us also point out the obvious, the theorem as stated does not produce a real dichotomy in the sense that the two alternatives are not mutually exclusive: for instance both $\mathcal{R}$ and $\mathcal{S}$ satisfy both alternatives. Should one be interested in a "true" dichotomy, it would be stated as follows

Corollary 3.4. *Let $\mathcal{I}$ be a Borel ideal. Then either $\mathcal{I} \restriction X \leq_K \mathsf{nwd}$ for every $\mathcal{I}$-positive set X or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$.*

4. Measure dichotomy

In this section we present a dichotomy for analytic P-ideals similar in form to the Category dichotomy. It is inspired by Christensen's result [7] linking the Fubini property to non-pathologicity for submeasures on atomless Boolean algebras.

Theorem 4.1 (Measure Dichotomy)**.** *Let $\mathcal{I}$ be an analytic P-ideal. Then, either $\mathcal{I} \leq_K \mathcal{Z}$ or there is $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.*

Before we delve into the proof we recall the connection there is between analytic P-ideals, and lower semicontinuous submeasures (lscsm).[6] To each lscsm φ on ω naturally correspond the following two ideals:

---

[6] Recall that a *submeasure on* ω is a function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ such that (1) $\varphi(\emptyset) = 0$, (2) if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$, and (3) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. To avoid trivialites, we also require that $\varphi(F) < \infty$ for all finite subsets of ω. The submeasure φ is *lower semicontinuous* if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for every $A \subseteq \omega$.

- $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ and
- $Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$.

It is immediate from the definition that $Exh(\varphi) \subseteq Fin(\varphi)$, $Fin(\varphi)$ is an F_σ-ideal and $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal. A theorem of Mazur [27] asserts that every F_σ-ideal is of the form $Fin(\varphi)$. For analytic P-ideals there is the following fundamental result:

Theorem 4.2 (Solecki [29, 30])**.** *Let $\mathcal{I}$ be an ideal on ω. Then:*

- *$\mathcal{I}$ is an analytic P-ideal if and only if there is a lscsm φ such that $\mathcal{I} = Exh(\varphi)$.*
- *$\mathcal{I}$ is an F_σ P-ideal if and only if there is a lscsm φ such that $\mathcal{I} = Exh(\varphi) = Fin(\varphi)$.*

A submeasure φ on a set X is *non-pathological* if for every $A \subseteq X$

$$\varphi(A) = \hat{\varphi}(A) =_{def} \sup\{\mu(A) : \mu \text{ is a measure on } X \text{ dominated by } \varphi\}.$$

Following Farah [9] we say that an analytic P-ideal $\mathcal{I}$ on ω is *non-pathological* if there is a lscsm φ such that $\mathcal{I} = Exh(\varphi) = Exh(\hat{\varphi})$.

We define the *degree of pathology* of a submeasure φ on X such that $\varphi(X) < \infty$ by

$$P(\varphi) = \frac{\varphi(X)}{\sup\{\mu(X) : \mu \text{ is a measure dominated by } \varphi\}}.$$

We shall be dealing with normalized submeasures. Recall, that a (sub)measure φ on a set X is *normalized* if $\varphi(X) = 1$.

Given a family $\mathcal{B} \subseteq \mathcal{P}(F)$ of subsets of a set F, the *Kelley's covering number* [23] of $\mathcal{B}$ is defined as follows: For any finite sequence $S = \langle S_0, \dots S_n\rangle$ of (not necessarily distinct) elements of $\mathcal{B}$ let

$$m(S) = \min\{|\{i \leq n : x \in S_i\}| : x \in F\}.$$

The *covering number* $C(\mathcal{B})$ is defined as

$$C(\mathcal{B}) = \sup\left\{\frac{m(S)}{|S|} : S \in \mathcal{B}^{<\omega}\right\}.$$

The theorem of Kelley which links the covering number with measures is the following.

Theorem 4.3 (Kelley [23])**.** *Given a family $\mathcal{B} \subseteq \mathcal{P}(F)$ of subsets of a set F, the covering number $C(\mathcal{B})$ is the minimum of the numbers $\sup\{\mu(A) : A \in \mathcal{B}\}$, where the minimum is taken over all normalized measures μ on $\mathcal{P}(F)$.* □

Using Kelley's theorem,we shall prove the following lemma, which can be seen as a finitary quantitative version of a theorem of Christensen [7] showing that a submeasure φ on an atomless Boolean algebra is pathological if and only if the Fubini theorem for φ fails.

Lemma 4.4. *Let F be a finite set, $\varepsilon > 0$, φ a normalized submeasure on $\mathcal{P}(F)$ and $\mathcal{A}_\varepsilon = \{A \subseteq F : \varphi(A) < \varepsilon\}$. Then*

$$C(\mathcal{A}_\varepsilon) \geq 1 - \frac{1}{\varepsilon P(\varphi)}.$$

Proof. Fix F, φ, and ε. By Kelley's theorem, it suffices to show that for all normalized measures μ on F there is a set $A \in \mathcal{A}_\varepsilon$ such that $\mu(A) \geq 1 - \frac{1}{\varepsilon P(\varphi)}$, i.e., given a normalized measure μ on F there is a set $A \in \mathcal{A}_\varepsilon$ such that $\mu(F \setminus A) \leq \frac{1}{\varepsilon P(\varphi)}$.

Having fixed such μ, let $\psi = \varphi - \varepsilon\mu$ and note that if A and B are disjoint subsets of F then $\psi(A \cup B) \leq \psi(A) + \psi(B)$. Let $\mathcal{F}$ be a maximal disjoint family of subsets B of F such that $\psi(B) < 0$, and let $A = \bigcup \mathcal{F}$. Then

(1) $\varepsilon\mu \restriction \mathcal{P}(F \setminus A) \leq \varphi \restriction \mathcal{P}(F \setminus A)$, and
(2) $\varphi(B) < \varepsilon\mu(B)$ for all $B \in \mathcal{F}$.

Let $\widehat{\varepsilon\mu}(C) = \varepsilon\mu(C \setminus A)$. $\widehat{\varepsilon\mu}$ is then a measure on F supported by $F \setminus A$ such that $\widehat{\varepsilon\mu} \leq \sup\{\nu(F) : \nu \text{ is a measure dominated by } \varphi\}$. By (1), $\varepsilon\mu(F \setminus A) \leq \frac{1}{P(\varphi)}$, while (2) implies that

$$\varphi(A) \leq \sum_{B \in \mathcal{F}} \varphi(B) < \sum_{B \in \mathcal{F}} \varepsilon\mu(B) = \varepsilon\mu(A) \leq \varepsilon.$$

□

In this context, the Kelley's covering number "measures" the failure of the Fubini theorem: $C(\mathcal{A}_\varepsilon) > \delta$ if and only if there is an $N < \omega$ and there is a set $A \subseteq F \times N$ such that all horizontal sections of A have submeasure $< \varepsilon$ while all vertical sections have normalized counting measure $> \delta$.

It can be easily seen, that the finite set N can be replaced by the Cantor set and the counting measure by the Haar measure. Interpreted in this way, the lemma says that "the more pathological is the submeasure, the worse the Fubini theorem for φ fails". We are now ready to prove the Measure Dichotomy.

Proof. Let $\mathcal{I}$ be an analytic ideal and let φ be a lower semicontinuous submeasure such that $\mathcal{I} = Exh(\varphi)$. Without loss of generality we can assume that $|\varphi| = \lim_{n\to\infty} \varphi(\omega \setminus n) > 1$. Partition ω into intervals $\langle F_n : n < \omega\rangle$ so that for any $n < \omega$, $\min(F_{n+1}) = \max(F_n) + 1$, $\varphi(F_n) \geq 1$, and $\varphi(F_n \setminus \{\max F_n\}) < 1$. Let φ_n be the normalization of $\varphi \restriction F_n$ (by multiplying $\varphi \restriction F_n$ by $\frac{1}{\varphi(F_n)}$) and let $r_n = P(\varphi_n) = P(\varphi \restriction F_n)$ be the degree of pathologicity of φ_n for $n < \omega$.

Case 1. *The sequence $\langle r_n : n < \omega\rangle$ is unbounded.*

Let $\langle r_{n_k} : k < \omega\rangle$ be a subsequence of the sequence $\langle r_n : n < \omega\rangle$ such that $r_{n_k} \geq 3 \cdot 2^{k+1}$ for every $k \in \omega$, and let $X = \bigcup_{k<\omega} F_{n_k}$. It is clear that $X \in \mathcal{I}^+$. We will show that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.

Let $\varepsilon_k = 2^{-k-1}$, and $\mathcal{A}_{\varepsilon_k} = \{A \subseteq F_{n_k} : \varphi_{n_k}(A) < \varepsilon_k\}$. By lemma 4.4,

$$C(\mathcal{A}_{\varepsilon_k}) \geq \frac{2}{3},$$

for any $k < \omega$. That is, for every $k < \omega$, there is a sequence $A^k = \langle A^k_0, \dots, A^k_{N_k-1}\rangle \subseteq \mathcal{A}_{\varepsilon_k}$ such that for all $x \in F_{n_k}$

$$|\{i < N_k : x \in A_i\}| \geq \frac{2}{3} N_k.$$

Now, for every k pick a pairwise disjoint sequence $\langle U^k_i : i < N_k\rangle$ of open subsets of 2^ω such that $\lambda(U^k_i) = \frac{1}{N_k}$. For any $x \in F_{n_k}$ define $W_x = \bigcup\{U^k_i : x \in A^k_i\}$. Note that $\mu(W_x) \geq \frac{2}{3}$. Every W_x contains infinitely many elements of Ω, so for every $x \in X$ one can choose an element U_x of Ω contained in W_x, such that $U_x \neq U_y$ for all $y \neq x$. Putting $f(x) = U_x$ defines then a one-to-one function from X to Ω such that for all $z \in 2^\omega$ and $k < \omega$ there is at most one $i < N_k$ such that $z \in U^k_i$. Hence, given $z \in 2^\omega$, we have that

$$\varphi_{n_k}(\{x \in F_{n_k} : z \in f(x)\}) \leq \varphi_{n_k}(\{x \in F_{n_k} : z \in W_x\}) \leq \\ \leq \varphi_{n_k}(A^k_i) < \frac{\varepsilon_k}{\varphi(F_{n_k})},$$

hence,

$$\varphi(f^{-1}[I_z]) \leq \sum_{k<\omega} \varphi(f^{-1}[I_z] \cap F_{n_k}) = \\ = \sum_{k<\omega} [\varphi(F_{n_k}) \cdot \varphi_{n_k}(f^{-1}[I_z] \cap F_{n_k})] < \sum_{k<\omega} \frac{1}{2^{k+1}} < \infty.$$

for every subbasic set $I_z = \{C \in \Omega : z \in C\}$. Actually, the formula shows that $f^{-1}[I_z] \in Exh(\varphi)$, therefore f witnesses $\mathcal{S} \leq_{KB} \mathcal{I} \restriction X$.

Case 2. *The sequence $\langle r_n : n < \omega\rangle$ is bounded.*

Assume that the sequence $\langle r_n : n < \omega\rangle$ is bounded by $r < \infty$. Then, for every $n < \omega$, there is a measure μ_n on F_n bounded by φ such that

$\frac{\varphi(F_n)}{\mu_n(F_n)} \leq r$. Given $A \subseteq \omega$, let:

$$\psi(A) = \sup\{\mu_n(A \cap F_n) : n < \omega\}.$$

ψ is then a lower semicontinuous submeasure on $\mathcal{P}(\omega)$ bounded by φ, hence, $Exh(\varphi) \subseteq Exh(\psi)$. Moreover, $\omega \notin Exh(\psi)$ since $\psi(\omega \setminus n) \geq \frac{1}{r}$ for all $n < \omega$.

We will show that there is $Y \in \mathcal{Z}^+$ such that $Exh(\psi) \leq_K \mathcal{Z} \upharpoonright Y$. Let $\langle M_n : n < \omega\rangle$ be a sequence of natural numbers such that $2^{M_n - n - 2} > |F_n|$ and let $\{A_x : x \in F_n\}$ be a family of pairwise disjoint subsets of $[2^{M_n}, 2^{M_n+1})$ such that for any $x \in F_n$:

$$2^{M_n}\frac{\mu_n(\{x\})}{\mu_n(F_n)} - \frac{2^{M_n-n-1}}{|F_n|} < |A_x| < 2^{M_n}\frac{\mu_n(\{x\})}{\mu_n(F_n)} + \frac{2^{M_n-n-1}}{|F_n|}.$$

Finding such a family is easy as $2^{M_n-n-2} > |F_n|$. Then, for any $x \in F_n$,

$$\left|\frac{|A_x|}{2^{M_n}} - \frac{\mu_n(\{x\})}{\mu_n(F_n)}\right| \leq 2^{-n-1},$$

and so, the normalized counting-measure in $[2^{M_n}, 2^{M_n+1})$ is a 2^{-n-1}-approximation to μ_n in F_n, for all $n < \omega$. Let $Y = \bigcup_{n<\omega}\bigcup_{x\in F_n} A_x$. Y is a $\mathcal{Z}$-positive set since $Y \setminus k$ is 2^{-m}-approximated to $\bigcup_{n\geq m}[2^{M_n}, 2^{M_n+1})$, where $m = \min\{l : k < 2^{M_l}\}$. Defining $f : Y \to \omega$ by putting

$$f(y) = x \text{ iff } y \in A_x.$$

The function f witnesses $\mathcal{I} \leq_K \mathcal{Z} \upharpoonright Y$. To see this, take $B \in Exh(\psi)$. Then $f^{-1}[B] = \bigcup_{x\in B} A_x$ intersects every interval $[2^{M_n}, 2^{M_n+1})$ in a set whose cardinality is 2^{-n-1}-approximated by $\varphi(B \cap F_n)$. Hence

$$\lim_{n\to\infty} \frac{|f^{-1}[B] \cap [2^{M_n}, 2^{M_n+1})|}{2^{M_n}} = 0.$$

Since $\mathcal{Z}$ is a K-uniform ideal (proposition 2.1) we conclude that

$$\mathcal{I} \leq_K \mathcal{Z} \upharpoonright Y \leq_K \mathcal{Z}.$$

□

A direct consequence of the proof is that $\mathcal{Z}$ is the largest non-pathological analytic P-ideal in the Katětov order. Again, if one wants a "true" dichotomy it would be phrased as

Proposition 4.5. *Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{I} \upharpoonright X \leq_K \mathcal{Z}$ for every $\mathcal{I}$-positive set X or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \upharpoonright X \geq_K \mathcal{S}$.*

with the following immediate consequence

Corollary 4.6. *For an analytic P-ideal $\mathcal{I}$ the following are equivalent:*

(a) $\mathcal{I} \restriction X \leq_K \mathcal{Z}$ *for every* $\mathcal{I}$*-positive set* X*,*
(b) $\mathcal{S} \not\leq_K \mathcal{I} \restriction X$*, for every* $\mathcal{I}$*-positive set* X*,*
(c) $\mathcal{I}$ *has the Fubini property and*
(d) $\mathcal{I}$ *is non-pathological.*

5. More on the Katětov order

The research on the structure of the Katětov order on Borel ideals is still only beginning. We do not know the answers to many fundamental questions. Perhaps the most important is the following:

Question 5.1. *Is there a tall Borel ideal Katětov-minimal among tall Borel ideals?*

This is, of course, equivalent to asking whether the Katětov order restricted to tall Borel ideals is $\mathfrak{c}$-downwards closed. We conjecture the answer to be negative, while the following question seems more likely to have a positive solution:

Question 5.2. *Is there a Borel tall ideal* $\mathcal{J}$ *such that for every Borel tall ideal* $\mathcal{I}$ *there is an* $\mathcal{I}$*-positive set* X *such that* $\mathcal{J} \leq_K \mathcal{I} \restriction X$*?*

We call such an ideal $\mathcal{J}$ *locally K-minimal.* There is a natural candidate, the ideal $\mathcal{R}$. Recall that for an ideal $\mathcal{I}$ there is an $\mathcal{I}$-positive set X such that $\mathcal{R} \leq_K \mathcal{I} \restriction X$, if and only if $\mathcal{I}^+ \not\to (\mathcal{I}^+)^2_2$, i.e., there is a coloring of pairs of elements of an $\mathcal{I}$-positive with all homogeneous subsets in $\mathcal{I}$.

A partial answer to the question is the following:

Theorem 5.3. [19] *Let* $\mathcal{I}$ *be a tall Borel ideal on* ω *such that* $\mathcal{P}(\omega)/\mathcal{I}$ *is proper. Then there is an* $\mathcal{I}$*-positive set* X *such that* $\mathcal{I} \restriction X \geq_K \mathcal{R}$.

In particular, $\mathcal{R}$ is locally K-minimal among tall F_σ ideals. On the other hand, there are even F_σ ideals which are not Katětov above $\mathcal{R}$, so $\mathcal{R}$ is not K-minimal among F_σ ideals. In [19] a co-analytic ideal $\mathcal{I}$ is described such that $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$, i.e., $\mathcal{R}$ is not locally K-minimal among co-analytic ideals.

Question 5.4. *Is there a tall Borel ideal* $\mathcal{I}$ *such that* $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$ *?*

For the record, we repeat the question mentioned in section 2.

Question 5.5. *Is* $\mathcal{R} \leq_K \mathcal{S}$*?*

Another problem of Ramsey theoretic flavour asks whether F_σ ideals are co-initial among tall Borel ideals:

Question 5.6. *Does every tall Borel ideal contain a tall* F_σ *sub-ideal?*

A somewhat dual problem is to recognize those Borel ideals that can be extended to F_σ ideals. This is in part answered by the following result [7]:

Theorem 5.7. [19] *Let $\mathcal{I}$ be a Borel ideal on ω. Then the following are equivalent:*

1. *there is an F_σ ideal $\mathcal{J}$ containing $\mathcal{I}$, and*
2. *there is a P^+-ideal $\mathcal{K}$ containing $\mathcal{I}$.*

Question 5.8. *Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\mathcal{I} \geq_K$ conv or there is an F_σ-ideal $\mathcal{J}$ containing $\mathcal{I}$?*

An approximation to this conjecture is the following result.

Theorem 5.9. [28] *Let $\mathcal{I}$ be a Borel ideal such that the forcing $\mathcal{P}(\omega)/\mathcal{I}$ is proper. Then, either there is an $\mathcal{I}$-positive set X such that $\mathsf{conv} \leq_K$ $\mathcal{I} \restriction X$ or there is an F_σ-ideal $\mathcal{J}$ containing $\mathcal{I}$.*

A similar problem is to characterize those Borel ideals that can be extended to an $F_{\sigma\delta}$ ideal:

Question 5.10. *Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\mathcal{I} \geq_K$ $\mathsf{fin} \times \mathsf{fin}$ or there is an $F_{\sigma\delta}$ ideal $\mathcal{J}$ containing $\mathcal{I}$?*

It is somewhat surprising that we do not have many examples of Borel ideals $\mathcal{I}$ such that the forcing $\mathcal{P}(\omega)/\mathcal{I}$ is not proper (see [16]), while known proofs of properness of the quotients $\mathcal{P}(\omega)/\mathcal{I}$ are somewhat *ad hoc* arguments. It would be good to understand the situation better.

Tall F_σ ideals tend to be rather non-homogeneous. In fact, so much so that even the following is not known:

Question 5.11. *Is $\mathcal{ED}_{fin}$ the only tall F_σ ideal which is K-uniform?*

The last several questions deal with J. Baumgartners notion of an $\mathcal{I}$-ultrafilter.

Question 5.12. *Is there a Borel ideal $\mathcal{I}$ such that $\mathcal{I}$-ultrafilters exist in* ZFC*? Is there a $\mathcal{Z}$-ultrafilter in* ZFC*?*

It is easy to describe a co-analytic ideal such that $\mathcal{I}$-ultrafilters exist in ZFC. Should the answer to the question be negative, one has to wonder whether even the following is possible

[7] Recall that an ideal $\mathcal{I}$ on ω is a P^+-*ideal* if for any decreasing sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+$ there is an $X \in \mathcal{I}^+$ such that $X \subseteq^* X_n$ for all $n \in \omega$, i.e. $X_n \setminus X$ is finite for all $n \in \omega$.

Question 5.13. *Is it consistent with* ZFC *that* $\mathcal{I} \leq_K \mathcal{U}^*$ *(*$\mathcal{I} \leq_{KB} \mathcal{U}^*$*) for every Borel ideal* $\mathcal{I}$ *and every ultrafilter* $\mathcal{U}$*?*

That is, is it consistent that the "stratification" of ultrafilters offered by the Katětov order on Borel ideals is vacuous.

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Centro de Ciencias Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México.
E-mail address: michael@matmor.unam.mx