Restricted MAD families

Osvaldo Guzmán Michael Hrušák Osvaldo Téllez*

Abstract

Let $\mathcal{I}$ be an ideal on ω. By $\mathsf{cov}^*(\mathcal{I})$ we denote the least size of a family $\mathcal{B} \subseteq \mathcal{I}$ such that for every infinite $X \in \mathcal{I}$ there is $B \in \mathcal{B}$ for which $B \cap X$ is infinite. We say that an AD family $\mathcal{A} \subseteq \mathcal{I}$ is a *MAD family restricted to* $\mathcal{I}$ if for every infinite $X \in \mathcal{I}$ there is $A \in \mathcal{A}$ such that $|X \cap A| = \omega$. Let $\mathfrak{a}(\mathcal{I})$ be the least size of an infinite MAD family restricted to $\mathcal{I}$. We prove that If $\mathsf{max}\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} = \omega_1$ then $\mathfrak{a}(\mathcal{I}) = \omega_1$. We conclude that if $\mathcal{I}$ is tall and $\mathfrak{c} \leq \omega_2$ then $\mathfrak{a}(\mathcal{I}) = \mathsf{max}\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\}$. We use these results to prove that if $\mathfrak{c} \leq \omega_2$ then $\mathfrak{o} = \overline{\mathfrak{o}}$ and that $\mathfrak{a}_s = \mathsf{max}\{\mathfrak{a}, \mathsf{non}(\mathcal{M})\}$.

1 Introduction and preliminaries

We say that $\mathcal{A} \subseteq \wp(\omega)$[1] is an *almost disjoint family* (*AD*) if the intersection of any two of its elements is finite and $\mathcal{A}$ is *MAD* if it is maximal with respect to this property. MAD families have played a very important role in set theory, functional analysis and topology (see [12]). It follows by Zorn's lemma that every AD family can be extended to a MAD family; however, we may still wonder how the extensions of an AD family might be. This has been previously studied by Leathrum in [18] and was the object of study in [21]. Understanding how AD families can be extended to MAD families is fundamental in order to study certain combinatorial aspects of MAD families. This is relevant in the study of *forcing indestructibility of MAD families.* Given a MAD family $\mathcal{A}$ and a forcing $\mathbb{P}$, we say that $\mathcal{A}$ is $\mathbb{P}$-*destructible* if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$. For example, it is known that if $\mathcal{A}$ is a MAD family on the rational numbers such that every element of $\mathcal{A}$ is nowhere dense, then it will be destroyed by Cohen forcing. The reader that wishes to learn more about destructibility of MAD families may consult [13], [7], [15], [8], [11] or [17].

In order to state the main results of the paper, we need the following notions:

Definition 1 *Let $\mathcal{I}$ be an ideal (in a countable set).*

* *keywords:* Almost disjoint families, MAD families, ideals, off-branch.
AMS Classification: 03E17, 03E35, 03E05
The second author gratefully acknowledges support received from the PAPIIT project IN100317.

[1] If X is a set, by $\wp(X)$ we denote its power set.

1. *We define* $\mathsf{cov}^*(\mathcal{I})$ *as the least size of a family* $\mathcal{B} \subseteq \mathcal{I}$ *such that for every infinite* $X \in \mathcal{I}$ *there is* $B \in \mathcal{B}$ *for which* $B \cap X$ *is infinite.*

2. *We say that an AD family* $\mathcal{A} \subseteq \mathcal{I}$ *is a* MAD family restricted to $\mathcal{I}$ *if for every infinite* $X \in \mathcal{I}$ *there is* $A \in \mathcal{A}$ *such that* $|X \cap A| = \omega$.

3. $\mathfrak{a}(\mathcal{I})$ *is the least size of an infinite MAD family restricted to* $\mathcal{I}$.

The outline of the paper is as follows:

In the first section, we will prove our main combinatorial lemma: If the maximum of $\mathfrak{a}$ and $\mathsf{cov}^*(\mathcal{I})$ is ω_1 then $\mathfrak{a}(\mathcal{I}) = \omega_1$. This is a simple, yet very useful result. In the rest of the paper, we will derive several applications of this theorem.

The second section deals with the *off-branch numbers* of Leathrum (see [18]):

Definition 2 1. *A set* $B \subseteq 2^{<\omega}$ *is called* off-branch *if it has finite intersection with every branch of* $2^{<\omega}$ *(i.e. if* $r \in 2^{\omega}$ *then* $B \cap \{r \upharpoonright n \mid n \in \omega\}$ *is finite).*

2. $\mathfrak{o}$ *is the smallest size of a maximal family of almost disjoint off-branch sets.*

3. $\overline{\mathfrak{o}}$ *is the smallest size of a maximal family of almost disjoint antichains of* $2^{<\omega}$.

It is easy to see that $\mathfrak{o} \leq \overline{\mathfrak{o}}$. It is an old open question of Leathrum if the inequality $\mathfrak{o} < \overline{\mathfrak{o}}$ is consistent. We do not know the answer to this question, but we will prove that $\mathfrak{o} = \omega_1$ implies that $\overline{\mathfrak{o}} = \omega_1$. In particular, it is not possible to get the inequality if the size of the continuum is at most ω_2.

In the third section, we study the cardinal invariant $\mathfrak{a}_s$, which is defined as the smallest size of a maximal family of eventually different partial functions. In [6] Brendle showed that it is consistent that $\mathsf{max}\{\mathfrak{a},\mathsf{non}(\mathcal{M})\} < \mathfrak{a}_s$ (where $\mathfrak{a}$ is smallest size of a MAD family and $\mathsf{non}(\mathcal{M})$ is the smallest size of a non-meager subset of the Baire space). In the model of Brendle, the continuum has size at least ω_3. This is no coincidence, we will prove that if the continuum has size at most ω_2, then $\mathfrak{a}_s = \mathsf{max}\{\mathfrak{a},\mathsf{non}(\mathcal{M})\}$.

In the fourth section, we study the cardinal invariants $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N}))$ and $\mathfrak{a}(\mathcal{NDN})$. We use our results to answer some open questions found in [21].

In the fifth section, we obtain a preservation theorem for tight MAD families. We will use this result in the following section, but we expect it to have further applications.

In the sixth section, we look at the ideal $\mathcal{K}$, which is the ideal generated by the finitely branching subtrees of $\omega^{<\omega}$. We compare $\mathfrak{a}(\mathcal{K})$ with the cardinal invariant $\mathfrak{a}_T$, which is defined as the smallest size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$. At first glance, $\mathfrak{a}(\mathcal{K})$ and $\mathfrak{a}_T$ seem very similar, however,we will prove that they are consistently different. In fact, we will show that it is consistent that $\mathfrak{a}(\mathcal{K}) < \mathfrak{a}_T$. We will use a forcing of Miller ([19]) that destroys witnesses of $\mathfrak{a}_T$ without adding dominating reals. We will use our preservation result from the previous section and our main combinatorial lemma.

The last section is about the cardinal invariant $\mathfrak{a}^+(\omega_1)$, (introduced in [21]) which is defined as the least κ such that every AD family of size ω_1 can be extended to a MAD family of size at most κ. In [21] it was proved that it is consistent that $\omega_2 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ (where $\mathfrak{c}$ denotes the cardinality of the continuum). Nevertheless, the following problem is still open:

Problem 3 ([21]) *Is $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ consistent? In other words, is the statement "Every AD family of size ω_1 can be extended to a MAD family of size ω_1" consistent with the negation of the Continuum Hypothesis?*

We do not know the answer to the problem, but we will derive some consequences from the assumption that $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$ and show that it fails in most of the known models of set theory.

Our notation is mostly standard. If $\mathcal{X}$ is a set of subsets of ω, we denote by $\mathcal{X}^{\perp}$ the set of all infinite $A \subseteq \omega$ that are almost disjoint from every element of $\mathcal{X}$. If $\mathcal{I}$ is an ideal in ω, we denote by $\mathcal{I}^+$ those subsets of ω that are not in $\mathcal{I}$. If $X \in \mathcal{I}^+$ then by $\mathcal{I} \restriction X$ we denote the restriction of $\mathcal{I}$ to X. We say that $\mathcal{I}$ is *tall* if for every infinite $X \subseteq \omega$ there is $A \in \mathcal{I}$ such that $A \subseteq X$. The relationship between MAD families and definable ideals (typically Borel of low complexity) has been an active area of research (see e.g. [12], [7]).

If $\mathcal{J}$ is a σ-ideal of a Polish space X, we denote by $\mathsf{cov}(\mathcal{J})$ the smallest size of a subfamily of $\mathcal{J}$ that covers X. By $\mathsf{non}(\mathcal{J})$ we denote the smallest size of a subset of X that it is not in $\mathcal{J}$. By $\mathcal{M}$ we denote the σ-ideal of all meager sets in 2^{ω}, and by $\mathcal{N}$ we denote the σ-ideal of all Lebesgue null subsets of 2^{ω}.

The size of the continuum is denoted by $\mathfrak{c}$. Let $f, g \in \omega^{\omega}$, define $f \leq g$ if $f(n) \leq g(n)$ for every $n \in \omega$, and $f \leq^* g$ if $f(n) \leq g(n)$ for almost all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^{\omega}$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family. We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is a *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family. The reader may consult the survey [4] for the main properties of the cardinal invariants used in this paper.

1.1 Main combinatorial result

Let $\mathcal{I}$ be an ideal. It is easy to see that $\mathsf{cov}^*(\mathcal{I}) \leq \mathfrak{a}(\mathcal{I})$. Furthermore, if $\mathcal{I}$ is a tall ideal, then every MAD family restricted to $\mathcal{I}$ is actually a MAD family in the usual sense; hence $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$. Hence, if $\mathcal{I}$ is tall, then $\mathsf{max}\{\mathfrak{a},\mathsf{cov}^*(\mathcal{I})\} \leq \mathfrak{a}(\mathcal{I})$. We will show that there is a deeper connection between these cardinals.

The following lemma is well known. We prove it for the sake of completeness:

Lemma 4 *Let $\mathcal{C} = \{C_n \mid n \in \omega\} \subseteq [\omega]^\omega$ be a partition of ω. There is an almost disjoint family $\mathcal{D}$ such that:*

1. *$\mathcal{D} \subseteq \mathcal{C}^\perp$,*
2. *$|\mathcal{D}| = \mathfrak{a}$, and*
3. *for every $X \in \mathcal{C}^\perp$ there is $D \in \mathcal{D}$ such that $|D \cap X| = \omega$.*

Proof. Let $\mathcal{A}$ be a MAD family of size $\mathfrak{a}$. We may assume there is $\mathcal{B} = \{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ that is a partition of ω. Let $f : \omega \longrightarrow \omega$ be a bijection such that $f[A_n] = C_n$. It is easy to see that $\mathcal{D} = f[\mathcal{A} \setminus \mathcal{B}]$ has the desired properties. $\blacksquare$

We can now prove the following:

Proposition 5 *If $\mathsf{max}\{\mathfrak{a},\mathit{cov}^*(\mathcal{I})\} = \omega_1$ then $\mathfrak{a}(\mathcal{I}) = \omega_1$.*

Proof. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\} \subseteq \mathcal{I}$ such that for every $X \in \mathcal{I} \cap [\omega]^\omega$, there is $\alpha \in \omega_1$ for which $X \cap B_\alpha$ is infinite. We may assume that $\{B_n \mid n \in \omega\} \subseteq \mathcal{B}$ is a partition of ω. We will recursively build a sequence of AD families $\langle \mathcal{A}_\alpha \rangle_{\alpha \in \omega_1}$ such that:

1. $\langle \mathcal{A}_\alpha \rangle_{\alpha \in \omega_1}$ is an increasing chain of almost disjoint families of size ω_1,
2. $\mathcal{A}_\alpha \restriction B_\alpha$ is a (possibly finite) MAD family in B_α for every $\alpha \in \omega_1$, and
3. $\overline{\mathcal{A}_\alpha} = \mathcal{A}_\alpha \setminus \bigcup\limits_{\xi < \alpha} \mathcal{A}_\xi \subseteq \wp(B_\alpha)$.

We start by choosing $\{\mathcal{A}_n \mid n \in \omega\}$ such that $\overline{\mathcal{A}_n}$ is MAD family of subsets of B_n of size ω_1 for every $n \in \omega$. Assume $\alpha \leq \omega_1$ is an infinite ordinal, and we have already build all the $\mathcal{A}_\xi$ for $\xi < \alpha$. We shall see how to find $\mathcal{A}_\alpha$. In case $\bigcup\limits_{\xi<\alpha} \mathcal{A}_\xi \restriction B_\alpha$ is already a MAD family in B_α, we define $\mathcal{A}_\alpha = \bigcup\limits_{\xi<\alpha} \mathcal{A}_\xi$. So we assume that $\bigcup\limits_{\xi<\alpha} \mathcal{A}_\xi \restriction B_\alpha$ is not maximal in B_α. Enumerate $\alpha = \{\alpha_n \mid n \in \omega\}$. Note that $B_\alpha \not\subseteq^* B_{\alpha_0} \cup ... \cup B_{\alpha_m}$ for all $m \in \omega$. If this was not the case, every infinite subset of B_α would have infinite intersection with some B_{α_i} and therefore $\mathcal{A}_{\alpha_0} \cup ... \cup \mathcal{A}_{\alpha_m}$ would be MAD in B_α.

Define $C_n = \left(B_{\alpha_n} \setminus \bigcup\limits_{i<n} B_{\alpha_i} \right) \cap B_\alpha$. By possibly taking a subsequence and making finite changes, we may assume all the C_n are infinite and form a partition

of B_α. Since $\mathfrak{a} = \omega_1$, we may find $\mathcal{D}$ an AD family in B_α of size ω_1 such that every $C_n \in \mathcal{D}^\perp$, and if $X \subseteq B_\alpha$ has finite intersection with every C_n, then there is $D \in \mathcal{D}$ such that $|X \cap D| = \omega$.

We define $\mathcal{A}_\alpha = \left(\bigcup_{\xi<\alpha} \mathcal{A}_\xi \right) \cup \mathcal{D}$, and prove that $\mathcal{A}_\alpha \restriction B_\alpha$ is a MAD family in B_α. To see this let $X \in [B_\alpha]^\omega$, and proceed by cases. In case that there is $n \in \omega$ such that $X \cap C_n$ is infinite, the result follows since $\mathcal{A}_{\alpha_n} \restriction B_{\alpha_n}$ is MAD. In case that $X \cap C_n$ is finite for every $n \in \omega$, the result follows by the way we chose $\mathcal{D}$.

Let $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. It is clear that it is an AD family contained in $\mathcal{I}$, and note that if $X \in \mathcal{I} \cap [\omega]^\omega$ then there is some B_α such that $B_\alpha \cap X$ is infinite and therefore (since $\mathcal{A} \restriction B_\alpha$ is MAD) then there is en element of $\mathcal{A}$ with infinite intersection with X, so $\mathcal{A}$ is be MAD. ■

From the result, we get the following corollary:

Corollary 6 *Assume $\mathfrak{c} \leq \omega_2$ and let $\mathcal{I}$ be an ideal.*

1. *If $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$, then $\mathfrak{a}(\mathcal{I}) = \max\{\mathfrak{a}, cov^*(\mathcal{I})\}$.*
2. *If $\mathcal{I}$ is tall, then $\mathfrak{a}(\mathcal{I}) = \max\{\mathfrak{a}, cov^*(\mathcal{I})\}$.*

Proof. Assume $\mathfrak{c} \leq \omega_2$ and that $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$. In case $\max\{\mathfrak{a}, \mathrm{cov}^*(\mathcal{I})\} = \omega_1$ we get $\mathfrak{a}(\mathcal{I}) = \omega_1$ by the last result, if $\max\{\mathfrak{a}, \mathrm{cov}^*(\mathcal{I})\} = \omega_2$ then $\mathfrak{a}(\mathcal{I}) = \omega_2$ because $\max\{\mathfrak{a}, \mathrm{cov}^*(\mathcal{I})\} \leq \mathfrak{a}(\mathcal{I})$ (recall that $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$).

Finally, if $\mathcal{I}$ is tall, then $\mathfrak{a} \leq \mathfrak{a}(\mathcal{I})$, so the second assertion follows from the first. ■

1.2 The off-branch numbers

We will get some applications of the results proved in the last section. Given $r \in 2^\omega$, denote $\widehat{r} = \{r \restriction n \mid n \in \omega\}$. By $\mathcal{BR}$ we denote the ideal on $2^{<\omega}$ generated by $\{\widehat{r} \mid r \in 2^\omega\}$. In this way, $X \subseteq 2^{<\omega}$ belongs to the ideal $\mathcal{BR}$ if and only if X can be covered by finitely many branches. The elements of $\mathcal{BR}^\perp$ are often called the *off-branch sets*. Note that every antichain is an off-branch set, but there are off-branch sets that are not the union of finitely many antichains. The cardinal invariant $\mathfrak{a}\left(\mathcal{BR}^\perp\right)$ was introduced by Leathrum in [18] and it is denoted by $\mathfrak{o}$. Witnesses of $\mathfrak{a}\left(\mathcal{BR}^\perp\right)$ are usually called *MOB families.* Although $\mathcal{BR}^\perp$ is not a tall ideal, the following result was proved by Leathrum:

Proposition 7 (Leathrum) $\mathfrak{a} \leq \mathfrak{o}$.

In this way, we may conclude the following:

Corollary 8 *If $\mathfrak{c} \leq \omega_2$, then $\mathfrak{o} = \max\{\mathfrak{a}, cov^*\left(\mathcal{BR}^\perp\right)\}$.*

Let $\mathcal{AT}$ be the ideal on $2^{<\omega}$ generated by antichains. The invariant $\mathfrak{a}(\mathcal{AT})$ was also studied by Leathrum and it is denoted as $\overline{\mathfrak{o}}$. In this way, $\overline{\mathfrak{o}}$ is the smallest size of a maximal almost disjoint family of antichains. Since $\mathcal{AT} \subseteq \mathcal{BR}^{\perp}$ and every infinite off-branch set contains an infinite antichain, it follows that $\mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) \leq \mathsf{cov}^*(\mathcal{AT})$ and $\mathfrak{o} \leq \overline{\mathfrak{o}}$. The following is the most interesting problem regarding the off-branch numbers:

Problem 9 ([18]) *Is $\mathfrak{o} = \overline{\mathfrak{o}}$?*

We do not know the answer to the problem, but we will prove that this is the case if size of the continuum is at most ω_2. We will need the following notions due to Kamburelis and Weglorz (see [16]):

Definition 10

1. *A family of open sets $\mathcal{U} \subseteq \wp(2^{\omega})$ is called an* open splitting family *if for every infinite antichain $\{s_n \mid n \in \omega\} \subseteq 2^{<\omega}$ there is $U \in \mathcal{U}$ such that both sets $\{n \mid \langle s_n \rangle \subseteq U\}$*[2] *and $\{n \mid \langle s_n \rangle \cap U = \emptyset\}$ are infinite.*
2. *$\mathfrak{s}(\mathcal{B}_0)$ is the smallest size of an open splitting family.*
3. *Given $x \in 2^{\omega}$ and $n \in \omega$ let $r(x,n)$ be the sequence of length $n+1$ that agrees with x in the first n places but disagrees in the last one.*
4. *Let $x \in 2^{\omega}$, $A \in [\omega]^{\omega}$ and $U \subseteq 2^{\omega}$ an open set. We say that U* separates *(x,A) if $x \notin U$ and there are infinitely many $n \in A$ such that $\langle r(x,n) \rangle \subseteq U$.*
5. *$\mathfrak{sep}$ is the smallest size of a family of open sets $\mathcal{U}$ such that for every (x,A) there is $U \in \mathcal{U}$ that separates (x,A).*

Kamburelis and Weglorz proved that $\mathfrak{s}(\mathcal{B}_0) = \max\{\mathfrak{s}, \mathfrak{sep}\}$. However, in [5] Brendle proved that this two cardinal invariants are equal. In fact he proved the following:

Proposition 11 ([5])

1. *$\mathsf{non}(\mathcal{M}) \leq \mathfrak{sep}$.*
2. *$\mathfrak{sep} = \mathfrak{s}(\mathcal{B}_0)$.*

Note that the second assertion follows from the first since $\mathfrak{s} \leq \mathsf{non}(\mathcal{M})$ and $\mathfrak{s}(\mathcal{B}_0) = \max\{\mathfrak{s}, \mathfrak{sep}\}$. In [8] the authors proved that $\mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) = \mathfrak{sep}$. The same argument shows that in fact $\mathsf{cov}^*(\mathcal{AT}) = \mathfrak{sep}$. We will provide the whole argument for completeness. First we will need a definition and a lemma:

Definition 12 *Let $U \subseteq 2^{\omega}$ be an open set. Define A_U as the set of all minimal $s \in 2^{<\omega}$ for which $\langle s \rangle \subseteq U$.*

[2] If $s \in 2^{<\omega}$, define $\langle s \rangle = \{f \in 2^{\omega} \mid s \subseteq f\}$.

It is easy to see that $A_U \subseteq 2^{<\omega}$ is an antichain and if U is not clopen, then A_U is infinite. We now have the following:

Lemma 13 *Let $\mathcal{U}$ be an open splitting family and $C \subseteq 2^{<\omega}$ an infinite antichain. There is $U \in \mathcal{U}$ such that there are infinitely many $t \in A_U$ for which there is $s \in C$ such that $\langle s \rangle \subseteq \langle t \rangle$.*

Proof. By a compactness argument, we can find $C_1 = \{s_n \mid n \in \omega\}$ and $r \in 2^\omega$ with the following properties:

1. $C_1 \subseteq C$,
2. $r \restriction n \subseteq s_m$ for every $n \in \omega$ and for almost every $m \in \omega$,
3. $r \notin \bigcup\limits_{n\in\omega} \langle s_n \rangle$.

Since $\mathcal{U}$ is an open splitting family, we know there is $U \in \mathcal{U}$ such that both sets $\{n \mid \langle s_n \rangle \subseteq U\}$ and $\{n \mid \langle s_n \rangle \cap U = \emptyset\}$ are infinite. We claim that U has the desired properties. First note that $r \notin U$, this is because if $r \in U$ then the set $\{n \mid \langle s_n \rangle \cap U = \emptyset\}$ would be finite. Since every set $\langle t \rangle$ is clopen, then for every $t \in A_U$, $\langle t \rangle$ can only contained finitely many elements of C_1. ■

We have the following strengthening of Proposition 4.11 of [8]:

Proposition 14 $\mathsf{cov}^* \left(\mathcal{BR}^\perp\right) = \mathsf{cov}^* (\mathcal{AT}) = \mathfrak{sep}$.

Proof. We will first prove that $\mathfrak{sep} \leq \mathsf{cov}^* \left(\mathcal{BR}^\perp\right)$. Take $\mathcal{B} \subseteq \mathcal{BR}^\perp$ a witness for $\mathsf{cov}^* \left(\mathcal{BR}^\perp\right)$, we may even assume that $\mathcal{B}$ is closed under finite changes. For every element $B \in \mathcal{B}$, define $\mathcal{U}_B = \bigcup \{\langle s \rangle \mid s \in B\}$. We will now prove that the family $\{\mathcal{U}_B \mid B \in \mathcal{B}\}$ is a witness for $\mathfrak{sep}$. Let $x \in 2^\omega$, $A \in [\omega]^\omega$. We first define the set $Y = \{r(x, n) \mid n \in A\}$ (note that Y is an off-branch family). we can now find $B \in \mathcal{B}$ such that $B \cap Y$ is infinite. We may assume no restriction of x is in B (this is because B is off-branch, so by substracting a finite subset of B if needed, we can get that no restriction of x is in B). It then follows that $\mathcal{U}_B$ separates (x, A).

We will now show that $\mathsf{cov}^* (\mathcal{AT}) \leq \mathfrak{s}(\mathcal{B}_0)$. This completes the proof since $\mathsf{cov}^* \left(\mathcal{BR}^\perp\right) \leq \mathsf{cov}^* (\mathcal{AT})$ and $\mathfrak{sep} = \mathfrak{s}(\mathcal{B}_0)$. Let $\{U_\beta \mid \beta < \mathfrak{s}(\mathcal{B}_0)\}$ be an open splitting family. By Bartoszyński's characterization of $\mathsf{non}(\mathcal{M})$ (see [3] Lemma 2.4.8) there is a family $\mathcal{F} = \{f_\alpha \mid \alpha < \mathsf{non}(\mathcal{M})\}$ with the following properties:

1. $f_\alpha : \omega \longrightarrow 2^{<\omega}$, and
2. for every $g : W \longrightarrow 2^{<\omega}$, where $W \in [\omega]^\omega$, there is $\alpha < \mathsf{non}(\mathcal{M})$ such that there are infinitely many $n \in W$ such that $f_\alpha(n) = g(n)$.

For every $\beta < \mathfrak{s}(\mathcal{B}_0)$ we fix an enumeration $A_{U_\beta} = \{s_n^\beta \mid n \in \omega\} \subseteq 2^{<\omega}$ (recall that A_{U_β} is the set of all minimal nodes of $\{s \mid \langle s \rangle \subseteq U_\beta\}$). For every $\alpha < \mathsf{non}(\mathcal{M})$ and $\beta < \mathfrak{s}(\mathcal{B}_0)$ we define $B(\alpha, \beta) = \{f_\alpha(n) \mid s_n^\beta \subseteq f_\alpha(n)\}$ which is an antichain since A_{U_β} is an antichain. Let $\mathcal{B}$ be the set of all $B(\alpha, \beta)$ where $\alpha < \mathsf{non}(\mathcal{M})$ and $\beta < \mathfrak{s}(\mathcal{B}_0)$. We will prove that that for every infinite antichain Y there is $B(\alpha, \beta) \in \mathcal{B}$ such that $B(\alpha, \beta) \cap Y$ is infinite.

Since $\mathcal{U}$ is an open splitting family, we know there is $\beta < \mathfrak{s}(\mathcal{B}_0)$ such that there are infinitely many $s_n^\beta \in A_{U_\beta}$ for which there is $t \in Y$ such that $s_n^\beta \subseteq t$. Let $W = \{n \mid \exists t \in Y (s_n^\beta \subseteq t)\}$ which is an infinite set. Define $g : W \longrightarrow 2^{<\omega}$ such that for every $n \in W$ the following hold:

1. $s_n^\beta \subseteq g(n)$, and
2. $g(n) \in Y$.

We can now find $\alpha < \mathsf{non}(\mathcal{M})$ such that there are infinitely many $n \in W$ for which $f_\alpha(n) = g(n)$. It is then clear that $B(\alpha, \beta) \cap Y$ is infinite. Finally since $\mathsf{non}(\mathcal{M}) \leq \mathfrak{s}(\mathcal{B}_0)$ by the theorem of Brendle, we conclude that $|\mathcal{B}| = \mathfrak{s}(\mathcal{B}_0)$ and we get the desired result. ■

From this we can conclude that both $\mathfrak{o}$ and $\overline{\mathfrak{o}}$ are equal to $\mathsf{max}\{\mathfrak{a}, \mathfrak{sep}\}$ in case $\mathfrak{c} \leq \omega_2$. We get the following:

Corollary 15 *1. $\mathfrak{o} = \omega_1$ implies $\overline{\mathfrak{o}} = \omega_1$.*

2. If $\mathfrak{c} \leq \omega_2$ then $\mathfrak{o} = \overline{\mathfrak{o}} = \mathsf{max}\{\mathfrak{a}, \mathfrak{sep}\}$.

As was mentioned before, it is still an open problem if $\mathfrak{o} = \overline{\mathfrak{o}}$. Getting the consistency of $\mathfrak{o} < \overline{\mathfrak{o}}$ will most likely be very hard. Our result shows that countable support iteration can not be used to solve this problem and long finite support iterations will not work either since $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{o}$.

1.3 Almost disjoint families of eventually different partial functions

For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$. Recall that $\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\}$ and (the graphs of) functions from ω to ω. It is easy to see that $\mathcal{ED}$ is a tall ideal. The invariant $\mathfrak{a}_s$ is defined as the smallest size of a maximal family of eventually different partial functions. In other words, $\mathfrak{a}_s$ is the smallest size of a family $\mathcal{B}$ with the following properties:

1. For every $f \in \mathcal{B}$ there is $A \in [\omega]^\omega$ such that $f : A \longrightarrow \omega$,
2. for every $f \neq g \in \mathcal{B}$ $\{n \in dom(f) \cap dom(g) \mid f(n) = g(n)\}$ is finite, and
3. for every function $h : A \longrightarrow \omega$ with $A \in [\omega]^\omega$ there is $f \in \mathcal{B}$ such that there are infinitely many $n \in \omega$ for which $f(n) = h(n)$.

We now have the following result:

Lemma 16 $\mathfrak{a}_s = \mathfrak{a}(\mathcal{ED})$.

Proof. Let $\mathcal{B}$ be a maximal family of eventually different partial functions. Define $\mathcal{D} = \mathcal{B} \cup \{C_n \mid n \in \omega\}$. Note that $\mathcal{D} \subseteq \mathcal{ED}$ and it is easy to see that $\mathcal{D}$ is a MAD family. In this way we conclude that $\mathfrak{a}(\mathcal{ED}) \leq \mathfrak{a}_s$.

For the other inequality, let $\mathcal{A} \subseteq \mathcal{ED}$ be a MAD family contained in $\mathcal{ED}$. Note that for every $A \in \mathcal{A}$, there is a pair (X_A, F_A) with the following properties:

1. There is $m \in \omega$ such that $X_A \subseteq C_0 \cup ... \cup C_m$,
2. F_A is a finite set of disjoint partial functions, and
3. $A = X_A \cup \{(n, h(n)) \mid h \in F_A \wedge n \in dom(h)\}$.

Let $\mathcal{B} = \bigcup_{A \in \mathcal{A}} F_A$. It is easy to see that $|\mathcal{B}| = |\mathcal{A}|$, and that $\mathcal{B}$ is a maximal family of eventually different partial functions. So $\mathfrak{a}_s \leq \mathfrak{a}(\mathcal{ED})$. ■

In [14] it was proved that $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$. We conclude the following:

Corollary 17 *If* $\mathfrak{c} \leq \omega_2$, *then* $\mathfrak{a}_s = \max\{\mathfrak{a}, non(\mathcal{M})\}$.

We should mention that the hypothesis $\mathfrak{c} \leq \omega_2$ is needed. In [6] Brendle used the technique of forcing along a template to prove the following:

Proposition 18 (Brendle) *The following is relatively consistent with the axioms of ZFC:* $\mathfrak{a} = \omega_1$, $non(\mathcal{M}) = \omega_2$ *and* $\mathfrak{a}_s = \mathfrak{c} = \omega_3$.

Our result shows that the theorem of Brendle can not be improved to get $\mathfrak{a} = non(\mathcal{M}) = \omega_1$ and $\mathfrak{a}_s = \mathfrak{c} = \omega_2$.

Definition 19 *If* $a \subseteq \omega^{<\omega}$ *we define* $\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \upharpoonright n \in a)\}$. *Let* $\mathcal{I}$ *be a* σ*-ideal on* ω^ω *(or* 2^ω*). We define* $tr(\mathcal{I})$ the trace ideal of $\mathcal{I}$ *(which will be an ideal on* $\omega^{<\omega}$ *or* $2^{<\omega}$*) where* $a \in tr(\mathcal{I})$ *if and only if* $\pi(a) \in \mathcal{I}$.

1.4 Trace ideals

Note that if $a \subseteq \omega^{<\omega}$ then $\pi(a)$ is a G_δ set (furthermore, every G_δ set is of this form). While both $tr(\mathcal{M})$ and $tr(\mathcal{N})$ are Borel, in general, the trace ideals are not Borel (see [15] for more information). By nwd we will denote the ideal of the nowhere dense subsets of the rational numbers. It is well known that $tr(\mathcal{M})$ is equivalent to nwd. By $\mathcal{NDN}$ we will denote the ideal $tr(\mathcal{M}) \cap tr(\mathcal{N})$. In [21] the cardinal invariants $\mathfrak{a}(tr(\mathcal{M}))$, $\mathfrak{a}(tr(\mathcal{N}))$ and $\mathfrak{a}(\mathcal{NDN})$ were studied. In that paper, the following results were proven:

Proposition 20 ([21])

1. $\mathsf{cov}(\mathcal{M}), \mathfrak{a} \leq \mathfrak{a}(\mathsf{nwd})$,

2. $\mathsf{cov}(\mathcal{N}), \mathfrak{a} \leq \mathfrak{a}(tr(\mathcal{N}))$, *and*

3. $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N})) \leq \mathfrak{a}(\mathcal{NDN})$.

and the following question was asked:

Problem 21 ([21]) *Are the inequalities between* $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N}))$, $\mathfrak{a}(\mathcal{NDN})$ *consistently strict and complete?*

We can readily provide the following:

Corollary 22 *Both* $\mathfrak{a}(\mathsf{nwd}) < \mathfrak{a}(tr(\mathcal{N}))$ *and* $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}(\mathsf{nwd})$ *are consistent.*

Proof. It is easy to see that both nwd and $tr(\mathcal{N})$ are tall ideals. It is a theorem of Keremedis that $\mathsf{cov}(\mathcal{M}) = \mathsf{cov}^*(\mathsf{nwd})$ (see e.g. [2] for a proof). In this way, if $\mathfrak{c} = \omega_2$ and $\mathfrak{a} = \omega_1$ then $\mathfrak{a}(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathfrak{a}(tr(\mathcal{N})) = \mathsf{cov}^*(tr(\mathcal{N}))$. Furthermore, in [9] it was proved that $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(tr(\mathcal{N})) \leq \mathsf{non}(\mathcal{M})$. From these results it is clear that in the Cohen model (the model obtained after adding ω_2 Cohen reals to a model of CH) the inequality $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}(\mathsf{nwd})$ holds and in the random model (the model obtained after adding ω_2 random reals to a model of CH) the inequality $\mathfrak{a}(\mathsf{nwd}) < \mathfrak{a}(tr(\mathcal{N}))$ holds. ■

Problem 23 *Is* $\mathfrak{a}(\mathcal{NDN})$ *the maximum of* $\mathfrak{a}(\mathsf{nwd})$ *and* $\mathfrak{a}(tr(\mathcal{N}))$*?*

Another problem from [21] is the following:

Problem 24 *Are* $\mathfrak{a}(\mathsf{nwd})$, $\mathfrak{a}(tr(\mathcal{N}))$, $\mathfrak{a}(\mathcal{NDN})$ *incomparable with* $\mathfrak{o}, \overline{\mathfrak{o}}$, $\mathfrak{a}_s$*?*

We will provide some partial answers to the question. We start with the following:

Proposition 25 $\mathfrak{a}(\mathsf{nwd})$ *and* $\mathfrak{a}_s$ *are incomparable.*

Proof. We know that if $\mathfrak{c} = \omega_2$ and $\mathfrak{a} = \omega_1$ then $\mathfrak{a}(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathfrak{a}_s = \mathsf{non}(\mathcal{M})$. The result follows since $\mathsf{cov}(\mathcal{M})$ and $\mathsf{non}(\mathcal{M})$ are independent (with $\mathfrak{c} = \omega_2$). ■

Regarding $\mathfrak{a}(tr(\mathcal{N}))$ and $\mathfrak{a}_s$ we have the following:

Proposition 26 *It is consistent that* $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}_s$.

Proof. By Proposition 4.1 of [15], we know that $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(tr(\mathcal{N})) \leq \max\{\mathfrak{d}, \mathsf{cov}(\mathcal{N})\}$. In this way, in order to obtain a model of $\mathfrak{a}(tr(\mathcal{N})) < \mathfrak{a}_s$, it is enough to find a model of $\mathfrak{c} = \mathsf{non}(\mathcal{M}) = \omega_2$, and $\mathfrak{a} = \mathfrak{d} = \mathsf{cov}(\mathcal{N}) = \omega_1$. The existence of such models is well known, for example, they can be obtained by iterating the Mathias forcing associated to the ideal $\mathcal{ED}$. ■

Finally, the next proposition follows from Corollary 6 and known inequalities between cardinal invariants:

Proposition 27 *If* $\mathfrak{c} \leq \omega_2$ *then the following hold:*

1. $\mathfrak{a}_s \leq \mathfrak{o}$,
2. $\mathfrak{a}(tr(\mathcal{N})) \leq \mathfrak{a}_s$, *and*
3. $\mathfrak{a}(\mathsf{nwd}) \leq \mathfrak{o}$.

1.5 A preservation theorem for tight MAD families

In this section, we will prove a preservation theorem for tight MAD families that will be needed in the following section. Recall that an AD family $\mathcal{A}$ is *tight* if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$. Note that tight AD families are MAD families. In [13] it was proved that tight MAD families exists assuming $\mathfrak{b} = \mathfrak{c}$ and that they are Cohen-indestructible (recall that if $\mathcal{A}$ is a MAD family and $\mathbb{P}$ is a partial order, then $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if $\mathcal{A}$ is still maximal after forcing with $\mathbb{P}$).

Definition 28 *Let* $\mathcal{A}$ *be a tight MAD family. We say that a proper forcing* $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ *if for every* $p \in \mathbb{P}$, M *a countable elementary submodel of* $H(\kappa)$ *(where* κ *is a large enough regular cardinal) such that* $\mathbb{P}, \mathcal{A}, p \in M$ *and* $B \in \mathcal{I}(\mathcal{A})$ *for which* $|B \cap Y| = \omega$ *for every* $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, *there is* $q \leq p$ *an* $(M, \mathbb{P})$*-generic condition such that* $q \Vdash$ "$\forall \dot{Z} \in \left(\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]\right)\left(\left|\dot{Z} \cap B\right| = \omega\right)$" *(where* $\dot{G}$ *denotes the name of the generic filter). We say that* q *is an* $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition.

It is easy to see that if $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$, then $\mathcal{A}$ is a tight MAD family after forcing with $\mathbb{P}$. We will need the following well known fact:

Lemma 29 *Let* $\mathcal{A}$ *be an AD family,* $\mathbb{P}$ *a partial order,* $\dot{B}$ *a* $\mathbb{P}$*-name for a subset of* ω *and* $p \in \mathbb{P}$ *such that* $p \Vdash$ "$\dot{B} \in \mathcal{I}(\mathcal{A})^+$"*. The set*

$$C = \{n \mid \exists q \leq p\ (q \Vdash \text{“}n \in \dot{B}\text{”})\} \in \mathcal{I}(\mathcal{A})^+ .$$

Proof. Since $\dot{B}$ is forced to be a subset of C, the result follows. ∎

We will prove that the countable support iteration of forcings that strongly preserve $\mathcal{A}$-tightness, also strongly preserve $\mathcal{A}$-tightness. Our proof will be a variation of the preservation of properness under countable support iteration by Shelah ([22]). First we do the two step iteration:

Lemma 30 *Let* $\mathcal{A}$ *be a tight MAD family. If* $\mathbb{P}$ *is a proper forcing that strongly preserves the tightness of* $\mathcal{A}$ *and* $\dot{\mathbb{Q}}$ *is a* $\mathbb{P}$*-name for a proper forcing such that* $\mathbb{P} \Vdash$"$\dot{\mathbb{Q}}$ *strongly preserves the tightness of* $\mathcal{A}$", *then* $\mathbb{P} * \dot{\mathbb{Q}}$ *strongly preserves the tightness of* $\mathcal{A}$. *Furthermore, if* $B \in \mathcal{I}(\mathcal{A})$, M *is a countable elementary submodel with* $\mathcal{A}, \mathbb{P}, \dot{\mathbb{Q}} \in M$, $p \in \mathbb{P}$, *is an* $(M, \mathbb{P}, \mathcal{A}, B)$*-generic condition and* $\dot{q}$ *is a* $\mathbb{P}$*-name for an element of* $\dot{\mathbb{Q}}$ *such that* $p \Vdash$ "$\dot{q}$ *is an* $(M\left[\dot{G}\right], \dot{\mathbb{Q}}, \mathcal{A}, B)$*-generic condition*", *then* $(p, \dot{q})$ *is an* $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$*-generic condition.*

Proof. Let $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a generic filter with $(p, \dot{q}) \in G$, denote $G_{\mathbb{P}}$ the projection of G to $\mathbb{P}$. Since p is an $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition, it follows that B has infinite intersection with every element of $M[G_P] \cap \mathcal{I}(\mathcal{A})^+$. Finally, since $\dot{q}$ is forced to be an $(M\left[\dot{G}\right], \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition, then B will have infinite intersection with every element of $M[G] \cap \mathcal{I}(\mathcal{A})^+$. Finally, note that $(p, \dot{q})$ is an $(M, \mathbb{P} * \dot{\mathbb{Q}})$-condition (see [1]). ■

We will now prove the "proper iteration lemma" ([1] Lemma 2.8) for $(M, \mathbb{P}, \mathcal{A})$-generic conditions. In the following, if $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings and $\alpha \leq \gamma$, by $\Vdash_\alpha$ we will denote $\Vdash_{\mathbb{P}_\alpha}$and by $\dot{G}_\alpha$ the canonical name for a $\mathbb{P}_\alpha$-generic filter.

Proposition 31 *Let $\mathcal{A}$ be a tight MAD family. Let $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ be a countable support iteration of proper forcings such that $\mathbb{P}_\alpha \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}$". Let $B \in \mathcal{I}(\mathcal{A})$, M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (where κ is a large enough regular cardinal) with $\mathcal{A}, \mathcal{P}, \gamma \in M$. For every $\alpha \in M \cap \gamma$ and an $(M, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic condition $p \in \mathbb{P}_\alpha$ the following holds:*

If $\dot{q}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_\alpha$ "$\dot{q} \in \mathbb{P}_\gamma \cap M$" and $p \Vdash_\alpha$ "$\dot{q} \restriction \alpha \in \dot{G}_\alpha$", then there is an $(M, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic condition $\overline{p} \in \mathbb{P}_\gamma$ such that $\overline{p} \restriction \alpha = p$ and $\overline{p} \Vdash_\gamma$ "$\dot{q} \in \dot{G}$".

Proof. We will prove the proposition by induction on γ. The case where γ is a successors follows easily by the last lemma, so we assume that γ is a limit ordinal and the proposition holds for every ordinal smaller that γ. Let $\langle \alpha_n \rangle_{n \in \omega}$ be an increasing sequence of ordinals in $M \cap \gamma$ such that $\alpha_0 = \alpha$ and $\bigcup \alpha_n = \bigcup M \cap \gamma$. We fix an enumeration $\{D_n \mid n \in \omega\}$ of all open dense sets of $\mathbb{P}_\gamma$ that are in M and fix $\left\{\dot{Z}_n \mid n \in \omega\right\}$ an enumeration of all $\mathbb{P}_\gamma$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M such that every name appears infinitely many times in the enumeration. We will recursively construct sequences $\langle \dot{q}_n \rangle_{n \in \omega}$, $\langle p_n \rangle_{n \in \omega}$ and $\langle \dot{m}_n \rangle_{n \in \omega}$ with the following properties:

1. $p_0 = p$, $\dot{q}_0 = \dot{q}$,
2. $p_n \in \mathbb{P}_{\alpha_n}$ is an $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic condition,
3. $p_{n+1} \restriction \alpha_n = p_n$,
4. $\dot{q}_n$ is a $\mathbb{P}_{\alpha_n}$-name such that $p_n \Vdash_{\alpha_n}$ "$\dot{q}_n \in \mathbb{P}_\gamma \cap M$" and $p_n \Vdash_{\alpha_n}$ "$\dot{q}_n \restriction \alpha_n \in \dot{G}_{\alpha_n}$",
5. $p_{n+1} \Vdash_{\alpha_{n+1}}$ "$\dot{q}_{n+1} \leq \dot{q}_n$" and $p_{n+1} \Vdash_{\alpha_{n+1}}$ "$\dot{q}_{n+1} \in D_n$", and
6. $\dot{m}_n$ is a $\mathbb{P}_\gamma$-name for a natural number such that $p_{n+1} \Vdash_{\alpha_n}$ "$\dot{q}_n \Vdash_\gamma$ "$\dot{m}_n \in \left(\dot{Z}_n \cap B\right) \setminus n$"".

Assume we have constructed $\dot{q}_n, p_n$ and $\dot{m}_n$. We will see how to construct $\dot{q}_{n+1}, p_{n+1}$ and $\dot{m}_{n+1}$. Let $G_{\alpha_n} \subseteq \mathbb{P}_{\alpha_n}$ be a generic filter with $p_n \in G_{\alpha_n}$. We know that $\dot{q}_n [G_{\alpha_n}] \in \mathbb{P}_\gamma \cap M$ and $\dot{q}_n [G_{\alpha_n}] \upharpoonright \alpha_n \in G_{\alpha_n}$. We now argue in $V [G_{\alpha_n}]$: Since p_n is an $(M, \mathbb{P}_{\alpha_n})$-generic condition, there is $r \in D_n \cap M$ such that $r \leq \dot{q}_n [G_{\alpha_n}]$ and $r \upharpoonright \alpha_n \in G_{\alpha_n}$. Let W be the set of all $m \in \omega$ such that there is $\overline{r} \in \mathbb{P}_\gamma$ such that the following holds:

1. $\overline{r} \leq r$,
2. $\overline{r} \upharpoonright \alpha_n \in G_{\alpha_n}$, and
3. $\overline{r} \Vdash_\gamma$ "$m \in \dot{Z}_n \setminus n$".

Clearly $W \in M [G_{\alpha_n}] \cap \mathcal{I}(\mathcal{A})^+$. Since p_n is an $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic condition, there is $m_n \in B$ and $q_{n+1} \in \mathbb{P}_\gamma$ such that $q_{n+1} \leq r$, $q_{n+1} \upharpoonright \alpha_n \in G_{\alpha_n}$ and $q_{n+1} \Vdash_\gamma$ "$m_n \in \dot{Z}_n \setminus n$". Back in V, let $\dot{q}_{n+1}$ and $\dot{m}_n$ be names for q_{n+1} and m_n that are forced by p_n to have all the properties above. We now apply the inductive hypothesis on γ_{n+1} and find p_{n+1} with the desired properties.

Let $\overline{p} = \bigcup_{n\in\omega} p_n$, it is easy to see that $\overline{p}$ is an $(M, \mathbb{P}_\gamma)$-generic condition and $\overline{p} \Vdash$ "$\dot{q}_n \in G_\gamma$" for every $n \in \omega$ (see the proof of Lemma 2.8 in [1] for more details). It is clear that $\overline{p}$ is a $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic condition. ■

We conclude the following:

Corollary 32 *Let $\mathcal{A}$ be a tight MAD family. If $\mathcal{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings such that $\mathbb{P}_\alpha \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}$", then $\mathbb{P}_{\omega_2} \Vdash$ "$\mathcal{A}$ is a tight MAD family".*

It is worth mentioning that this result can be used to prove that $\mathfrak{a} = \omega_1$ in the Sacks and Miller models, since such forcings strongly preserve the tightness of MAD families.

1.6 AD families of finitely branching trees

Next we consider the ideal $\mathcal{K}$, i.e. the ideal generated by the finitely branching subtrees of $\omega^{<\omega}$ Regarding the cardinal invariant $\mathfrak{a}(\mathcal{K})$, we have the following:

Proposition 33 $\mathfrak{a} \leq \mathfrak{a}(\mathcal{K})$.

Proof. For every $n \in \omega$, let $z_n : n \longrightarrow \omega$ be the constant 0 function. Define $X_n = \{z_n{}^\frown i \mid i \in \omega\}$ and $X = \bigcup_{n\in\omega} X_n$. Let $f : X \longrightarrow \omega$ be a bijection and define $A_n = f[X_n]$. We now find a family $\mathcal{B} = \{K_\alpha \mid \omega \leq \alpha < \mathfrak{a}(\mathcal{K})\} \subseteq \mathcal{K}$ such that for every infinite $Y \in \mathcal{K}$ there is α such that $|K_\alpha \cap Y| = \omega$. For every $\omega \leq \alpha < \mathfrak{a}(\mathcal{K})$ let $A_\alpha = f[X \cap K_\alpha]$ and $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{a}(\mathcal{K})\} \setminus [\omega]^{<\omega}$. We claim that $\mathcal{A}$ is a MAD family.

Note that if $K_\alpha \in \mathcal{B}$ and $n \in \omega$, then $X_n \cap K_\alpha$ is finite, this implies that $\mathcal{A}$ is an almost disjoint family. In order to prove that $\mathcal{A}$ is maximal, note that for every $Z \subseteq \omega$, if Z is almost disjoint with every A_n for $n \in \omega$, then $f^{-1}(Z) \in \mathcal{K}$ and hence there is $K_\alpha \in \mathcal{B}$ such that $f^{-1}(Z) \cap K_\alpha$ is infinite, which implies that $Z \cap A_\alpha$ is infinite. ■

The ideal $\mathcal{WF}$ is defined as the ideal generated by well-founded subtrees of $\omega^{<\omega}$. It is easy to see that $\mathcal{K}^\perp = \mathcal{WF}$. Recall that an ideal $\mathcal{I}$ on ω is called *Fréchet* (or *nowhere tall*) if for every $A \in \mathcal{I}^+$ there is $B \in [A]^\omega$ such that $B \in \mathcal{I}^\perp$. It is not hard to see that $\mathcal{BR}$, $\mathcal{BR}^\perp$, $\mathcal{WF}$ and $\mathcal{K}$ are Fréchet ideals.

It is easy to see that $\mathcal{I} \subseteq \mathcal{I}^{\perp\perp}$ for any ideal $\mathcal{I}$, while $\mathcal{I}$ is Fréchet if and only if $\mathcal{I} = \mathcal{I}^{\perp\perp}$. It follows that $\mathcal{I}^\perp$ is a Fréchet ideal for any ideal $\mathcal{I}$.

Lemma 34 *1.* $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$.

2. $\mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$.

3. $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b}$.

Proof. We will first prove that $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$. Let $\kappa < \mathfrak{c}$ and $\mathcal{X} = \{B_\alpha \mid \alpha < \kappa\}$ a subset of $\mathcal{BR}$. For every $\alpha < \kappa$ there is a set $F_\alpha \in [2^\omega]^{<\omega}$ such that $B_\alpha \subseteq \bigcup\limits_{\alpha<\kappa} \widehat{r}$. Since $\kappa < \mathfrak{c}$ there is $x \notin \bigcup\limits_{\alpha<\kappa} F_\alpha$. Clearly $\widehat{x} \in \mathcal{BR}$ and it is almost disjoint with B_α. It follows that $\mathsf{cov}^*(\mathcal{BR}) = \mathfrak{c}$.

We will now prove that $\mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$. It is easy to see that for every finitely branching tree $T \subseteq \omega^{<\omega}$ there is $f \in \omega^\omega$ such that $[T] \subseteq \{h \in \omega^\omega \mid h \leq f\}$. It follows from this fact that $\mathsf{cov}^*(\mathcal{K}) \leq \mathfrak{d}$. We will now prove that $\mathfrak{d} \leq \mathsf{cov}^*(\mathcal{K})$. Let $\mathcal{D} \subseteq \mathcal{K}$ such that for every infinite $X \in \mathcal{K}$ there is $D \in \mathcal{D}$ such that $X \cap D$ is infinite. We may assume that every element of $\mathcal{D}$ is a finitely branching tree. Since $\omega^\omega \subseteq \mathcal{K}$, it follows that $\omega^\omega = \bigcup\limits_{T\in\mathcal{D}} [T]$, which implies that $\mathfrak{d} \leq \mathsf{cov}^*(\mathcal{K})$.

Finally, we will show that $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b}$. We will first prove that $\mathsf{cov}^*(\mathcal{WF})$ is at most $\mathfrak{b}$. Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family of increasing functions with $|\mathcal{B}| = \mathfrak{b}$. For every $s \in \omega^{<\omega}$, let $\langle s\rangle_{<\omega} = \{t \in \omega^{<\omega} \mid s \subseteq t\}$. Fix an enumeration $\omega^{<\omega} = \{t_n \mid n \in \omega\}$.

If $f \in \mathcal{B}$, $s \in \omega^{<\omega}$ and $n \in \omega$, define $X_n(s,f) = \left\{t_i \in \langle s^\frown n\rangle_{<\omega} \mid i \leq f(n)\right\}$ and let $X(s,f) = \bigcup\limits_{n\in\omega} X_n(s,f)$. It is easy to see that $\{X(s,f) \mid f \in \mathcal{B} \wedge s \in \omega^{<\omega}\}$ is a witness for $\mathsf{cov}^*(\mathcal{WF})$, so $\mathsf{cov}^*(\mathcal{WF}) \leq \mathfrak{b}$.

We will now prove that $\mathfrak{b} \leq \mathsf{cov}^*(\mathcal{WF})$. Let $\mathcal{D} \subseteq \mathcal{WF}$ such that for every infinite $A \in \mathcal{WF}$ there is $W \in \mathcal{D}$ such that $A \cap W$ is infinite and $|\mathcal{D}| = \mathsf{cov}^*(\mathcal{WF})$. For every $n \in \omega$, let $r_n \in \omega^\omega$ such that $r_n(0) = n$ and $r_n(m) = 0$ for every $m > 0$. Let $W \in \mathcal{D}$, since W is contained in a well-founded tree, we can find a function $g_W : \omega \longrightarrow \omega$ such that $\widehat{r_n} \cap W \subseteq \omega^{g_W(n)}$ for every $n \in \omega$. It is easy to see that $\{g_W \mid W \in \mathcal{D}\}$ is an unbounded family. ■

It follows that if $\mathfrak{c} \leq \omega_2$ then $\mathfrak{a}(\mathcal{K}) = \max\{\mathfrak{a}, \mathfrak{d}\}$. The cardinal invariant $\mathfrak{a}_T$ is defined as the smallest size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$ (or $2^{<\omega}$). This cardinal invariant has been studied by Miller ([19]) and Newelski ([20]). It is easy to see that $\mathfrak{a}_T$ is the smallest cardinality of a partition of ω^ω into disjoint compact sets. It follows that $\mathfrak{d} \leq \mathfrak{a}_T$. Spinas ([24]) proved that the inequality $\mathfrak{d} < \mathfrak{a}_T$ is consistent, answering a question on [10]. The invariants $\mathfrak{a}_T$ and $\mathfrak{a}(\mathcal{K})$ are very similar, it would be tempting to conjecture that in fact $\mathfrak{a}(\mathcal{K}) = \mathfrak{a}_T$. We will now prove that this is not the case.

Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and t is a *splitting node of* p (i.e. $t^\frown 0, t^\frown 1 \in p$). Recall that Sacks forcing is the set of Sacks trees ordered by inclusion. The following forcing notion was introduced by Miller in [19]:

Definition 35 *Let $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a partition of 2^ω into compact sets. $\mathbb{P}(\mathcal{C})$ is the collection of all p such that the following hold:*

1. *$p \subseteq 2^{<\omega}$ is a Sacks tree, and*
2. *if $\alpha < \omega_1$, then $C_\alpha \cap [p]$ is nowhere dense in $[p]$ (i.e. for every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and $\langle t \rangle \cap [p] \cap C_\alpha = \emptyset$).*

If $p, q \in \mathbb{P}(\mathcal{C})$, then $p \leq q$ if $p \subseteq q$.

In [19] Miller proved that $\mathbb{P}(\mathcal{C})$ is proper, has the Laver property and forces that $\mathcal{C}$ is no longer a partition of 2^ω. In [24] Spinas showed that $\mathbb{P}(\mathcal{C})$ is ω^ω-bounding. It follows by the results of Miller and Spinas that $\mathbb{P}(\mathcal{C})$ even has the Sacks property. We will prove that $\mathbb{P}(\mathcal{C})$ does not increase $\mathfrak{a}$. We fix $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ a partition of 2^ω into compact sets. We will need some basic results about the forcing $\mathbb{P}(\mathcal{C})$:

Definition 36 *We say that $X = \{x_s \mid s \in \omega^{<\omega}\} \subseteq 2^\omega$ is* nice *if the following conditions hold:*

1. *For every $s \in \omega^{<\omega}$, the sequence $\langle x_{s^\frown n} \rangle_{n\in\omega}$ converges to x_s; furthermore, $\Delta(x_s, x_{s^\frown n}) < \Delta(x_s, x_{s^\frown n+1})$[3],*
2. *for every $s, t, z \in \omega^{<\omega}$, if $s \subseteq t \subseteq z$, then $\Delta(x_s, x_z) < \Delta(x_t, x_z)$, and*
3. *for every $s \in \omega^{<\omega}$, let $\alpha_s < \omega_1$ such that $x_s \in C_{\alpha_s}$, and*
4. *If $s \subseteq t$ then $\alpha_s \neq \alpha_t$.*

The following was proved implicitly in [24]:

Lemma 37 *Let p be a Sacks tree. If there is a nice $X = \{x_s \mid s \in \omega^{<\omega}\}$ that is dense in $[p]$, then $p \in \mathbb{P}(\mathcal{C})$.*

[3] If $x, y \in 2^\omega$ and $x \neq y$, denote $\Delta(x, y) = \min\{n \mid x(n) \neq y(n)\}$.

Proof. We need to prove that every C_β is nowhere dense in $[p]$. Let $\beta < \omega_1$ and $t \in p$. Since X is nice and dense in $[p]$, we can find $s \in \omega^{<\omega}$ such that $t \subseteq x_s$ and $\alpha_s \neq \beta$. Since $x_s \notin C_\beta$ and C_β is closed, there is $z \in p$ such that $t \subseteq z \subseteq x_s$ and $\langle z \rangle \cap p \cap C_\beta = \emptyset$. ■

If p is a Sacks tree and $s \in p$, let $p_s = \{t \in p \mid s \subseteq t \vee t \subseteq s\}$.

Proposition 38 *If $\mathcal{A}$ is a tight MAD family and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ is a partition of 2^ω in compact sets, then $\mathbb{P}(\mathcal{C})$ strongly preserves the tightness of $\mathcal{A}$.*

Proof. Let $p \in \mathbb{P}(\mathcal{C})$, M a countable elementary submodel of $\mathsf{H}(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathcal{C}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. Let $\{D_n \mid n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}(\mathcal{C})$ that are in M and fix $\left\{\dot{Z}_n \mid n \in \omega\right\}$ an enumeration of all $\mathbb{P}_\gamma$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M such that every name appears infinitely many times in the enumeration. We will recursively construct $\langle p_n \rangle_{n \in \omega}$ and $X = \{x_s \mid s \in \omega^{<\omega}\}$ such that the following conditions hold:

1. $p_0 = p$,
2. $\langle p_n \rangle_{n \in \omega}$ is a decreasing sequence and $p_n \in M$ for every $n \in \omega$,
3. X is nice and $X \subseteq 2^\omega \cap M$,
4. $X \subseteq [p_n]$ for every $n \in \omega$, and
5. For every $s \in \omega^n$ and $i, m \in \omega$ if $m = \Delta(x_s, x_{s^\frown i})$ and $t = (x_{s^\frown i}) \restriction m$ then $(p_{n+1})_t \in D_n$ and $(p_{n+1})_t \Vdash$ "$\left(\dot{Z}_n \cap B\right) \setminus n \neq \emptyset$".

To start let $p_0 = p$, and let $x_\emptyset$ be any element of $[p_0] \cap M$. Assume we have defined p_n and $\left\{x_s \mid s \in \omega^{\leq n}\right\}$, we will define p_{n+1} and $\left\{x_s \mid s \in \omega^{n+1}\right\}$.

Let $s \in \omega^n$ and choose $l \in \omega$ such that $l > \Delta(x_s, x_{s'})$ for all $s' \subsetneq s$. Define Y_s as the set of all $m > l$ such that $x_s \restriction m$ is a splitting node of p_n. For every $m \in Y_s$, let $t_m = (x_s \restriction m)^\frown (1 - x_s(m))$ (which is a node p_n) and let $p^s_m = (p_n)_{t_m}$, clearly $p^s_m \in M$. Let $C_m = \{j \mid \exists r \leq p^s_m (r \Vdash \text{"} j \in \dot{Z}_n \text{"})\}$. Since $C_m \in \mathcal{I}(\mathcal{A})^+$, there is $j \in C_m \cap B$ such that $j > n$. We choose $r^s_m \leq p^s_m$ such that $r^s_m \in M$ and $r^s_m \Vdash$ "$j \in \dot{Z}_n$". We may further assume that $r^s_m \in D_n$ and $[r^s_m] \cap C_{\alpha_z} = \emptyset$ for every $z \subseteq s$ (recall that if $z \in \omega^{<\omega}$, α_z denoted the unique ordinal such that $x_z \in C_{\alpha_z}$). Let $Y_s = \left\{m^i_s \mid i \in \omega\right\}$. For every $i \in \omega$, choose $x_{s^\frown i}$ be any branch in $r^s_{m^i_s}$ and let $p_{n+1} = \bigcup\{r^s_{m^i_s} \mid s \in \omega^n \wedge i \in \omega\}$.

We now let $q = \bigcap\limits_{n \in \omega} p_n$. It is easy to see that X is a dense subset of $[q]$ and q is a Sacks tree, so $q \in \mathbb{P}(\mathcal{C})$. Moreover, it is not hard to see that q is an $(M, \mathbb{P}(\mathcal{C}), \mathcal{A}, B)$*-generic condition.* ■

It follows that the forcings of the type $\mathbb{P}(\mathcal{C})$ preserve tight MAD families, even in the iteration. We can now prove the consistency result:

Proposition 39 *It is consistent that* $\mathfrak{a}(\mathcal{K}) < \mathfrak{a}_T$.

Proof. We start with a model of GCH and perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a forcing of the type $\mathbb{P}(\mathcal{C})$. Furthermore, with a suitable bookkeeping we make sure that $\mathfrak{a}_T = \mathfrak{c} = \omega_2$ holds in the final extension. Since $\mathbb{P}_{\omega_2}$ strongly preserves the tightness of $\mathcal{A}$, it follows that there is a (tight) MAD family of size ω_1. Moreover, $\mathfrak{d} = \omega_1$ holds in the extension since each $\mathbb{P}_\alpha$ is ω^ω-bounding. In this way, $\mathfrak{a} = \mathfrak{d} = \omega_1$ hence $\mathfrak{a}(\mathcal{K}) = \omega_1$. ■

We do not, however, know the answer to the following question:

Problem 40 *Is* $\mathfrak{a}(\mathcal{K}) \leq \mathfrak{a}_T$*?*

It seems difficult to produce a model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$. A model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$ and $\mathfrak{c} = \omega_2$ would be a model of $\mathfrak{d} = \omega_1$ and $\mathfrak{a} = \omega_2$, which would answer a famous open problem of Roitman. In fact, in all known models of $\mathfrak{c} = \omega_2$ the equality $\mathfrak{d} = \mathfrak{a}(\mathcal{K})$ holds. It is possible to build models of $\mathfrak{d} < \mathfrak{a}(\mathcal{K})$ by template iterations (see [23] and [6]) but this approach does not seem to help in order to build a model of $\mathfrak{a}_T < \mathfrak{a}(\mathcal{K})$.

1.7 Remarks on $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$

Recall that $\mathfrak{a}^+(\omega_1)$ is defined as the least κ such that every AD family of size ω_1 can be extended to a MAD family of size at most κ. In this way, $\omega_1 = \mathfrak{a}^+(\omega_1)$ is equivalent to the assertion that every AD of size ω_1 can be extended to a MAD family of size ω_1. This is obviously true under CH, but it is unknown if it is consistent with the failure of the Continuum Hypothesis:

Problem 41 ([21]) *Is it consistent that* $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$*?*

In [21] it was proved that it is consistent that $\omega_2 = \mathfrak{a}^+(\omega_1) < \omega_3 = \mathfrak{c}$, so at least $\mathfrak{a}^+(\omega_1)$ is consistently less than $\mathfrak{c}$. One "rule of thumb" which one learns when working on cardinal invariants, is that if an invariant is consistently less than $\mathfrak{c}$, then this will already happen in the Sacks model. This intuition is formalized by the following interesting theorem of Zapletal: (see [25] chapter 6).

Proposition 42 (LC) *If* $\mathfrak{j}$ *is a tame invariant*[4] *such that* $\mathfrak{j} < \mathfrak{c}$ *is consistent, then "*$\mathfrak{j} = \omega_1$*" holds in the Sacks model.*

Unfortunately, the theorem of Zapletal can not be applied to $\mathfrak{a}^+(\omega_1)$. Furthermore, it follows by the results on [21] that $\mathfrak{a}^+(\omega_1) = \mathfrak{c}$ holds in the Sacks model. In this section, we will derive some consequences of $\omega_1 = \mathfrak{a}^+(\omega_1) < \mathfrak{c}$. Our main tool is the following result:

Proposition 43 *The following are equivalent:*

[4]The reader may consult [25] for the definition of tame invariant.

1. $\mathfrak{a}^{+}(\omega_1) = \omega_1$.

2. $\mathfrak{a} = \omega_1$ *and for every ideal* $\mathcal{I}$ *on* ω, *if* $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$, *then* $\mathsf{cov}^*\left(\mathcal{I}^{\perp}\right) \leq \omega_1$.

Proof. We first assume that $\mathfrak{a}^{+}(\omega_1) = \omega_1$. Let $\mathcal{I}$ be an ideal on ω such that $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$. Clearly $\mathfrak{a}^{+}(\omega_1) = \omega_1$ implies $\mathfrak{a} = \omega_1$, so $\max\{\mathfrak{a}, \mathsf{cov}^*(\mathcal{I})\} = \omega_1$ which we already know implies that $\mathfrak{a}(\mathcal{I}) = \omega_1$. In this way, there is $\mathcal{A} \subseteq \mathcal{I}$ a MAD family restricted to $\mathcal{I}$ such that $|\mathcal{A}| = \omega_1$. Since $\mathfrak{a}^{+}(\omega_1) = \omega_1$, we can find an AD family $\mathcal{B} \subseteq \mathcal{A}^{\perp}$ such that $\mathcal{A} \cup \mathcal{B}$ is a MAD family and $|\mathcal{B}| \leq \omega_1$. By the maximality of $\mathcal{A}$, it follows that $\mathcal{B} \subseteq \mathcal{I}^{\perp}$. Since $\mathcal{A} \cup \mathcal{B}$ is a MAD family, we have that $\mathcal{B}$ is a MAD family restricted to $\mathcal{I}^{\perp}$, so $\mathsf{cov}^*\left(\mathcal{I}^{\perp}\right) \leq |\mathcal{B}| \leq \omega_1$.

We now assume that $\mathfrak{a} = \omega_1$ and if $\mathcal{I}$ is a ideal on ω such that $\mathsf{cov}^*(\mathcal{I}) \leq \omega_1$, then $\mathsf{cov}^*\left(\mathcal{I}^{\perp}\right) \leq \omega_1$. We will prove that $\mathfrak{a}^{+}(\omega_1) = \omega_1$. Let $\mathcal{A}$ be an AD family of size ω_1, we must prove that $\mathcal{A}$ can be extended to a MAD family of the same size. It suffices to show that $\mathfrak{a}\left(\mathcal{I}(\mathcal{A})^{\perp}\right) \leq \omega_1$. If $\mathfrak{a}\left(\mathcal{I}(\mathcal{A})^{\perp}\right) = \omega$ there is nothing to prove, so we assume that $\omega_1 \leq \mathfrak{a}\left(\mathcal{I}(\mathcal{A})^{\perp}\right)$, which implies that $\mathfrak{a} \leq \mathfrak{a}\left(\mathcal{I}(\mathcal{A})^{\perp}\right)$. It is easy to see that $\mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = \omega_1$, hence $\mathsf{cov}^*\left(\mathcal{I}(\mathcal{A})^{\perp}\right) \leq \omega_1$. ■

We can now prove the following:

Proposition 44 *If* $\mathfrak{a}^{+}(\omega_1) = \omega_1 < \mathfrak{c}$ *then*

1. $\omega_1 < \mathfrak{sep}$, *and*

2. $\mathfrak{d} = \omega_1$.

Proof. First assume that $\mathfrak{sep} = \omega_1$. Since $\mathfrak{sep} = \mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) = \omega_1$ and we are assuming that $\mathfrak{a}^{+}(\omega_1) = \omega_1$ holds, we conclude that $\mathfrak{c} = \mathsf{cov}^*\left(\mathcal{BR}^{\perp}\right) = \mathsf{cov}^*\left(\mathcal{BR}^{\perp\perp}\right) = \omega_1$ which is in contradiction with our hypothesis.

Since $\mathfrak{a}^{+}(\omega_1) = \omega_1$ implies $\mathfrak{a} = \omega_1$, we conclude that $\mathsf{cov}^*(\mathcal{WF}) = \mathfrak{b} = \omega_1$, which implies that $\omega_1 = \mathsf{cov}^*\left(\mathcal{WF}^{\perp}\right) = \mathsf{cov}^*(\mathcal{K}) = \mathfrak{d}$. ■

It follows from the result that $\mathfrak{a}^{+}(\omega_1) = \omega_1$ fails in the Sacks, Cohen, Hechler, Laver, Mathias and Miller models. We will now prove that it also fails in the random model. By μ we will denote the standard measure on 2^{ω} and μ^* denotes the exterior measure.

Lemma 45 *There is a set* $A \subseteq 2^{\omega}$ *such that* $\mu^*(A) = 1$ *and* $|A| = \mathsf{non}(\mathcal{N})$.

Proof. Let $S \subseteq [0,1]$ be the set of all $x \in [0,1]$ such that there is $B \subseteq 2^{\omega}$ with $|B| = \mathsf{non}(\mathcal{N})$ for which $x \leq \mu^*(B)$. Let z be the supremum of S (note that $S \neq \emptyset$). We claim that $z \in S$. Since z is the supremum of S, there is an increasing sequence $\langle z_n \rangle_{n\in\omega}$ of elements of S that converges to z. For every $n \in \omega$, we choose $B_n \subseteq 2^{\omega}$ such that $z_n \leq \mu^*(B_n)$ and $|B_n| = \mathsf{non}(\mathcal{N})$. Clearly $B = \bigcup\limits_{n\in\omega} B_n$ has size $\mathsf{non}(\mathcal{N})$ and $z = \mu^*(B)$. We now claim that $z = 1$. We argue by contradiction. Assume that $z < 1$. Let $B \subseteq 2^\omega$ such that $|B| = \mathsf{non}(\mathcal{N})$ and $\mu^*(B) = z$. Since $\mu^*(B) < 1$, there is a non-null compact set $C \subseteq 2^\omega$ such that $B \cap C = \emptyset$. Let $A \subseteq C$ such that $A \notin \mathcal{N}$ and $|A| = \mathsf{non}(\mathcal{N})$. Let $D = A \cup B$, clearly D has size $\mathsf{non}(\mathcal{N})$ and $z < \mu^*(D)$, which is a contradiction. ■

We can now prove the following:

Proposition 46 **($\mathfrak{a}^+(\omega_1) = \omega_1$)** *If $\mathsf{non}(\mathcal{N}) = \omega_1$, then $\mathsf{cov}^*(tr(\mathcal{N})) = \omega_1$.*

Proof. Assume that $\mathsf{non}(\mathcal{N}) = \omega_1$. Let $X \subseteq 2^\omega$ such that $\mu^*(X) = 1$ and $|X| = \mathsf{non}(\mathcal{N}) = \omega_1$. Define $\mathcal{A} = \{\widehat{r} \mid r \in X\}$, clearly $\mathcal{A}$ is an AD family of size ω_1 and $\mathcal{A} \subseteq tr(\mathcal{N})$. We claim that $\mathcal{A}^\perp \subseteq tr(\mathcal{N})$. Let $B \in tr(\mathcal{N})^+$, thus, $\pi(B)$ is a non-null G_δ set. Since $\mu^*(X) = 1$, there is an $r \in X \cap \pi(B)$, which implies that $\widehat{r} \cap B$ is infinite, so $B \notin \mathcal{A}^\perp$.

Since $\mathfrak{a}^+(\omega_1) = \omega_1$, there is a MAD family $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $|\mathcal{B}| = \omega_1$. By the comment above, we know that $\mathcal{B} \subseteq tr(\mathcal{N})$, Since $\mathcal{B}$ is a MAD family, it follows that $\mathsf{cov}^*(tr(\mathcal{N})) \leq |\mathcal{B}| = \omega_1$. ■

Since $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(tr(\mathcal{N}))$ and $\mathsf{non}(\mathcal{N}) = \omega_1$ holds in the random model, we can conclude the following:

Corollary 47 *$\mathfrak{a}^+(\omega_1) = \omega_1$ fails in the random model.*

Acknowledgement 48 *The authors would like to thank the generous referee for all his/her careful corrections and comments.*

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Osvaldo Guzmán
Centro de Ciencias Matemáticas, UNAM, Mexico
oguzman@matmor.unam.mx

Michael Hrušák
Centro de Ciencias Matemáticas, UNAM, Mexico
michael@matmor.unam.mx

Osvaldo Téllez
Instituto Tecnológico y de Estudios Superiores de Monterrey.
Campus Ciudad de México. Departamento de Matemáticas, CCM.
Calle del Puente 222 Col. Ejidos de Huipulco, Tlalpan C.P. 14380, CDMX
osvaldot@itesm.mx