CONVERGENT SEQUENCES IN TOPOLOGICAL GROUPS

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Introduction

This paper is part highly selective survey and part research announcement. It reviews recent developments concerning the role convergent sequences play in the theory of topological groups, concentrating on phenomena of set-theoretic nature with important contributions from members of the *Toronto Set Theory Seminar*: A. Dow, P. Koszmider, Y. Peng, J. Steprāns, P. Szeptycki, A. Tomita, S. Todorčević and S. Watson.

It deals with two extreme cases, on one hand it studies groups whose topology is determined by their convergent sequences, i.e. *Fréchet* and *sequential* topological groups, and on the other hand, it looks at topological groups which contain no non-trivial convergent sequences at all.

It has long been known that both phenomena trivialize in the realm of compact (even locally compact) topological groups as

- (Ivanovskiĭ-Vilenkin-Kuz'minov [51, 108, 53]/Cleary and Morris [16]) Every (locally) compact topological group contains a non-trivial convergent sequence, and
- (Hagler-Gerlits-Efimov [44, 38, 30]/Arhangel'skii and Malykhin [8]) Every (locally) compact sequential topological group is metrizable.

The major developments central to the paper are the relatively recent solutions to *Malykhin's problem* concerning metrizability of separable Fréchet groups ([48]), and *Nyikos's problem* about intermediate sequential order in topological groups [81] on one hand, and the solution of *van Douwen's problem* asking about the existence of an infinite countably compact topological group without non-trivial convergent sequences, and, consequently, also the proof of existence (in ZFC) of a pair of countably compact topological groups whose product is not countably compact, providing a solution to *Comfort's problem* in [49] on the other hand.

The paper will therefore be naturally split into two parts.

In the first part, which can be viewed as an update on Shakhmatov's excellent survey [74], we concentrate on Fréchet and sequential topological groups and present a new *Invariant Ideal Axiom* (IIA) extracted from the above mentioned solutions to Malykhin's and Nyikos's problems, which we propose as a paradigm axiom for the study of convergence properties in topological groups. We show that the axiom successfuly settles many of the problems mentioned in [74] and present a list of the questions that remain open.

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The second part reviews the history of constructions of topological groups without non-trivial convergent sequences, presents the use of *iterated ultrapowers* in constructions of countably compact groups and presents outstanding open problems in the area.

All topological spaces and groups considered are completely regular. To see more about topological groups in general, and about the place these problems occupy in the general theory consult [6, 9, 18, 19, 24, 74, 91].

1. Fréchet and sequential groups

1.1. **Notation ans standard facts.** Recall that a topological space X is *Fréchet* if for any $x \in \overline{A} \subseteq X$ there is a sequence $S \subseteq A$ such that $S \to x$. A space X is called *sequential* if for every $A \subseteq X$ which is not closed there is a $C \subseteq A$ such that $C \to x \notin A$. The term Fréchet space appears to have been coined by Arkhangel'skii in [3], while the term sequential seems to appear for the first time in Franklin's [31], where the following notion is defined:

Given $A \subseteq X$, define the *sequential closure* of A as

$$[A]' = \{\, x \in X : C \to x \text{ for some } C \subseteq A \,\}, \text{ and}$$

$$[A]_\alpha = \cup\{[A_\beta]' : \beta < \alpha\} \text{ for } \alpha \leq \omega_1.$$

Then X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subseteq X$, and the *sequential order* of X is defined as

$$\mathfrak{so}(X) = \min\{\, \alpha \leq \omega_1 : [A]_\alpha = \overline{A} \text{ for every } A \subseteq X\}.$$

Arkhangel'skii [4] introduced the hierarchy of *α_i-properties* to study the behaviour of Fréchet spaces and products.Given a topological space X and i = 1, 1.5, 2, 3, or 4 we say that X is an *α_i-space* provided for every countable family $\{\, S_n : n \in \omega \,\}$ of sequences converging to some point $x \in X$ (usually called a *sheaf* at x) there exists a 'diagonal' sequence S converging to x such that:

(α_1) $S_n \setminus S$ is finite for all $n \in \omega$,
(α_2) $S_n \cap S$ is infinite for all $n \in \omega$,
(α_3) $S_n \cap S$ is infinite for infinitely many $n \in \omega$,
(α_4) $S_n \cap S \neq \varnothing$ for infinitely many $n \in \omega$.

There is a short list of important simple sequential *test spaces*:

- **Arens space** ([2])- $S_2 = [\omega]^{\leq 2}$ where $U \subseteq S_2$ is open if and only if for every $s \in U$ the set $\{\, s \cup \{n\} \in S_n : s \cup \{n\} \notin U \,\}$ is finite,
- **Arkhangel'skii-Franklin space** ([7])- $S_\omega = [\omega]^{<\omega}$ with $U \subseteq S_\omega$ open if and only if for every $s \in U$ the set $\{\, n \in \omega : s \cup \{n\} \notin U \,\}$ is finite.
- **sequential fan** ([1][1]) - the quotient $S(\omega) = S_2/[\omega]^{\leq 1}$, and finally
- **convergent sequence of discrete set** ([25])- $D(\omega) = \omega \times \omega \cup \{(\omega, \omega)\} \subseteq (\omega+1)^2$ in the natural product topology.

The sequential fan $S(\omega)$ and the convergent sequence of discrete sets $D(\omega)$ are both Fréchet spaces (in the language of ideals they correspond to the ideals $\emptyset \times Fin$ and $Fin \times \emptyset$ (see [46, 107]). $D(\omega)$ is metrizable, $S(\omega)$ is not α_4. The spaces S_2 and S_ω are sequential, and $\mathfrak{so}(S_2) = 2$ while $\mathfrak{so}(S_\omega) = \omega_1$, S_ω is homogeneous but not a topological group ([96, Theorem 6.1]). The following facts summarize known

[1]This is the earliest reference to the space we could find, though possibly it appeared earlier. The earliest mention of the name *sequential fan* we could find is in [32].

simple observations concerning the role of the test spaces in topological groups, we attribute them according to where we think they appeared explicitly in print for the first time:

Proposition 1. *(1)* (Nyikos [62]) *Every Fréchet group is α_4.*

(2) (Nyikos [62]) *A topological group contains a copy of S_2 if and only if it contains a copy of $S(\omega)$.*

(3) (Tanaka [88]) *A countable sequential topological group is Fréchet if and only if it does not contain a closed copy of $S(\omega)$.*

(4) (Banakh and Zdomsky [10]) *Let $\mathbb{G}$ be a sequential group in which every point is G_δ. If G contains a closed copy of $D(\omega)$ then it is Fréchet.*

Finally, let us mention that while Fréchet spaces which are α_2 and not first countable exist in ZFC ([68, 63, 41]), each of the following statements is relatively consistent with ZFC that *all Fréchet α_2-spaces are α_1* ([27]) and *all Fréchet α_1-spaces are first countable* ([28]).

1.2. **Products of Fréchet and sequential groups.** It was observed by van Douwen [25] (implicitly also in [29, 31]) that the product $D(\omega)\times S(\omega)$ is not sequential. In particular, the products of countable Fréchet spaces (one of them metric) need not be sequential, but also products of compact Fréchet spaces need not be Fréchet [85]. The α_i-properties were introduced to study the behavior of Fréchet spaces under products (see e.g. [5, 42, 65, 67]). Let us mention a recent solution by Todorčevic [95] to a conjecture of Nogura [67]: Assume OCA. If x and Y are both Fréchet and α_4, and $X\times Y$ is Fréchet then $X\times Y$ is α_4.

The question of preservation of sequentiality and the Fréchet property under products in groups also has a long history [42, 74]). We shall survey the main development here.

In general the problem was settled by Todorčević [94] who showed that

Theorem 1 (Todorčević [94])**.** *There are Fréchet groups $\mathbb{G}$ and $\mathbb{H}$ such that $\mathbb{G}\times\mathbb{H}$ is not sequential, in fact, has uncountable tightness*[2]*.*

Whether this can be done with one single group remains an open problem:

Question 1 (Shakhmatov [74])**.** *Is there a Fréchet group $\mathbb{G}$ such that $\mathbb{G}\times\mathbb{G}$ is not Fréchet?*

Consistent examples were constructed by Malykhin [56], Malykhin and Shakhmatov [57], Shibakov [77, 80] and Peng-Todorčevic [70], usually assuming CH or MA. Consistent negative partial answers follow from the results of the next section

In fact, the groups constructed by Todorčević in [94] are of the form $C_p(X)$ and hence their square is Fréchet.

Theorem 2 (Tkachuk [93])**.** *If $C_p(X)$ is Fréchet then $C_p(X)^\omega$ is Fréchet.*

The theorem below also implies that $\mathbb{H}\times\mathbb{G}$ cannot be sequential.

Theorem 3 (Scheepers [72])**.** *Let X be a topological space. Then α_2, α_3, and α_4 are equivalent in spaces of the form $C_p(X)$.*

Much of this based on the work of Gerlits and Nagy:

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[2] Recall that a topological space X has uncountable tightness if there is a point x in the closure of a set $A\subseteq X$ which is not in the closure of any countable subset of A.

Theorem 4 (Gerlits-Nagy [39]). *The following are equivalent for a space X:*
$C_p(X)$ is Fréchet.
$C_p(X)$ is sequential.

Gerlits and Nagy [39] also gave a combinatorial translation in terms of open covers (see also [34]).

An analogous question for sequential groups seems very much open. Recall that the product of a countable sequential group which not Fréchet and a countable metrizable group is never sequential by the above mentioned observation of van Douwen.

Question 2. *Is there a non-Fréchet sequential group $\mathbb{G}$ such that $\mathbb{G} \times \mathbb{G}$ is not sequential?*

Consistent negative partial answers to both questions (they consistently fail for countable groups) follow from the results mentioned in the next section.

1.3. **Malykhin and Nyikos problems solved.** The two prominent results concerning Fréchet and sequential groups of the last decade, which build on [11] and [14], answer old problems of Malykhin [6] and Nyikos [62] respectively, were:

Theorem 5 (Hrušák-Ramos [48]). *It is consistent that every countable Fréchet group is metrizable.*

Theorem 6 (Shibakov [81]). *It is consistent that every sequential group is either metrizable or has sequential order ω_1.*

We have recently [50] discovered a succinct way of expressing key properties of these forcing models in the *Invariant Ideal Axiom*. Recall that an *ideal* is a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ closed under taking subsets and finite unions, and it is invariant if both $g \cdot I = \{g \cdot h : h \in I\}$ and $I \cdot g = \{h \cdot g : h \in I\}$, and $I^{-1} = \{h^{-1} : h \in I\}$ are in $\mathcal{I}$ for every $I \in \mathcal{I}$ and $g \in \mathbb{G}$. Recall also that $\mathcal{I}^+ = \mathcal{P}(\mathbb{G}) \setminus \mathcal{I}$. To state the axiom we introduce the following two notions: A topological group $\mathbb{G}$ is *remotely Fréchet* if every dense subset of $\mathbb{G}$ contains an infinite C convergent in $\mathbb{G}$, and an ideal is *small* if for every $X \in \mathcal{I}^+$ and every $f : X \to \omega$ there is a partition $\{P_n : n \in \omega\}$ of ω into infinite pieces such that for every $I \in \mathcal{I}$ there is an $n \in \omega$ such that $P_n \cap f[I] = \emptyset$.

IIA: *For every countable remotely Fréchet topological group $\mathbb{G}$ and every small invariant ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ one of the following holds:*

1. *there is a countable $\mathcal{S} \subseteq \mathcal{I}$ such that for every infinite sequence C convergent in $\mathbb{G}$ there is an $I \in \mathcal{S}$ such that $C \cap I$ is infinite,*
2. *there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every non-empty open $U \subseteq \mathbb{G}$ there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$.*

Theorem 7 ([50]). *The Invariant Ideal Axiom* IIA *is relatively consistent with* ZFC.

It should be mentioned here that some extra conditions on the group $\mathbb{G}$ and ideal $\mathcal{I}$ are necessary as without any restrictions on $\mathbb{G}$ or $\mathcal{I}$ the axiom is false. In fact, the current formulation of the axiom is only provisional and is likely to be changed for the final version of [50].

The *Invariant Ideal Axiom* suffices to entail the above results and more. Recall that a k_ω space is a quotient image of a sum of countably many compact spaces or, equivalently, there is a countable family $\mathcal{K}$ of compact subsets of the space such that a set U is open if and only if its intersection with every $K \in \mathcal{K}$ is relatively open in K.

Theorem 8 ([50])**.** *Assume* IIA*. Then every countable sequential group* $\mathbb{G}$ *is either metrizable or* k_ω*.*

Countable k_ω groups have a very well understood structure: their topologies are $F_{\sigma\delta}$, and are completely determined by their *compact scatteredness rank* defined as the supremum of the Cantor-Bendixson index of their compact subspaces:

Theorem 9 (Zelenyuk [110])**.** *Countable* k_ω *groups of the same compact scatteredness rank are homeomorphic.*

So, in particular, there is a complete list of k_ω group topologies on countable groups of length ω_1, hence Theorem provides the (perhaps) ultimate answer to Malykhin and Nyikos type problems.

Theorem 8 has the following definable analogue due to the second author which answered a question of Todorčević and Uzcátegui [96] as a precursor:

Theorem 10 ([82])**.** *All countable sequential analytic*[3] *groups are either* k_ω *or metrizable.*

Theorem 8 entails, in particular:

(1) Every separable Fréchet group is metrizable.
(2) Every Fréchet group is α_1,
(3) The product of two countable Fréchet groups is Fréchet,
(4) Every countable sequential group is either metrizable or has sequential order ω_1,
(5) The product of two countable sequential groups which are not Fréchet is sequential.

The existence of consistent counterexamples to all of the above statements have a long history (see e.g. [42, 60, 65, 74]).

We wish to make a case for IIA to be considered a *canonical axiom for the study of convergence properties in topological groups* as it seems to get rid of many (if not all) undesirable pathologies.

We close this section with an open problem:

Question 3. *Is* IIA *true for definable groups and ideals? I.e., let* $\mathbb{G}$ *be a countable group with analytic topology and let* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ *be an invariant analytic ideal. Is it true that one of the following holds?*

(1) there is a countable $\mathcal{S} \subseteq \mathcal{I}$ *such that for every infinite sequence* C *convergent in* $\mathbb{G}$ *there is an* $I \in \mathcal{S}$ *such that* $C \cap I$ *is infinite,*
(2) there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ *such that for every non-empty open* $U \subseteq \mathbb{G}$ *there is an* $H \in \mathcal{H}$ *such that* $H \setminus U \in \mathcal{I}$*.*

Is the same true for all countable groups and all invariant ideals assuming AD*?*

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[3] Here a topological space (X, τ) is *analytic* if X is countable and τ is an analytic subset of $\mathcal{P}(X)$.

1.4. **The α_i-properties in Fréchet groups.** As we saw in the last section, consistently, all countable Fréchet groups are metrizable, and hence all Fréchet groups are α_1. By the result of Nyikos every Fréchet group is α_4 in ZFC.

On the other hand, it has been shown by Shakhmatov [73] that consistently there are countable Fréchet groups which are not α_3 and by Shibakov [80] countable Fréchet groups which are α_3 but not α_2

To study the gap between α_3 and α_4, Nogura [66] introduced a refined hierarchy: A sheaf $\Upsilon' = \{ S_n' : n \in \omega \}$ is called a *subsheaf* of a sheaf $\Upsilon = \{ S_n : n \in \omega \}$ if there exists an injection $j : \omega \to \omega$ so that $\operatorname{ran} S_n' \subseteq \operatorname{ran} S_{j(n)}$ for each $n \in \omega$. We call a sheaf $\Upsilon = \{ S_n : n \in \omega \}$ at x a *cross-sheaf of* Υ provided $\bigcup \Upsilon' \subseteq \bigcup \Upsilon$ and each $\operatorname{ran} S_n'$ meets infinitely many S_m. For $k \in \omega$ we define by induction what it means for Υ to be *k-nice* as follows: Υ is 0-nice if $\bigcup \Upsilon \subseteq S$ for some convergent sequence S, and Υ is $(k+1)$-nice if each cross-sheaf of Υ has a k-nice subsheaf. For $k \in \omega$ we call x an *α^k-point* if each sheaf at x has a k-nice subsheaf. Finally, call $x \in X$ an *α^∞-point* if every sheaf at x contains a k-nice subsheaf for some $k \in \omega$. Finally, call X an *α^k-space*, $k \in \omega \cup \{\infty\}$ if every $x \in X$ is an α^k-point.

It is clear from the definitions that $\alpha_3 = \alpha^0 \Rightarrow \alpha^1 \Rightarrow \cdots \Rightarrow \alpha^\infty$.

Question 4 ([73])**.** *Is every (countable) Fréchet topological group an α^∞ space? Is $\alpha_3 = \alpha^\infty$ for Fréchet groups?*

1.5. **Vive la différence.** While in Section 1.3 we were looking for the most "regular" behaviour of Fréchet and sequential groups, in this section we shall ask about the other extreme. The first such question asks to what extent does the existence of a non-metrizable Fréchet topology depend on the algebraic structure of the group. Recall that a group $\mathbb{G}$ is called *topologizable* if it admits a non-discrete Hausdorff group topology. It is a surprising result of Ol'shanski and Shelah, independently, that there are infinite groups which are not topologizable [69, 76].

Question 5 (Hrušák-Ramos [47])**.** *Is it consistent that countable Fréchet non-metrizable topological groups exist while some countable topologizable group does not admit a non-metrizable Fréchet topology?*

A similar question can be asked about the contribution of algebra in the behavior of the sequential order.

Question 6 ([81])**.** *Is it consistent that there exist countable groups of any sequential orders while some countable topologizable group does not admit any sequential topology of sequential order α for some $\alpha \in \omega_1$?*

Question 7 ([81])**.** *Is it consistent that there exist (countable) sequential groups of every sequential order other than α for some $\alpha \in \omega_1$?*

It is consistently possible that countable and uncountable sequential groups behave differently.

Theorem 11 ([84])**.** *In the model obtained by adding ω_2 Cohen reals to a model of $\diamondsuit$ there are no countable groups of sequential order other than ω_1, 0, or 1 but there are uncountable sequential groups of every sequential order.*

Question 8 (Hrušák-Ramos [48])**.** *Is it consistent with ZFC that there is a countable Fréchet group of weight $\aleph_2$ but no countable Fréchet group of weight $\aleph_1$?*

1.6. **Convergent sequences in pre-compact and countably compact groups.** The Σ-product of ω_1 copies of $\mathbb{Z}_2$ is a standard example of a non-metrizable Fréchet group. It is countably compact and not separable, of course.

Recall that a group is *precompact* if it is a subgroup of a compact group. All countably compact groups are precompact.

Theorem 12 (Hrušák-Ramos [47]). *It is consistent that every countable abelian group admits a precompact non-metrizable Fréchet topology.*

The paper containing the result contains a combinatorial condition characterizing the existence of nonmetrizable precompact Fréchet group topologies in the spirit of Gerlits-Nagy [39].

Question 9. *Is it consistent that all sequential precompact (pseudocompact) topological groups are Fréchet? Are all countable sequential precompact topological groups Fréchet in* ZFC*?*

It turns out that if G is a countable sequential group with a base of neighborhoods of 1 consisting of open subgroups then G is Fréchet (such topologies are sometimes called *linear*). I particular, every abelian precompact group of finite exponent has a base of neighborhoods of 0 consisting of open subgroups of finite index.

Proposition 2. *Every countable pre-compact sequential group of finite exponent (in particular, every boolean group) is Fréchet.*

Another partial result is:

Theorem 13 ([84]). *In the Cohen model all countable precompact sequential groups are Fréchet. There also exists a model of ZFC in which all countable precompact sequential groups are metrizable.*

Next we, briefly discuss recent developments concerning countably compact sequential groups, i.e. if we strengthen pre-compact to countably compact in the above questions. Shakhmatov in [74] asked explicitly if every countably compact sequential group is Fréchet. It turns out the answer is independent of the axioms of ZFC. On the one hand:

Theorem 14 (Shakhmatov-Shibakov [75]). *Assuming $\diamondsuit$, there exists a countably compact sequential boolean group of sequential order α for every $\alpha \leq \omega_1$*

but is also consistent that there are no non Fréchet countably compact sequential groups.

Theorem 15 ([81]). *It is consistent with* ZFC *that every sequential countably compact group is Fréchet.*

We present two more questions about sequential countaby compact groups:

Question 10. *Is it consistent that all sequential countably compact groups are Fréchet while there exists a sequential precompact (pseudocompact) topological group that is not Fréchet?*

Question 11. *Is it consistent that there exist countably compact Fréchet groups $\mathbb{G}$ and $\mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not Fréchet? A single countably compact Fréchet group $\mathbb{G}$ such that $\mathbb{G} \times \mathbb{G}$ is not Fréchet?*

The following question asked privately by A. Dow serves nicely as a bridge between the two parts of the text:

Question 12 (Dow)**.** *Does every countably tight countably compact group contain a convergent sequence?*

2. Countably compact groups without convergent sequences

2.1. **Groups without non-trivial convergent sequences.** The second part of this survey deals with groups which contain no non-trivial convergent sequences whatsoever, i.e. every convergent sequence in the group is eventually constant. It is not clear to us who first constructed a non-discrete topological group without convergent sequences, but such quest can be traced at least to Leptin [54] and Glicksberg [40] who independently discovered that an abelian group $\mathbb{G}$ endowed with the *Bohr topology*, i.e. the weakest topology which makes all group homomorphisms $\varphi : \mathbb{G} \to \mathbb{T}$ continuous[4], has no non-trivial convergent sequences. The first pseudocompact example in ZFC was constructed by Sirota [86], and further research on compact group with dense pseudocompact subgroups without convergent sequences thrived (see [21, 23, 33, 36, 58]), while the first consistent countably compact example was given by Hajnal and Juhász [43] under CH. The research on countably compact groups without convergent sequences gained on importance with van Douwen's [26] consistent solution to *Comfort's problem*:

Question 13 (Comfort [20, 19])**.** *Are there countably compact groups* $\mathbb{G}_0, \mathbb{G}_1$ *such that* $\mathbb{G}_0 \times \mathbb{G}_1$ *is not countably compact?*

The problem was motivated by the Comfort-Ross theorem[5]:

Theorem 16 (Comfort-Ross [22])**.** *Any product of pseudocompact topological groups is pseudo-compact.*

In his paper [26] van Douwen showed that every Boolean countably compact group without non-trivial convergent sequences contains two countably compact subgroups whose product is not countably compact, constructed an example assuming MA and asked:

Question 14 (van Douwen [26])**.** *Is there a countably compact group without non-trivial convergent sequences?*

More consistent examples and variations quickly followed [35, 36, 37, 45, 52, 87, 90, 99, 100, 101, 102, 103, 106] and the problem was finally solved in the positive in [49]. We shall comment on the solution in the next two sections.

2.2. **Iterated ultrapowers and p-compact groups.** Throughout this section let p be a free ultrafilter on ω.

Recall ([12]) that a point $x \in X$ is the *p-limit* of a sequence $\{x_n : n \in \omega\} \subseteq X$ ($x = p\text{-}\lim_{n\in\omega} x_n$) if $\{n \in \omega \colon x_n \in U\} \in p$ for every open $U \subseteq X$ containing x. The space X is *p-compact* if for every sequence $\{x_n : n \in \omega\} \subseteq X$ there is an $x \in X$ such that $x = p\text{-}\lim_{n\in\omega} x_n$.

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[4]By $\mathbb{T}$ we denote the 1-dimensional torus.

[5]Recall that for topological space this fails badly as there are even countably compact spaces whose product fails to be pseudocompact [61, 89].

Given a group $\mathbb{G}$ we denote by

$$\mathsf{ult}_p(\mathbb{G}) = \mathbb{G}^\omega / \equiv \text{ , where } f \equiv g \text{ iff } \{n : f(n) = g(n)\} \in p$$

the *ultrapower* of $\mathbb{G}$ w.r.t. p. The *Theorem of Lós* states that $\mathsf{ult}_p(\mathbb{G})$ is a group with the same first order properties as $\mathbb{G}$, and $\mathbb{G}$ embedds into $\mathsf{ult}_p(\mathbb{G})$ via constant functions, so we can consider $\mathbb{G}$ as a subgroup of $\mathsf{ult}_p(\mathbb{G})$. Let $\mathsf{ult}_p^0(\mathbb{G}) = \mathbb{G}$. Given an ordinal α with $\alpha > 0$, let (slightly abusing the notation)

$$\mathsf{ult}_p^\alpha(\mathbb{G}) = \mathsf{ult}_p\left(\bigcup_{\beta<\alpha} \mathsf{ult}_p^\beta(\mathbb{G})\right),$$

The Bohr topology $(\mathbb{G}, \tau_{\mathrm{Bohr}})$ *lifts* to a topology on $\mathsf{ult}_p(\mathbb{G})$ as follows: Every $\Phi \in \mathrm{Hom}(\mathbb{G}, \mathbb{T})$ naturally extends to a homomorphism $\overline{\Phi} \in \mathrm{Hom}(\mathsf{ult}_p(\mathbb{G}), \mathbb{T})$ by letting

$$\overline{\Phi}([f]) = p\text{ - }\lim_{n\in\omega} \Phi(f(n)).$$

$\overline{\Phi}$ is then a homomorphism from $\mathsf{ult}_p(\mathbb{G})$ to $\mathbb{T}$. This process can be iterated all the way to ω_1, extending Φ to $\overline{\Phi} : \mathsf{ult}_p^{\omega_1}(\mathbb{G}) \to \mathbb{T}$ and then defining $\tau_{\overline{\mathrm{Bohr}}}$ as the weakest topology making every $\overline{\Phi}$ continuous, for all $\Phi \in \mathrm{Hom}(\mathbb{G}, \mathbb{T})$.

The iterated ultrapower with this topology is not Hausdorff, so we let

$$\mathsf{Ult}_p^{\omega_1}(\mathbb{G}) = \mathsf{ult}_p^{\omega_1}(\mathbb{G})/K,$$

where $K = \bigcap_{\Phi\in\mathrm{Hom}(\mathbb{G},\mathbb{T})} \mathrm{Ker}(\overline{\Phi})$.

The group $(\mathsf{Ult}_p^{\omega_1}(\mathbb{G}), \tau_{\overline{\mathrm{Bohr}}})$ is a Hausdorff p-compact topological group, with $[f] = p\text{-}\lim f(n)$.

It turns out that if $\mathbb{G}$ is the countable Boolean group and p a selective ultrafilter then the iterated ultrapower construction produces a p-compact group without non-trivial convergent sequences.

Theorem 17 ([49])**.** *Let $p \in \omega^*$ be a selective ultrafilter. Then $(\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega}), \tau_{\overline{Bohr}})$ is a Hausdorff p-compact topological Boolean group without non-trivial convergent sequences.*

On the other hand, there is always an ultrafilter p for which $(\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega}), \tau_{\overline{\mathrm{Bohr}}})$ does contain non-trivial convergent sequences and, assuming CH this ultrafilter can even be a P-point.

This suggest the following interesting questions:

Question 15 ([37, 49])**.** *Is it consistent with* ZFC *that every Hausdorff p-compact topological group contains a non-trivial convergent sequence?*

Question 16 ([49])**.** *Is the existence of an ultrafilter p such that $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains no non-trivial convergent sequences equivalent to the existence of a selective ultrafilter? Is it consistent with* ZFC *that $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains a non-trivial convergent sequence for every ultrafilter $p \in \omega^*$?*

C. Corral in [17] investigates the ultrapower construction for topological spaces in general, and proves that there are severe limitations on when an ultrapower of a topological group is a topological group, in particular.

2.3. **Comfort's and van Douwen's problems solved.** The main theorem of [49] is the existence in ZFC of a countably compact subgroup of $2^{\mathfrak{c}}$ without convergent sequences.

Theorem 18 ([49])**.** *There is a dense countably compact subgroup of* $2^{\mathfrak{c}}$ *without non-trivial convergent sequences.*

The proof of the theorem is very similar to the proof of Theorem 17: one again starts with the Bohr topology on $[\omega]^{<\omega}$ and recursively extends homomorphisms using ultrafilters, only here one uses a carfully constructed $\mathfrak{c}$-sized family of ultrafilters rather than a single one.

The van Douwen's method then produces two countably compact subgroups whose product is not countably compact:

Corollary 1 ([49])**.** *There are countably compact groups* $\mathbb{G}_0, \mathbb{G}_1$ *such that* $\mathbb{G}_0 \times \mathbb{G}_1$ *is not countably compact.*

It is very likely that the same idea helps solve the following version of Comfort's question:

Question 17 (Comfort [19])**.** *Is there for every (not necessarily infinite* $\kappa < 2^{\mathfrak{c}}$ *a countably compact group* $\mathbb{G}$ *such that* $\mathbb{G}^{\kappa}$ *is countably compact but* $\mathbb{G}^{\kappa^+}$ *is not?*

Recently Tomita [105], building on [49], answered the question positively for $\kappa \leq \omega_1$.

The last problem we mention here is due to Castro-Pererira and Tomita:

Question 18 (Castro-Pereira–Tomita [15])**.** *Are there arbitrarily large countably compact groups without non-trivial convergent sequences?*

They showed [15] that the existence of a single selective ultrafilter is sufficient.

2.4. **Wallace problem.** A closely related problem is an old problem of Wallace:

Question 19 (Wallace [109])**.** *Is every both-sided cancellative countably compact topological semigroup necessarily a group?*

It is well-known that by a method very similar to van Douwen's one can encounter examples in every countably compact topological group without non-trivial convergent sequences which contains a copy of $\mathbb{Z}$, and such examples have been constructed assuming CH [71], weak versions of MA [98] or existence of many selective ultrafilters [55].

It is, however, not straightforward that the method of either Theorem 17 or Theorem 18 works for groups which are not torsion. We suspect they do, but new ideas are needed.

Question 20 (Tkachenko [92])**.** *Is there, in* ZFC*, a non-torsion countably compact topological group without non-trivial convergent sequences?*

Tkachenko [92] actually asks a formally stronger question, whether the free abelian group on $\mathfrak{c}$-many generators can be given a countably compact group topology without convergent sequences, a task that has been accomplished by him assuming CH in [90]. It was, however, noticed recently by Tomita [105] that any non-torsion countably compact group without non-trivial convergent sequences contains a countably compact free abelian subgroup, hence the two questions are equivalent.

Also recently Boero, Pereira and Tomita [13] constructed an example from a single selective ultrafilter. We conjecture that the"canonical" construction works:

Question 21 ([49]). *Assume $p \in \omega^*$ is a selective ultrafilter. Does $(\mathsf{Ult}_p^{\omega_1}(\mathbb{Z}), \tau_{\overline{Bohr}})$ contain no non-trivial convergent sequence?*

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References

[1] P. Alexandroff und H. Hopf, *Topologie I.* Berlin (1935)

[2] R. Arens, *Note on convergence in topology.* Math. Magazine **23** (1950), 229–234.

[3] A. V. Arhangel'skii, *Some types of factor mappings and the relations between classes of topological spaces.* Dokl. Akad. Nauk SSSR **153** 1963 743–746.

[4] A. V. Arhangel'skii, *The spectrum of frequences of topological spaces and their classification.* Dokl. Akad. Nauk SSSR, **206** (1972), 265–268.

[5] A. V. Arhangel'skii, *The spectrum of frequences of a topological space and the product operation.* Trudy Moskov. Mat. Obšč., **40** (1979), 171–206.

[6] A. V. Arhangel'skii, *Classes of topological groups.* Uspekhi Mat. Nauk **36** (1981), no. 3(219), 127–146, 255.

[7] A. V. Arhangel'skii and S. P. Franklin, *Ordinal invariants for topological spaces.* Michigan Math. J. **15** (1968), 313–320.

[8] A. V. Arhangel'skii and V. I. Malykhin, *Metrizability of topological groups.* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1996, no. 3, 13–16, 91; translation in Moscow Univ. Math. Bull. **51** (1996), no. 3, 9–11.

[9] A. V. Arhangel'skii and M. Tkachenko, *Topological Groups and Related Structures.* Atlantis Press, Paris, and World Sci., Hackensack, NJ, 2008.

[10] T. Banakh and L. Zdomskyĭ, *The topological structure of (homogeneous) spaces and groups with countable cs-character.* Appl. Gen. Topol. **5** (2004), no. 1, 25–48.

[11] D. Barman and A. Dow, *Proper forcing axiom and selective separability.* Topology Appl. **159** (2012), no. 3, 806–813.

[12] A. R. Bernstein, *A new kind of compactness for topological spaces.* Fund. Math. **66** (1969/1970), 185–193.

[13] A. C. Boero, I. C. Pereira, A. H. Tomita, *Countably compact group topologies on the free Abelian group of size continuum (and a Wallace semigroup) from a selective ultrafilter.* Acta Math. Hungar. **159** (2019), no. 2, 414–428.

[14] J. Brendle and M. Hrušák, *Countable Fréchet Boolean groups: an independence result.* J. Symbolic Logic **74** (3) (2009) 1061–1068.

[15] I. Castro-Pereira, A. H. Tomita, *Abelian torsion groups with a countably compact group topology.* Topology Appl. **157** (2010), no. 1, 44–52.

[16] J. Cleary and S. Morris, *Locally dyadic topological groups.* Bull. Austral. Math. Soc. **40** (1989), no. 3, 417–419.

[17] C. I. Corral Rojas, *Ultrapowers of topological spaces.* Pre-print (2019).

[18] W. W. Comfort, *Topological groups.* Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 1143–1263. MR 776643

[19] W. W. Comfort, *Problems on topological groups and other homogeneous spaces.* Open problems in topology, North-Holland, Amsterdam, 1990, 313–347.

[20] W. W. Comfort, *Letter to K. A. Ross.* (August 1, 1966).

[21] W. W. Comfort, S. U. Raczkowski, F. J. Trigos-Arrieta, *Making group topologies with, and without, convergent sequences.* Appl. Gen. Topol. **7** (2006), no. 1, 109–124.

[22] W.W. Comfort and K.A. Ross, *Pseudocompactness and uniform continuity in topological groups.* Pacific J. Math. **16** (1966) 483–496.

[23] J.J.Dijkstra-J.van Mill, *Groups without convergent sequences.* Topology and its Applications **74** (1996) 275-282.

[24] D. Dikranjan and D. Shakhmatov, *Selected topics from the structure theory of topological groups.* In (E. Pearl, ed.) Open problems in topology II, Elsevier, 2007, 389—406.

[25] E. K. van Douwen, *The product of a Fréchet space and a metrizable space.* Topology Appl. **47** (1992), no. 3, 163–164.

[26] E. K. van Douwen, *The product of two countably compact topological groups.* Trans. Amer. Math. Soc. **262** (1980) 417-427.

[27] A. Dow, *Two classes of Fréchet-Urysohn spaces.* Proc. Amer. Math. Soc. **108** (1990), no. 1, 241–247.
[28] A. Dow, J. Steprāns, *Countable Fréchet α_1-spaces may be first countable.* Arch. Math. Logic **32** (1992), no. 1, 33—50.
[29] R. M. Dudley, *On sequential convergence.* Trans. Amer. Math. Sot. **112** (1964) 483–507.
[30] B. A. Efimov, *Mappings and imbeddings of dyadic spaces.* Mat. Sb. (N.S.) **103** (145) (1977), no. 1, 52–68, 143.
[31] S. P. Franklin, *Spaces in which sequences suffice.* Fundamenta Mathematicae **57** (1965) 107–115.
[32] S. P. Franklin, B. V. Smith Thomas, *On the metrizability of $_\omega$-spaces.* Pacific J. Math. **72** (1977), no. 2, 399–402.
[33] J. Galindo and S. García-Ferreira, *Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences.* Topology Appl. **154** (2007), no. 2, 476–490.
[34] F. Galvin and A. Miller, *γ-sets and other singular sets of real numbers.* Topol. Appl., 17 (1984), 145–155.
[35] S. García-Ferreira and A. H. Tomita, *Countably compact groups and p-limits.* Bol. Soc. Mat. Mexicana (3) **9** (2003), no. 2, 309–321.
[36] S. García-Ferreira and A. H. Tomita, *Pseudocompact dense subgroups without non-trivial convergent sequences of some compact groups.* Topology Proc. **31** (2007), no. 1, 97–114.
[37] S. García-Ferreira and A. H. Tomita and S. Watson, *Countably compact groups from a selective ultrafilter.* Proc. Amer. Math. Soc. **133** (2005), no. 3, 937—943.
[38] J. Gerlits, *On subspaces of dyadic compacta.* Studia Sci. Math. Hungar. **11** (1976), no. 1-2, 115–120 (1978).
[39] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I.* Topol. Appl., **14** (1982), 151–161.
[40] I. Glicksberg, *Uniform boundedness for groups.* Canad. J. Math., **14** (1962), 269–276.
[41] G. Gruenhage, *Infinite games and generalizations of first-countable spaces.* Gen. Topology Appl. **6** (1976) 339–352.
[42] G. Gruenhage, *Products of Fréchet spaces.* Topology Proc. **30** (2006) no.2, 475–499.
[43] A. Hajnal and I. Juhász, *A separable normal topological group need not be Lindelöf.* Gen. Topology Appl. **6** (1976), no. 2, 199–205.
[44] J. Hagler, *On the structure of S and $C(S)$ for S dyadic.* Trans. Amer. Math. Soc. **214** (1975), 415–428.
[45] K. P. Hart and J. van Mill, *A countably compact topological group H such that $H \times H$ is not countably compact.* Trans. Amer. Math. Soc. **323** (1991), no. 2, 811–821.
[46] M. Hrušák,*Combinatorics of filters and ideals.* Set theory and its applications, 29–69, Contemp. Math., **533**, Amer. Math. Soc., Providence, RI, 2011.
[47] M. Hrušák, U. A. Ramos-García, *Precompact Fréchet topologies on Abelian groups.* Topology Appl. **159** (2012), no. 17, 3605–3613.
[48] M. Hrušák, U. A. Ramos-García, *Malykhin's problem.* Adv. Math. **262** (2014), 193–212.
[49] M. Hrušák, U. A. Ramos-García, J. van Mill and S. Shelah,*Countably compact groups without convergent sequences*, pre-print (2019).
[50] M. Hrušák and A. Shibakov, *Invariant Ideal Axiom*, pre-print (2019).
[51] L. N. Ivanovskiĭ, *On a conjecture of P. S. Alexandrov.* Dokl. Akad. Nauk SSSR **123** (1958) 785–786.
[52] P. Koszmider, A. H. Tomita and S. Watson *Forcing countably compact group topologies on a larger free abelian group.* Proceedings of the 15th Summer Conference on General Topology and its Applications/1st Turkish International Conference on Topology and its Ap- plications (Oxford, OH/Istanbul, 2000), vol. 25, 2000, 563–574 (2002).
[53] V. Kuz'minov, *Alexandrov's hypothesis in the theory of topological groups.* Dokl. Akad. Nauk SSSR **125** (1959), 727–729.
[54] H. Leptin, *Abelsche Gruppen mit kompakten Charaktergruppen und Dualitätstheorie gewisser linear topologischer abelscher Gruppen*, Abh. Math. Sem. Univ. Hamburg **19** (1955), 244–263.
[55] R. E. Madariaga-Garcia, A. H. Tomita, *Countably compact topological group topologies on free abelian groups from selective ultrafilters.* Topology Appl. **154** (2007), no. 7, 1470–1480.
[56] V. I. Malykhin, *Consistency results in topology.* Trudy Moskov. Mat. Obshch. **49** (1986), 141–166, 240.
[57] V. I. Malykhin and D. B. Shakhmatov, *Cartesian products of Fréchet topological groups and function spaces.* Acta Math. Hungar. **60** (1992), no. 3-4, 207–215.

[58] V. I. Malykhin and L. B. Shapiro, *Pseudocompact groups without convergent sequences.* Mat. Zametki **37** (1985), no. 1, 103–109, 139.

[59] E. Michael, $\aleph_0$*-spaces.* J. Math. Mech. **15** (1966), 983–1002.

[60] J. T. Moore and S. Todorčević, *The metrization problem for Fréchet groups.* In: (E. Pearl, ed.) Open Problems in Topology II. Elsevier, 2007, 201–2006.

[61] J. Novák, *On the Cartesian product of two compact spaces.* Fund. Math. **40** (1953), 106–112.

[62] P. J. Nyikos, *Metrizability and the Fréchet-Urysohn property in topological groups.* Proc. Amer. Math. Soc. 83 (1981), no. 4, 793–801.

[63] P. J. Nyikos, *The Cantor tree and the Fréchet-Urysohn property.* Papers on general topology and related category theory and topological algebra (New York, 1985/1987), 109–123, Ann. New York Acad. Sci., 552, New York Acad. Sci., New York, 1989.

[64] P. J. Nyikos, *Subsets of* ω^ω *and the Fréchet-Urysohn and* $\alpha - i$*-properties.* Topology Appl. **48** (1992), no. 2, 91–116.

[65] P. J. Nyikos, *Workshop lecture on products of Fréchet spaces.* Topology Appl. **157** (2010), no. 8, 1485–1490.

[66] T. Nogura, *Products of sequential convergence properties*, Czechoslov. Math. J. 39 (114) (1989), 262–279

[67] T. Nogura, *The product of* $\langle\alpha_i\rangle$*space.*, Topology and its Appl. 21(1985), 251–259.

[68] R. C. Olson, *Bi-quotient maps, countably bi-sequential spaces, and related topics.* Gen. Topology Appl. **4** (1974), 1–28.

[69] A. Yu. Olshansii, *A remark on countable non-topologizable groups.* Vestnik Mosk. Gos. Univ. Mat. Mekh., **3**:103, 1980.

[70] Y. Peng and S. Todorčević, *Powers of countable Fréchet spaces.* Fund. Math. **245** (2019), no. 1, 39–54.

[71] D. Robbie and S. Svetlichny, *An answer to A.D. Wallace's question about countably compact cancellative semigroups.* Proc. Amer. Math. Soc. **124** (1996) 325–330.

[72] M. Scheepers, $C_(X)$ *and Arhangelskiĭ's* α_i*-spaces.* Topology Appl. **89** (1998), no. 3, 265–275.

[73] D. B. Shakhmatov, α_i*-properties in Fréchet-Urysohn topological groups.* Topology Proc. **15** (1990), 143–183.

[74] D. B. Shakhmatov, *Convergence in the presence of algebraic structure.* In: (M. Hušek and J. van Mill, eds) Recent Progress in General Topology II, North-Holland, Amsterdam, 2002, 463–484.

[75] D. B. Shakhmatov and A. Shibakov, *Countably compact groups and sequential order.*, preprint (2019).

[76] S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications. Word problems, II.* (Conf. on Decision Problems in Algebra, Oxford, 1976), 373–394, Stud. Logic Foundations Math., **95**, North-Holland, Amsterdam-New York, 1980.

[77] A. Shibakov, *Examples of sequential topological groups under the continuum hypothesis.* Fund. Math. **151** (1996), no. 2, 107–120.

[78] A. Shibakov, *Sequential group topology on rationals with intermediate sequential order.* Proc. Amer. Math. Soc. **124** (1996), no. 8, 2599–2607.

[79] A. Shibakov, *Sequential topological groups of any sequential order under CH*, Fundam. Math. **155** (1) (1998) 79–89.

[80] A. Shibakov, *Countable Fréchet topological groups under CH.* Topology Appl. **91** (1999), no. 2, 119–139.

[81] A. Shibakov, *No interesting sequential groups.* Topology Appl. **228** (2017), 125–138.

[82] A. Shibakov, *On sequential analytic groups.* Proc. Amer. Math. Soc. 145 (2017), no. 9, 4087–4096.

[83] A. Shibakov, *On large sequential groups.* Fund. Math. **243** (2018), no. 2, 123–141.

[84] A. Shibakov, *Convergence in topological groups and the Cohen reals.* Topology Appl. **252** (2019), 81–89.

[85] P. Simon, *A compact Fréchet space whose square is not Fréchet.* Comment. Math. Univ. Carolin. **21** (1980), no. 4, 749–753.

[86] S. M. Sirota, *The product of topological groups and extremal disconnectedness*, Math. USSR Sbornik **8** (1969) 169–180.

[87] P. J. Szeptycki and A. H. Tomita, *HFD groups in the Solovay model.* Topology Appl. **156** (2009), no. 10, 1807–1810.

[88] Y. Tanaka, *Products of sequential spaces.* Proc. Amer. Math. Soc. **54** (1976), 371–375.

[89] H. Terasaka, *On Cartesian product of compact spaces.* Osaka Math. J. **4** (1952), 11–15.
[90] M. G. Tkachenko, *Countably compact and pseudocompact topologies on free abelian groups.* Izv. Vyssh. Uchebn. Zaved. Mat. (1990), no. 5, 68–75.
[91] M. G. Tkachenko, *Topological features of topological groups.* Handbook of the history of general topology, Vol. 3, Hist. Topol., Kluwer Acad. Publ., Dordrecht, 2001, 1027–1144.
[92] M. G. Tkachenko, *Topological groups: Between Compactness and $\aleph_0$-boundedness.* In (M. Hušek and J. van Mill eds.) Recent progress in general topology II., North Holland, 2002, 515–543.
[93] V. V. Tkačuk, *On multiplicativity of some properties of mapping spaces equipped with the topology of pointwise convergence.* Vestnik Moskov. Univ. Set. I Mat. Mekh., **6** (1984), 36–39.
[94] S. Todorčević, *Some applications of S and L combinatorics.* The work of Mary Ellen Rudin (Madison, WI, 1991), 130–167, Ann. New York Acad. Sci., **705**, New York Acad. Sci., New York, 1993.
[95] S. Todorčević, *A proof of Nogura's conjecture.* Proc. Amer. Math. Soc. **131** (2003), no. 12, 3919–3923.
[96] S. Todorčević and C. Uzcátegui, *Analytic k-spaces.* Topology Appl. **146/147** (2005), 511–526.
[97] S. Todorčević and C. Uzcátegui, *Analytic topologies over countable sets.* Topology Appl. **111** (2001), no. 3, 299–326.
[98] A. H. Tomita, *The Wallace problem: a counterexample from $MA_{countable}$ and p-compactness.* Canad. Math. Bull. **39** (1996), no. 4, 486–498.
[99] A. H. Tomita, *On finite powers of countably compact groups.* Comment. Math. Univ. Carolin. **37** (1996), no. 3, 617–626.
[100] A. H. Tomita, *A group under $MA_{countable}$ whose square is countably compact but whose cube is not.* Topology Appl. **91** (1999), no. 2, 91–104.
[101] A. H. Tomita, *Two countably compact topological groups: one of size $\aleph_\omega$ and the other of weight $\aleph_\omega$ without non-trivial convergent sequences.* Proc. Amer. Math. Soc. **131** (2003), no. 8, 2617–2622.
[102] A. H. Tomita, *A solution to Comfort's question on the countable compactness of powers of a topological group.* Fund. Math. **186** (2005), no. 1, 1–24.
[103] A. H. Tomita, *Countable compactness and finite powers of topological groups without convergent sequences.* Topology Appl. **146/147** (2005), 527–538.
[104] A. H. Tomita, *The weight of a countably compact group whose cardinality has countable confinality.* Topology Appl. **150** (2005), no. 1-3, 197–205.
[105] A. H. Tomita, *A van Douwen-like ZFC theorem for small powers of countably compact groups without non-trivial convergent sequences.* Topology Appl. **259** (2019), 347–364.
[106] A. H. Tomita and S. Watson, *Ultraproducts, p-limits and antichains on the Comfort group order.* Topology Appl. **143** (2004), no. 1-3, 147–157.
[107] C. Uzcátegui Aylwin, *Ideals on countable sets: a survey with questions* Revista Integracíń, temas de matemt́icas, Escuela de Matemáticas Universidad Industrial de Santander, **Vol. 37**, N 1, 2019, 167–198.
[108] N. Ja. Vilenkin, *Dyadicity of the group space of bicompact commutative groups.* Uspehi Mat. Nauk **13** (1958) no. 6 (84), 79–80.
[109] A. D. Wallace, *The structure of topological semigroups.* Bull. Amer. Math. Soc. **61** (1955), 95–112.
[110] E. G. Zelenyuk, *Topologies on groups, defined by compacta.* Mat. Stud. **5** (1995), 5–16.

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