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Forcing indestructibility of MAD families

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Abstract

Let $\mathcal{A} \subseteq [\omega]^\omega$ be a maximal almost disjoint family and assume $\mathbb{P}$ is a forcing notion. Say $\mathcal{A}$ is $\mathbb{P}$-indestructible if $\mathcal{A}$ is still maximal in any $\mathbb{P}$-generic extension. We investigate $\mathbb{P}$-indestructibility for several classical forcing notions $\mathbb{P}$. In particular, we provide a combinatorial characterization of $\mathbb{P}$-indestructibility and, assuming a fragment of MA, we construct maximal almost disjoint families which are $\mathbb{P}$-indestructible yet $\mathbb{Q}$-destructible for several pairs of forcing notions $(\mathbb{P}, \mathbb{Q})$. We close with a detailed investigation of iterated Sacks indestructibility.

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1. Introduction

Almost disjoint families (AD families for short) and, in particular, maximal almost disjoint families of sets of natural numbers (MAD families for short) play an important role in set theory as well as in its applications, for example in general topology. Let us mention but two sample examples, namely, the technique of almost disjoint coding in

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forcing theory and the construction of the Isbell-Mrówka space from an almost disjoint family in set theoretic topology.

A fundamental question about MAD families is whether they survive forcing extensions in which new real numbers are adjoined, and, until recently, surprisingly little was known about this. In particular, the relationship of $\mathfrak{a}$, the size of the smallest MAD family, with other cardinal invariants of the continuum was little understood. This has changed drastically with the advent of Shelah's theory of iteration along templates (see [19,6]) which provided a method of destroying MAD families with minimal damage. For example, his technique allows for killing arbitrary MAD families while preserving dominating families, and he thus obtained the consistency of $\mathfrak{d} < \mathfrak{a}$, solving a long-standing problem about cardinal invariants of the continuum. Shelah's results spurred new interest in the question under which condition is a MAD family (in)destructible by a given forcing notion.

This question may be seen, in a broader context, as an attempt to classify MAD families, and, ultimately, to arrive at some structural theory of MAD families. Note in this context that one of the most basic constructions of a MAD family starts with a perfect tree T the branches of which can be considered an AD family, and extends it to a maximal AD family $\mathcal{A}$ using Zorn's lemma. Since adjoining a new real naturally adds a new branch to T, such a MAD family is necessarily destroyed by **any** forcing adding reals. On the other hand, Kunen [14] constructed a Cohen-indestructible MAD family $\mathcal{B}$ assuming CH, and his method of construction was later extended in various directions by many people. This means the families $\mathcal{A}$ and $\mathcal{B}$ are fundamentally different.

Hrušák [10] and Kurilić [16] independently characterized Cohen-indestructibility of MAD families by using a combinatorial reformulation which doesn't mention forcing or models. Hrušák [10] also investigated Sacks forcing and Miller forcing and, in joint work with García Ferreira [11], showed that Cohen-indestructibility and random-indestructibility are incomparable notions. The latter work also provided a more general framework for indestructibility using the Katětov ordering.

In Section 2 of the present work, we shall continue this line of research and provide a combinatorial characterization of forcing indestructibility of MAD families and, more generally, tall ideals, for many classical forcing notions. The main new idea is that we work with the **G_δ-closure** G_A of a subset A of $2^{<\omega}$ (or $\omega^{<\omega}$), namely, the set of all $x \in 2^\omega$ (ω^ω) such that infinitely many initial segments of x belong to A. The advantage of this approach is that it allows us to treat many rather distinct forcing notions adding real numbers (e.g. tree-like forcings like Sacks forcing **as well as** Cohen and random forcing) in one general framework. Accordingly, we first set up this framework, prove a general characterization theorem saying when a tall ideal is $\mathbb{P}$-indestructible for a given forcing notion $\mathbb{P}$ which falls into this framework (Theorem 2.2.2), and then show that all forcings we consider do indeed satisfy the conditions of the framework. The price we have to pay for this is that these conditions are rather technical, and that it is sometimes rather tedious to verify a given forcing notion satisfies them (this is in particular true for forcing notions adjoining dominating reals). On the other hand, the verification of the latter is trivial in other cases and, furthermore, it is quite clear that the framework also works for many other forcing notions which we have not studied in detail. The actual characterizations, then, are mere corollaries. As an instance we mention:

Theorem 2.4.9. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is random-indestructible.*
(ii) *$\forall B \subseteq 2^{<\omega}$ such that G_B is not null, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I}$ is not null.*
(iii) *$\forall B \subseteq 2^{<\omega}$ such that G_B is not null, $\forall f : B \to \omega$ finite-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I}$ is not null.*

This answers a question of Hrušák [10].[1] An immediate consequence of these characterizations is that we get the following diagram about implications between forcing indestructibility (see 2.1 for the definitions of the forcing notions):

$$\begin{tikzcd}
& \mathbb{B}\text{-indestructible} \arrow[r] & \mathbb{S}\text{-indestructible} \\
\mathbb{L}\text{-indestructible} \arrow[r] & \mathbb{M}\text{-indestructible} \arrow[ur] & \\
\mathbb{D}\text{-indestructible} \arrow[r] \arrow[u] & \mathbb{C}\text{-indestructible} \arrow[u] &
\end{tikzcd}$$

Fig. 1. Diagram of forcing indestructibility.

A natural question is whether any of these arrows is reversible or whether there are any other arrows. An even more fundamental question is whether we can always build a $\mathbb{P}$-indestructible MAD family (or, more generally, tall ideal) for a given forcing notion $\mathbb{P}$ and, if so, whether this construction can be done in **ZFC** alone.

We shall investigate this in Section 3 and show that, indeed, there are no other arrows in the diagram than those shown above. Furthermore, for tall ideals $\mathcal{I}$, constructions of counterexamples to possible further arrows can be done in **ZFC**. This is more tricky for MAD families: first, adding a dominating real destroys all MAD families so that there are no $\mathbb{P}$-indestructible MAD families for forcing notions $\mathbb{P}$ adjoining a dominating real. Furthermore, as of now, even the construction of an $\mathbb{S}$-indestructible MAD family (which is weaker than all of the others) requires hypotheses beyond **ZFC** (see Conjecture 4.4.3) though we do not know whether they are really necessary. Such hypotheses are usually of the form $\mathfrak{j} = \mathfrak{c}$ where $\mathfrak{j}$ is one of the standard cardinal invariants of the continuum. Wherever possible, we shall construct a MAD family of the required kind (which is the more difficult task). Sample results include:

Theorem 3.6.1. *Assume $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$. Then there is a random-indestructible Miller-destructible MAD family of size $\mathfrak{c}$.*

Theorem 3.7.3. *There is a tall ideal $\mathcal{I}$ which is Laver-indestructible yet Cohen-destructible.*

[1] Some characterizations can be done in terms of the Katětov order. We do not know whether this is possible for Theorem 2.4.9, however. See Section 2.4 for more details.

Section 4 provides an in-depth investigation of iterated forcing indestructibility in the case of Sacks forcing $\mathbb{S}$. Apart from characterizing iterated Sacks indestructibility (Sections 4.2, 4.3 and 4.5), we prove:

Theorem 4.4.1. *Assume either* $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ *or* $\mathfrak{b} = \mathfrak{c}$. *There is a MAD family* $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ *which is* $\mathbb{S}$*-indestructible yet* $\mathbb{S}_2$*-destructible.*

Here $\mathbb{S}_2$ denotes the two step iteration of Sacks forcing $\mathbb{S}$.

Theorem 4.7.1. *In the Sacks model (the extension of a model of CH by iteratively adding* ω_2 *Sacks reals with countable support), every iterated Sacks-indestructible MAD family has size* $\aleph_1$.

These results show that the case of Sacks forcing differs from that of Cohen or random forcing while still being somewhat similar; namely, in the case of $\mathbb{C}$ or $\mathbb{B}$, the two step iteration is the same as the single step, so there is no result like 4.4.1; on the other hand, it is well-known that after adding many Cohen (or random) reals to a model of CH, any Cohen (random, respectively) indestructible MAD family must have size $\aleph_1$ (Theorem 4.1.1).

Some of the proofs in this section are rather sketchy (or even left out) because, on the one hand, they are quite technical while, on the other hand, they are rather standard arguments (mostly fusion arguments) in iterated Sacks forcing. In particular, the arguments should be easy to follow for anybody familiar with the representation of iterated Sacks forcing $\mathbb{S}_\alpha$ as Borel sets in $(2^\omega)^\alpha$. See for example, the recent work of Ciesielski and Pawlikowski [7] and of Zapletal [22] for an in-depth investigation of iterated Sacks forcing. The main arguments (4.2, 4.4 and 4.7), however, are done in detail.

1.1. *Notation and basic facts*

Our notation is fairly standard. See [12] or [14] for set theory in general and forcing theory in particular. $\exists^\infty$ means "there are infinitely many $n \in \omega$" and $\forall^\infty$ stands for "for all but finitely many $n \in \omega$". By the reals $\mathbb{R}$, we usually mean the elements of the Cantor space 2^ω, of the Baire space ω^ω, or of $[\omega]^\omega$, the infinite subsets of the natural numbers ω. $\mathcal{B}$ (or $\mathcal{B}(2^\omega)$, $\mathcal{B}(\omega^\omega)$), then, denotes the Borel subsets of $\mathbb{R}$ (of 2^ω or ω^ω, respectively).

Given $s \in 2^{<\omega}$ (or $\omega^{<\omega}$), let $[s] = \{x \in 2^\omega : s \subseteq x\}$, the clopen set given by s. Given a tree $T \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$), let $[T] = \{x \in 2^\omega : x|_n \in T \text{ for all } n \in \omega\}$ denote the set of its **branches**. $stem(T)$ denotes the **stem** of T, that is, the unique $s \in T$ which has at least two immediate successors and is comparable with any $t \in T$. For $s \in T$, $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$ is the **restriction** of T to s.

For $x, y \in \omega^\omega$, say y **eventually dominates** x and write $x \leq^* y$ if $x(n) \leq y(n)$ holds for all but finitely many $n \in \omega$. The **(un)bounding number** $\mathfrak{b}$ is the smallest size of an unbounded family in the structure $(\omega^\omega, \leq^*)$ while the **dominating number** $\mathfrak{d}$ is the least size of a cofinal family in $(\omega^\omega, \leq^*)$. $\mathfrak{c}$ denotes the cardinality of the continuum. A family $\mathcal{A} \subseteq [\omega]^\omega$ is **almost disjoint** (**AD** for short) if $A \cap B$ is finite for any distinct $A, B \in \mathcal{A}$. $\mathcal{A}$ is a **maximal almost disjoint** (**MAD**) family if, additionally, for any $X \in [\omega]^\omega$, there is $A \in \mathcal{A}$ such that $X \cap A$ is infinite. For simplicity, we shall assume through the paper that AD families satisfy $\cup\mathcal{A} = \omega$. The **almost disjointness number** $\mathfrak{a}$ is the least size of a MAD family. We assume familiarity with basic cardinal invariants of the continuum like those mentioned here, as well as with their order relationship. See [1] or [3] for details.

For $A, B \in [\omega]^\omega$, say A is **almost contained** in B and write $A \subseteq^* B$ if $A \setminus B$ is finite. $X \in [\omega]^\omega$ is a **pseudo-intersection** of $\mathcal{F} \subseteq [\omega]^\omega$ if $X \subseteq^* A$ for any $A \in \mathcal{F}$. All ideals $\mathcal{I} \subseteq \mathcal{P}(\omega)$ we will consider in this paper are proper and contain the finite subsets of ω (so $\cup\mathcal{I} = \omega$), that is, they are **free ideals**. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a **tall ideal** if the dual filter $\mathcal{I}^* = \{\omega \setminus A : A \in \mathcal{I}\}$ doesn't have a pseudo-intersection.

It is clear tall ideals are a generalization of MAD families.

Definition 1.1.1. $\mathcal{I}(\mathcal{A}) = \{X \in \mathcal{P}(\omega) : (\exists n)(\exists \langle A_i : i < n\rangle \subseteq \mathcal{A}) X \subseteq^* \bigcup_{i<n} A_i\}$ is the ideal generated by $\mathcal{A}$.

Fact 1.1.2. *For an almost disjoint family $\mathcal{A}$, $\mathcal{A}$ is MAD iff $\mathcal{I}(\mathcal{A})$ is a tall ideal.*

We proceed to argue that families of infinite subsets of ω are closely connected to families of subsets of reals.

Definition 1.1.3 (*G_δ Closure*). For any $A \subseteq 2^{<\omega}$ or $\omega^{<\omega}$, the G_δ closure of A is

$$G_A = \{f \in 2^\omega (\text{ or } \omega^\omega) : (\exists^\infty n \in \omega)\, f|_n \in A\}.$$

Clearly any G_A is a G_δ-set.

Lemma 1.1.4. *The following are equivalent for an ideal $\mathcal{I}$.*

1. *$\mathcal{I}$ is a tall ideal.*
2. *For any $B \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) and any $f : B \to \omega$, $\mathcal{F} = \{G_{f^{-1}``D} : D \in \mathcal{I}\}$ is a covering of G_B.*
3. *For any $f : 2^{<\omega}$(or $\omega^{<\omega}$) $\to \omega$ one-to-one, $\mathcal{F} = \{G_{f^{-1}``D} : D \in \mathcal{I}\}$ is a covering of the real line.*

Proof. First we show 1 implies 2. Assume $\mathcal{F}$ is not a covering, i.e. there is a $g \in G_B \setminus \left(\bigcup_{D\in\mathcal{I}} G_{f^{-1}``D}\right)$. Let $A = \{f(g|_n) : n \in \omega \wedge g|_n \in B\}$. We first argue that $A$ is infinite. Otherwise $A \in \mathcal{I}$, and $g \in G_{f^{-1}``A}$ follows immediately, a contradiction. Next choose $D \in \mathcal{I}$ arbitrarily. If $A \cap D$ was infinite, $g \in G_{f^{-1}``A\cap D} \subseteq G_{f^{-1}``D}$, again a contradiction. Therefore $A \cap D$ is finite. This shows that $\mathcal{I}$ is not tall, the final contradiction.

2 implies 3 is trivial.

To show 3 implies 1, assume $\mathcal{I}$ was not tall and choose $A \in [\omega]^\omega$ such that $A \cap D$ is finite for all $D \in \mathcal{I}$. Fix $g \in 2^\omega$. Define a bijection $f : 2^{<\omega} \to \omega$ such that f maps $\{g|_n : n \in \omega\}$ to A and $2^{<\omega} \setminus \{g|_n : n \in \omega\}$ to $\omega \setminus A$. It is straightforward to see that $g \notin G_{f^{-1}``D}$ for all $D \in \mathcal{I}$, a contradiction to 3. □

For MAD families we additionally have:

Lemma 1.1.5. *Let $\mathcal{A} \subseteq [\omega]^\omega$ and assume for any $f : 2^{<\omega}$(or $\omega^{<\omega}$) $\to \omega$ one-to-one $\mathcal{F} = \{G_{f^{-1}``D} : D \in \mathcal{A}\}$ is a disjoint covering of the real line. Then $\mathcal{A}$ is a MAD family.*

Proof. We first argue that $\mathcal{A}$ is an almost disjoint family. Assume $D_0, D_1 \in \mathcal{A}$, $D_0 \neq D_1$, and $|D_0 \cap D_1| = \aleph_0$. Fix $g \in 2^\omega$. Define a bijection $f : 2^{<\omega} \to \omega$ such that f

maps $\{g|_n : n \in \omega\}$ to $D_0 \cap D_1$ and $2^{<\omega} \setminus \{g|_n : n \in \omega\}$ to $\omega \setminus (D_0 \cap D_1)$. Clearly $g \in G_{f^{-1}\text{''}D_0} \cap G_{f^{-1}\text{''}D_1}$, a contradiction. To see that $\mathcal{A}$ is maximal, use 1.1.2 and 1.1.4. $\square$

In 1.1.5, the converse is false in general.

2. Characterization of forcing indestructibility

Hrušák [10] and Kurilić [16] have characterized forcing indestructibility of MAD families for Cohen forcing $\mathbb{C}$. Using the concept of "G_δ-closure", we will prove analogous results for many classical forcing notions (Sections 2.2–2.4). We start with briefly reviewing the definitions of, and basic facts about, these forcing notions (Section 2.1).

2.1. *Forcing notions and corresponding ideals*

Let $\mathcal{B}(\mathbb{R})$, or $\mathcal{B}$ for short, be the family of Borel sets in $\mathbb{R}$, where $\mathbb{R} = 2^\omega$ or ω^ω. Also assume $I_\mathbb{P} \subseteq \mathcal{B}$ is a σ-ideal. We consider forcing notions of the form $\mathbb{P} = \mathcal{B}/I_\mathbb{P}$, ordered by inclusion modulo $I_\mathbb{P}$. Notice this is forcing equivalent to $\mathcal{B} \setminus I_\mathbb{P}$ ordered by inclusion, and we shall use this description to avoid having to work with equivalence classes. Call such forcing notions **real forcings**. The following is well-known (see [22, Lemma 2.1.1]).

Lemma 2.1.1 (*Zapletal*). *If $G \subseteq \mathbb{P} = \mathcal{B} \setminus I_\mathbb{P}$ is a generic filter, then there is a real $r \in V[G]$ such that a Borel set B coded in V belongs to G iff $r \in B^{V[G]}$.*

We are going to investigate the ideal $I_\mathbb{P}$ corresponding to several famous proper forcing notions.

Sacks forcing Sacks forcing $\mathbb{S}$ is the set of all perfect trees in $2^{<\omega}$ ordered by inclusion.

The perfect set theorem says:

Fact 2.1.2. *For every analytic set $X \subseteq 2^\omega$:*
- *either X is countable,*
- *or X contains a perfect subset.*

Let *cntble* be the ideal of (at most) countable sets of reals. So the above fact shows $\mathbb{S}$ is a dense subset of $\mathcal{B}(2^\omega)/\mathit{cntble}$; these are forcing equivalent.

It is clear that any countable set coded in the ground model doesn't contain a Sacks real (it doesn't contain any new real). In this sense we can say Sacks forcing is the "weakest" forcing which adds a new real.

Miller forcing Miller forcing $\mathbb{M}$ is the set of all rational perfect trees ordered by inclusion.

Definition 2.1.3. 1. A tree $T \subseteq \omega^{<\omega}$ is rational perfect iff T is a tree such that $(\forall t \in T)(\exists s \in T)t \subseteq s \wedge (\exists^\infty n)s\hat{}\langle n\rangle \in T$.
2. A set of reals $B \subseteq \omega^\omega$ is σ-bounded iff there is a countable set $\{x_n \in \omega^\omega : n \in \omega\}$ such that $(\forall y \in B)(\exists k)y \leq x_k$.

An s with $s\hat{}\langle n\rangle \in T$ for infinitely many n as in 1 of the definition is called an ω-**splitting node** of T. We denote by $\mathit{split}(T)$ the set of ω-splitting nodes of T.

It is well-known that

Fact 2.1.4. *For every analytic set $X \subseteq \omega^\omega$:*
- *either X is σ-bounded,*
- *or X contains a rational perfect subset.*

Let $\mathcal{K}_\sigma$ be the ideal of σ-bounded sets. Then the above fact shows $\mathbb{M}$ is a dense subset of $\mathcal{B}(\omega^\omega)/\mathcal{K}_\sigma$; they are forcing equivalent.

Laver forcing Laver forcing $\mathbb{L}$ is the set of all Laver trees ordered by inclusion, where

Definition 2.1.5. 1. $T \subseteq \omega^{<\omega}$ is a Laver tree iff $(\forall t \in T)(\exists^\infty n \in \omega) t\hat{}\langle n\rangle \in T$,
2. A set of reals $B \subseteq \omega^\omega$ is strongly dominating iff for any $\phi : \omega^{<\omega} \to \omega$, there is a $g \in B$ such that $(\forall^\infty n \in \omega) g(n) \geq \phi(g|_n)$.

For more details, see [4,9,20]. Then we have

Fact 2.1.6. *For every analytic set $X \subseteq \omega^\omega$:*
- *either X is not strongly dominating,*
- *or X contains a Laver tree.*

Let *not-dominating* be the ideal of not strongly dominating sets. Then the above fact shows $\mathbb{L}$ is a dense subset of $\mathcal{B}(\omega^\omega)/$*not-dominating*; they are forcing equivalent.

Cohen forcing Cohen forcing $\mathbb{C}$ is the set of finite partial functions $\omega \to 2$ ordered by inclusion. More generally, $\mathbb{C}_\kappa$ is the set of finite partial functions $\kappa \to 2$.

Let $\mathcal{M}$ be the ideal of meager sets of reals. It is well-known $\mathbb{C}$ is a dense subset of $\mathcal{B}(2^\omega)/\mathcal{M}$; they are forcing equivalent.

Fact 2.1.7. *Every analytic set $X \subseteq 2^\omega$ has Baire property; that is, there is an open set U such that $U \triangle X$ is meager.*

Note that Cohen forcing adds an unbounded real.

Random forcing Random forcing $\mathbb{B}$ is the measure algebra $\mathcal{B}(2^\omega)/\mathcal{N}$ where $\mathcal{N}$ is the ideal of null sets of reals. More generally, $\mathbb{B}_\kappa$ is the measure algebra on 2^κ. It is well-known as ω^ω-bounding forcing.

Fact 2.1.8. *Every analytic set is Lebesgue measurable.*

We use μ to denote Lebesgue measure.

Hechler forcing Hechler forcing $\mathbb{D}$ is the following poset.

Definition 2.1.9.

$$\mathbb{D} = \{\langle s, f\rangle : s \in \omega^{\uparrow<\omega} \wedge f \in \omega^{\uparrow\omega} \wedge s \subseteq f\}$$

ordered by

$$\langle s, f\rangle \leq \langle t, g\rangle \iff t \subseteq s \wedge (\forall k)[g(k) \leq f(k)].$$

By definition, $\mathbb{D}$ adds a dominating real. Here we define $\omega^{\uparrow\omega}$ to be the set of all strictly increasing functions from ω to ω. Similarly $\omega^{\uparrow<\omega}$ is the set of all strictly increasing finite sequences. Clearly, $\omega^{\uparrow\omega}$ is homeomorphic to the Baire space ω^ω.

Following [17], we will define the dominating topology $\mathcal{D}$ on $\omega^{\uparrow\omega}$ corresponding to Hechler forcing.

Definition 2.1.10 (*Dominating Topology*). 1. For any $\langle s, f\rangle \in \mathbb{D}$,
$$U_{\langle s,f\rangle} = \{x \in \omega^{\uparrow\omega} : s \subseteq x \wedge f \leq x\}.$$
2. The dominating topology $\mathcal{D}$ is the topology on $\omega^{\uparrow\omega}$ whose base is $\{U_{\langle s,f\rangle} : \langle s, f\rangle \in \mathbb{D}\}$.
3. Let $X_{\mathcal{D}}$ be the topological space $\langle \omega^{\uparrow\omega}, \mathcal{D}\rangle$.

We can define $\mathcal{D}$-meager sets, $\mathcal{D}$-Borel sets $\mathcal{B}_{\mathcal{D}}$, etc. as for the usual topology. It is trivial that $\mathcal{B} \subset \mathcal{B}_{\mathcal{D}}$, etc.

Let $\mathcal{M}_{\mathcal{D}}$ be the ideal of $\mathcal{D}$-meager sets. As in the case of Cohen forcing, $\mathbb{D}$ is a dense subset of $\mathcal{B}(\omega^{\uparrow\omega})/\mathcal{M}_{\mathcal{D}}$; they are forcing equivalent.

For more details, see [17].

In any case, we clearly have

Theorem 2.1.11. *Assume $\mathbb{P}$ is one of the above proper real forcings. Let $\dot{r}_{gen}$ be the name of the $\mathbb{P}$-generic real. Let B be a Borel set coded in the ground model. Then*

- *either $B \in I_{\mathbb{P}}$ (then $\Vdash$ "$\dot{r}_{gen} \notin B$"),*
- *or $B \notin I_{\mathbb{P}}$ (then $B \Vdash$ "$\dot{r}_{gen} \in B$").*

2.2. *Weak fusion*

All forcing notions $\mathbb{P}$ we will consider have a dense set of G_δ's in the following sense:

$$\text{If } B \subseteq 2^{<\omega} \text{ or } \omega^{<\omega} \text{ with } G_B \in \mathbb{P} \text{ (i.e. } G_B \notin I_{\mathbb{P}}), \text{ and } E \leq G_B, \\ \text{then there is } B' \subseteq B \text{ with } G_{B'} \in \mathbb{P} \text{ and } G_{B'} \leq E. \qquad (*)$$

It is clear that $\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}, \mathbb{B}$ and $\mathbb{D}$ have this property with respect to the corresponding ideal.

Definition 2.2.1. Let $\mathbb{P} = \mathcal{B} \setminus I_{\mathbb{P}}$ be a real forcing. Say $\mathbb{P}$ has **weak fusion** if given $E \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{C}$ such that $E \Vdash$ "$\dot{C} \in [\omega]^\omega$", there are

- pairwise disjoint antichains $B_n \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$),
- antichains $\mathcal{A}_n \subseteq \mathbb{P}$,
- one-to-one functions $h_n : B_n \to \mathcal{A}_n$ for $n \in \omega$,
- and a one-to-one function $g : \{(n, A) : n \in \omega \wedge A \in \mathcal{A}_n\} \to \omega$ with $g(n, A) \geq n$

such that

(1) $G_B \leq E$ (in particular $G_B \notin I_{\mathbb{P}}$),
(2) $(\forall B' \subseteq B \text{ with } G_{B'} \in \mathbb{P})(\forall k)(\exists n \geq k)(\exists s \in B_n \cap B')$.
  - $[s] \cap G_{B'} \in \mathbb{P}$, and
  - $[s] \cap G_{B'}$ is compatible with $h_n(s)$,

(3) $(\forall n)(\forall A \in \mathcal{A}_n)\, A \Vdash$ "$g(n, A) \in \dot{C}$",

where $B = \bigcup_{n\in\omega} B_n$.

Let us first check this is enough to get the characterization of $\mathbb{P}$-indestructibility we are heading for. Recall that all ideals $\mathcal{I} \subseteq \mathcal{P}(\omega)$ we consider here are free ideals (i.e. they contain all finite sets).

Theorem 2.2.2. *Assume $\mathbb{P} = \mathcal{B} \setminus I_{\mathbb{P}}$ is a real forcing with weak fusion. Let $\mathcal{I}$ be a tall ideal. Then the following are equivalent:*

(1) *$\mathcal{I}$ is $\mathbb{P}$-indestructible.*
(2) *$\forall B \subseteq 2^{<\omega}$(or $\omega^{<\omega}$) such that $G_B \notin I_{\mathbb{P}}$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin I_{\mathbb{P}}$.*
(3) *$\forall B \subseteq 2^{<\omega}$(or $\omega^{<\omega}$) such that $G_B \notin I_{\mathbb{P}}$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin I_{\mathbb{P}}$.*

Proof. To show (1) implies (2), suppose not (2). Let $B \subseteq 2^{<\omega}$ be such that $G_B \notin I_{\mathbb{P}}$ and $\exists f : B \to \omega$ function, $\forall I \in \mathcal{I}$, $G_{f^{-1}"I} \in I_{\mathbb{P}}$.

Let r be a $\mathbb{P}$-generic real such that $r \in G_B$. Such r exists by the preceding discussion, see Lemma 2.1.1. In particular, if $\dot{r}$ is the name for the generic from 2.1.1, $G_B \Vdash$ "$\dot{r} \in G_B$". Note that $r \notin G_{f^{-1}"I}$ for all $I \in \mathcal{I}$. Namely, since $G_{f^{-1}"I} \in I_{\mathbb{P}}$, $2^\omega \setminus G_{f^{-1}"I}$ belongs to the generic filter, and so $r \in 2^\omega \setminus G_{f^{-1}"I}$.

Since $r \in G_B$, $(\exists^\infty n) r|_n \in B$. Since $\mathcal{I}$ contains all finite sets, f must be finite-to-one on $B \cap \{r|_n : n \in \omega\}$. Therefore $A = \{f(r|_n) : r|_n \in B\}$ is infinite, yet $A \cap I$ is finite for all $I \in \mathcal{I}$.

(2) implies (3) is trivial.

We can show (3) implies (1) by using a fusion argument. Let $E \in \mathbb{P}$, $\dot{C}$ be a $\mathbb{P}$-name such that $E \Vdash$ "$\dot{C} \in [\omega]^\omega$". Let $B_n, \mathcal{A}_n, h_n$ and g be as in the definition of "weak fusion". Then $G_B \le E$ where $B = \bigcup_{n\in\omega} B_n$.

Define $f : B \to \omega$ by $f(s) = g(n, h_n(s))$ for all $s \in B_n$. This makes sense because the B_n are pairwise disjoint. Since h_n and g are one-to-one, so is f. By the hypothesis (3), there is $I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin I_{\mathbb{P}}$, i.e. $G_{f^{-1}"I} \in \mathbb{P}$. Clearly $G_{f^{-1}"I} \le G_B$. To complete the proof, it suffices to show that $G_{f^{-1}"I} \Vdash$ "$|I \cap \dot{C}| = \aleph_0$". For this, it is enough to prove

$$(\forall D \le G_{f^{-1}"I})(\forall k)(\exists l \ge k)(\exists D' \le D)\, D' \Vdash \text{“} l \in I \cap \dot{C} \text{”}.$$

To see this, fix $D \le G_{f^{-1}"I}$ and k. By $(*)$ there is $B' \subseteq f^{-1}"I$ such that $G_{B'} \subseteq D$. By (2) there are $n \ge k$ and $s \in B_n \cap B'$ such that $[s] \cap G_{B'} \in \mathbb{P}$ and $[s] \cap G_{B'}$ is compatible with $h_n(s)$. Let D' be a common extension of $[s] \cap G_{B'}$ and $h_n(s)$, and let $l = g(n, h_n(s)) = f(s) \ge n \ge k$. Since $s \in B'$, $l \in I$. By 3, $D' \Vdash$ "$l \in \dot{C}$", and we are done. $\square$

Note that (1) implies (2) is true for every real forcing. Indeed, "weak fusion" was used only for (3) implies (1).

If we don't care about f being one-to-one, we can get away with a notion which is somewhat simpler than "weak fusion". However, it turns out that having f one-to-one makes the constructions in Section 3 much more lucid, and this is the reason for (3) in Theorem 2.2.2.

We proceed to show that most of our forcing notions satisfy weak fusion.

Lemma 2.2.3. *Sacks forcing $\mathbb{S}$, Miller forcing $\mathbb{M}$, and Laver forcing $\mathbb{L}$ have weak fusion.*

Proof. Since the proofs are all very similar, we do it only for Laver forcing $\mathbb{L}$ which is, in fact, the most difficult case. Here, as well as in a number of subsequent proofs, we shall freely use rank arguments which have become a standard tool in the combinatorial investigation of forcing notions adjoining a dominating real since they have been introduced for Hechler forcing by Baumgartner and Dordal [2].

Fix $E = [T] \in \mathbb{L}$ and an $\mathbb{L}$-name $\dot{C}$ for an element of $[\omega]^\omega$. As usual, we think of Laver forcing as forcing with trees, that is, we identify $[T]$ with T, and consider $T \in \mathbb{L}$. Recursively construct antichains $B'_n \subseteq T$, antichains $\mathcal{A}'_n \subseteq \mathbb{L}$, one-to-one functions $h'_n : B'_n \to \mathcal{A}'_n$ and one-to-one functions $g'_n : \mathcal{A}'_n \to \omega$ with $g'_n(A) \geq n$ for all $A \in \mathcal{A}'_n$ such that

- if $n < m$ and $\sigma \in B'_m$ then $\sigma|_k \in B'_n$ for some $k < |\sigma|$,
- h'_n is onto $\mathcal{A}'_n$ and if $\sigma \in B'_n$, then $h'_n(\sigma)$ is a Laver subtree of T with stem σ,
- for all n and $\sigma \in B'_n$, $\bigcup\{h'_{n+1}(\tau) : \sigma \subseteq \tau, \tau \in B'_{n+1}\}$ is a Laver subtree of $h'_n(\sigma)$ with stem σ,
- $h'_n(\sigma) \Vdash$ "$g'_n(h'_n(\sigma)) \in \dot{C}$".

Let us argue that this construction can be carried out. Suppose $n \in \omega$ and for all $m < n$, $B'_m, \mathcal{A}'_m, h'_m$ and g'_m have been constructed as required (possibly $n = 0$). We construct $B'_n, \mathcal{A}'_n, h'_n$ and g'_n. Fix $\sigma \in B'_{n-1}$ (where we put $B'_{-1} = \{stem(T)\}$). $h'_{n-1}(\sigma) \in \mathcal{A}'_{n-1}$ is a Laver subtree of T with stem σ by inductive assumption (where $h'_{-1}(stem(T)) = T$ and $\mathcal{A}'_{-1} = \{T\}$). For $\tau \in h'_{n-1}(\sigma)$, $|\tau| > |\sigma|$, define the rank function $rk(\tau)$ by recursion as follows.

- $rk(\tau) = 0 \iff \exists h'_n(\tau)$ a subtree of $h'_{n-1}(\sigma)$ with stem τ and $\exists g'_n(h'_n(\tau)) \geq \tau(|\sigma|)$ such that
$$h'_n(\tau) \Vdash \text{“}g'_n(h'_n(\tau)) \in \dot{C}\text{”}.$$
- $rk(\tau) \leq \alpha \iff \exists^\infty l \in \omega$ such that $\tau\hat{\ }\langle l\rangle \in h'_{n-1}(\sigma)$ and $rk(\tau\hat{\ }\langle l\rangle) < \alpha$.

A standard rank argument shows that all $\tau \in h'_n(\sigma)$, $|\tau| > |\sigma|$, have rank $< \infty$. Therefore we may find $B'_{n,\sigma} \subseteq h'_{n-1}(\sigma)$ such that $B'_{n,\sigma}$ is an antichain, $\tau \in B'_{n,\sigma}$ implies $rk(\tau) = 0$, and $\bigcup\{h'_n(\tau) : \sigma \subseteq \tau, \tau \in B'_{n,\sigma}\}$ is a Laver subtree of $h'_{n-1}(\sigma)$ with stem σ. Let $\mathcal{A}'_{n,\sigma}$ be the image of $B'_{n,\sigma}$ under h'_n. Clearly $h'_n|_{B'_{n,\sigma}}$ is one-to-one.

By pruning $B'_{n,\sigma}$ (and thus $\mathcal{A}'_{n,\sigma}$) but keeping the remaining properties, we may assume g'_n is one-to-one on $\mathcal{A}'_{n,\sigma}$. The point is that whenever $rk(\tau) = 1$, and there are infinitely many l such that $\tau\hat{\ }\langle l\rangle \in B'_{n,\sigma}$, then g'_n must be finite-to-one on $\{h'_n(\tau\hat{\ }\langle l\rangle) : \tau\hat{\ }\langle l\rangle \in B'_{n,\sigma}\}$ for otherwise $rk(\tau) = 0$, a contradiction. This means that we can make g'_n one-to-one, simultaneously for all such τ, and still keep infinitely many l with $\tau\hat{\ }\langle l\rangle \in B'_{n,\sigma}$.

Now unfix σ, and let $B'_n = \bigcup\{B'_{n,\sigma} : \sigma \in B'_{n-1}\}$, $\mathcal{A}'_n = \bigcup\{\mathcal{A}'_{n,\sigma} : \sigma \in B'_{n-1}\}$. Clearly, $h'_n|_{B'_n}$ is still one-to-one, and a further pruning argument along the same lines shows we may assume that so is $g'_n|_{\mathcal{A}'_n}$. Clearly all of the required properties are satisfied, and the construction is complete.

Clearly, if $B' = \bigcup_{n\in\omega} B'_n$, then properties (1), (2), and (3) in Definition 2.2.1 are satisfied for $B'_n, \mathcal{A}'_n, h'_n$ and g' given by $g'(n, A) = g'_n(A)$. However, g' may not be one-to-one. Yet it is easy to see that a simultaneous pruning argument yields $B_n \subseteq B'_n$, $\mathcal{A}_n = h'_n{}''B_n$, $h_n = h'_n|_{B_n}$, $g = g'|_{\bigcup_n \{n\}\times\mathcal{A}_n}$ which still have the properties exhibited in the above recursive construction and such that g is one-to-one. This completes the proof. $\square$

We leave the following proof to the reader (in fact, this is similar to, but much simpler than, Lemma 2.2.5 below).

Lemma 2.2.4. *Cohen forcing $\mathbb{C}$ has weak fusion.*

Note that both in 2.2.3 and in 2.2.4 one in fact proves

(2') $(\forall n)(\forall s \in B_n)[s] \cap G_B \leq h_n(s)$

instead of (2) in Definition 2.2.1 of "weak fusion". To see (2') implies (2), it suffices to note that whenever $B' \subseteq B$ with $G_{B'} \in \mathbb{P}$ and $k \in \omega$ are given, then there are indeed $n \geq k$ and $s \in B_n \cap B'$ with $[s] \cap G_{B'} \in \mathbb{P}$. For, if $[s] \cap G_{B'} \in I_{\mathbb{P}}$ for all such s, then $G_{B'} = \bigcup\{[s] \cap G_{B'} : s \in B' \cap \bigcup_{n \geq k} B_n\} \in I_{\mathbb{P}}$ because $I_{\mathbb{P}}$ is a σ-ideal, a contradiction.

Lemma 2.2.5. *Hechler forcing $\mathbb{D}$ has weak fusion.*

Proof. Recall that we think of $\mathbb{D}$ as $\mathcal{B} \setminus \mathcal{M}_{\mathcal{D}}$ (i.e. as a Boolean algebra) where $\mathcal{M}_{\mathcal{D}}$ is the family of meager sets in the dominating topology. We will use the rank analysis of $\mathbb{D}$ due to Baumgartner and Dordal [2].

Fix $E = \langle s, x \rangle \in \mathbb{D}$ (which we identify with $U_{\langle s,x \rangle}$) and a $\mathbb{D}$-name $\dot{C}$ for an element of $[\omega]^\omega$. Recall that conditions in a dense subset of $\mathbb{D}$ are of the form $\langle s, x \rangle$ where $s \subseteq x, s \in \omega^{\uparrow<\omega}$ and $x \in \omega^{\uparrow\omega}$ are strictly increasing. Say t is **compatible with** $\langle s, x \rangle$ if $t \in \omega^{\uparrow<\omega}$, $s \subseteq t$ and $t(i) \geq x(i)$ for all $i \in |t|$. Recursively construct sets $X_n \subseteq \omega^{\uparrow<\omega}$, $Y_n \subseteq \omega^{\uparrow<\omega}$ and $\langle t_i^{t'} : i \in \omega \rangle$, $\langle m_i^{t'} : i \in \omega \rangle$, $\langle A_i^{t'} : i \in \omega \rangle$ for $t' \in Y_n$ such that

1. X_n is a maximal antichain of $t \in \omega^{\uparrow<\omega}$ compatible with $\langle s, x \rangle$,
2. Y_n is an antichain of $t' \in \omega^{\uparrow<\omega}$ compatible with $\langle s, x \rangle$,
3. for all $t \in X_n$ and all $y \in \omega^{\uparrow\omega}$ with $t \subseteq y$, there is $t' \in Y_{n+1}$ compatible with $\langle t, y \rangle$,
4. for all $t' \in Y_{n+1}$ there is $l \leq |t'|$ with $t'|_l \in X_n$,
5. for all $t \in X_{n+1}$ there is $l \leq |t|$ with $t|_l \in X_n$,
6. for all $t' \in Y_n$ there is $l > |t'|$ such that for all i, $t' \subseteq t_i^{t'}$, and $|t_i^{t'}| = l$, $t_i^{t'}(|t'|) \geq i$,
7. for $t' \in Y_n$, if $i \neq j$ then $m_i^{t'} \neq m_j^{t'}$,
8. $t_i^{t'} \in X_n$ whenever $t' \in Y_n$ and $i \in \omega$,
9. the $A_i^{t'}$, $i \in \omega$, $t' \in Y_n$, are an antichain in $\mathbb{D}$,
10. $A_i^{t'} \Vdash$ "$m_i^{t'} \in \dot{C}$",
11. $A_i^{t'}$ is compatible with any condition of the form $\langle t_i^{t'}, y \rangle$.

Set $X_{-1} = \{s\}$. Assume X_n has been constructed ($n \geq -1$). We describe how to produce Y_{n+1} and X_{n+1}. Fix $t \in X_n$. Let $\dot{m}_t$ be a name for natural number such that

$$\Vdash \text{“}\dot{m}_t \in \dot{C} \wedge \dot{m}_t \geq \dot{d}(|t|)\text{”}$$

where $\dot{d}$ is the name for the $\mathbb{D}$-generic real. For $t' \supseteq t$ compatible with $\langle t, y \rangle$ where $t \subseteq y$ and $y(i) = \max\{y(i-1)+1, x(i)\}$ for $i \geq |t|$, and $m \in \omega$ define $rk_t^m(t')$ by recursion as follows.

- $rk_t^m(t') = 0 \iff (\exists x' \supseteq t')\, \langle t', x' \rangle \Vdash$ "$\dot{m}_t = m$",
- $rk_t^m(t') \le \alpha \iff (\exists l > |t'|)(\exists \langle t_n : n \in \omega \rangle)\, t' \subseteq t_n, |t_n| = l, t_n(|t'|) \ge n, rk_t^m(t_n) < \alpha$.

Note that $rk_t^m(t) = \infty$ for all m. (For if we had $rk_t^m(t) < \infty$ for some m, we could find $t' \supseteq t$ compatible with $\langle t, y \rangle$ and $x' \supseteq t'$ such that $t(|t|) > m$ and $\langle t', x' \rangle \Vdash$ "$\dot{m}_t = m$". This contradicts $\Vdash$ "$\dot{m}_t \ge \dot{d}(|t|)$".)

Next define $rk_t(t')$ for such t' by:

- $rk_t(t') = 0 \iff (\exists m)\, rk_t^m(t') < \infty$,
- $rk_t(t') \le \alpha \iff (\exists l > |t'|)(\exists \langle t_n : n \in \omega \rangle)\, t' \subseteq t_n, |t_n| = l, t_n(|t'|) \ge n, rk_t(t_n) < \alpha$.

A standard argument shows that $rk_t(t') < \infty$ for all t' compatible with $\langle t, y \rangle$. In particular $0 < rk_t(t) < \infty$. Unfix t. Let

$$Y_{n+1} = \{t' : \text{ for some } t \in X_n, rk_t(t') = 1 \wedge rk_t(t'|_l) > 1 \text{ for all } |t| \le l < |t'|\}.$$

The above conditions 2 and 4 are immediate and 3 can be shown by a standard rank argument.

For $t' \in Y_{n+1}$, choose $\langle t_i^{t'} : i \in \omega \rangle$ and $\langle m_i^{t'} : i \in \omega \rangle$ such that for some $l > |t'|, t' \subseteq t_i^{t'}, |t_i^{t'}| = l, t_i^{t'}(|t|) \ge i$ and $rk_t^{m_i^{t'}}(t_i^{t'}) < \infty$, and such that the $m_i^{t'}$ are pairwise distinct (this is clearly possible: choose $t_i^{t'}$ such that $rk_t(t_i^{t'}) = 0$ and $m_i^{t'}$ such that $rk_t^{m_i^{t'}}(t_i^{t'}) < \infty$; since $rk_t(t') > 0$ the $m_i^{t'}$'s are distinct without loss of generality). This gives us 6 and 7. Let X_{n+1} be any maximal antichain satisfying 5 and containing the $t_i^{t'}$ for $t' \in Y_n$ and $i \in \omega$. So 1 and 8 hold. For $t' \in Y_{n+1}$, let $A_i^{t'}$ be the union of all conditions of the form $\langle t'', x'' \rangle$ where $t'' \supseteq t_i^{t'}$ is compatible with $\langle t, y \rangle, rk_t^{m_i^{t'}}(t'') = 0$ and $\langle t'', x'' \rangle \Vdash$ "$\dot{m}_t = m_i^{t'}$". 9, 10 and 11 are immediate. This completes the recursive construction.

By shrinking the collection of $t_i^{t'}$ and $m_i^{t'}$ if necessary (but preserving the X_n and Y_n), we may assume without loss of generality that the $m_i^{t'}$ are pairwise distinct for all i and all t'. Now let $B_n = \{t_i^{t'} : i \in \omega \wedge t' \in Y_n\}, \mathcal{A}_n = \{A_i^{t'} : i \in \omega \wedge t' \in Y_n\}, h_n(t_i^{t'}) = A_i^{t'}$ and $g(n, A_i^{t'}) = m_i^{t'}$. Put $B = \bigcup_n B_n$. By 1, 3 and 6 we see that G_B is $\mathcal{M}_{\mathcal{D}}$-dense in $E = \langle s, x \rangle$, so in fact $G_B = E$ modulo $\mathcal{M}_{\mathcal{D}}$ and (1) in Definition 2.2.1 holds. Let $B' \subseteq B$ with $G_{B'} \in \mathbb{D}$ and $k \in \omega$. For some $n \ge k$, there must be $t \in B_n \cap B'$ such that $[t] \cap G_{B'}$ is $\mathcal{M}_{\mathcal{D}}$-dense in $\langle t, y \rangle$ for some y. Say $t = t_i^{t'}$ where $t' \in Y_n$ and $i \in \omega$. By 11 we know $[t_i^{t'}] \cap G_{B'}$ is compatible with $A_i^{t'}$. So 2.2.1(2) holds. Finally, 2.2.1(3) follows from 10. □

2.3. Random forcing

It is easy to see that random forcing does not satisfy weak fusion in the sense of the preceding section. However, we get the following result which is only a slight weakening of "weak fusion".

Lemma 2.3.1. *Random forcing $\mathbb{B}$ satisfies all clauses in the definition of "weak fusion" except for the assumption that g be one-to-one. However, we may require g is finite-to-one.*

Proof. Let $E \in \mathbb{B}$, and let $\dot{C}$ be a $\mathbb{B}$-name such that $E \Vdash$ "$\dot{C} \in [\omega]^\omega$". Also let μ be Lebesgue measure on 2^ω.

Recursively build

- finite antichains $B_n \subseteq 2^{<\omega}$,
- conditions $E_n \in \mathbb{B}$,
- finite antichains $\mathcal{A}_n \subseteq \mathbb{B}$,
- bijections $h_n : B_n \to \mathcal{A}_n$,
- and a function $g : \{(n, A) : n \in \omega \wedge A \in \mathcal{A}_n\} \to \omega$

such that

1. $n < m$ and $\sigma \in B_m$ implies $\sigma|_k \in B_n$ for some $k < |\sigma|$,
2. $\mu(E_n) \geq \mu(E) \cdot (\frac{1}{2} + \frac{1}{2^{n+2}})$, $E_{n+1} \leq E_n \leq E$,
3. $E_n = \bigcup \mathcal{A}_n$,
4. for $\sigma \in B_n$, $h_n(\sigma) = [\sigma] \cap E_n$,
5. for $A \in \mathcal{A}_n$, $A \Vdash$ "$g(n, A) \in \dot{C}$",
6. if $n < m$, $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_m$, then $g(n, A) < g(m, B)$.

Fix n, and assume $B_m, E_m, \mathcal{A}_m, h_m, g$ have been constructed for $m < n$. Consider E_{n-1} with the convention $E_{-1} = E$. Set $l_{n-1} := \max\{g(n-1, A) : A \in \mathcal{A}_{n-1}\}$ with $l_{-1} = 0$. For each $l > l_{n-1}$, let $E^l = \|l \in \dot{C}\| \cap E_{n-1}$. Since $\mu(E_{n-1}) \geq \mu(E) \cdot (\frac{1}{2} + \frac{1}{2^{n+1}})$, we can find l_n such that $\mu\left(\bigcup\{E^l : l_{n-1} < l \leq l_n\}\right) \geq \mu(E) \cdot (\frac{1}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}})$. Since every measurable set can be approximated by a basic clopen set, we may find $B^l \subseteq 2^{<\omega}$ such that $B_n = \bigcup\{B^l : l_{n-1} < l \leq l_n\}$ is an antichain satisfying 1 and such that if we let $h_n(\sigma) = [\sigma] \cap E^l$ for $\sigma \in B^l$, $\mathcal{A}_n = \{h_n(\sigma) : \sigma \in B_n\}$, and $E_n = \bigcup \mathcal{A}_n$, then $\mu(E_n) \geq \mu(E) \cdot (\frac{1}{2} + \frac{1}{2^{n+2}})$. So 2, 3 and 4 hold. For $\sigma \in B^l$, we let $g(n, h_n(\sigma)) = l$, and 5 and 6 follow. This completes the recursive construction.

By 6, g is finite-to-one. By 2, $E_\infty = \bigcap_n E_n$ satisfies $\mu(E_\infty) \geq \frac{\mu(E)}{2}$, and therefore $E_\infty \in \mathbb{B}$, and $E_\infty \leq E$. $E_\infty = G_B$ is easy to see, and thus property 2.2.1(1) is satisfied. By 4, for all n and all $\sigma \in B_n$, $[\sigma] \cap E_\infty \leq [\sigma] \cap E_n = h_n(\sigma)$ so that (2') (see after 2.2.4) and, a fortiori, 2.2.1(2) holds. Condition 5 is property 2.2.1(3). $\square$

2.4. Characterizations

Before explicitly stating the characterizations of $\mathbb{P}$-indestructibility of MAD families for our forcing notions $\mathbb{P}$, we briefly consider the following notion which simplifies the characterization in several cases.

Definition 2.4.1. Say an ideal $I_\mathbb{P} \subseteq \mathcal{B}$ is **strongly homogeneous** if for all $B \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) with $G_B \notin I_\mathbb{P}$, there is an injection $h : 2^{<\omega}$(or $\omega^{<\omega}$) $\to B$ such that for all $C \subseteq B$, if $G_{h^{-1}``C} \notin I_\mathbb{P}$ then $G_C \notin I_\mathbb{P}$.

Proposition 2.4.2. *Let $\mathcal{I}$ be a tall ideal, and assume $I_\mathbb{P} \subseteq \mathcal{B}$ is strongly homogeneous. Then the following are equivalent.*

1. $\forall B \subseteq 2^{<\omega}$*(or* $\omega^{<\omega}$*) with* $G_B \notin I_\mathbb{P}$, $\forall f : B \to \omega$ *(one-to-one),* $\exists I \in \mathcal{I}$ *with* $G_{f^{-1}``I} \notin I_\mathbb{P}$,
2. $\forall f : 2^{<\omega}$*(or* $\omega^{<\omega}$*)* $\to \omega$ *(one-to-one),* $\exists I \in \mathcal{I}$ *with* $G_{f^{-1}``I} \notin I_\mathbb{P}$.

Proof. 1 implies 2 is trivial.

To show 2 implies 1, let $B \subseteq 2^{<\omega}$ such that $G_B \notin I_{\mathbb{P}}$, $f : B \to \omega$ (one-to-one). Let h be as in the definition of strong homogeneity. In case f is one-to-one, $f \circ h$ is also one-to-one. So there is $I \in \mathcal{I}$ such that $G_{(f\circ h)^{-1}"I} \notin I_{\mathbb{P}}$. Therefore $G_{f^{-1}"I} \notin I_{\mathbb{P}}$. $\square$

We leave it to the reader to verify that *cntble*, $\mathcal{K}_\sigma$, and $\mathcal{M}$ are strongly homogeneous (think of $\mathcal{M}$ as an ideal on the Baire space ω^ω). The proofs are straightforward.

Definition 2.4.3 (*Hrušák and García Ferreira [11]*). Let $\mathcal{J}, \mathcal{I}$ be ideals on ω. Say $\mathcal{J} \leq_K \mathcal{I}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}"I \in \mathcal{I}$ for every $I \in \mathcal{J}$. $\leq_K$ is called the Katětov ordering.

Put $\mathcal{I}_{\mathbb{P}} = \{I \subseteq 2^{<\omega}(\text{ or } \omega^{<\omega}) : G_I \in I_{\mathbb{P}}\}$. By 2.2.2, $\mathcal{I}_{\mathbb{P}}$ is $\mathbb{P}$-destructible. In fact,

Proposition 2.4.4 (*Hrušák, Private Communication*). *Assume $\mathbb{P}$ has weak fusion and $I_{\mathbb{P}}$ is strongly homogeneous. The following are equivalent for a tall ideal $\mathcal{J}$.*

1. *$\mathcal{J}$ is $\mathbb{P}$-destructible,*
2. *$\mathcal{J} \leq_K \mathcal{I}_{\mathbb{P}}$.*

Proof. First we show 1 implies 2. If $\mathcal{J}$ is $\mathbb{P}$-destructible, then by 2.4.2 and 2.2.2 there is $f : 2^{<\omega} \to \omega$ such that $G_{f^{-1}"J} \in I_{\mathbb{P}}$ for all $J \in \mathcal{J}$. So $f^{-1}"J \in \mathcal{I}_{\mathbb{P}}$ for all $J \in \mathcal{J}$. Thus $\mathcal{J} \leq_K \mathcal{I}_{\mathbb{P}}$.

To show 2 implies 1 is analogous. $\square$

Note that 2 implies 1 uses neither of the assumptions on $\mathbb{P}$ (because it uses only the easy direction of 2.2.2). Putting together everything we proved so far, we get:

Theorem 2.4.5. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{S}$-indestructible.*
(ii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin$ cntble, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin$ cntble.*
(iii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin$ cntble, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin$ cntble.*
(iv) *$\forall f : 2^{<\omega} \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin$ cntble.*
(v) *$\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{S}} = \{I \subseteq 2^{<\omega} : G_I \in$ cntble$\}$.*
(vi) *$\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ which adds a new real.*

Proof. The equivalence from (i) through (iii) follows from Theorem 2.2.2 and Lemma 2.2.3. (iv) is Proposition 2.4.2 and the comment after the proposition. (v) is Proposition 2.4.4. Concerning (vi) note that (i) implies (vi) is trivial, and the proof of (vi) implies (ii) is identical to the first part of Theorem 2.2.2. To see this, simply note that any forcing adding a new real in fact adds a new real belonging to a given uncountable Borel set coded in the ground model and that any new real must avoid any countable set coded in the ground model. $\square$

A few remarks concerning this theorem are in order. The equivalence of (i) and (vi) is due to Hrušák [10]. The basic pattern of the above result is also due to Hrušák: he attempted a characterization along the same line, but there is a gap in his argument. Namely, instead of considering the G_δ-closure G_B of a set $B \subseteq 2^{<\omega}$, he considered the closure $\bar{B}$, that is, the set of branches through the tree defined from B by closing B under initial segments. Clearly $G_B \subseteq \bar{B}$, but the converse inclusion doesn't hold in general.

In fact, it can be shown by a tedious though not difficult argument that assuming, say, CH there is a MAD family $\mathcal{A}$ on ω which satisfies (iv) with G_B replaced by $\bar{B}$ while being $\mathbb{S}$-destructible. So his characterization is ultimately incorrect.

A similar remark applies to Hrušák's characterization of $\mathbb{M}$-indestructibility a correct version of which we present in the next theorem whose proof is exactly analogous.

Theorem 2.4.6. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{M}$-indestructible.*
(ii) *$\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{K}_\sigma$.*
(iii) *$\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{K}_\sigma$.*
(iv) *$\forall f : \omega^{<\omega} \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{K}_\sigma$.*
(v) *$\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{M}} = \{I \subseteq \omega^{<\omega} : G_I \in \mathcal{K}_\sigma\}$.*
(vi) *$\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ which adds an unbounded real.*

Theorem 2.4.7. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{L}$-indestructible.*
(ii) *$\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin$ not-dominating, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin$ not-dominating.*
(iii) *$\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin$ not-dominating, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin$ not-dominating.*
(iv) *$\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ which adds a dominating real.*

Proof. The equivalence from (i) through (iii) follows again from Theorem 2.2.2 and Lemma 2.2.3. (i) implies (iv) is trivial, and for (iv) implies (ii) argue as follows. If $G_B \notin$ not-dominating, then G_B contains a Laver tree $[T]$ which is homeomorphic to ω^ω. Call a real $g \in \omega^\omega$ **strongly dominating** if for any $\phi : \omega^{<\omega} \to \omega$ in the ground model, $(\forall^\infty n \in \omega) g(n) \geq \phi(g|_n)$. Clearly any strongly dominating real is dominating while the converse fails in general. However, it is well-known (and easy to see) that whenever there is a dominating real over some model V of **ZFC**, then there is also a strongly dominating real over V. Moreover, a strongly dominating real must avoid all sets from not-dominating coded in the ground model. Therefore, the argument in the first half of the proof of Theorem 2.2.2 applies, and we get (iv) implies (ii). □

Theorem 2.4.8 (*Hrušák [10], Kurilić [16]*). *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{C}$-indestructible.*
(ii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{M}$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{M}$.*
(iii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{M}$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{M}$.*
(iv) *$\forall f : 2^{<\omega} \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{M}$.*
(v) *$\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{C}} = \{I \subseteq 2^{<\omega} : G_I \in \mathcal{M}\}$.*

The proof is analogous to the proof of Theorem 2.4.5, using 2.2.2, 2.2.4 and 2.4.1. The above result is phrased somewhat differently in Kurilić's and Hrušák's work, but is essentially the same. For example, in Hrušák's work, the stipulation in (iv) above is that $\overline{f^{-1}"I}$ is not nowhere dense. This is, however, the same as saying that $G_{f^{-1}"I} \notin \mathcal{M}$ for, clearly, $G_{f^{-1}"I} \subseteq \overline{f^{-1}"I}$ and, on the other hand, if $[s]$ is a clopen subset of $\overline{f^{-1}"I}$, then $[s] \cap G_{f^{-1}"I}$ is dense in $[s]$. A similar comment applies to Kurilić's characterization.

Theorem 2.4.9. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{B}$-indestructible.*
(ii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{N}$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin \mathcal{N}$.*
(iii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{N}$, $\forall f : B \to \omega$ finite-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin \mathcal{N}$.*

This is clear by Theorem 2.2.2 and Lemma 2.3.1. This characterization answers a question of Hrušák [10, Question 9].

Finally we have, by 2.2.2 and Lemma 2.2.5:

Theorem 2.4.10. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{D}$-indestructible.*
(ii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{M}_{\mathcal{D}}$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin \mathcal{M}_{\mathcal{D}}$.*
(iii) *$\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{M}_{\mathcal{D}}$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}"I} \notin \mathcal{M}_{\mathcal{D}}$.*

3. The hierarchy of forcing indestructibility

After reviewing some basic notions as well as some known results about the existence of indestructible MAD families (Sections 3.1–3.3), we prove a number of theorems saying there are MAD families (or, at least, tall ideals) which are $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ while being destructible for other forcing notions (Sections 3.4–3.7).

3.1. *The covering and additivity number of ideals*

Here we introduce covering numbers and additivity numbers related to ideals. We will see they are deeply connected with forcing indestructibility.

Definition 3.1.1 (*Covering and Additivity Number*). We define two basic cardinal invariants as follows:

1. $\mathsf{cov}(I) = \min\{|\mathcal{A}| : \mathbb{R} = \bigcup_{A\in\mathcal{A}} A \wedge \mathcal{A} \subseteq I\}$,
2. $\mathsf{add}(I) = \min\{|\mathcal{A}| : \bigcup_{A\in\mathcal{A}} A \notin I \wedge \mathcal{A} \subseteq I\}$.

It is easy to see that $\mathsf{add}(I) \leq \mathsf{cov}(I)$ for any ideal I.

First we will investigate covering numbers of the ideals which correspond to forcing notions.

Sacks forcing $\mathsf{cov}(\mathit{cntble}) = \mathfrak{c}$ and $\mathsf{add}(\mathit{cntble}) = \omega_1$.
Miller forcing $\mathsf{cov}(\mathcal{K}_\sigma) = \mathfrak{d}$ and $\mathsf{add}(\mathcal{K}_\sigma) = \mathfrak{b}$ (see [1]).
Laver forcing $\mathsf{cov}(\mathit{not\text{-}dominating}) = \mathsf{add}(\mathit{not\text{-}dominating}) = \mathfrak{b}$ (see [23] and [22]).
Cohen forcing This is just $\mathsf{cov}(\mathcal{M})$ and $\mathsf{add}(\mathcal{M})$.
Random forcing Similarly, this is just $\mathsf{cov}(\mathcal{N})$ and $\mathsf{add}(\mathcal{N})$.
Hechler forcing $\mathsf{cov}(\mathcal{M}_\mathcal{D}) = \mathsf{add}(\mathcal{M})$ and $\mathsf{add}(\mathcal{M}_\mathcal{D}) = \omega_1$ (see [17]).

These numbers are important when we construct examples of $\mathbb{P}$-indestructible MAD families. Especially, we will use the following lemma.

Lemma 3.1.2. *Assume we have the following characterization of $\mathbb{P}$-indestructibility of MAD families: for any MAD family $\mathcal{A}$, $\mathcal{A}$ is $\mathbb{P}$-indestructible iff for any $B \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) such that $G_B \notin I_\mathbb{P}$ and any $f : B \to \omega$ there exists some $D \in \mathcal{A}$ such that $G_{f^{-1}\text{"}D} \notin I_\mathbb{P}$. Also assume $\mathbb{P}$ is homogeneous in the sense that $\mathsf{cov}(I_\mathbb{P}|_G) = \mathsf{cov}(I_\mathbb{P})$ for all Borel sets $G \notin I_\mathbb{P}$.*

Let $\mathcal{A}$ be a MAD family of size less than $\mathsf{cov}(I_\mathbb{P})$. Then $\mathcal{A}$ is $\mathbb{P}$-indestructible.

Proof. Fix any B and any function $f : B \to \omega$. Assume our hypothesis about the characterization, and $\mathcal{A}$ is a MAD family of size less than $\mathsf{cov}(I_\mathbb{P})$. By Lemma 1.1.4, we have our $\{G_{f^{-1}\text{"}D} : D \in \mathcal{A}\}$ is a covering family of G_B of size less than $\mathsf{cov}(I_\mathbb{P})$. Then there must be some $D \in \mathcal{A}$ such that $G_{f^{-1}\text{"}D} \notin I_\mathbb{P}$. □

Corollary 3.1.3. *Assume we are in the situation of Lemma 3.1.2 and $\mathfrak{a} < \mathsf{cov}(I_\mathbb{P})$ holds. Then there is a $\mathbb{P}$-indestructible MAD family.*

Proof. Any MAD family of size $\mathfrak{a}$ is $\mathbb{P}$-indestructible. □

Hrušák [10] used this lemma implicitly when he constructed an $\mathbb{S}$-indestructible MAD family. We shall use it below in the proof of 3.4.1 and 4.6.1. In the special case $\mathbb{P} = \mathbb{C}$ (Cohen forcing) and $I_\mathbb{P} = \mathcal{M}$, 3.1.3 was proved by Hrušák [10, Proposition 6] and Kurilić [16, Corollary 3] (independently). Note that, even in the situation the assumption of Corollary 3.1.3 holds, it still remains a problem whether a $\mathbb{P}$-indestructible MAD family of size continuum exists or not.

3.2. The existence of indestructible MAD families

First we address: is the diagram (Fig. 1) really meaningful: can we construct a $\mathbb{P}$-indestructible MAD family or tall ideal, for any forcing $\mathbb{P}$?

Cohen forcing One can construct indestructible MAD families by forcing or under some cardinal invariant hypothesis.

For example,

Theorem 3.2.1. (1) *(Kunen) Assume CH. Then there exists a MAD family of size $\aleph_1$ which is $\mathbb{C}_\kappa$-indestructible for any κ.*
(2) *(Steprāns) After adding $\aleph_1$ many Cohen reals, there is a MAD family of size $\aleph_1$ which is $\mathbb{C}$-indestructible.*

Proof. See [14] and [21]. □

Note that by the product lemma, $\mathbb{C}$-indestructible and $\mathbb{C}_\kappa$-indestructible is the same thing.

Lemma 3.2.2 (*Hrušák, Kurilić*). *$\mathfrak{b} = \mathfrak{c}$ implies the existence of a $\mathbb{C}$-indestructible MAD family.*

For more details, see [10, Proposition 6] and [16, Theorem 6].

Steprāns [21] raised a question: can we construct a $\mathbb{C}$-indestructible MAD family under **ZFC**? Throughout this paper, we will see forcing indestructibility and the covering number of the corresponding ideal are closely related, so his question seems to have a negative answer.

Random forcing The following theorem is well-known.

Theorem 3.2.3. *Assume CH. There is a $\mathbb{B}_\kappa$-indestructible MAD family of size $\aleph_1$ for any κ.*

Proof. One can show $\mathfrak{a} = \aleph_1$ in the random model, by adapting the proof of Theorem 3.2.1(1). The MAD family witnessing this is in fact $\mathbb{B}_\kappa$-indestructible for any κ. For more details, see [3, Section 11.4]. □

Laver and Hechler These forcings add a dominating real, so there are no $\mathbb{L}$- (and $\mathbb{D}$-) indestructible MAD families.

In the following subsections, we will construct $\mathbb{P}$-indestructible MAD families (or tall ideals), for any forcing $\mathbb{P}$.

3.3. The hierarchy of forcing indestructibility

We can easily see, for example, any not σ-bounded subset of reals is uncountable. So it is clear any $\mathbb{M}$-indestructible MAD family is also $\mathbb{S}$-indestructible.

Using the characterizations of Section 2, we can build a hierarchy of forcing indestructibility, see Fig. 1.

Looking at that diagram, we may ask: do the converses of these implications hold? In other words, for example, does $\mathbb{M}$-destructibility imply $\mathbb{S}$-destructibility? Or, is there an $\mathbb{S}$-indestructible, $\mathbb{M}$-destructible MAD family?

It is known $\mathbb{C}$-indestructibility doesn't imply $\mathbb{B}$-indestructibility.

Definition 3.3.1. 1. Two partial functions $f, g \in \omega^\omega$ are eventually different iff $|f \cap g| < \aleph_0$.

2. A family $\mathcal{A}$ is a maximal family of eventually different partial functions iff $\mathcal{A}$ is a family of eventually different partial functions which is maximal.

Theorem 3.3.2. *Assume CH. There is a maximal family of eventually different partial functions of size $\aleph_1$ which is $\mathbb{C}$-indestructible.*

Proof. One can construct such a family by adapting the proof of Theorem 3.2.1(1). For more details, see [24, Theorem 4.2] or [11, Proposition IV.1]. □

Note that we can think of any maximal family of eventually different partial functions $\mathcal{A}$ as a MAD family on $\omega \times \omega$.

It is clear the maximality of such a maximal family of eventually different partial functions is destroyed by random forcing because the latter adds eventually different reals. Note that this result also shows neither $\mathbb{M}$-indestructibility nor $\mathbb{S}$-indestructibility imply $\mathbb{B}$-indestructibility.

On the other hand, Hrušák and García Ferreira [11, Proposition IV.2] proved that under CH, given any ω^ω-bounding proper forcing $\mathbb{P}$ of size $\mathfrak{c} = \aleph_1$, there is a $\mathbb{P}$-indestructible, $\mathbb{C}$-destructible MAD family. In particular, there is a $\mathbb{B}$-indestructible, $\mathbb{C}$-destructible MAD family. This also follows from our Theorem 3.6.1 below.

Clearly any MAD family is destroyed by adding a dominating real. So this shows $\mathbb{C}$-indestructibility, $\mathbb{B}$-indestructibility and $\mathbb{M}$-indestructibility imply neither $\mathbb{L}$-indestructibility nor $\mathbb{D}$-indestructibility.

We will show analogous results for other forcings, and we will see this diagram forms really a hierarchy. Wherever possible, we will construct MAD families of the required kind. Note, however, that such constructions usually need hypotheses beyond **ZFC** (like 3.4.1, 3.5.4 and 3.6.1), while tall ideals of the same kind can always be constructed in **ZFC** (see 3.7.3 and 3.7.4; this is also true for the results in Sections 3.4–3.6).

3.4. *Construction of an $\mathbb{S}$-indestructible MAD family*

We can construct an $\mathbb{S}$-indestructible MAD family by using the characterization in the previous section (Theorem 2.4.5). Originally, the existence of such a family under **ZFC** was claimed by Hrušák [10]; however, his construction was based on his false characterization of $\mathbb{S}$-indestructibility (see the discussion after Theorem 2.4.5); we still do not know whether this argument can indeed be carried out solely in **ZFC** (Conjecture 4.4.3). During the *Set theory and Analysis Program* at the Fields Institute (Toronto) in fall 2002, Hrušák and the first author of the present paper obtained the existence of an $\mathbb{S}$-indestructible MAD family under $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. We provide the argument below (Theorem 3.4.1) and thank Hrušák for allowing us to include it here.

Note that by Lemma 3.2.2 (and Theorem 2.4.5), the existence of an $\mathbb{S}$-indestructible MAD family also follows from $\mathfrak{b} = \mathfrak{c}$, an assumption which is well-known to be independent from $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ [1]. We shall exploit this below in Section 4 where we will carry out related constructions which also can be done either under $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ or under $\mathfrak{b} = \mathfrak{c}$ (Theorems 4.4.1 and 4.6.1). Finally, we remark that 3.4.1 is well-known under CH and follows from Spinas' result [3, Section 11.5] that $\mathfrak{a} = \aleph_1$ in the iterated Sacks model (see the comments after the statement of Theorem 4.6.1).

Theorem 3.4.1. *Assume $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. There is an $\mathbb{S}$-indestructible MAD family.*

Proof. If $\mathfrak{a} < \mathfrak{c}$, we already know any MAD family of size $\mathfrak{a}$ is $\mathbb{S}$-indestructible (see Corollary 3.1.3, see also [10]).

Assume $\mathfrak{a} = \mathfrak{c}$. First enumerate all one-to-one functions $2^{<\omega} \to \omega$ as $\{ f_\alpha : 2^{<\omega} \to \omega$ one-to-one; $\alpha < \mathfrak{c}\}$. We are going to construct an $\mathbb{S}$-indestructible MAD family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ by induction in $\mathfrak{c}$ steps as follows: for any $\alpha < \mathfrak{c}$,

- A_α is almost disjoint from any A_β such that $\beta < \alpha$,
- if $G_{f_\alpha^{-1}``A_\beta}$ is countable for all $\beta < \alpha$, $G_{f_\alpha^{-1}``A_\alpha}$ is uncountable.

This is sufficient by Theorem 2.4.5.

step $\alpha < \mathfrak{c}$ We consider two cases.

$(\exists\beta < \alpha)G_{f_\alpha^{-1}\text{"}A_\beta}$ **is uncountable** In this case fix any set A as A_α such that A is almost disjoint from any A_β where $\beta < \alpha$. Since $\alpha < \mathfrak{a} = \mathfrak{c}$, $\{A_\beta : \beta < \alpha\}$ is not MAD, so there exists such a set A. We need the hypothesis $\mathfrak{a} = \mathfrak{c}$ only in this case.

Otherwise In this case we have $(\forall\beta < \alpha)\quad G_{f_\alpha^{-1}\text{"}A_\beta}$ is countable. Since $\mathsf{cov}(\mathit{cntble}) = \mathfrak{c}$, we have $|\bigcup_{\beta<\alpha} G_{f_\alpha^{-1}\text{"}A_\beta}| < \mathfrak{c}$, so we can fix a perfect tree T such that

$$(\forall\beta < \alpha)G_{f_\alpha^{-1}\text{"}A_\beta} \cap [T] = \emptyset.$$

Note that even if the intersection with the G_δ-closure is empty, it is possible that $f_\alpha^{-1}\text{"}A_\beta \cap [T]$ is infinite for some β. This happens for example when the intersection forms an anti-chain in T. However, we can prove the following lemma by the assumption:

Lemma 3.4.2. *Assume $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. For any $\alpha < \mathfrak{c}$, assume $\{A_\beta : \beta < \alpha\}$ is a AD family such that $(\forall\beta < \alpha)\ G_{f_\alpha^{-1}\text{"}A_\beta} \in \mathit{cntble}$. Then we can find $A \subseteq \omega$ such that*
- *A is almost disjoint from A_β for any $\beta < \alpha$,*
- *$G_{f_\alpha^{-1}\text{"}A} \notin \mathit{cntble}$.*

It is enough to let $A_\alpha = A$ and the proof of this theorem is complete. □

Proof of Lemma 3.4.2. We will find $B \subseteq T$ such that $|f_\alpha\text{"}B \cap A_\beta| < \omega$ holds.

Note that $\{f_\alpha^{-1}\text{"}A_\beta \cap T : \beta < \alpha\}$ forms an off-branch family (see Definition 3.5.1 (2)) of T. Therefore for any $\beta < \alpha$ the set of nodes $T \setminus f_\alpha^{-1}\text{"}A_\beta$ contains a subset which is open dense in T: if there is no open dense subset, then we can fix a branch of T which has an infinite intersection with $f_\alpha^{-1}\text{"}A_\beta$. Let D_β be such an open dense set.

Next we consider an elementary submodel of $\mathbf{V}$. Let $\mathbf{M}$ be a model of **ZFC** such that $\{A_\beta : \beta < \alpha\} \subseteq \mathbf{M}$ and $|\mathbf{M}| = |\alpha| < \mathfrak{c}$ in $\mathbf{V}$. Then

Claim 3.4.3. *There is a Cohen real $c \in \mathbb{R} \cap \mathbf{V}$ over $\mathbf{M}$.*

Proof of Claim 3.4.3. $\mathbf{V} \models$ "$|\mathbf{M}| = |\alpha| < \mathfrak{c}$" means, in $\mathbf{V}$ there are at most $|\alpha|$ many meager sets whose Borel codes are in $\mathbf{M}$. Since $\mathbf{V} \models$ "$\alpha < \mathsf{cov}(\mathcal{M}) = \mathfrak{c}$" there is a real $c \in \mathbb{R} \cap \mathbf{V}$ which isn't included in any such meager set; it is Cohen over $\mathbf{M}$. □

It is well-known that once we have a Cohen real, then we have a perfect set of Cohen reals: if we define a forcing notion $\mathbb{P}$ by

- $S \in \mathbb{P}$ iff $S \subseteq T$ is a finite subtree of T such that all of its top nodes have the same length: $(\exists n)$ if $t \in S$ is a maximal node then $|t| = n$,
- $S_0 \le S_1$ iff $S_0 \supseteq S_1$ and S_0 is an end-extension of S_1, i.e. $S_0 \cap 2^{\le m} = S_1$ where m is the height of S_1.

Let G be a $\mathbb{P}$-generic filter, and let us work in $\mathbf{M}[G]$. Clearly $\mathbb{P}$ is a countable forcing notion, so it is essentially the same as Cohen forcing. So we may assume $G \in \mathbf{V}$ and $\mathbf{M}[G] \subseteq \mathbf{V}$.

Claim 3.4.4. 1. *$S^G = \bigcup\{S \in \mathbb{P} : S \in G\}$ is a perfect subtree of T,*
2. *For any $r \in \mathbb{R}$, if $r \in \mathbf{M}[G] \cap [S^G]$ then r is a Cohen real (in the relative topology of $[T]$) over $\mathbf{M}$.*

Proof of Claim 3.4.4. 1. By easy density argument: for any $S \in \mathbb{P}$ and for all $t \in S$ there is $S' \leq S$ such that S' has a splitting node above t.
2. This is clear from the genericity: in fact for any $f \in [S^G]$ we can construct a $\mathbb{C}$-generic filter G_f such that $G_f = \{p \in \mathbb{C} : p \subseteq f\}$. $\square$

Claim 3.4.5. *$S^G \cap f_\alpha^{-1}$"A_β is finite for all $\beta < \alpha$.*

Proof. We prove this claim by a density argument. Fix any $S \in \mathbb{P}$, and let m be the height of S. Fix any $\beta < \alpha$. We construct $S_0 \leq S$ such that for any maximal node $\tau \in S_0$, for any $\sigma \supseteq \tau$, $\sigma \notin f_\alpha^{-1}$"$A_\beta$. Then clearly we have $S_0 \Vdash$ "$|S^G \cap f_\alpha^{-1}$"$A_\beta| < \omega$", so we are done.

Recall D_β is open dense in T. For any top-node σ of S, there is a $\tau_\sigma \in D_\beta$ such that $\sigma \subseteq \tau_\sigma$. Clearly there is no node which is a member of f_α^{-1}"A_β above τ_σ because D_β is open.

Therefore it is enough to let

$$S_0 = \bigcup\{T_{\tau_\sigma} : \sigma \text{ is a top-node of } S\} \cap 2^{\leq n}$$

where $n = \max\{|\tau_\sigma| : \sigma \text{ is a top-node of } S\}$. $\square$

Let $A_\alpha = f_\alpha$"S^G, and we are done. $\square$

3.5. Construction of an $\mathbb{M}$-indestructible MOB family

Definition 3.5.1. (1) $B \subseteq [\omega^{<\omega}]^\omega$ is a branch iff $(\exists f \in \omega^\omega) B = \{\sigma : \sigma \subseteq f\}$.
(2) $\mathcal{A} \subseteq [\omega^{<\omega}]^\omega$ is an off-branch family iff for any $A \in \mathcal{A}$ and for any branch B, $|A \cap B| < \aleph_0$ and $\mathcal{A}$ is almost disjoint,
(3) $\mathcal{A} \subseteq [\omega^{<\omega}]^\omega$ is a maximal off-branch family (MOB) iff $\mathcal{A}$ is off-branch and maximal with respect to inclusion,
(4) $\mathcal{A} \subseteq [\omega^{<\omega}]^\omega$ is a maximal antichain family iff $\mathcal{A}$ is a maximal almost disjoint family of antichains of $\omega^{<\omega}$.
(5) $\mathfrak{o} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a MOB}\}$,
(6) $\bar{\mathfrak{o}} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a maximal antichain family }\}$.

These notions are due to Leathrum [18]. Clearly any maximal antichain family is MOB; so $\mathfrak{o} \leq \bar{\mathfrak{o}}$. Furthermore $\mathfrak{a} \leq \mathfrak{o}$ is well-known [18].

It is known that both $\mathbb{C}$ and $\mathbb{B}$ destroy any MOB family (for more details, see [18,5]). So to show neither $\mathbb{C}$ nor $\mathbb{B}$-destructibility implies $\mathbb{M}$-destructibility, it is enough to construct an $\mathbb{M}$-indestructible MOB family. In fact, the existence of a $\mathbb{M}$-indestructible MOB family is well-known under CH. Namely, Shelah and Spinas (unpublished) proved that $\bar{\mathfrak{o}} = \omega_1$ in the Miller model (the model obtained by iterating $\mathbb{M}$ ω_2 times with countable support over a model for CH). This was used to show the consistency of $\bar{\mathfrak{o}} < \mathfrak{d}$, a result obtained independently around the same time by the first-named author of the present paper via a ccc forcing argument which turned out to be much simpler than investigating the combinatorics of the Miller model (see [5] for details). The result of Shelah and Spinas necessarily involved constructing under CH a maximal antichain family which is iterated Miller-indestructible (more explicitly, which is $\mathbb{M}_\alpha$-indestructible for all countable ordinals α, where $\mathbb{M}_\alpha$ denotes the α-stage countable support iteration of $\mathbb{M}$; see Section 4, in particular 4.5, for the analogous discussion in case of Sacks forcing $\mathbb{S}$). Now, in general, $\mathbb{M}_\alpha$-indestructibility is stronger than mere $\mathbb{M}$-indestructibility (see 4.4 for the corresponding result on Sacks forcing $\mathbb{S}$). In any case, under CH, Theorem 3.5.4 below is due to Shelah and Spinas.

In view of recent work of Zapletal [23], there is another way to look at the Shelah–Spinas result. Namely, [23] says that the Miller model is a "minimal model" for making $\mathfrak{d}$ large in the sense that for every cardinal invariant $\mathfrak{j}$ of the continuum which has a reasonably easy definition, if $\mathfrak{j} < \mathfrak{d}$ is consistent, then $\mathfrak{j} = \omega_1$ in the Miller model. Since $\bar{\mathfrak{o}}$ falls into Zapletal's framework, we may argue as follows: *by [5], $\bar{\mathfrak{o}} < \mathfrak{d}$ is consistent; ergo, by Zapletal's work, $\bar{\mathfrak{o}} = \omega_1$ in the Miller model; ergo, there exists an $\mathbb{M}$-indestructible maximal antichain family under CH.*

For the remainder of this subsection, we will consider only maximal antichain families. As in Section 2, we get the following characterization (see, in particular, Theorem 2.4.6).

Lemma 3.5.2. *The following are equivalent: for any maximal antichain family $\mathcal{A}$,*

(1) *$\mathcal{A}$ is $\mathbb{M}$-indestructible.*
(2) *$\mathcal{A}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ which adds an unbounded real.*
(3) *$\forall A \subseteq \omega^{<\omega}$ such that $G_A \notin \mathcal{K}_\sigma$, $\forall f : A \to \omega^{<\omega}$ such that f"A is an antichain, $\exists B \in \mathcal{A}$ such that the $G_{f^{-1}"B} \notin \mathcal{K}_\sigma$.*
(4) *$\forall A \subseteq \omega^{<\omega}$ such that $G_A \notin \mathcal{K}_\sigma$, $\forall f : A \to \omega^{<\omega}$ one-to-one such that f"A is an antichain, $\exists B \in \mathcal{A}$ such that the $G_{f^{-1}"B} \notin \mathcal{K}_\sigma$.*
(5) *$\forall f : \omega^{<\omega} \to \omega^{<\omega}$ one-to-one such that range(f) is an antichain, $\exists B \in \mathcal{A}$ such that $G_{f^{-1}"B} \notin \mathcal{K}_\sigma$.*

Proof. First we will show (2) implies (3). Assume there is an $A \subseteq \omega^{<\omega}$ such that $G_A \notin \mathcal{K}_\sigma$, $g : A \to \omega^{<\omega}$ such that g"A is an antichain and $(\forall B \in \mathcal{A})G_{g^{-1}"B}$ is σ-bounded. Say $f_B \in \omega^\omega$ is an eventually dominating real for $G_{g^{-1}"B}$. In the generic extension, let x be a new unbounded real in G_A (so it is unbounded by all f_B), then we have $x \notin G_{g^{-1}"B}$ for any $B \in \mathcal{A}$. As in the proof of Theorem 2.2.2, define

$$D = \{\sigma \in \omega^{<\omega} : (\exists n)\sigma = g(x|_n)\}.$$

Clearly this is an infinite antichain and almost disjoint from any $B \in \mathcal{A}$.

(1) implies (2), (3) implies (4), and (4) implies (5): they are trivial.

The proof (4) implies (1) is also similar, except that we use the following lemma.

Lemma 3.5.3 (*Main Lemma for $\mathbb{M}$-Indestructible Maximal Antichain Family*). *Assume $T \in \mathbb{M}$, $\dot{C}$ is an $\mathbb{M}$-name such that $T \Vdash$ "$\dot{C} \subseteq \omega^{<\omega}$ is an antichain".*

Then we can find a tree $T' \leq T$, a set $A \subseteq \omega^{<\omega}$, and a one-to-one function $g : A \to \omega^{<\omega}$ such that

- *$[T'] = G_A$,*
- *$(\forall\sigma \in A)T'_\sigma \Vdash$ "$g(\sigma) \in \dot{C}$",*
- *g"A is an antichain in $\omega^{<\omega}$.*

Similarly the reason (5) implies (4) is that $\mathcal{K}_\sigma$ is strongly homogeneous. $\square$

Proof of Lemma 3.5.3. The proof is best characterized as a "diagonal" fusion argument on a Miller tree. Assume $T \in \mathbb{M}$, $\dot{C}$ is a $\mathbb{M}$-name such that $T \Vdash$ "$\dot{C} \subseteq \omega^{<\omega}$ is an antichain". Then by a proof similar to the one of Lemma 2.2.3, we may assume without loss of generality there is $B \subseteq T$ such that

- $G_B = [T]$,
- $(\forall t \in B)(\exists \tau_t \in \omega^{<\omega}) T_t \Vdash$ "$\tau_t \in \dot{C}$",
- the correspondence $t \to \tau_t$ is one-to-one.

We may assume $B \subseteq split(T)$. Note for any branch $h \in [T]$ we have $\{\tau_t : t \subseteq h\}$ is an antichain.

So all we have to do is finding $A \subseteq B$ infinite and defining g such that g"A is an antichain and G_A is still a Miller tree. Let $g(t) = \tau_t$ for all $t \in B$. Let $\{s_n : n \in \omega\}$ be an enumeration of $\omega^{<\omega}$ such that $s_n \subseteq s_m$ implies $n \leq m$.

Construction of A**:** We construct a system $\sigma_n, \tau_n \in \omega^{<\omega}$, $B_n \subseteq \omega^{<\omega}$ and $P_n \in \mathbb{M}$ such that, for any $n \in \omega$,

(i) $P_0 = T$, $B_0 = B$, $\sigma_0 = stem(T)$,
(ii) $B_n \subseteq split(P_n)$ and $G_{B_n} = [P_n]$,
(iii) $g(\sigma_n) = \tau_n$,
(iv) $\{\sigma_i : i \leq n\} \subseteq B_n$ and $B_{n+1} \subseteq B_n$,
(v) $P_{n+1} \leq_n P_n$ (where $\leq_n$ denotes the fusion order on $\mathbb{M}$ and means that the first n splitting nodes $\{\sigma_i : i \leq n\}$ of P_n also belong to P_{n+1}),
(vi) $\sigma_i \cap \sigma_j = \sigma_n$ iff $s_i \cap s_j = s_n$ (this means that the common initial segment of σ_i and σ_j is σ_n iff the common initial segment of s_i and s_j is s_n),
(vii) $\forall s \in B_n \setminus \{\sigma_n\} : g(s) \perp \tau_n$.

If this is possible, it suffices to put

- $A = \{\sigma_n : n \in \omega\}$,
- $T' = \bigcap_{n \in \omega} P_n$.

By clause (v), $T' \in \mathbb{M}$; by (ii) and (iv), $G_A \subseteq [T']$. In fact, by clause (vi), G_A still contains a rational perfect tree so that we may assume $G_A = [T']$ without loss of generality. By (iii), (iv) and (vii), g"A is indeed an antichain. Hence it suffices to check we can carry out the recursive construction.

step 0: $P_0 = T$, $B_0 = B$, $\sigma_0 = stem(T)$ and $\tau_0 = g(\sigma_0)$. Then all clauses are satisfied. Note, in particular, that (vii) holds because $\sigma_0 \subseteq s$ for all $s \in B_0$.

step n for $n > 0$: Let $\delta_n \in split(B_{n-1})$ such that for all $i, j < n$, $\delta_n \cap \sigma_j = \sigma_i$ iff $s_n \cap s_j = s_i$.

For notation, let $B_{n-1}|_t = \{s \in B_{n-1} : s \supset t\}$ for any $t \in B_{n-1}$. For simplicity we will write τ_s instead of $g(s)$ (and τ_i for $g(\sigma_i)$).

We shall use the following well-known partition result for rational perfect trees: if $S \in \mathbb{M}$, $C \subseteq split(S)$, $G_C = [S]$ and $h : C \to 2$, then there are $S' \leq S$ and $C' \subseteq C$ such that $h|_{C'}$ is constant and $stem(S') = stem(S)$, $C' \subseteq split(S')$ and $G_{C'} = [S']$.

Recursively construct $\langle \sigma^j : j < n \rangle$ such that $\delta_n \supseteq \sigma^0 \supseteq \cdots \supseteq \sigma^{j-1} \supseteq \sigma^j \supseteq \cdots \supseteq \sigma^{n-1} \supseteq \sigma_n$ as follows.

the initial step Let $\sigma^{-1} = \delta_n = g(\sigma_n)$.

step $j < n$ If there are $\sigma \supseteq \sigma^{j-1}, \sigma \in B_{n-1}$, and a subtree $S^j \leq (P_{n-1})_{\sigma_j}$ and $C^j \subseteq B_{n-1}$ such that $stem(S^j) = \sigma_j, C^j \subseteq split(S^j), G_{C^j} = [S^j]$ and $\tau_\sigma \subseteq \tau_s$ for all $s \in C^j$, then we let σ^j be such a σ.

Otherwise we let $\sigma^j = \sigma^{j-1}$.

step n Choose σ_n such that $\sigma^{n-1} \subset \sigma_n$ and $\sigma_n \in B_{n-1}$.

Let $\tau_n = \tau_{\sigma_n}$.

For each $j < n$ for which the first alternative holds, also fix S^j and C^j as above.

For any $j < n$ for which the second alternative is true define $h^j : B_{n-1} \cap (P_{n-1})_{\sigma_j} \to 2$ by

$$h^j(s) = \begin{cases} 0 & \text{if } \tau_n \perp \tau_s \\ 1 & \text{if } \tau_n \| \tau_s \end{cases}$$

for any $s \in B_{n-1} \cap (P_{n-1})_{\sigma^j}$. By the partition result mentioned above, we may find $S^j \leq (P_{n-1})_{\sigma^j}, stem(S^j) = \sigma^j$, and $C^j \subseteq B_{n-1}$ such that $h^j|_{C^j}$ is constant on $C^j \subseteq split(S^j)$ and $G_{C^j} = [S^j]$. Now note that $h^j|_{C^j} = 1$ is impossible because if it was true then σ_n would witness the first alternative in the above construction, a contradiction. Therefore, $h^j|_{C^j} = 0$, and $\tau_n \perp \tau_s$ for all $s \in C^j$.

Similarly, for each $j < n$ for which the first alternative holds, $\tau_n \perp \tau_{\sigma^j}$ because $\tau_n = \tau_{\sigma_n}$ and $\sigma^j \subset \sigma_n$. Therefore $\tau_n \perp \tau_s$ for all $s \in C^j$.

This means, however, we can put

- $B_n = \{\sigma_i : i \leq n\} \cup \bigcup_{j<n} C^j \cup B_{n-1}|_{\sigma_n}$,
- $P_n = \bigcup_{j<n} S^j \cup P_{n-1}|_{\sigma_n}$.

Then $P_n \leq_{n-1} P_{n-1}, [P_n] = G_{B_n}$, the $\sigma_j, j \leq n$, are splitting nodes of P_n, and we also have that $\tau_s = g(s) \perp \tau_n$ for all $s \in B_n \setminus \{\sigma_n\}$ so that (ii) to (vii) are indeed satisfied.

This completes the proof. □

Using the previous characterization, we can construct an $\mathbb{M}$-indestructible MOB family under certain hypotheses. Recall $\mathfrak{b} = \mathsf{add}(\mathcal{K}_\sigma)$.

Theorem 3.5.4. *Assume $\mathfrak{b} = \mathfrak{c}$. Then there is an $\mathbb{M}$-indestructible maximal antichain family of size $\mathfrak{c}$.*

Proof. Let us enumerate all one-to-one functions $\{g_\alpha : \omega^{<\omega} \to \omega^{<\omega}; \alpha < \mathfrak{c}\}$ such that $range(g_\alpha)$ is an antichain for any α. By $\mathfrak{c}$-step induction, we are going to construct a maximal antichain family $\{A_\alpha : \alpha < \mathfrak{c}\}$ such that if $(\forall \beta < \alpha)\ G_{g_\alpha^{-1}``A_\beta}$ is $\sigma$-bounded then $G_{g_\alpha^{-1}``A_\alpha}$ contains a rational perfect set.

step $\alpha < \mathfrak{c}$**:** We have two cases.

case 1: $(\exists \beta < \alpha)\ G_{g_\alpha^{-1}``A_\beta} \notin \mathcal{K}_\sigma$. Recall that **ZFC** implies $\mathfrak{b} \leq \mathfrak{a} \leq \bar{\mathfrak{o}} \leq \mathfrak{c}$. So $\{A_\beta : \beta < \alpha\}$ is not maximal. Therefore we can find an infinite antichain $A_\alpha \subseteq \omega^{<\omega}$ such that

$$|A_\alpha \cap A_\beta| < \aleph_0.$$

case 2: otherwise i.e. $(\forall\beta < \alpha)\ G_{g_\alpha^{-1}\text{“}A_\beta} \in \mathcal{K}_\sigma$.

We have $\bigcup_{\beta<\alpha} G_{g_\alpha^{-1}\text{“}A_\beta} \in \mathcal{K}_\sigma$ because $\alpha < \mathfrak{b} = \mathsf{add}(\mathcal{K}_\sigma)$. So there is $h \in \omega^\omega$ such that $x \le^* h$ for all $x \in \bigcup_{\beta<\alpha} G_{g_\alpha^{-1}\text{“}A_\beta}$. By Lemma 3.5.2 we need to construct a rational perfect tree $T \subseteq \omega^{<\omega}$ and $A \subseteq T$ such that

- $G_A = [T]$,
- $(g_\alpha^{-1}\text{“}A_\beta) \cap A$ is finite for all $\beta < \alpha$.

First recursively build $\langle \sigma_s : s \in \omega^{<\omega}\rangle \subseteq \omega^{<\omega}$ such that

- $s \subseteq t$ implies $\sigma_s \subseteq \sigma_t$,
- $n < m$ implies $\sigma_{s\hat{\ }\langle n\rangle}(|\sigma_s|) < \sigma_{s\hat{\ }\langle m\rangle}(|\sigma_s|)$,
- for all s and all $\beta < \alpha$, $\{n : g_\alpha(\sigma_{s\hat{\ }\langle n\rangle}) \in A_\beta\}$ is finite,
- for all $i < |\sigma_s|$, $\sigma_s(i) \ge h(i)$.

Note that the first two conditions imply that the σ_s will generate a rational perfect tree.

Assume σ_s has been constructed. Let $\{X_n : n \in \omega\} \subseteq \omega^\omega$ such that $\sigma_s \subseteq X_n$, $X_n(i) \ge h(i)$ for all i, and $X_n(|\sigma_s|) < X_m(|\sigma_s|)$ whenever $n < m$. $X_n \ge h$ implies in particular that $X_n \notin \bigcup_{\beta<\alpha} G_{g_\alpha^{-1}\text{“}A_\beta}$.

Therefore, for each n and each $\beta < \alpha$, the set $\{m : g_\alpha(X_n|_m) \in A_\beta\}$ is finite. Hence, for each n and for each $\beta < \alpha$, we can find $k_\beta(n) \in \omega$ such that $g_\alpha(X_n|_m) \notin A_\beta$ for any $m \ge k_\beta(n)$. Since $\alpha < \mathfrak{b}$, there is $k \in \omega^\omega$ such that for all $\beta < \alpha$, the set $\{n : g_\alpha(X_n|_{k(n)}) \in A_\beta\}$ is finite. Therefore, letting $\sigma_{s\hat{\ }\langle n\rangle} = X_n|_{k(n)}$, all requirements are satisfied. This completes the recursive construction.

For $\beta < \alpha$, define a function $l_\beta : \omega^{<\omega} \to \omega$ such that for all $m \ge l_\beta(s)$, $g_\alpha(\sigma_{s\hat{\ }\langle m\rangle}) \notin A_\beta$. Since $\alpha < \mathfrak{b}$, there is $l \in \omega^\omega$ such that $l_\beta <^* l$ for all $\beta < \alpha$. This means that for all $\beta < \alpha$, the set $\{s : s(i) \ge l(s|_i)$ for all $i < |s|$ and $g_\alpha(\sigma_s) \in A_\beta\}$ is finite. So we let

- $A = \{\sigma_s : s(i) \ge l(s|_i)$ for all $i < |s|\}$,
- $T = \{\sigma_s|_j : \sigma_s \in A$ and $j \le |\sigma_s|\}$.

Clearly $[T] = G_A$ and $g_\alpha^{-1}\text{“}A_\beta \cap A$ is finite for all $\beta < \alpha$.

Now let $A_\alpha = g_\alpha\text{“}A$. Then $|A_\alpha \cap A_\beta| < \aleph_0$ for all $\beta < \alpha$ and $g_\alpha^{-1}\text{“}A_\alpha = A$ so that $G_{g_\alpha^{-1}\text{“}A_\alpha}$ contains a rational perfect tree. This completes the proof of the theorem. □

3.6. *Construction of a $\mathbb{B}$-indestructible, $\mathbb{M}$-destructible MAD family*

Theorem 3.6.1. *Assume $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$. Then there is a $\mathbb{B}$-indestructible $\mathbb{M}$-destructible MAD family of size $\mathfrak{c}$.*

Proof. Let us enumerate finite-to-one functions $\{g : B \to \omega; B \subseteq 2^{<\omega}$ and $G_B \notin \mathcal{N}\}$ as $\{g_\alpha : \alpha < \mathfrak{c}\}$. Fix any bijection $f : \omega^{<\omega} \to \omega$. By $\mathfrak{c}$-step induction, we are going to construct a MAD family $\{A_\alpha : \alpha < \mathfrak{c}\}$ such that

- if $(\forall\beta < \alpha) G_{g_\alpha^{-1}\text{“}A_\beta} \in \mathcal{N}$ then $G_{g_\alpha^{-1}\text{“}A_\alpha} \notin \mathcal{N}$.
- $G_{f^{-1}\text{“}A_\alpha} \in \mathcal{K}_\sigma$,

for any $\alpha < \mathfrak{c}$. Notice that this is sufficient, by Theorems 2.4.6 and 2.4.9.

step $\alpha < \mathfrak{c}$: We consider two cases.

case 1: $(\exists\beta < \alpha)G_{g_\alpha^{-1}\text{''}A_\beta} \notin \mathcal{N}$. In this case it suffices to fix $A \subseteq \omega$ such that $G_{f^{-1}\text{''}A} \in \mathcal{K}_\sigma$ and A is almost disjoint from any A_β. $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$ implies $\mathfrak{a} = \mathfrak{c}$, so we can easily get an infinite set A with $|A \cap A_\beta| < \aleph_0$ and $G_{f^{-1}\text{''}A} \in \mathcal{K}_\sigma$. Let $A_\alpha = A$ as above.

case 2: otherwise By $\alpha < \mathfrak{c} = \mathsf{add}(\mathcal{N})$, we have $\bigcup_{\beta<\alpha} G_{g_\alpha^{-1}\text{''}A_\beta} \in \mathcal{N}$. Recall the fact for any $X \subseteq 2^\omega$ measurable, $\epsilon > 0$, we can get a closed subset $C \subseteq X$ such that $\mu(X \setminus C) < \epsilon$. Therefore we may fix a tree $T \subseteq 2^{<\omega}$ such that $[T] \notin \mathcal{N}$ and $[T] \subseteq G_{B_\alpha} \setminus (\bigcup_{\beta<\alpha} G_{g_\alpha^{-1}\text{''}A_\beta})$ where $B_\alpha = dom(g_\alpha)$. We use

Lemma 3.6.2 (*Main Lemma for $\mathbb{B}$-Indestructible $\mathbb{M}$-Destructible MAD Family*). *For any $A \subseteq 2^{<\omega}$ such that $G_A \notin \mathcal{N}$ and for any $g : A \to \omega^{<\omega}$ finite-to-one, there exists a $B \subseteq A$ such that*

- $\mu(G_B) > 0$,
- $G_{g\text{''}B} \in \mathcal{K}_\sigma$.

It is enough to apply Lemma 3.6.2 to $A = T \cap B_\alpha$ and $g = f^{-1} \circ g_\alpha$, then we get $A_\alpha = g_\alpha\text{''}B$ as required. This completes the proof of the theorem. □

Proof of Lemma 3.6.2. For $\sigma \in \omega^{<\omega}$ define

$$\begin{aligned} X_\sigma &= \{f \in 2^\omega : (\exists^\infty n)[f|_n \in A \wedge \sigma \subseteq g(f|_n)]\} \\ &= \bigcap_{m\in\omega} \bigcup_{n\geq m} \{f \in 2^\omega : f|_n \in A \wedge \sigma \subseteq g(f|_n)\} \subseteq G_A \end{aligned}$$

so X_σ is a G_δ set. For $\sigma \in \omega^{<\omega}$ and $i \in \omega$, we take the difference

$$\begin{aligned} Y_{\sigma,i} &= X_\sigma \setminus X_{\sigma\hat{}\langle i\rangle} \\ &= \{f \in 2^\omega : (\exists^\infty n)[f|_n \in A \wedge \sigma \subseteq g(f|_n)] \\ &\quad \wedge \{n : f|_n \in A \wedge g(f|_n) \supseteq \sigma\hat{}\langle i\rangle\} \text{ is finite}\}. \end{aligned}$$

Since $Y_{\sigma,i}$ is an intersection of a $\boldsymbol{\Pi}^0_2$ and a $\boldsymbol{\Sigma}^0_2$ set, it is a $\boldsymbol{\Delta}^0_3$ set. For $\sigma \in \omega^{<\omega}$, let $Y_\sigma = \bigcap_{i\in\omega} Y_{\sigma,i}$. So Y_σ is a $\boldsymbol{\Pi}^0_3$ set. Note $X_\sigma = X_{\sigma\hat{}\langle i\rangle} \dot\cup Y_{\sigma,i}$ for all $i \in \omega$. Therefore

$$X_\sigma = \bigcup_{i\in\omega} X_{\sigma\hat{}\langle i\rangle} \dot\cup Y_\sigma.$$

case 1: $\mu(Y_\sigma) > 0$ for some $\sigma \in \omega^{<\omega}$. There is a closed set contained in Y_σ which still has positive measure. So let $T \subseteq 2^{<\omega}$ be a tree such that $[T] \subseteq Y_\sigma$ and $[T] \notin \mathcal{N}$.

First note $[T] = G_{A\cap T}$, so $G_{A\cap T} \notin \mathcal{N}$. Similarly $[T] = G_{A'}$ for $A' = \{t \in A \cap T : g(t) \supseteq \sigma\}$; to show $[T] \subseteq G_{A'}$, note that any $f \in [T] \subseteq Y_\sigma$ satisfies $(\exists^\infty n)[f|_n \in A$ and $g(f|_n) \supseteq \sigma]$, so $(\exists^\infty n) f|_n \in A'$. The converse is clear.

To get $B \subseteq A'$ such that $G_{g\text{''}B} \in \mathcal{K}_\sigma$, let us consider $\{t \in A \cap T : g(t) \supseteq \sigma\hat{}\langle i\rangle\}$. Since $[T] \subseteq Y_\sigma$, we know that $(\forall f \in [T])(\forall i \in \omega)$ there are finitely many $n \in \omega$ such that $f|_n \in A$ and $g(f|_n) \supseteq \sigma\hat{}\langle i\rangle$.

Construction of B: So we are going to construct recursively finite sets $B_i \subseteq A \cap T \subseteq 2^{<\omega}$ and numbers l_i such that

- if $i < j$, then $\max\{|t| : t \in B_i\} < l_i \leq \min\{|t| : t \in B_j\}$,

- if $t \in B_i$, then $\sigma\hat{}\langle j\rangle \subseteq g(t)$ for some j with $l_{i-1} \leq j < l_i$ for $i > 0$, $0 \leq j < l_0$ for $i = 0$,
- $\mu(\{f \in [T] : (\forall j \leq i)(\exists n) f|_n \in B_j\}) \geq \mu([T]) \cdot (1 - \Sigma_{j \leq i} \frac{1}{2^{j+2}})$.

step $i < \omega$**:** Assume we already have B_j for $j < i$. Since $\forall f \in [T] \forall j < l_{i-1}$ there are only finitely many $n \in \omega$ such that $g(f|_n) \supseteq \sigma\hat{}\langle j\rangle$ and $\forall f \in [T]$ there are infinitely many n with $g(f|_n) \supseteq \sigma$,

$$[T] = \bigcup_{l \in \omega, l \geq l_{i-1}} \{f \in [T] : (\exists n, j)[l_{i-1} \leq n < l \wedge \\ l_{i-1} \leq j < l \wedge g(f|_n) \supseteq \sigma\hat{}\langle j\rangle]\}$$

and the union on the right-hand side is increasing.

Therefore we can find l_i such that

$$\mu(\{f \in [T] : (\exists n, j)[l_{i-1} \leq n < l_i \wedge l_{i-1} \leq j < l_i \wedge g(f|_n) \supseteq \sigma\hat{}\langle j\rangle]\}) \\ \geq \mu([T]) \cdot \left(1 - \frac{1}{2^{i+2}}\right).$$

Put $B_i = \{t : l_{i-1} \leq |t| < l_i \wedge (\exists j)[l_{i-1} \leq j < l_i \wedge g(t) \supseteq \sigma\hat{}\langle j\rangle]\}$. Clearly B_i is as required.

step ω**:** Let $B = \bigcup_{i \in \omega} B_i$.

We will check G_B satisfies the requirements.

By the above $G_B \subseteq [T]$. Similarly by construction $\mu(G_B) \geq \mu\left(\frac{[T]}{2}\right) > 0$.

We claim that $G_{g\text{"}B}$ is empty. To see this note that

- $\sigma \subseteq g(t)$ for all $t \in B$,
- for all $j \in \omega$ there are finitely many $t \in B$ with $\sigma\hat{}\langle j\rangle \subseteq g(t)$ (namely $t \in B_i$ with $l_{i-1} \leq j < l_i$ in this case).

So if $x \in \omega^\omega$ then there are only finitely many n such that $x|_n = g(t)$ for some $t \in B$. Therefore we are done.

case 2: $\mu(Y_\sigma) = 0$ **for all** $\sigma \in \omega^{<\omega}$**.** This means $\mu(X_\sigma) = \mu(\bigcup_{i \in \omega} X_{\sigma\hat{}\langle i\rangle})$ for all $\sigma \in \omega^{<\omega}$.

Construction of B**:** We can recursively construct sets $U_j, Z_j \subseteq 2^\omega$, finite sets $B_j \subseteq 2^{<\omega}$, $D_j, E_j \subseteq \omega^{<\omega}$ and numbers l_j such that

- $U_j = \{f \in G_A : (\forall l \leq j)(\exists n) f|_n \in B_l\}$,
- $\mu(U_j \cap Z_j) \geq \mu(G_A) \cdot (1 - \Sigma_{l \leq j} \frac{1}{2^{l+2}})$,
- $Z_j = \bigcup_{\sigma \in E_j} X_\sigma$ and for all $\sigma \in E_j$, X_σ has positive measure,
- $E_j \subseteq \omega^{l_j}$,
- $(\forall \tau \in D_j) \tau|_{l_{j-1}} \in E_{j-1}$,
- $(\forall \tau \in E_j) \tau|_{l_{j-1}} \in E_{j-1}$,
- $g|_{B_j} : B_j \to D_j$ onto,
- if $\sigma \in D_j$ then $l_{j-1} < |\sigma| < l_j$,
- if $t \in B_j$ then $l_{j-1} < |t| < l_j$.

step $j < \omega$**:** Assume we already have $U_l, Z_l, B_l, D_l, E_l, l_l$ for $l < j$.

First choose B_j such that $|t| > l_{j-1}$ for all $t \in B_j$, such that $\mu(\{f \in U_{j-1} \cap Z_{j-1} : (\exists n) f|_n \in B_j\})$ is close enough to $\mu(U_{j-1} \cap Z_{j-1})$, and such that for all $t \in B_j$, $g(t) \supset \sigma$ holds for some $\sigma \in E_{j-1}$.

Let $D_j = \{g(t) : t \in B_j\}$. Choose l_j such that $l_j > |t|, |g(t)|$ for any $t \in B_j$.

Finally we can find E_j by assumption of case 2: choose E_j so that $Z_j = \bigcup_{\sigma\in E_j} X_\sigma$ and $\mu(Z_j \cap U_j)$ is close enough to $\mu(Z_{j-1} \cap U_j)$.

step ω**:** Let $B = \bigcup_{j\in\omega} B_j$, and we will check G_B satisfies the requirements.

Clearly $\{f \in G_A : (\forall j)(\exists n) f|_n \in B_j\} \subseteq G_B$ and the first set has measure $\geq \frac{\mu(G_A)}{2} > 0$ by construction. So $G_B \notin \mathcal{N}$.

Let $D = \bigcup_{j\in\omega} D_j$. Then $g"B = D$, so we have $G_{g"B} = G_D$.

It suffices to show $G_D \subseteq [T]$ where $[T] = \{x \in \omega^\omega : (\forall j) x|_{l_j} \in E_j\}$ (note T is a compact tree). To show this, assume $x \in G_D$. Then $(\exists^\infty n) x|_n \in D$, so $(\exists^\infty j)(\exists n) x|_n \in D_j$, therefore $(\exists^\infty j) x|_{l_{j-1}} \in E_{j-1}$. So we have $(\forall j) x|_{l_j} \in E_j$; this means $x \in [T]$. $\square$

3.7. Destructibility and indestructibility of ideals

In Section 2, we characterized the ideals which are $\mathbb{P}$-indestructible for a given forcing notion $\mathbb{P}$. Conversely, we may fix some (definable) tall ideal and ask which forcing notions destroy it. Natural candidates for such ideals are those derived from the forcing notions we are studying. Recall (see Section 2.4) that $\mathcal{I}_\mathbb{P} = \{I \subseteq 2^{<\omega} (\text{ or } \omega^{<\omega}) : G_I \in I_\mathbb{P}\}$ where $\mathbb{P} = \mathcal{B} \setminus I_\mathbb{P}$ is a real forcing. So, e.g. $\mathcal{I}_\mathbb{C} = \{I \subseteq 2^{<\omega} : G_I \text{ is meager}\}$, etc. By Theorem 2.4.5 ((vi)→(v)), $\mathcal{I}_\mathbb{S} = \{I \subseteq 2^{<\omega} : G_I \in \mathsf{cntble}\}$ is destroyed by any forcing notion $\mathbb{P}$ which adds a new real, that is, the ideal generated by $\mathcal{I}_\mathbb{S}^\mathbf{V}$ in $\mathbf{V}^\mathbb{P}$ is not tall. Similarly, we get, as pointed out by the referee:

Proposition 3.7.1. *The following are equivalent for a forcing notion* $\mathbb{P}$*:*

(i) $\mathbb{P}$ *adds an unbounded real,*
(ii) $\mathbb{P}$ *destroys* $\mathcal{I}_\mathbb{M} = \{I \subseteq \omega^{<\omega} : G_I \in \mathcal{K}_\sigma\}$.

Proof. First we show (i) implies (ii). This is the same as (vi) implies (v) in Theorem 2.4.6.

The proof of the converse is as follows. Assume $\mathbb{P}$ is ω^ω-bounding. We need to show $\mathcal{I}_\mathbb{M}^\mathbf{V}$ still generates a tall ideal in $\mathbf{V}^\mathbb{P}$. So let $A \in [\omega^{<\omega}]^\omega$. We need to find $I \in \mathcal{I}_\mathbb{M}^\mathbf{V}$ such that $|I \cap A| = \aleph_0$.

<u>**case 1**</u>**:** Assume A contains a branch. That is, there is $x \in \omega^\omega$ such that $x|_n \in A$ for infinitely many n. Since $\mathbb{P}$ is ω^ω-bounding, there is $G \in \mathcal{K}_\sigma{}^\mathbf{V}$ such that $x \in G$. In fact, there is $I \in \mathcal{I}_\mathbb{M}^\mathbf{V}$ such that $x|_n \in I$ for all n. Thus $|A \cap I| = \aleph_0$ as required.

<u>**case 2**</u>**:** Assume A has no branch. Then, by a compactness argument, there must be $\sigma \in \omega^{<\omega}$ such that for infinitely many $n \in \omega$ there is τ_n such that $\sigma\hat{}\langle n\rangle\hat{}\tau_n \in A$. Since $\mathbb{P}$ is ω^ω-bounding, there is $g : \omega \to [\omega^{<\omega}]^{<\omega}$ in $\mathbf{V}$ such that for all n, if there is τ such that $\sigma\hat{}\langle n\rangle\hat{}\tau \in A$, then there is such τ with $\tau \in g(n)$. Let $I = \{\sigma\hat{}\langle n\rangle\hat{}\tau : \tau \in g(n)\} \in \mathbf{V}$. Clearly $G_I = \emptyset \in \mathcal{K}_\sigma{}^\mathbf{V}$. Thus $I \in \mathcal{I}_\mathbb{M}^\mathbf{V}$. Since $|A \cap I| = \aleph_0$ we are done. $\square$

Proposition 3.7.2. *The following are equivalent for a forcing notion* $\mathbb{P}$*:*

(i) $\mathbb{P}$ *adds a Cohen real,*
(ii) $\mathbb{P}$ *destroys* $\mathcal{I}_\mathbb{C} = \{I \subseteq \omega^{<\omega} : G_I \in \mathcal{M}\}$,
(iii) $\mathbb{P}$ *destroys* $\mathcal{I}_{\mathsf{nwd}} = \{I \subseteq \omega^{<\omega} : G_I \in \mathsf{nwd}\}$.

Proof. First we prove (i) implies (ii). This is the same as (i) implies (v) in Theorem 2.4.8.

(ii) implies (iii) is trivial.

For (iii) implies (i) we argue as follows. Let $A \in [2^{<\omega}]^\omega$ be such that $|A \cap I| < \aleph_0$ for all $I \in \mathcal{I}_{nwd}^{\mathbf{V}}$. By compactness of 2^ω, A has a branch x, i.e. $x|_n \in A$ for infinitely many $n \in \omega$. We claim that x is Cohen over $\mathbf{V}$. For assume this were not the case. Then, for some closed nowhere dense tree $T \subseteq 2^{<\omega}$ belonging to $\mathbf{V}$, $x \in [T]$. Clearly $[T] = G_T$ so $T \in \mathcal{I}_{nwd}^{\mathbf{V}}$. Since $x|_n \in T$ for all n, we get $|A \cap T| = \aleph_0$, a contradiction. □

We believe an analogous result holds for random forcing $\mathbb{B}$, but we were unable to prove it.

Theorem 3.7.3. *There is an $\mathbb{L}$-indestructible and $\mathbb{C}$-destructible tall ideal. Namely, $\mathcal{I}_{\mathbb{C}} = \{I \subseteq \omega^{<\omega} : G_I \in \mathcal{M}\}$ and $\mathcal{I}_{nwd} = \{I \subseteq \omega^{<\omega} : G_I \in nwd\}$ are such ideals.*

Proof. This is immediate from Proposition 3.7.2. □

Theorem 3.7.4. *There is a $\mathbb{D}$-indestructible and $\mathbb{B}$-destructible tall ideal. Namely, $\mathcal{I}_{\mathbb{B}} = \{I \subseteq 2^{<\omega} : G_I \in \mathcal{N}\}$ is such an ideal.*

Proof. By Theorem 2.4.9, $\mathcal{I} = \mathcal{I}_{\mathbb{B}}$ is $\mathbb{B}$-destructible. Thus it is enough to show that $\mathcal{I}$ is $\mathbb{D}$-indestructible. By Theorem 2.4.10, it suffices to prove that for all one-to-one partial functions $f : \omega^{\uparrow<\omega} \to 2^{<\omega}$ with $G_{dom(f)} \notin \mathcal{M}_{\mathcal{D}}$, there is $I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathcal{M}_{\mathcal{D}}$. We shall even establish there is $I \subseteq 2^{<\omega}$ such that $|I \cap 2^n| \le 1$ for all $n$ with $G_{f^{-1}``I} \notin \mathcal{M}_{\mathcal{D}}$. Clearly $I \in \mathcal{I}$ for such I.

So fix f as required. Since $G_{dom(f)} \notin \mathcal{M}_{\mathcal{D}}$, there exists $\langle s_0, h_0\rangle \in \mathbb{D}$ such that $U_{\langle s_0,h_0\rangle} \setminus G_{dom(f)} \in \mathcal{M}_{\mathcal{D}}$ (i.e. $G_{dom(f)}$ is $\mathcal{D}$-comeager in $U_{\langle s_0,h_0\rangle}$).

Let $T_0 \subseteq \omega^{\uparrow<\omega}$ be the collection of all $s \in \omega^{\uparrow<\omega}$ compatible with $\langle s_0, h_0\rangle$; i.e. $s \in T_0$ iff $s_0 \subseteq s$ and $s(i) \ge h_0(i)$ for all $i \in |s|$. For $s \in T_0$ define the rank function $rk(s)$ by

$$\begin{aligned} rk_k(s) = 0 &\text{ iff } (\exists m > |s|)(\exists \langle t_n : n \in \omega\rangle \subseteq T_0)|t_n| \\ &= m \wedge s \subseteq t_n \wedge t_n(|s|) \ge n \wedge t_n \in dom(f) \\ rk_k(s) \le \beta &\text{ iff } (\exists m > |s|)(\exists \langle t_n : n \in \omega\rangle \subseteq \omega^m)|t_n| \\ &= m \wedge s \subseteq t_n \wedge t_n(|s|) \ge n \wedge rk(t_n) < \beta. \end{aligned}$$

As is usual for rank arguments, $rk(s)$ is either $< \omega_1$, or undefined (in which case we write $rk(s) = \infty$).

Claim 3.7.5. *For any $s \in T_0$, $rk(s) < \infty$.*

Proof of Claim 3.7.5. Assume $rk(s)$ is undefined for some $s \in T_0$. We recursively define $h \in \omega^\omega$ such that $s \subseteq h$, $h(i) \ge h_0(i)$ for all $i \in \omega$, and

$$\begin{aligned} &\text{whenever } t \in T_0, s \subsetneq t, \text{ is compatible with } \langle s, h\rangle, \\ &\qquad\qquad \text{then } rk(t) = \infty \text{ and } t \notin dom(f). \end{aligned} \tag{$*$}$$

Since $rk(s) = \infty$, we find $h(|s|) \ge h_0(|s|)$ such that whenever $s \subsetneq t$, $|t| = |s|+1$, $t(|s|) \ge h(|s|)$, then $rk(t) = \infty$ and $t \notin dom(f)$.

Assume $m > |s|$ and $h|_m$ has been defined such that $(*)$ holds for all $t \in T_0$ of length $\le m$. We need to define $h(m)$ such that $(*)$ still holds for all $t \in T_0$ of length $\le m + 1$.

Assume this is impossible. Then there is a sequence $\langle t_n : n \in \omega\rangle \subseteq T_0$ such that

$|t_n| = m + 1$, $t_n(m) \geq n$, $s \subseteq t_n$, $t_n(i) \geq h(i)$ for all $i < m$, and

- either $rk(t_n) < \infty$,
- or $t_n \in dom(f)$

for all $n \in \omega$. By pruning the sequence $\langle t_n : n \in \omega\rangle$, we may assume there is $s' \in T_0$, $s' \supseteq s$, $|s'| \leq m$, such that $s' \subseteq t_n$ and $t_n(|s'|) \geq n$ for all n. Note $rk(s') = \infty$ and either $s' = s$ or $s' \notin dom(f)$ by inductive hypothesis ($*$). By definition of rank we must have $rk(t_n) = \infty$ and $t_n \notin dom(f)$ for almost all n, a contradiction.

Therefore the recursive construction can be carried out. Clearly $U_{\langle s,h\rangle} \subseteq U_{\langle s_0,h_0\rangle}$ and, by ($*$), $G_B \cap U_{\langle s,h\rangle} = \emptyset$. This contradicts the fact that $G_{dom(f)}$ is $\mathcal{D}$-comeager in $U_{\langle s_0,h_0\rangle}$. □

For $s \in T_0$ with $rk(s) = 0$, fix $m^s \in \omega$, $\langle t^s_n : n \in \omega\rangle \subseteq T_0$ with $|t^s_n| = m^s$, $s \subseteq t^s_n$, $t^s_n(|s|) \geq n$ and $t^s_n \in dom(f)$. Using that f is one-to-one and pruning the sequences $\langle t^s_n : n \in \omega\rangle$ if necessary, we may assume that, if we let

$$I = \{ f(t^s_n) : s \in T_0, n \in \omega, rk(s) = 0\}$$

then for each m there is at most one $t \in 2^m$ such that $t \in I$. So $I \in \mathcal{I}$, and we need to check that $G_{f^{-1}``I} \notin \mathcal{M}_{\mathcal{D}}$. In fact, we shall argue that $G_{f^{-1}``I}$ is still $\mathcal{D}$-comeager in $U_{\langle s_0,h_0\rangle}$.

Choose $\langle s, h\rangle$ such that $U_{\langle s,h\rangle} \subseteq U_{\langle s_0,h_0\rangle}$; that is, $s \in T_0$ and $h(i) \geq h_0(i)$ for all i. By the claim, $rk(s) < \infty$. By a standard induction on $rk(s)$ we argue that there is $s' \in T_0$ compatible with $\langle s, h\rangle$ such that $rk(s') = 0$ (if $rk(s) = 0$, $s' = s$ works. If $rk(s) > 0$, find $s' \in T_0$ compatible with $\langle s, h\rangle$ such that $rk(s') < rk(s)$ and use the inductive assumption). Then, by definition of rank and choice of the $t^{s'}_n$, we find n such that $t^{s'}_n$ is compatible with $\langle s, h\rangle$. So $t^{s'}_n \in f^{-1}``I$. Since the $\langle s, h\rangle$ was arbitrary, this argument in fact shows $G_{f^{-1}``I} \cap U_{\langle s,h\rangle} \neq \emptyset$, and we are done. □

4. Iterated Sacks forcing indestructibility

We generalize the results about single-step Sacks forcing indestructibility to iterated Sacks forcing.

4.1. Product forcing and isomorphisms of names arguments

Here we summarize the known results about iterated forcing indestructibility and product forcing indestructibility of MAD families. Using "isomorphism of names" arguments, one can prove the following theorem:

Theorem 4.1.1 (*Kunen*). *Let κ be any uncountable cardinal such that $\kappa^\omega = \kappa$.*

1. *In the model obtained by adding κ many Cohen reals over a model of CH, the size of any MAD family is either $\aleph_1$ or κ. Furthermore there is a Cohen indestructible MAD family of size $\aleph_1$, and no Cohen indestructible MAD family of size $\mathfrak{c}$ in this model.*
2. *In the model obtained by adding κ many random reals over a model of CH, the size of any MAD family is either $\aleph_1$ or κ. Furthermore there is a random indestructible MAD family of size $\aleph_1$, and no random indestructible MAD family of size $\mathfrak{c}$ in this model.*

For more details, see [3,14] (see also [10, Proposition 7]). In Sections 4.4 and 4.7 below, we shall investigate to which extent similar results can be proved for iterated Sacks forcing.

It is easy to prove Theorem 4.1.1, because these forcings can be thought of as large products and they satisfy the factor lemma [15]. Moreover finite support product and finite support iteration of Cohen forcing is the same.

For non-c.c.c. forcings, we can't use a finite support product, because it collapses $\aleph_1$. The "isomorphism of names" argument also works for countable support products. So we can prove a similar result for Sacks forcing. For any $\kappa \geq \aleph_2$, we have

Theorem 4.1.2 (*Folklore*). *Let κ be any uncountable cardinal such that $\kappa^\omega = \kappa$.*

Any infinite MAD family is either of size $\aleph_1$ or of size κ in the model obtained by adding κ-many Sacks reals by countable support product over a model of CH.

Since there is no factor lemma for side-by-side Sacks forcing, we can't argue that a MAD family of size $\mathfrak{c}$ is $\mathbb{S}$-destructible in this model (see also Conjecture 4.4.4).

However, the countable support product of tree forcings whose conditions are isomorphic to $\omega^{<\omega}$, $\mathbb{M}$ and $\mathbb{L}$, collapses $\aleph_1$. So we have no analogue of 4.1.2 in this case.

4.2. Characterization of $\mathbb{S}_2$-indestructibility

Our goal in this and the next sections is to characterize iterated Sacks indestructibility in the same vein as the characterization of $\mathbb{S}$-indestructibility from Theorem 2.4.5. For simplicity, let us first consider $\mathbb{S}_2 = \mathbb{S} * \dot{\mathbb{S}}$, the two step iteration of Sacks forcing. It is well-known that $\mathbb{S}_2$ is forcing equivalent to $\mathcal{B}((2^\omega)^2) \setminus \mathit{cntble}^2$ where $\mathcal{B}((2^\omega)^2)$ are the Borel sets in the plane and cntble^2 is the Fubini power of the ideal of countable sets (see [13,22,7] for details).

Here, for any ideal $I \subseteq \mathcal{P}(2^\omega)$, we define its **Fubini power** $I^2 \subseteq \mathcal{P}((2^\omega)^2)$ by

$$X \in I^2 \Leftrightarrow \{x \in 2^\omega : X_x \notin I\} \in I.$$

For $X \in \mathcal{P}((2^\omega)^2)$, $X_x = \{y : (x, y) \in X\}$ denotes the **vertical section** at $x \in 2^\omega$.

Lemma 4.2.1. *$\mathbb{S}_2$ has weak fusion.*

Proof. Let $E \in \mathbb{S}_2 = \mathcal{B}((2^\omega)^2) \setminus \mathit{cntble}^2$ and an $\mathbb{S}_2$-name $\dot{C}$ for an infinite subset of ω be given. Without loss, we may assume $E = [T]$ where $T \subseteq (2^{<\omega})^2$ is a tree such that

- $(\forall s \in p(T))$ if t_0, t_1 are such that $(s, t_0), (s, t_1) \in T$, then $p(T^{(s,t_0)}) = p(T^{(s,t_1)})$ is perfect,
- $(\forall x \in [p(T)])$ the vertical section $T_x = \{t : (x|_{|t|}, t) \in T\}$ is perfect,

where

- $p(T) = \{s : (\exists t)\,(s, t) \in T\}$ denotes the projection of T onto the first coordinate and $[p(T)]$ is of course the set of branches through $p(T)$,
- $T^{(s,t)} = \{(s', t') \in T : (s, t) \subseteq (s', t') \vee (s', t') \subseteq (s, t)\}$ is the subtree of T defined by (s, t), for $(s, t) \in T$.

Call such trees T **nice**.

(It is well-known that for every analytic subset A of $\mathbb{R}^2$ which does not belong to *cntble*^{2}, we may find a nice perfect tree T such that $[T] \subseteq A$. See [22] or [7] for details.)

We construct, by recursion on $n \in \omega$,

- finite antichains $B_n = \{(s_\sigma, t_{\sigma,\tau}) : \sigma \in 2^n, \tau \in 2^n\} \subseteq (2^{<\omega})^2$,
- nice trees $T_n \in \mathbb{S}_2$,
- finite antichains $\mathcal{A}_n \subseteq \mathbb{S}_2$,
- bijections $h_n : B_n \to \mathcal{A}_n$,
- a one-to-one function $g : \bigcup_{n\in\omega}\{n\} \times \mathcal{A}_n \to \omega$,

such that

(a) if $n < m$ and $\sigma, \tau \in 2^n$, $\sigma', \tau' \in 2^m$, $\sigma \subseteq \sigma'$, $\tau \subseteq \tau'$, then $s_\sigma \subseteq s_{\sigma'}$ and $t_{\sigma,\tau} \subseteq t_{\sigma',\tau'}$,
(b) $T_{n+1} \leq_n T_n \leq_{n-1} \cdots \leq_0 T_0 \leq T$,
(c) $(s_\sigma, t_{\sigma,\tau}) \in T_n$ for $\sigma, \tau \in 2^n$, and $T_n \leq \bigcup_{\sigma,\tau\in 2^n}[(s_\sigma, t_{\sigma,\tau})]$,
(d) $h_n(s_\sigma, t_{\sigma,\tau}) \geq [(s_\sigma, t_{\sigma,\tau})] \cap T_n$ for $\sigma, \tau \in 2^n$,
(e) $A \Vdash$ "$g(n, A) \in \dot{C}$" for $A \in \mathcal{A}_n$,

where $\leq_n$ denotes the standard fusion order on $\mathbb{S}_2$ (in its representation as $\mathcal{B}((2^\omega)^2) \setminus$ *cntble*^{2}). Namely, $S \leq_n T$ if $S \leq T$ and all $(2^n)^2$ nodes on the n-th splitting level of T belong to S.

Fix n, and assume $B_m, T_m, \mathcal{A}_m, h_m, g$ have been constructed for $m < n$. Consider T_{n-1} (with the convention that $T_{-1} = T$).

<u>$n = 0$</u> In this case simply let $B_n = B_0 = \{(s_{\langle\rangle}, t_{\langle\rangle,\langle\rangle})\}$ where $(s_{\langle\rangle}, t_{\langle\rangle,\langle\rangle})$ is the stem of T.
The rest of the construction is as in the general case.

<u>$n > 0$</u> Fix $\sigma \in 2^{n-1}$. Consider $\{(s_\sigma, t_{\sigma,\tau}) : \tau \in 2^{n-1}\}$. Since T_{n-1} is nice, there is a tree $S \subseteq 2^{<\omega}$ such that $p((T_{n-1})^{(s_\sigma,t_{\sigma,\tau})}) = S$ for all $\tau \in 2^{n-1}$. Let $x_0 \neq x_1$ belong to $[S]$. Again by niceness, $(T_{n-1})_{x_i}$ is perfect for $i \in 2$, and, a fortiori, all $((T_{n-1})^{(s_\sigma,t_{\sigma,\tau})})_{x_i}$ are perfect. This means we may find $s_{\sigma\hat{}\langle i\rangle}$ and $t_{\sigma\hat{}\langle i\rangle,\tau\hat{}\langle j\rangle}$ all of the same length for $i, j \in 2$ and for $\tau \in 2^{n-1}$ such that

- $s_{\sigma\hat{}\langle i\rangle} \subseteq x_i$,
- $s_{\sigma\hat{}\langle 0\rangle} \perp s_{\sigma\hat{}\langle 1\rangle}$,
- $t_{\sigma\hat{}\langle i\rangle,\tau\hat{}\langle j\rangle} \in ((T_{n-1})^{(s_\sigma,t_{\sigma,\tau})})_{x_i}$ which means $(s_{\sigma\hat{}\langle i\rangle}, t_{\sigma\hat{}\langle i\rangle,\tau\hat{}\langle j\rangle}) \in T_{n-1}$,
- $t_{\sigma\hat{}\langle i\rangle,\tau\hat{}\langle 0\rangle} \perp t_{\sigma\hat{}\langle i\rangle,\tau\hat{}\langle 1\rangle}$.

Fix $i \in 2$. Since T_{n-1} is nice, there is $S' \subseteq 2^{<\omega}$ such that $p((T_{n-1})^{(s_{\sigma\hat{}\langle i\rangle},t_{\sigma\hat{}\langle i\rangle,\tau})}) = S'$ for all $\tau \in 2^n$. List $\{\tau_k : k \in 2^n\} = 2^n$. By recursion on $k < 2^n$ we construct perfect trees $S_k \subseteq 2^{<\omega}$, nice trees $T^{\sigma\hat{}\langle i\rangle,k} \subseteq T_{n-1}$ and natural numbers n_k such that

- $S_{k+1} \subseteq S_k \subseteq \cdots \subseteq S_0 \subseteq S'$,
- $S_k = p(T^{\sigma\hat{}\langle i\rangle,k})$,
- the stem of $T^{\sigma\hat{}\langle i\rangle,k}$ extends $(s_{\sigma\hat{}\langle i\rangle}, t_{\sigma\hat{}\langle i\rangle,\tau_k})$,
- $T^{\sigma\hat{}\langle i\rangle,k} \Vdash$ "$n_k \in \dot{C}$",
- the n_k are all distinct.

This is clearly possible. At stage k simply consider the tree $(T_{n-1})^{(s_{\sigma\hat{}\langle i\rangle},t_{\sigma\hat{}\langle i\rangle,\tau_k})} \cap (S_{k-1} \times 2^{<\omega})$ (where $S_{-1} = S'$ in case $k = 0$). This is a nice tree, and we may find a nice subtree

$T^{\sigma\hat{}\langle i\rangle,k}$ forcing a number n_k to belong to $\dot{C}$ which is distinct from the previously by chosen numbers. Finally let $S_k = p(T^{\sigma\hat{}\langle i\rangle,k})$. This completes the k-recursion.

Let

- $h_n(s_{\sigma\hat{}\langle i\rangle}, t_{\sigma\hat{}\langle i\rangle,\tau_k}) = T^{\sigma\hat{}\langle i\rangle,k}$,
- $g(n, T^{\sigma\hat{}\langle i\rangle,k}) = n_k$.

Finally define

$$T^{\sigma\hat{}\langle i\rangle} = \left(\bigcup\{T^{\sigma\hat{}\langle i\rangle,k} : k \in 2^n\}\right) \cap (S_{2^n-1} \times 2^{<\omega}).$$

$T^{\sigma\hat{}\langle i\rangle}$ is easily seen to be a nice tree with $p(T^{\sigma\hat{}\langle i\rangle}) = S_{2^n-1}$.

Unfix i and σ, and put

$$T_n = \bigcup\{T^{\sigma\hat{}\langle i\rangle} : \sigma \in 2^{n-1} \wedge i \in 2\}.$$

Clearly T_n is still a nice tree and $T_n \leq_{n-1} T_{n-1}$ is immediate. Next,

- $B_n = \{(s_{\sigma\hat{}\langle i\rangle}, t_{\sigma\hat{}\langle i\rangle,\tau_k}) : \sigma \in 2^{n-1}, i \in 2, k \in 2^n\}$ is an antichain,
- $\mathcal{A}_n = h_n[B_n]$ is an antichain,
- h_n is a bijection,

and properties (a) through (e) are obvious by construction.

Finally, if we carry out the above construction by going recursively through all pairs $(\sigma, i) \in 2^{n-1} \times 2$ (instead of dealing with them simultaneously), we may also assume that $g(n,\cdot)$ is one-to-one, and that the range of $g(n,\cdot)$ is disjoint from the range of $g(m,\cdot)$ for $m < n$. This shows g will be one-to-one, and completes the n-recursion.

We are left with showing that (a) through (e) above imply that $\mathbb{S}_2$ has weak fusion. However, if we let

$$T_\infty = \bigcap_{n\in\omega} T_n$$

then $[T_\infty] = G_B$ is clear by (c), and $[T_\infty] \notin \mathsf{cntble}^2$ by (b). So 1 in Definition 2.2.1 holds. (2') (and hence 2) is immediate by (d), and 3 is property (e). This completes the proof of the lemma. □

Theorem 4.2.2. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{S}_2$-indestructible.*

(ii) *$\forall B \subseteq (2^{<\omega})^2$ such that $G_B \notin \mathsf{cntble}^2$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^2$.*

(iii) *$\forall B \subseteq (2^{<\omega})^2$ such that $G_B \notin \mathsf{cntble}^2$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^2$.*

(iv) *$\forall f : (2^{<\omega})^2 \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^2$.*

Proof. The equivalence of (i) to (iii) is immediate from Lemma 4.2.1 and Theorem 2.2.2. Concerning (iv), note that cntble^2 is strongly homogeneous, and use 2.4.2. □

4.3. Characterization of $\mathbb{S}_\alpha$-indestructibility

Theorem 4.2.2 can be generalized to an analogous result for $\mathbb{S}_\alpha$ for countable ordinals α, where $\mathbb{S}_\alpha$ is the α-stage iteration of Sacks forcing. This is done as follows.

For any ideal $I \subseteq \mathcal{P}(2^\omega)$, define the **$\alpha$-th Fubini power** $I^\alpha \subseteq \mathcal{P}((2^\omega)^\alpha)$ to consist of sets $X \subseteq (2^\omega)^\alpha$ such that there is a family $\{A_x \in I : x \in (2^\omega)^{<\alpha}\}$ such that for all $y \in X$ there is a $\beta < \alpha$ such that $y(\beta) \in A_{y|_\beta}$. This notion is due to Zapletal [22]. His definition is in terms of infinite games and is easily seen to be equivalent to ours. For another equivalent definition, see [7].

It is obvious that $I^1 = I$ and that the present definition of I^2 is equivalent to the one in the last section. It is well-known that $\mathbb{S}_\alpha$ is forcing equivalent to $\mathcal{B}((2^\omega)^\alpha) \setminus \mathsf{cntble}^\alpha$ where $\mathcal{B}((2^\omega)^\alpha)$ are the Borel subsets of $(2^\omega)^\alpha$ (see [22] and [7] for details).

For countable α, let $Fn(\alpha, 2^{<\omega}) = \{\varphi : dom(\varphi) \subseteq \alpha \text{ finite} \wedge ran(\varphi) \subseteq 2^n \text{ for some } n\}$. Note that $Fn(\alpha, 2^{<\omega})$ plays the same role for countable α as do $2^{<\omega}$ or $(2^{<\omega})^n$ in the finite case. For any $A \subseteq Fn(\alpha, 2^{<\omega})$, put

$$G_A = \left\{ x \in (2^\omega)^\alpha : (\forall F \in [\alpha]^{<\omega})(\forall n)(\exists m \geq n)(\exists \varphi \in A) \begin{array}{c} dom(\varphi) \supseteq F, ran(\varphi) \subseteq 2^m, \\ (\forall \beta \in dom(\varphi)) x(\beta)|_m = \varphi(\beta) \end{array} \right\}$$

the **G_δ-closure of** A (note it is obvious G_A is a G_δ-subset of $(2^\omega)^\alpha$).

We omit the details of the following natural generalization of Theorem 4.2.2.

Theorem 4.3.1. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(i) *$\mathcal{I}$ is $\mathbb{S}_\alpha$-indestructible.*
(ii) *$\forall B \subseteq Fn(\alpha, 2^{<\omega})$ such that $G_B \notin \mathsf{cntble}^\alpha$, $\forall f : B \to \omega$, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^\alpha$.*
(iii) *$\forall B \subseteq Fn(\alpha, 2^{<\omega})$ such that $G_B \notin \mathsf{cntble}^\alpha$, $\forall f : B \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^\alpha$.*
(iv) *$\forall f : Fn(\alpha, 2^{<\omega}) \to \omega$ one-to-one, $\exists I \in \mathcal{I}$ such that $G_{f^{-1}``I} \notin \mathsf{cntble}^\alpha$.*

4.4. Construction of an $\mathbb{S}$-indestructible, $\mathbb{S}_2$-destructible MAD family

In this subsection we prove the following strengthening of Theorem 3.4.1.

Theorem 4.4.1. *Assume either* $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ *or* $\mathfrak{b} = \mathfrak{c}$. *There is a MAD family $\mathcal{A}$ such that*

1. *$\mathcal{A}$ is $\mathbb{S}$-indestructible,*
2. *$\mathcal{A}$ is $\mathbb{S}_2$-destructible.*

Proof. Let $\{f_\alpha : 2^{<\omega} \to (2^{<\omega})^2; \alpha < \mathfrak{c}\}$ be an enumeration of one-to-one functions.

We are going to construct a MAD family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}((2^{<\omega})^2)$ by recursion in $\mathfrak{c}$ steps such that

(a) if $(\forall \beta < \alpha) G_{f_\alpha^{-1}``A_\beta} \in \mathsf{cntble}$ then $G_{f_\alpha^{-1}``A_\alpha} \notin \mathsf{cntble}$,
(b) $G_{A_\alpha} \in \mathsf{cntble}^2$.

By (a) and Theorem 2.4.5, $\mathcal{A}$ is $\mathbb{S}$-indestructible, and by (b) and Theorem 4.2.2, $\mathcal{A}$ is $\mathbb{S}_2$-destructible.

In fact, instead of (b), we shall guarantee the following stronger condition:

(b') (b'1) either $(\exists x \in 2^{\omega}) G_{A_\alpha} \subseteq \{x\} \times 2^{\omega}$,
(b'2) or $(\forall x \in 2^{\omega}) |(G_{A_\alpha})_x| \leq 1$.

It is obvious (b') implies (b).

We consider two cases in stage α.

case 1: $(\exists \beta < \alpha) G_{f_\alpha^{-1}\text{“}A_\beta}$ **is uncountable** Then (a) is trivially satisfied, and we need to find A_α almost disjoint from any A_β, $\beta < \alpha$, such that (b') holds. Since $|\alpha| < \mathfrak{c}$ there is $x \in 2^{\omega}$ such that for all $\beta < \alpha$, if $G_{A_\beta} \subseteq \{y\} \times 2^{\omega}$ for some y (i.e. (b'1) holds), then $x \neq y$. Next find $y \in 2^{\omega}$ such that for all $\beta < \alpha$, if (b'2) holds for β, then $y \notin (G_{A_\beta})_x$. So $(x, y) \notin G_{A_\beta}$ for all $\beta < \alpha$. Let $A_\alpha = \{(x|_n, y|_n) : n \in \omega\}$. Then $G_{A_\alpha} = \{(x, y)\}$ and $|A_\alpha \cap A_\beta| < \aleph_0$ for all $\beta < \alpha$.

case 2: otherwise i.e. $(\forall \beta < \alpha) G_{f_\alpha^{-1}\text{“}A_\beta} \in \mathsf{cntble}$.

We proceed in two steps. First we show

Lemma 4.4.2. *Let $f : 2^{<\omega} \to (2^{<\omega})^2$ be any one-to-one function. There exists a set $A \in [(2^{<\omega})^2]^{\omega}$ such that*

1. $G_{f^{-1}\text{“}A} \notin \mathsf{cntble}$,
2.

(b"1) *either* $(\exists x \in 2^{\omega}) G_A \subseteq \{x\} \times 2^{\omega}$,
(b"2) *or* $(\forall x \in 2^{\omega}) |(G_A)_x| \leq 1$.

Let us first argue how the proof of case 2 is completed using Lemma 4.4.2. Apply 4.4.2 with $f = f_\alpha$ to get A. Clearly A satisfies (a) and (b'), but it need not be almost disjoint from the A_β, $\beta < \alpha$.

If $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, apply Lemma 3.4.2 with f_α replaced by $f_\alpha|_{f_\alpha^{-1}\text{“}A}$ (it is easy to see 3.4.2 also applies in this more general case), and get $A_\alpha \subseteq A$ almost disjoint from A_β, $\beta < \alpha$, such that $G_{f^{-1}\text{“}A_\alpha} \notin \mathsf{cntble}$. Since A_α is a subset of A, (b') still holds, and we are done.

If $\mathfrak{b} = \mathfrak{c}$, either argue directly that the analogue of 3.4.2 holds or use the argument of Hrušák [10] or Kurilić [16] in the proof that $\mathfrak{b} = \mathfrak{c}$ implies the existence of a $\mathbb{C}$-indestructible MAD family. (In fact, under $\mathfrak{b} = \mathfrak{c}$, the strengthening of 3.4.2 obtained by replacing cntble by $\mathcal{M}$ holds. See [10] or [16] for details.) Then proceed as in the case $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$.

This completes the proof of the theorem. □

Proof of Lemma 4.4.2. Let $f : 2^{<\omega} \to (2^{<\omega})^2$ one-to-one be given. Write $f = \langle f_0, f_1 \rangle$ where f_i denotes the i-th coordinate of f (So $f(t) = \langle f_0(t), f_1(t) \rangle$ for all $t \in 2^{<\omega}$ and $f_i(t) \in 2^{<\omega}$ for all $t \in 2^{<\omega}$ and $i \in 2$).

case 1: $(\exists x \in 2^{\omega})(\exists s_0 \in 2^{<\omega})(\forall n)(\forall s \supseteq s_0)(\exists t \supseteq s)\ f_0(t) \supseteq x|_n$. Fix such x and s_0. It is straightforward to construct $B \subseteq 2^{<\omega}$ such that $s_0 \subseteq t$ for all $t \in B$ and

1. G_B is perfect,
2. $(\forall n)(\forall^\infty t \in B)\ f_0(t) \supseteq x|_n$.

Let $A = f``B$. So $B = f^{-1}``A$. By 2, it is immediate that $G_A \subseteq \{x\} \times 2^\omega$. So we are in case (b"1).

case 2: otherwise; $(\forall x \in 2^\omega)(\forall s_0 \in 2^{<\omega})(\exists n)(\exists s \supseteq s_0)(\forall t \supseteq s)\ f_0(t) \not\supseteq x|_n$. In this case we recursively construct $\{s_\sigma : \sigma \in 2^{<\omega}\}$ and $\{t_\sigma : \sigma \in 2^{<\omega}\}$ such that

(i) $\sigma \subseteq \tau$ implies $s_\sigma \subseteq t_\sigma \subseteq s_\tau$,
(ii) $\sigma \perp \tau$ implies $s_\sigma \perp s_\tau$,
(iii) for all n, if $\sigma \neq \tau \in 2^n$, then $f_0(s_\sigma) \perp f_0(s_\tau)$ and $(\forall t \supseteq t_\sigma)\ f_0(t) \perp f_0(s_\tau)$,
(iv) $|\sigma| < |\tau|$ implies $|f_0(s_\sigma)| < |f_0(s_\tau)|$.

Note first that as a consequence we get

(v) $f_0(s_\sigma) \subset f_0(s_\tau)$ implies $\sigma \subset \tau$.

(By (iv), we must have $|\sigma| \leq |\tau|$. Let $n = |\sigma|$. If $\tau|_n \neq \sigma$, then by (iii) and (i), $f_0(s_\sigma) \perp f_0(s_\tau)$, a contradiction. Hence $\tau|_n = \sigma$, and $\sigma \subset \tau$ follows.)

Let us check that the recursive construction can be carried out. $s_{\langle\rangle} = \langle\rangle = t_{\langle\rangle}$.

Assume $n > 0$ and $\{s_\sigma : \sigma \in 2^{n-1}\}$ and $\{t_\sigma : \sigma \in 2^{n-1}\}$ have been constructed so that (i) through (iv) hold. List 2^n as $\{\sigma_i : i \in 2^n\}$. First fix $u_{\sigma_i} \supseteq t_{\sigma_i|_{(n-1)}}$ pairwise incompatible. Then construct, by recursion on i, $x_i \in 2^\omega, n_i \in \omega, s_{\sigma_i} \in 2^{<\omega}, t^i_j \in 2^{<\omega} (j \in 2^n)$ such that

- $f_0(s_{\sigma_i}) \supseteq x_i|_{n_i}$,
- $(\forall j \neq i)(\forall t \supseteq t^i_j)\ f_0(t) \not\supseteq x_i|_{n_i}$,
- $u_{\sigma_i} \subseteq t^0_i \subseteq \cdots \subseteq t^{i-1}_i \subseteq t^i_i = s_{\sigma_i} \subseteq t^{i+1}_i \subseteq \cdots \subseteq t^{2^n-1}_i$.

Note that t^i_j will be produced simultaneously for all j (for fixed i).

step $i = 0$ Let $x_0 \in \overline{\{f_0(t) : u_{\sigma_0} \subseteq t\}}$, that is, x_0 is a limit point of the $f_0(t)$, $u_{\sigma_0} \subseteq t$. That such a limit exists follows from compactness and from the fact that f is one-to-one. By assumption, we can find n_0 and $t^0_j \supseteq u_{\sigma_j}$ $(j \neq 0)$ such that $(\forall j \neq 0)(\forall t \supseteq t^0_j)\ f_0(t) \not\supseteq x_0|_{n_0}$. Let $t^0_0 = s_{\sigma_0}$ be such that $u_{\sigma_0} \subseteq s_{\sigma_0}$ and $f_0(s_{\sigma_0}) \supseteq x_0|_{n_0}$.

step $i + 1$ This is almost identical. Let $x_{i+1} \in \overline{\{f_0(t) : t^i_{i+1} \subseteq t\}}$. Find n_{i+1} and $t^{i+1}_j \supseteq t^i_j$ $(j \neq i+1)$ such that $(\forall j \neq i+1)(\forall t \supseteq t^{i+1}_j)\ f_0(t) \not\supseteq x_{i+1}|_{n_{i+1}}$. Let $t^{i+1}_{i+1} = s_{\sigma_{i+1}}$ be such that $t^i_{i+1} \subseteq s_{\sigma_{i+1}}$ and $f_0(s_{\sigma_{i+1}}) \supseteq x_{i+1}|_{n_{i+1}}$.

In the end, let $t_{\sigma_i} = t^{2^n-1}_i$. Then property (iii) is satisfied (this is the main point of the above construction). (i) and (ii) are also clear by choice of the u_{σ_i} and by construction. Concerning (iv), we can easily make it hold by choosing the above n_i large enough. This completes the recursive construction.

It is relatively easy to further prune the family $\{s_\sigma : \sigma \in 2^{<\omega}\}$ such that

(vi) if $\sigma \subset \tau$ and there is $\theta \supset \tau$ such that $f_1(s_\sigma) \subset f_1(s_\theta)$, then $f_1(s_\sigma) \subset f_1(s_\tau)$

(the t_σ are no longer relevant).

Set $B = \{s_\sigma : \sigma \in 2^{<\omega}\}$ and $A = f``B$. So $B = f^{-1}``A$. By (i) and (ii), G_B is perfect. So it suffices to check that G_A satisfies (b"2). Fix $x \in 2^\omega$, and assume there are $y_0 \neq y_1$ with $(x_0, y_0), (x_1, y_1) \in G_A$. This means there are infinite sets $Y_i (i \in 2)$ and σ^i_n for $n \in Y_i$ such that

$$\langle x|_n, y_i|_n\rangle = f(s_{\sigma^i_n}) = \langle f_0(s_{\sigma^i_n}),\ f_1(s_{\sigma^i_n})\rangle$$

for all $n \in Y_i$ and $i \in 2$. Note that if $n < m$ and $n, m \in Y_0 \cup Y_1$, then $\sigma_n^i \subset \sigma_m^j$ by property (v) (where i (j respectively) is such that $n \in Y_i$ ($m \in Y_j$ respectively)). Fix k such that $y_0|_k \neq y_1|_k$. Find $k \leq n_0 < n_1 < m_0$ such that $n_0, m_0 \in Y_0$ and $n_1 \in Y_1$. Thus, by the preceding remark, $\sigma_{n_0}^0 \subset \sigma_{n_1}^1 \subset \sigma_{m_0}^0$. Also $f_1(s_{\sigma_{n_0}^0}) = y_0|_{n_0} \subset y_0|_{m_0} = f_1(s_{\sigma_{m_0}^0})$. Therefore, by (vi), $y_0|_{n_0} = f_1(s_{\sigma_{n_0}^0}) \subset f_1(s_{\sigma_{n_1}^1}) = y_1|_{n_1}$.

This contradiction finishes the proof of Lemma 4.4.2. □

Notice that the MAD family of Theorem 4.4.1 necessarily has size $\mathfrak{c}$: since $\mathrm{cov}(\mathit{cntble}^2) = \mathfrak{c}$, by Lemma 3.1.2, any MAD family of size $<\mathfrak{c}$ is $\mathbb{S}_2$-indestructible.

Our original motivation to prove 4.4.1 came from a question of Hrušák [10, Question 3]. Namely, he asked whether it is consistent that no MAD family of size $\mathfrak{c}$ is $\mathbb{S}$-indestructible (more exactly, he conjectured there is no such MAD family in the Sacks model). This, in turn, was motivated by his incorrect argument showing there is an $\mathbb{S}$-indestructible MAD family in **ZFC**.

Originally, Theorem 4.4.1 was intended to give a negative answer to Hrušák's question. However, since the result builds on assumptions beyond **ZFC** ($\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{c}$), we were not able to achieve this. Still, the way 4.4.1 is proved from 3.4.1 strongly suggests that if there is an $\mathbb{S}$-indestructible MAD family in **ZFC**, then there is also an $\mathbb{S}$-indestructible MAD family *of size* $\mathfrak{c}$ in **ZFC**. Note, in particular, that Lemma 4.4.2 is a **ZFC**-result. We believe that both are true.

Conjecture 4.4.3. *There is an $\mathbb{S}$-indestructible MAD family in* **ZFC**.

Conjecture 4.4.4. *There is an $\mathbb{S}$-indestructible MAD family of size $\mathfrak{c}$ in* **ZFC**. *More explicitly, there is an $\mathbb{S}$-indestructible, $\mathbb{S}_2$-destructible MAD family in* **ZFC**.

Hrušák's original conjecture was motivated by the fact (see Theorem 4.1.1) that there is no $\mathbb{C}$-indestructible MAD family of size $\mathfrak{c}$ in the Cohen model. However, the situation with $\mathbb{C}$ and $\mathbb{S}$ is basically different, for $\mathbb{C}_\alpha$ (the α-stage finite support iteration or finite support product of $\mathbb{C}$) is forcing equivalent to $\mathbb{C}$ for countable α while $\mathbb{S}, \mathbb{S}_2, \mathbb{S}_3, \ldots, \mathbb{S}_\alpha$ are all different. Accordingly, we shall see in Section 4.7 that Hrušák's conjecture is correct in the sense there is no $\mathbb{S}_{\omega_1}$-indestructible MAD family of size $\mathfrak{c}$ in the Sacks model (Theorem 4.7.1).

4.5. $\mathbb{S}_{\omega_1}$-*indestructibility*

In Section 4.3, we characterized $\mathbb{S}_\alpha$-indestructibility for countable ordinals α (see Theorem 4.3.1). We now briefly consider uncountable α. In fact, by the following result which seems to be well-known (see, for example, the comment in Blass's survey article [3, Section 11.5]), this boils down to the countable case. We include a sketch of the proof for the sake of completeness.

Theorem 4.5.1. *The following are equivalent for a tall ideal $\mathcal{I}$.*

(i) *$\mathcal{I}$ is $\mathbb{S}_\alpha$-indestructible for all α.*
(ii) *$\mathcal{I}$ is $\mathbb{S}_\alpha$-indestructible for some uncountable α.*
(iii) *$\mathcal{I}$ is $\mathbb{S}_{\omega_1}$-indestructible.*
(iv) *$\mathcal{I}$ is $\mathbb{S}_\alpha$-indestructible for all countable α.*

Proof. Clearly (i) implies (ii), (ii) implies (iii), and (iii) implies (iv). In fact, since no new reals arise in limit stages of uncountable cofinality of the iteration, the equivalence of (iii) and (iv) is easy to see. We shall sketch the argument for (iv) implies (i).

Lemma 4.5.2. *Let α be an ordinal. Assume $p_0 \in \mathbb{S}_\alpha$ and $\dot{A}$ is any $\mathbb{S}_\alpha$-name for a subset of ω. Then there is $p \leq p_0$ such that for all $B \subseteq \omega$, if there is $q \leq p$ with $q \Vdash$ "$B \cap \dot{A} = \emptyset$", then there is $r \leq p$ with*

- *$supt(r) = supt(p)$,*
- *r is compatible with q,*
- *$r \Vdash$ "$B \cap \dot{A} = \emptyset$",*

where $supt(p) = \{\beta < \alpha : p(\beta) \neq \dot{1}\}$ denotes the **support** *of p.*

Sketch of Proof of Lemma 4.5.2. This is a canonical fusion argument. Namely, one builds a decreasing sequence $\langle q_n : n \in \omega\rangle$ of conditions and auxiliary $\langle F_n \subseteq \alpha$ finite: $n \in \omega\rangle$ such that

- $p_0 \geq q_0 \geq_{F_0,0} \geq q_1 \geq \cdots \geq_{F_{n-1},n-1} q_n \geq_{F_n,n} \cdots$,
- $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} supt(q_n)$,
- q_n can be thought of as a finite union of conditions deciding "$n \in \dot{A}$".

Essentially, at stage n, we consider the n-th splitting levels of the $q_n(\beta)$ where $\beta \in F_n$. Since this is a standard argument, we leave out the details.

Let $p = \bigcap_{n\in\omega} q_n$ be the fusion. Then $supt(p) = \bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} supt(q_n)$. Now let $B \subseteq \omega$ and $q \leq p$ such that $q \Vdash$ "$B \cap \dot{A} = \emptyset$". Note there are finite maximal antichains H_n of conditions below p such that $r \in H_n$ decides "$n \in \dot{A}$" and such that $supt(r) = supt(p)$ for all $r \in H_n$. This means $p = \Sigma H_n$ for all n. For $n \in B$ let $G_n \subseteq H_n$ be such that for $r \in G_n$, $r \Vdash$ "$n \notin \dot{A}$" and for $r \in H_n \setminus G_n$, $r \Vdash$ "$n \in \dot{A}$". Clearly, $q \leq \Sigma G_n$ for all $n \in B$. Therefore $q \leq \bigcap_{n\in B} \Sigma G_n =: r$. $supt(r) = supt(p)$ is straightforward by construction. $\square$

The above means that whatever is decided about $\dot{A}$ by a condition below p is in fact already decided by a condition with support $= supt(p)$.

A similar fusion argument shows that given any $p_0 \in \mathbb{S}_\alpha$ there is $p \leq p_0$ such that for all $\beta \in supt(p)$, whatever is decided about $p(\beta)$ by a condition below $p|_\beta$ is in fact already decided by a condition with support $= supt(p|_\beta) = supt(p) \cap \beta$. Call such p **canonical**. Informally, we may think of such a condition as an element of $\mathbb{S}_{supt(p)}$. In particular, if we let $\alpha_0 = otp(supt(p)) < \omega_1$, then there is a projection mapping π sending p to $\pi(p) \in \mathbb{S}_{\alpha_0}$ such that for each $\beta_0 < \alpha_0$, $\pi(p)(\beta_0)$ is the $\mathbb{S}_{\beta_0}$-name for a condition in $\mathbb{S}$ corresponding to the $\mathbb{S}_\beta$-name $p(\beta)$ where β is the β_0-th element of $supt(p)$. This makes sense because $p(\beta)$ depends only on coordinates in $supt(p|_\beta)$. Note that π is an order-isomorphism between canonical conditions in $\mathbb{S}_{supt(p)}$ (below p) and canonical conditions in $\mathbb{S}_{\alpha_0}$ (below $\pi(p)$). Some care has to be taken because while $\mathbb{S}_{supt(p)}$ is a suborder of $\mathbb{S}_\alpha$, it is *not* completely embedded in $\mathbb{S}_\alpha$. Still, given an $\mathbb{S}_\alpha$-name $\dot{A}$ for a subset of ω such that the pair $(p, \dot{A})$ satisfies the condition of 4.5.2, we may think of $\dot{A}$ as an $\mathbb{S}_{supt(p)}$-name (below p), and we may also canonically project it to an $\mathbb{S}_{\alpha_0}$-name $\pi(\dot{A})$.

This projection has the property that for all canonical $q \le p, q \in \mathbb{S}_{supt(p)}$ and for all $n \in \omega$, $q \Vdash_{\mathbb{S}_\alpha}$ "$n \in \dot{A}$" $\iff \pi(q) \Vdash_{\mathbb{S}_{\alpha_0}}$ "$n \in \pi(\dot{A})$".

Now assume $\mathcal{I}$ is $\mathbb{S}_\alpha$-indestructible for all countable α, yet there is an uncountable α such that $\mathcal{I}$ is $\mathbb{S}_\alpha$-destructible. We shall reach a contradiction. Let $p \in \mathbb{S}_\alpha$ and $\dot{A}$ be such that $p \Vdash_{\mathbb{S}_\alpha}$ "$|\dot{A} \cap I| < \aleph_0$ for any $I \in \mathcal{I}$". Without loss p is canonical and the pair $(p, \dot{A})$ satisfies Lemma 4.5.2. Let $\alpha_0 = otp(supt(p))$, and let $\pi : \mathbb{S}_{supt(p)} \to \mathbb{S}_{\alpha_0}$ be as above. Without loss $supt(p) \cap \omega_1 \in \omega_1$. Let $\alpha_1 = supt(p) \cap \omega_1$. Then $\pi|_{\mathbb{S}_{\alpha_1}} = id$, and p and $\pi(p)$ are canonically compatible with common extension $p \cup \pi(p)$; in fact, any condition in $\mathbb{S}_{supt(p)}$ below p is compatible with $\pi(p)$ and any condition in $\mathbb{S}_{\alpha_0}$ below $\pi(p)$ is compatible with p. Let $\dot{B} = \pi(\dot{A})$.

By assumption, find $q \le \pi(p), q \in \mathbb{S}_{\alpha_0}$ and $I \in \mathcal{I}$ such that $q \Vdash_{\mathbb{S}_{\alpha_0}}$ "$|\dot{B} \cap I| = \aleph_0$". Then $\pi^{-1}(q) \in \mathbb{S}_{supt(p)}, \pi^{-1}(q) \le p$ and q and $\pi^{-1}(q)$ are compatible with common extension $q \cup \pi^{-1}(q)$. Find $r_0 \in \mathbb{S}_\alpha, r_0 \le q \cup \pi^{-1}(q)$ and $n \in \omega$ such that $r_0 \Vdash_{\mathbb{S}_\alpha}$ "$\dot{A} \cap I \subseteq n$". For simplicity assume that $r_0 \Vdash_{\mathbb{S}_\alpha}$ "$\dot{A} \cap I = \emptyset$". By Lemma 4.5.2, there is $r \le p, r \in \mathbb{S}_{supt(p)}$ such that r is compatible with r_0 and $r \Vdash$ "$\dot{A} \cap I = \emptyset$". Since r and r_0 are compatible, also r and $\pi^{-1}(q)$ are compatible, and we may assume without loss that $r \le \pi^{-1}(q)$. Then $\pi(r) \le q, \pi(r) \in \mathbb{S}_{\alpha_0}$. Find $s \le \pi(r), s \in \mathbb{S}_{\alpha_0}$, and $n \in I$ such that $s \Vdash_{\mathbb{S}_{\alpha_0}}$ "$n \in \dot{B}$". Then $\pi^{-1}(s) \in \mathbb{S}_{supt(p)}, \pi^{-1}(s) \le r$, and $\pi^{-1}(s) \Vdash_{\mathbb{S}_{\alpha_0}}$ "$n \in \dot{A}$". This is a contradiction, and the proof of the theorem is complete. $\square$

4.6. Construction of an iterated Sacks indestructible MAD family

In this section, we sketch the proof of the following strengthening of Theorem 3.4.1.

Theorem 4.6.1. *Assume either* $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ *or* $\mathfrak{b} = \mathfrak{c}$. *Then there is a MAD family* $\mathcal{A}$ *which is* $\mathbb{S}_\alpha$*-indestructible for any* α.

Before starting out with the proof, a few comments are in order. First notice that in view of Theorem 4.5.1, it suffices to consider countable α when doing the construction. Next, as mentioned already when discussing 3.4.1, this result is well-known in the case CH holds. Namely, to show that $\mathfrak{a} = \aleph_1$ in the iterated Sacks model one must construct a MAD family which is $\mathbb{S}_\alpha$-indestructible for all countable α. The former, however, was proved by Spinas [3, Section 11.5] (or see [7,8] for an alternative proof).

Sketch of Proof. If $\mathfrak{a} < \mathfrak{c}$, we are done: for all countable α, $\mathsf{cov}(\mathit{cntble}^\alpha) = \mathfrak{c}$ so that by Corollary 3.1.3 any MAD family of size $\mathfrak{a}$ will be $\mathbb{S}_\alpha$-indestructible for all α. Therefore assume $\mathfrak{a} = \mathfrak{c}$.

In view of 4.3.1 and 4.5.1, list all one-to-one functions $f : Fn(\alpha, 2^{<\omega}) \to \omega$ $(\alpha < \omega_1)$ as $\{f_\beta : \beta < \mathfrak{c}\}$. We need to construct pairwise almost disjoint $\{A_\beta : \beta < \mathfrak{c}\}$ such that

- for all $\beta < \mathfrak{c}$, if $f_\beta : Fn(\alpha, 2^{<\omega}) \to \omega$ and $G_{f_\beta^{-1}``A_\gamma} \in \mathit{cntble}^\alpha$ for all $\gamma < \beta$, then $G_{f_\beta^{-1}``A_\beta} \notin \mathit{cntble}^\alpha$.

If the antecedent of this clause fails, stage β of the construction is trivial. If it holds, as in the proof of 3.4.1, we may find a tree $T \subseteq Fn(\alpha, 2^{<\omega})$ (where α is such that $f_\beta : Fn(\alpha, 2^{<\omega}) \to \omega$) such that $[T] \notin \mathit{cntble}^\alpha$ and $G_{f_\beta^{-1}``A_\gamma} \cap [T] = \emptyset$ for all $\gamma < \beta$.

If $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, a real Cohen over the objects constructed so far easily yields a set $B \subseteq T$ with $G_B \notin \mathit{cntble}^\alpha$ and $|B \cap G_{f_\beta^{-1}\text{``}A_\gamma}| < \aleph_0$ for all $\gamma < \beta$, so that $A_\beta = f_\beta\text{''}B$ works (see Lemma 3.4.2 and its proof).

If $\mathfrak{b} = \mathfrak{c}$, a dominating real does the same. (See the comments in the proof of Theorem 4.4.1, as well as [10] and [16].) This completes the argument. □

4.7. There are no iterated Sacks indestructible MAD families of size $\mathfrak{c}$ in the Sacks model

Finally we show:

Theorem 4.7.1. *In the Sacks model (the extension of a model of CH by forcing with $\mathbb{S}_{\omega_2}$), any MAD family which is $\mathbb{S}_\alpha$-indestructible for all α has size $\aleph_1$.*

We mentioned already at the end of Section 4.4 that this gives a positive answer to a modified version of a question of Hrušák [10, Question 3], and that this can be considered an analogue of the corresponding results on Cohen and random forcing in Theorem 4.1.1.

Proof. Assume the theorem was false and there is a MAD family $\mathcal{A} = \{A_\alpha : \alpha < \omega_2\}$ which is $\mathbb{S}_{\omega_1}$-indestructible (this is the same by Theorem 4.5.1). Let $\dot{\mathcal{A}} = \{\dot{A}_\alpha : \alpha < \omega_2\}$ be an $\mathbb{S}_{\omega_2}$-name for $\mathcal{A}$. By standard arguments, there exists an ω_1-club $C \subseteq \omega_2$ such that for all $\alpha \in C$,

$$\Vdash_{\mathbb{S}_\alpha} \text{“}\dot{\mathcal{A}}_\alpha = \{\dot{A}_\beta : \beta < \alpha\} \text{ is a MAD family”}$$

(in particular, this means that for all $\beta < \alpha$, $\dot{A}_\beta$ is an $\mathbb{S}_\alpha$-name). Since no $\mathcal{A}_\alpha$ is maximal in the ultimate extension, we clearly have

$$\Vdash_{\mathbb{S}_\alpha} \text{“}\dot{\mathcal{A}}_\alpha \text{ is } \mathbb{S}_{\omega_2}\text{-destructible”}$$

for all $\alpha \in C$. Hence, by Theorem 4.5.1,

$$\Vdash_{\mathbb{S}_\alpha} \text{“}\dot{\mathcal{A}}_\alpha \text{ is } \mathbb{S}_{\omega_1}\text{-destructible”}$$

for all $\alpha \in C$. In the $\mathbb{S}_\alpha$-extension, there are an ordinal $\gamma^\alpha < \omega_1$, a condition $q^\alpha \in \mathbb{S}_{\gamma^\alpha}$ and a $\mathbb{S}_{\gamma^\alpha}$-name $\dot{A}^\alpha$ for a subset of ω such that

$$q^\alpha \Vdash_{\mathbb{S}_{\gamma^\alpha}} \text{“}\dot{A}^\alpha \text{ is almost disjoint from all } A_\beta, \beta < \alpha\text{”}.$$

Let $B^\alpha = \dot{A}^\alpha$. Back in the ground model, we have names $\dot{\gamma}^\alpha, \dot{q}^\alpha, \dot{B}^\alpha$ for these objects, i.e.

$$\Vdash_{\mathbb{S}_\alpha} \text{“}q^\alpha \Vdash_{\dot{\mathbb{S}}_{\dot{\gamma}^\alpha}} \text{“}\dot{B}^\alpha \text{ is almost disjoint from all } \dot{A}_\beta, \beta < \alpha\text{”\,”}.$$

Since $cf(\alpha) = \omega_1$ for all $\alpha \in C$, we may find $\beta^\alpha < \alpha$ and $p^\alpha \in \mathbb{S}_{\beta^\alpha}$ such that $\dot{\gamma}^\alpha, \dot{q}^\alpha$ and $\dot{B}^\alpha$ are $\mathbb{S}_{\beta^\alpha}$-names and, in fact, we may also assume that p^α decides the value of $\dot{\gamma}^\alpha$, say

$$p^\alpha \Vdash_{\mathbb{S}_{\beta^\alpha}} \text{“}\dot{\gamma}^\alpha = \gamma^\alpha\text{”}$$

for some ordinal $\gamma^\alpha < \omega_1$. Note that the function $\alpha \to \beta^\alpha$ is regressive, so there are $\beta < \omega_2$ and a stationary set $S \subseteq C$ such that $\beta^\alpha = \beta$ for all $\alpha \in S$.

By further pruning, we may then also assume there are $p \in \mathbb{S}_\beta$, $\gamma < \omega_1$, an $\mathbb{S}_\beta$-name $\dot{q}$ for a condition in $\dot{\mathbb{S}}_\gamma$ and an $\mathbb{S}_\beta$-name $\dot{B}$ for an $\dot{\mathbb{S}}_\gamma$-name for a subset of ω such that for all $\alpha \in S$, $p^\alpha = p$, $\gamma^\alpha = \gamma$, $\dot{q}^\alpha = \dot{q}$ and $\dot{B}^\alpha = \dot{B}$. So, in particular,

$$p \Vdash_{\mathbb{S}_\alpha} \text{“}\dot{q} \Vdash_{\dot{\mathbb{S}}_\gamma} \text{“}\dot{B} \text{ is almost disjoint from all } \dot{A}_\delta, \delta < \alpha\text{”}\text{”}$$

for all $\alpha \in S$. Notice that while $\dot{B}$ is a $\mathbb{S}_\beta$-name and is always interpreted as the same $\mathbb{S}_\gamma$-name $B = \dot{A}$ in the $\mathbb{S}_\beta$-extension, the interpretation of $\dot{A}$ depends on α, namely, on the interval $[\alpha, \alpha + \gamma)$ in which $\dot{A}$ is adjoined, in the above formula.

Claim 4.7.2. *$p \Vdash_{\mathbb{S}_{\omega_2}}$ “$\dot{q} \Vdash_{\mathbb{S}_\gamma}$ “$\dot{B}$ is almost disjoint from all $\dot{A}_\delta$, $\delta < \omega_2$” ”.*

Again $\dot{B}$ arises in the $\mathbb{S}_\beta$-extension as $B = \dot{A}$, and $\dot{A}$ is then adjoined by forcing with $\mathbb{S}_\gamma$ over the $\mathbb{S}_{\omega_2}$-extension. Clearly the claim finishes the proof of the theorem. □

Proof of Claim 4.7.2. By stepping into the $\mathbb{S}_\beta$-extension with p belonging to the generic filter, if necessary, we may assume without loss of generality that $\beta = 0$. Then we have q and $B = \dot{A}$. We use

Lemma 4.7.3. *Let V be a model of* **ZFC**. *Let γ be an ordinal and let $q_0 \in V$, $\dot{A}$ an $\mathbb{S}_\gamma$-name for a subset of ω with $\dot{A} \in V$. Then there is $q \le q_0, q \in V$, such that whenever $W \supseteq V$ is a model of* **ZFC**, *$r \le q, r \in W$ and $n \in \omega$ with $r \Vdash_{\mathbb{S}_\gamma}$ “$n \in \dot{A}$” in W are given, then there is $s \le q, s \in V$, s compatible with r, with $s \Vdash_{\mathbb{S}_\gamma}$ “$n \in \dot{A}$” (in V).*

Proof. This follows readily from properness. Namely, let $N \subseteq V$ be an elementary substructure containing γ, q_0 and $\dot{A}$, and let $q \le q_0, q \in V$, be $(\mathbb{S}_\gamma, N)$-generic. Clearly q is as required: if $r \le q, r \in W, n \in \omega$ with $r \Vdash_{\mathbb{S}_\gamma}$ “$n \in \dot{A}$” are given, there is $s_0 \in N$ compatible with r such that $s_0 \Vdash_{\mathbb{S}_\gamma}$ “$n \in \dot{A}$”, and $s = s_0 \cdot q$ is as required.

Alternatively, this can be shown directly with a fusion argument very similar to the proof of the related Lemma 4.5.2. □

Assume the claim was false, and let $\delta < \omega_2$ and $r_0 \le q, r_0 \in \mathbb{S}_{\omega_2+\gamma}$, be such that

$$r_0 \Vdash_{\mathbb{S}_{\omega_2+\gamma}} \text{“}|\dot{A} \cap \dot{A}_\delta| = \aleph_0\text{”}.$$

Find $\alpha \in S, \alpha > \delta$, such that $r_0|_{\omega_2} \in \mathbb{S}_\alpha$ (i.e. $supt(r_0) \cap \omega_2 \subseteq \alpha$) and $r_0|_{[\omega_2,\omega_2+\gamma)}$ is an $\mathbb{S}_\alpha$-name. Step into the $\mathbb{S}_\alpha$-extension with $r_0|_{\omega_2}$ belonging to the generic filter. Since $r_0|_{[\omega_2,\omega_2+\gamma)} \in V^{\mathbb{S}_\alpha}$ can be thought of as a condition of $\mathbb{S}_\gamma$, we can find $s_0 \le r_0|_{[\omega_2,\omega_2+\gamma)}$, $s_0 \in \mathbb{S}_\gamma$ and $n_0 \in \omega$ such that

$$s_0 \Vdash_{\mathbb{S}_\gamma} \text{“}\dot{A} \cap A_\delta \subseteq n_0\text{”}$$

(in $V^{\mathbb{S}_\alpha}$). Without loss of generality, we may assume the pair $(s_0, \dot{A})$ satisfies 4.7.3 (with V being $V^{\mathbb{S}_\alpha}$). Since

$$\Vdash_{\mathbb{S}_{\omega_2}} \text{“}s_0 \Vdash_{\mathbb{S}_\gamma} |\dot{A} \cap A_\delta| = \aleph_0\text{”}$$

we find, in $W = V^{\mathbb{S}_{\omega_2}}$, a condition $r \le s_0, r \in \mathbb{S}_\gamma$, and $n \in A_\delta, n \ge n_0$, such that

$$r \Vdash_{\mathbb{S}_\gamma} \text{“}n \in \dot{A}\text{”}$$

(in W). By Lemma 4.7.3, there is $s \leq s_0$, $s \in V^{\mathbb{S}_\alpha}$, such that

$$s \Vdash_{\mathbb{S}_\gamma} \text{“}n \in \dot{A}\text{”}$$

(in $V^{\mathbb{S}_\alpha}$). This contradiction completes the proof of Claim 4.7.2. $\square$

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References

[1] Tomek Bartoszynski, Haim Judah, Set Theory. On the Structure of the Real Line, A.K. Peters, 1995.
[2] James Baumgartner, Peter Dordal, Adjoining dominating functions, J. Symbolic Logic 50 (1985) 94–101.
[3] Andreas Blass, Combinatorial cardinal characteristics of the continuum, Handbook of Set Theory (in press).
[4] Jörg Brendle, Greg Hjorth, Otmar Spinas, Regularity properties for dominating projective sets, Ann. Pure Appl. Logic 72 (1995) 291–307.
[5] Jörg Brendle, Mob families and mad families, Arch. Math. Logic 37 (1998) 183–197.
[6] Jörg Brendle, Mad families and iteration theory, in: Y. Zhang (Ed.), Logic and Algebra, Contemporary Mathematics, vol. 302, 2002, pp. 1–31.
[7] Krzysztof Ciesielski, Janusz Pawlikowski, The Covering Property Axiom, CPA: A Combinatorial Core of the Iterated Perfect Set Model, Cambridge Tracts in Mathematics, vol. 164, Cambridge University Press, 2004.
[8] Krzysztof Ciesielski, Janusz Pawlikowski, Crowded and selective ultrafilters under the covering property axiom, J. Appl. Anal. 9 (2003) 19–55.
[9] Martin Goldstern, Miroslav Repický, Saharon Shelah, Otmar Spinas, On tree ideals, Proc. Amer. Math. Soc. 123 (5) (1995) 1573–1581.
[10] Michael Hrušák, Mad families and the rationals, Comment. Math. Univ. Carolin. 42 (2) (2001) 345–352.
[11] Michael Hrušák, Salvador García Ferreira, Ordering MAD families à la Katětov, J. Symbolic Logic 68 (2003) 1337–1353.
[12] Thomas Jech, Set Theory, The third millenium edition, Springer, Berlin, 2002.
[13] Vladimir Kanovei, On non-well founded iterations of the perfect set forcing, J. Symbolic Logic 64 (2) (1999) 551–574.
[14] Kenneth Kunen, Set Theory, North-Holland, 1980.
[15] Kenneth Kunen, Random and Cohen reals, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 897–911.
[16] Miloš S. Kurilić, Cohen-stable families of subsets of integers, J. Symbolic Logic 66 (1) (2001) 257–270.
[17] Grzegorz Łabedzki, Miroslav Repický, Hechler reals, J. Symbolic Logic 60 (2) (1995) 444–458.
[18] Thomas E. Leathrum, A special class of almost disjoint families, J. Symbolic Logic 60 (3) (1995) 879–891.
[19] Saharon Shelah, Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing, Acta Math. (in press).
[20] Otmar Spinas, Dominating projective sets in the Baire space, Ann. Pure Appl. Logic 68 (1994) 327–342.
[21] Juris Steprāns, Combinatorial consequences of adding Cohen reals, in: Haim Judah (Ed.), Set Theory of The Reals, Israel Mathematical Conference Proceedings, vol. 6, 1993, pp. 583–617.
[22] Jindřich Zapletal, Descriptive set theory and definable forcing, Mem. AMS 167 (2004).
[23] Jindřich Zapletal, Isolating cardinal invariants, J. Math. Logic 3 (1) (2003) 143–162.
[24] Yi Zhang, Towards a problem of E. van Douwen and A. Miller, Math. Logic Quart. 45 (2) (1999) 183–188.