Almost disjoint families and topology

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1 Introduction

An infinite family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *almost disjoint (AD)* if the intersection of any two distinct elements of $\mathcal{A}$ is finite. It is *maximal almost disjoint (MAD)* if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$.

Almost disjoint families and, in particular, MAD families with special combinatorial or topological properties are notoriously difficult to construct, yet there are also only very few known negative consistency results. There is the classical construction of a Luzin gap [109], the proof, due to Simon [145], that there is a MAD family which can be partitioned into two nowhere MAD families, and the construction of Mrówka of a MAD family the Ψ-space of which has a unique compactification.

Recently, there have been several fundamental new developments in the study of structural properties of almost disjoint families. The longstanding problem of whether the minimal size of a MAD family $\mathfrak{a}$ can be strictly larger than the dominating number $\mathfrak{d}$ was solved by Shelah in [140] using a novel forcing technique of *iterations along templates*. The method was further developed by Brendle [28–30] who used it to show that the cardinal invariant $\mathfrak{a}$ can consistently have countable cofinality [29].

Another important result and a new technique for constructing MAD families were presented by Shelah in [141]. Building on work of Balcar and Simon [8, 10, 144], he showed that completely separable MAD families exist assuming $\mathfrak{c} < \aleph_\omega$. The technique was further developed and the original proof was simplified by Mildenberger, Steprāns and Raghavan in [119] (see also [136]). We present a version of the argument here.

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Almost disjoint families of graphs of functions were studied in various contexts [37, 99, 133, 165] though many fundamental problems remain open. An important contribution to the subject has recently been made by Raghavan [134] who showed that in ZFC there is an AD family of graphs of functions which is a MAD family when augmented by the vertical sections. This answered a longstanding problem of van Douwen.

An attempt to classify MAD families via Katětov order was initiated in [88] and continued in [34, 68, 92] and [4]. One of the basic problems of [88] was recently solved in [4] by consistently constructing a MAD family maximal in the Katětov order. This is an ongoing project with many fundamental problems open.

Almost disjoint families are also one of the natural combinatorial tools used in topology, often via the corresponding Mrówka-Isbell spaces. These spaces have proved to be rather flexible and versatile sources of examples in many areas of topology ranging from the study of Fréchet and sequential spaces [43, 45, 47, 145], hyperspaces [89] and continuous selections [85, 90] to C_p-theory [51, 84] and functional analysis [59, 60, 101, 113, 114, 142]. For instance, in [90] a Ψ-space which admits a continuous weak selection but is not weakly orderable is constructed, answering an old problem of van Mill and Wattel [162].

Many important contributions to the study of structural properties of almost disjoint families and their applications to topology were made by Dow. In [47] he showed that, assuming PFA, every MAD family contains a Luzin subfamily. This fact was then used to bound the sequential order of scattered sequential spaces for which the scattered height and sequential order coincide. In [46] he constructed a consistent example of a MAD family which can not be partitioned into two nowhere maximal AD families. In [54] together with Zhou they used a variant of PFA to prove the existence of a partitioner algebra which has a subalgebra not representable as a partitioner algebra.

The article is divided into eleven sections. It contains almost no proofs with four notable exceptions: We present the constructions of a Luzin family (Theorem 2), Mrówka family (Theorem 16), and Shelah's construction of a completely separable MAD family from $\mathfrak{c} < \aleph_\omega$ (Theorem 13), as well as Simon's proof of existence of a partitionable MAD family (Theorem 46) in order to illustrate different techniques involved in the study of almost disjoint families.

2 Notation, definitions and basic facts

Our set-theoretic notation is mostly standard and follows [104].

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal* on a set X if it is non-empty, closed under taking subsets and finite unions of its elements and proper (i.e. $X \notin \mathcal{I}$). Unless otherwise specified all ideals are assumed to contain all finite subsets of X.

Given an ideal $\mathcal{I}$ on X we denote by $\mathcal{I}^*$ the *dual filter*, consisting of complements of the sets in $\mathcal{I}$. Similarly, if $\mathcal{F}$ is a filter on X, $\mathcal{F}^*$ denotes the dual ideal.

We say that an ideal $\mathcal{I}$ on X is *tall* if for each $Y \in [X]^\omega$ there is an $I \in \mathcal{I}$ such that $I \cap Y$ is infinite. Given an ideal $\mathcal{I}$ on a set X, we denote by $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets, i.e. subsets of X which are not in $\mathcal{I}$. If $\mathcal{I}$ is an ideal on X and $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on Y.

Most of our ideals will be ideals on ω. We consider $\mathcal{P}(\omega)$ equipped with the natural topology induced by identifying subsets of ω with their characteristic functions, hence identifying $\mathcal{P}(\omega)$ with 2^ω with the product topology. We call an ideal or filter Borel (analytic, co-analytic,...) if it is Borel (analytic, co-analytic,...) in this topology. Several Borel ideals will be considered in the text. Some of them will not be ideals on ω but rather ideals on some other countable set ($\omega \times \omega$, $\mathbb{Q}$, $\omega^{<\omega}$, ...).

- $\overline{\mathsf{ctbl}} = \{A \subseteq \mathbb{Q} : A \text{ has countable closure}\}$,
- $\mathsf{scattered} = \{A \subseteq \mathbb{Q} : A \text{ is scattered (as a topological space)}\}$,
- $\mathsf{nwd} = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense }\}$.
- $tr(\mathsf{ctbl}) = \{A \subseteq \omega^{<\omega} : |\{r \in \omega^\omega : \exists^\infty n \in \omega\ r \restriction n \in A\}| \leq \omega\}$.
- $\mathcal{ED} = \{A \subseteq \omega \times \omega : \exists m \in \omega \forall n \geq m\ |\{k \in \omega : (n,k) \in A\}| \leq m\}$.
- $\mathcal{ED}_{fin} = \mathcal{ED} \restriction \Delta$, where $\Delta = \{(m,n) : n \leq m\}$.
- $\mathsf{fin} \times \mathsf{fin} = \{A \subseteq \omega \times \omega : \forall^\infty n \in \omega\ \{m \in \omega : (n,m) \in A\} \text{ is finite}\}$.

The ideal $\mathsf{fin} \times \mathsf{fin}$ is the result of a *Fubini product* of ideals: Given $\mathcal{I}, \mathcal{J}$ ideals on ω we let

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : (n,m) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}.$$

It should be clear that all of these ideals are Borel of a low Borel complexity, all F_σ, $F_{\sigma\delta}$ or $F_{\sigma\delta\sigma}$.

Given an almost disjoint family $\mathcal{A}$ we denote by $\mathcal{I}(\mathcal{A}) = \{X \subseteq \omega : X \subseteq^* \bigcup \mathcal{B}$ for some finite $\mathcal{B} \subseteq \mathcal{A}\}$ the *ideal generated by* $\mathcal{A}$ and by $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ the family of $\mathcal{I}(\mathcal{A})$-positive sets. The following is the most basic observation about almost disjoint families. We are not quite sure who proved it first, though it is often attributed to Sierpiński. It appears in Sierpiński's book [143] but it had definitely been (at least implicitly) known a lot earlier (see e.g. [3, 74, 109, 155]):

Proposition 1. *1. There is an AD family of size* $\mathfrak{c}$.
2. Every MAD family is uncountable.
3. Every AD family can be extended to a maximal one.

Part (1) can be proved as follows: Denote (for future reference) by $A_f = \{f \restriction n : n \in \omega\}$ the branch through the tree $2^{<\omega}$ corresponding to a function $f \in 2^\omega$. For $X \subseteq 2^\omega$ let

$$\mathcal{A}_X = \{A_f : f \in X\}.$$

It is immediate that $\mathcal{A}_X$ is an AD family of size X. In particular, $\mathcal{A}_{2^\omega}$ has size $\mathfrak{c}$. To see (2) let $\{A_n : n \in \omega\}$ be an AD family an let $k_n \in A_n \setminus \bigcup_{j<n} A_j$ then $\{k_n : n \in \omega\}$ is almost disjoint from all A_n. (3) follows directly from the Kuratowski-Zorn lemma.

We will also denote by $\mathcal{A}_{\omega^\omega}$ the AD family $\{\{f \restriction n : n \in \omega\} : f \in \omega^\omega\}$ of subsets of $\omega^{<\omega}$.

The minimal size of a MAD family is denoted by $\mathfrak{a}$. Other combinatorial characteristics of the continuum mentioned here are the *dominating number* $\mathfrak{d}$, the *bounding number* $\mathfrak{b}$, the *distributivity number* $\mathfrak{h}$, the *splitting number* $\mathfrak{s}$, the *tower number* $\mathfrak{t}$, and the *pseudo-intersection number* $\mathfrak{p}$. It is well-known [16, 23] that $\mathfrak{p} \le \mathfrak{t} \le \mathfrak{h} \le \mathfrak{s} \le \mathfrak{d}$ and $\mathfrak{h} \le \mathfrak{b} \le \min\{\mathfrak{d}, \mathfrak{a}\}$.

A fundamental structural theorem for $\beta\omega$ is the so called *base tree* theorem of Balcar, Pelant and Simon [9]:

Theorem 1 ([9]). *There is a tree $T \subseteq [\omega]^\omega$ ordered by $\supseteq^*$ of height $\mathfrak{h}$ such that*

1. *each $A \in T$ has $\mathfrak{c}$-many immediate successors, and*
2. *for every $B \in [\omega]^\omega$ there is an $A \in T$ such that $A \subseteq B$.*

In other words, there is a collection $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ of MAD families such that (1) $\mathcal{A}_\alpha$ refines $\mathcal{A}_\beta$ for $\beta < \alpha$ and (2) $\bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ is dense in $[\omega]^\omega$.

See e.g. [16, 23] for definitions and further information about cardinal invariants.

Arguably the most useful basic combinatorial property of AD families is presented by the following proposition which we prove.

Proposition 2 ([8, 116]). *Given an AD family $\mathcal{A}$ and a decreasing sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \subseteq^* X_n$ for all $n \in \omega$.*

Proof. Let an AD family $\mathcal{A}$ and a decreasing sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ be given. Without loss of generality, $\mathcal{A}$ is MAD (if not, extend $\mathcal{A}$ to a MAD family so that each X_n remains positive). Choose for each $n \in \omega$ infinite sets Y_n and A_n so that $Y_n \subseteq^* X_i$ for all $i \in \omega$, $A_n \in \mathcal{A} \setminus \{A_m : m < n\}$, $Y_n \subseteq A_n \cap X_n$ and so that $Y_n \cap \bigcup\{A_i : i < n\} = \emptyset$. Then $X = \bigcup_{n\in\omega} Y_n$ is the desired set. □

It readily implies that $\mathcal{I}^+(\mathcal{A})$ is a *happy family* in the language of [116] or a *selective co-ideal* in the language of [57] for every AD family $\mathcal{A}$:

Proposition 3 ([8, 116]). *Let $\mathcal{A}$ be an AD family:*

1. *For every $\varphi : [\omega]^2 \to 2$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ which is φ-homogeneous, i.e $|\varphi''[X]^2| = 1$.*

2. *For every $h : \omega \to \omega$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $h \upharpoonright X$ is constant or strictly increasing.*

A stronger selective property of AD families was considered in [80]: An almost disjoint family $\mathcal{A}$ is *+-Ramsey* if every tree $T \subseteq \omega^{<\omega}$ such that $\{n \in \omega : t^\frown n \in T\} \in \mathcal{I}^+(\mathcal{A})$ for every $t \in T$ has a branch $b \in [T]$ such that $rng(b) \in \mathcal{I}^+(\mathcal{A})$. Here $[T] = \{b \in \omega^\omega : \forall n \in \omega \ b \upharpoonright n \in T\}$. It is easy to see that the property is indeed stronger, that is there is a MAD family $\mathcal{A}$ which is not +-Ramsey.

Problem 1 ([80]). Is there a +-Ramsey MAD family in ZFC?

The definition of a *Mrówka-Isbell space* [71, 93, 125, 126, 161] was without a doubt motivated by the *Niemytski plane*. The terminology comes from [71, 161].

Definition 1. Given an AD family $\mathcal{A}$, define a space $\Psi(\mathcal{A})$ as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a separable, scattered, zero-dimensional, first countable, locally compact Moore space. The extent of $\Psi(\mathcal{A})$ is equal to the cardinality of $\mathcal{A}$. The space $\Psi(\mathcal{A})$ is metrizable if and only if $\mathcal{A}$ is at most countable. If $\mathcal{A}$ is infinite then $\Psi(\mathcal{A})$ is not countably compact and $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is a MAD family [125].

An example of a Ψ-space appears in [3][chapter V., paragraph 1.3]: A topology on the real line is refined by declaring all rational points isolated. To each irrational point a convergent sequence is chosen and the cofinite subsets of the given convergent sequence are declared basic open neighbourhoods of the irrational number. This, of course, also presents another proof of the fact that there are AD families of size $\mathfrak{c}$.

A curious fact was noticed by Kannan and Rajagopalan in [98] and can be taken as a short definition of a Ψ-space:

Proposition 4 ([98]). *A separable space X is homeomorphic to $\Psi(\mathcal{A})$ for some, not necessarily infinite, AD family $\mathcal{A}$ if and only if X is hereditarily locally compact.*

3 Luzin families and separation

Given an almost disjoint family $\mathcal{A}$ and two subfamilies $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$ we say that a set $X \subseteq \omega$ *separates* $\mathcal{B}$ and $\mathcal{C}$ if $A \subseteq^* X$ for every $A \in \mathcal{B}$ and $A \cap X =^* \emptyset$ for every $A \in \mathcal{C}$.

3.1 Luzin families

One of the first contributions to the study of structural properties of families of almost disjoint sets was the construction of Luzin [109] probably influenced by the construction of the *Hausdorff gap* in [75].

Theorem 2 ([109]). *There is an uncountable almost disjoint family such that no two uncountable subfamilies can be separated.*

Proof. Recursively construct an AD family $\{A_\alpha : \alpha < \omega_1\}$ so that

$$\forall \alpha < \omega_1 \ \forall n \in \omega \ \{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\} \text{ is finite.}$$

To do this let $\{A_n : n \in \omega\}$ be a partition of ω into infinite pieces. Having constructed $\{A_\beta : \beta < \alpha\}$ enumerate it as $\{B_n : n \in \omega\}$ and for each n choose $a_n \subseteq B_n \setminus \bigcup_{j<n} B_j$ of size n. Let $A_\alpha = \bigcup_{n\in\omega} a_n$.

Now, assume that $\mathcal{B}, \mathcal{C}$ are uncountable families of $\mathcal{A}$ which can be separated. Then, for some $m \in \omega$ there are uncountable $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcup \mathcal{B}' \cap \bigcup \mathcal{C}' \subseteq m$. However, as both families are uncountable, there is an $A_\alpha \in \mathcal{B}'$ such that there are infinitely many $\beta < \alpha$ such that $A_\beta \in \mathcal{C}'$. Then, however, $A_\alpha \cap A_\beta \notin m$ for one of these β, which is a contradiccion. □

Definition 2. An almost disjoint family$\mathcal{A}$ is *Luzin* if it can be enumerated as $\{A_\alpha : \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1 \ \forall n \in \omega \ \{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}$ is finite.

Luzin families are fundamental examples of almost disjoint families. Recently there has been a flurry of activity concerning consistency results involving Luzin families. Abraham and Shelah [1] call an almost disjoint family $\mathcal{A}$ *inseparable* if no two uncountable subfamilies can be separated and they call $\mathcal{A}$ *Luzin** if it can be enumerated as $\{A_\alpha : \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1 \ \forall n \in \omega \ \{\beta < \alpha : |A_\alpha \cap A_\beta| < n\}$ is finite. Obviously, every Luzin* AD family is Luzin and every Luzin family is inseparable.

Theorem 3 ([1]).

1. (CH) *There is an inseparable AD family which contains no Luzin subfamily.*
2. ($\mathsf{MA} + \neg\mathsf{CH}$) *Every inseparable AD family is a countable union of Luzin* subfamilies.*

Roitman and Soukup in [138] introduced the notion of an anti-Luzin family: An AD family $\mathcal{A}$ is an *anti-Luzin* family if for every $\mathcal{B} \in [\mathcal{A}]^{\aleph_1}$ there are $\mathcal{C}, \mathcal{D} \in [\mathcal{B}]^{\aleph_1}$ which can be separated.

Typical examples of anti-Luzin AD families are the families $\mathcal{A}_X$ for $X \subseteq 2^\omega$.

Theorem 4 ([138]). ($\mathsf{MA} + \neg\mathsf{CH}$) *Every AD family is either anti-Luzin or contains an uncountable Luzin subfamily.*

Theorem 5 ([138]). *Assuming* $\upharpoonleft$,[1] *there is an uncountable almost disjoint family which contains no uncountable anti-Luzin and no uncountable Luzin subfamilies.*

More recently, Dow [47] showed:

Theorem 6 ([47]). (PFA) *Every MAD family contains an uncountable Luzin subfamily.*

and Dow and Shelah in [50] showed that Martin's Axiom does not suffice:

Theorem 7 ([50]). *It is relatively consistent with* $\mathsf{MA} + \neg\mathsf{CH}$ *that there is a maximal almost disjoint family which is* ω_1*-separated, i.e. any disjoint pair of* $\leq \omega_1$*-sized subfamilies are separated.*

A connection between Luzin families and Hausdorff gaps was studied by Kalemba and Plewik in [97].

3.2 Normality and related properties

Separation characterizes normality of Ψ-spaces:

Proposition 5 ([154]). $\Psi(\mathcal{A})$ *is normal if and only if* $\mathcal{B}$ *and* $\mathcal{A} \setminus \mathcal{B}$ *can be separated for every* $\mathcal{B} \subseteq \mathcal{A}$.

and leads to the solution of the normal Moore space problem for separable spaces:

Theorem 8 ([154]). *The following are equivalent:*

1. *There is a non-metrizable separable normal Moore space.*
2. *There is an uncountable AD family* $\mathcal{A}$ *such that* $\Psi(\mathcal{A})$ *is normal.*
3. *There is an uncountable* Q*-set, i.e. an uncountable set of reals every subset of which is relatively* G_δ*.*

The *Jones' Lemma* implies that if $\Psi(\mathcal{A})$ is normal then $2^{|\mathcal{A}|} = \mathfrak{c}$, so, in particular, the Continuum Hypothesis implies that normality of separable Moore spaces implies metrizability. On the other hand, Silver has shown that every set of reals of size $< \mathfrak{p}$ is a Q-set.

Luzin families provide examples of non-normal Ψ-spaces of size ω_1. In fact, they also provide examples of Ψ-spaces which are not countably paracompact [82]. Historically, Luzin families have been often referred to as Luzin gaps. However, the use of that term has recently shifted [58, 157–159]:

[1] Recall that $\upharpoonleft$ is the following weakening of CH: There is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ of size $\aleph_1$ such that every uncountable subset of ω_1 contains an element of $\mathcal{S}$.

Definition 3 ([158]). A pair $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$, $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ of subfamilies of $[\omega]^\omega$ is called a *Luzin gap* if there is an $m \in \omega$ such that

1. $A_\alpha \cap B_\alpha \subseteq m$ for all $\alpha < \omega_1$, and
2. $A_\alpha \cap B_\beta$ is finite yet $(A_\alpha \cap B_\beta) \cup (A_\beta \cap B_\alpha) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

Obviously, every Luzin family $\mathcal{A}$ contains Luzin gaps. If $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$, $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ form a Luzin gap then (as essentially proved by Luzin) the two families $\mathcal{A}$ and $\mathcal{B}$ can not be separated, hence, the space $\Psi(\mathcal{A} \cup \mathcal{B})$ is not normal. In fact, assuming PFA this (together with Jones' Lemma) characterizes normality:

Theorem 9 ([73]). (PFA) *Let $\mathcal{A}$ be an AD family. Then $\Psi(\mathcal{A})$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain a Luzin gap.*

It is not known at the moment whether MA suffices (see [73]).

A weaker form of separation was considered by Dow in [44] and Brendle in [27]. Given an AD family $\mathcal{A}$ and two subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ we say that a set $D \subseteq \omega$ *weakly separates* $\mathcal{B}$ and $\mathcal{C}$ if $D \cap B$ is finite for every $B \in \mathcal{B}$ and $D \cap C$ is infinite for every $C \in \mathcal{C}$. Dow [44] used the notion to produce a model of ZFC in which all compact separable radial spaces are Fréchet, while Brendle [27] considered the cardinal invariant $\mathfrak{ap}$ defined as the minimal size of an AD family $\mathcal{A}$ which has a subfamily $\mathcal{B}$ such that $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can not be weakly separated and compared it to other, related, cardinal invariants. In particular, he showed that it is consistent with ZFC that every set of reals of size $\mathfrak{ap}$ is a Q-set.

Szeptycki and Vaughan [152, 153] (see also [40]) introduced the notion of a soft AD family by declaring an almost disjoint family $\mathcal{A}$ *soft* if there is a set intersecting each element of $\mathcal{A}$ in a finite but non-empty set. The reason for this was

Proposition 6. *The space $\Psi(\mathcal{A})$ satisfies property (a)*[2] *if and only if the family $\{A \setminus f(A) : A \in \mathcal{A}\}$ is soft for every $f : \mathcal{A} \to \omega$.*

They showed that there is a close connection between normality of $\Psi(\mathcal{A})$, weak separation and softness of $\mathcal{A}$.

Theorem 10 ([152]). *Let $\mathcal{A}$ be an AD family of size $< \mathfrak{d}$. If for every $\mathcal{B} \subseteq \mathcal{A}$ the pair $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be weakly separated, then $\mathcal{A}$ is soft.*

It is an open question, whether the assumption on cardinality of $\mathcal{A}$ is necessary.

Problem 2 ([152]). Is every AD family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is normal soft?

[2] A space X has property (a) if for every open cover $\mathcal{U}$ of X and a dense set $D \subseteq X$ there is a closed discrete $F \subseteq D$ such that $st(F, \mathcal{U}) = X$.

Separated and weakly separated almost disjoint families are often used for coding, see e.g. [7, 106, 107, 115].

A stronger condition was introduced by Steprāns in [149] (attributed there to Szymański and Zhou):

Definition 4 ([149]). An almost disjoint family $\mathcal{A}$ is called a *strong Q-sequence* if for every family $\{B_A : A \in \mathcal{A}\}$ such that $B_A \subset A$ there is an $X \subseteq \omega$ such that $X \cap A =^* B_A$ for every $A \in \mathcal{A}$.

Steprāns [149] proved that while the existence of a strong Q-sequence contradicts Martin's Axiom, it is relatively consistent with MA(σ-centered). The notion of a strong Q-sequence is closely related to the well known open *Katowice problem*:

Problem 3. Can $\omega^* = \beta\omega \setminus \omega$ and $\omega_1^* = \beta\omega_1 \setminus \omega_1$ ever be homeomorphic?

Among the consequences of a positive solution are: $\mathfrak{d} = \omega_1$ and there is an uncountable strong Q-sequence (see [127]). Recently, Chodounský [39] has shown that the two consequences are mutually consistent.

4 Completely separable MAD families

A MAD family $\mathcal{A}$ on ω is *completely separable*, if for every $M \in \mathcal{I}^+(\mathcal{A})$ there is an $A \in \mathcal{A}$ such that $A \subseteq M$. The notion of completely separable MAD family was introduced in 1971 by Hechler [77] who showed that such families exist assuming Martin's Axiom. In 1972, Erdös and Shelah [55] asked whether completely separable MAD families exist in ZFC.

The question of existence of completely separable MAD families is closely tied to the *disjoint refinement property* [8, 10–13]: *Does $\mathcal{I}^+(\mathcal{A})$ have an almost disjoint refinement for every MAD family $\mathcal{A}$?* This question, formally stronger than the question of existence of a completely separable MAD family, has natural topological reformulations:

Theorem 11 ([10, 146]). *The following are equivalent:*

1. *$\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement for every MAD family $\mathcal{A}$.*
2. *Every nowhere dense subset of ω^* is a $\mathfrak{c}$-set, i.e. for every nowhere dense $N \subseteq \omega^*$ there is a family of $\mathfrak{c}$-many pairwise disjoint open subsets of ω^* each containing N in its closure.*
3. *There are no maximal elements in the Veksler order on nowhere dense subsets of ω^*. (The* Veksler order *is defined by $M < N$ if M is a nowhere dense subset of N).*

For more information consult [10] or [91]. We will present a proof of a recent theorem of Shelah that assuming $\mathfrak{c} < \aleph_\omega$ there is a completely separable MAD family. A minor modification of the proof shows that under the same assumption, $\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement for every MAD family $\mathcal{A}$.

Shelah's original proof from [141] was split into three parts: $\mathfrak{s} < \mathfrak{a}$, $\mathfrak{s} = \mathfrak{a}$ and $\mathfrak{s} > \mathfrak{a}$ where the second and third case needed a mild covering lemma type requirement (see lemma 2) the failure of which requires large cardinals and which can easily be shown to hold below $\aleph_\omega$. Then Mildenberger, Raghavan and Steprāns [119] eliminated the use of the special assumption in the case $\mathfrak{s} = \mathfrak{a}$ and presented a unified proof covering the $\mathfrak{s} < \mathfrak{a}$ and $\mathfrak{s} = \mathfrak{a}$ cases and hence have also given one single proof for all cases known previously: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$ [8,10,144].

Theorem 12 ([119]). *Assuming $\mathfrak{s} \leq \mathfrak{a}$ there is a completely separable MAD family.*

Throughout the rest of this section we will use the following convention: Given a set $x \subseteq \omega$ we let $x^0 = x$ and $x^1 = \omega \setminus x$.

Lemma 1 ([141]). *Given a decreasing sequence $\{B_n : n \in \omega\}$ of infinite subsets of ω there is a family $\mathcal{P} = \{c_\alpha : \alpha < \mathfrak{b}\}$ of subsets of ω such that for any AD family $\mathcal{A}$ such that for infinitely many n, $B_n \setminus B_{n+1}$ almost contains an $A_n \in \mathcal{A}$ and any set $X \in \mathcal{I}^+(\mathcal{A})$ which intersects each A_n in an infinite set there is an $\alpha < \mathfrak{b}$ such that $X \cap c^i_\alpha \in \mathcal{I}^+(\mathcal{A})$ for $i \in \{0,1\}$.*

Proof. It suffices to consider only MAD families. Given the sequence $\{B_n : n \in \omega\}$ consider $C_n = B_n \setminus B_{n+1}$. The lemma has a non-trivial content only if there are infinitely many n such that C_n is infinite. Fix $\{f_\alpha : \alpha < \mathfrak{b}\}$ an $<^*$-increasing unbounded chain of strictly increasing functions from ω to ω. For $\alpha < \mathfrak{b}$ let

$$c_\alpha = \bigcup_{n\in\omega} f_\alpha(n) \cap C_n.$$

Now, let $\mathcal{A}$ be a MAD family such that $B_n \setminus B_{n+1}$ almost contains an $A_n \in \mathcal{A}$ for infinitely many n and a let $X \in \mathcal{I}^+(\mathcal{A})$ intersect each A_n in an infinite set. Consider $\{D_k : k \in \omega\}$ an infinite family of elements of $\mathcal{A}$ disjoint from $\{A_n : n \in \omega\}$ such that each D_k intersects $X \cap \bigcup_{n\in\omega} A_n$ in an infinite set, i.e for each k there are infinitely many n such that $D_k \cap A_n \cap X$ is non-empty (and finite). For $k, j \in \omega$ let $m^k_j = \min\{l \geq j : D_k \cap A_l \cap X \text{ is non-empty}\}$ and let

$$g_k(j) = \min\{m \in \omega : D_k \cap A_{m^k_j} \cap X \subseteq m\}.$$

Let $g \in \omega^\omega$ dominate all g_k, $k \in \omega$ and let $\alpha < \mathfrak{b}$ be such that $f_\alpha \nleq^* g$. Then $X \cap c^0_\alpha \in \mathcal{I}^+(\mathcal{A})$ as $X \cap c^0_\alpha$ intersects all of the D_k in an infinite set, and $X \cap c^1_\alpha \in \mathcal{I}^+(\mathcal{A})$ as $X \cap c^0_\alpha$ intersects all of the A_n in an infinite set. □

Lemma 2. *Let $\mathfrak{b} \leq \kappa < \aleph_\omega$. There is a sequence $\{u_\alpha : \omega \leq \alpha < \kappa\}$ such that:*

1. *u_α is subset of α of order type ω,*
2. *for every $X \subseteq \kappa$ of order type $\mathfrak{b}$ there is an $\alpha < \sup X$ such that $X \cap u_\alpha$ is infinite.*

Proof. It is well-known and easy to see that for $\kappa < \aleph_\omega$ there is a family $\mathcal{W}$ of countable subsets of κ which is cofinal, i.e every countable subset of κ is contained in an element of $\mathcal{W}$. On the other hand, it is known and just as easy to prove (by induction on the order type of w), that for any infinite set w of order type $\alpha < \omega_1$ there is a family $\mathcal{O}_w$ of size $\mathfrak{b}$ of subsets of w of order type ω such that for any $x \in [w]^\omega$ there is an $o \in \mathcal{O}_w$ such that $o \cap x$ is infinite. Thus by replacing, $\mathcal{W}$ by $\bigcup_{w \in \mathcal{W}} \mathcal{O}_w$ one gets

Claim. For every ordinal $\mathfrak{b} \leq \alpha < \aleph_\omega$ there is a family $\mathcal{O}_\alpha$ of size $|\alpha|$ consisting of sets of order type ω such that for any infinite $X \subseteq \alpha$ there is an $o \in \mathcal{O}_\alpha$ such that $o \cap X$ is infinite.

We now proceed by induction on n, where $\kappa = \omega_n$.

For $\kappa = \mathfrak{b} = \omega_{n_0}$ just enumerate $\mathcal{O}_\kappa$ as $\{u_\alpha : \omega \leq \alpha < \kappa\}$ so that u_α is subset of α.

Assuming the lemma was proved for ω_n fix the corresponding sequence $\{u_\alpha : \omega \leq \alpha < \omega_n\}$ and also fix $\mathcal{O}_\beta$ for every ordinal $\beta < \omega_{n+1}$. For an ordinal $\delta = \omega_n \cdot \beta + 2 \cdot \gamma + 1$ ($\beta > 0$) let $u_\delta = \{\omega_n \cdot \beta + \xi : \xi \in u_\gamma\}$, and for $\beta > 0$ enumerate $\mathcal{O}_{\omega_n \cdot \beta}$ as $\{u_{\omega_n \cdot \beta + 2 \cdot \gamma} : \gamma \in \omega_n\}$. It should be obvious that this works. □

Now we are ready to prove the result.

Theorem 13 ([141]). *Assuming $\mathfrak{c} < \aleph_\omega$ there is a completely separable MAD family.*

Proof. Fix a sequence $\{u_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ as in lemma 2. Slightly abusing notation we denote by $u_\alpha(n)$ the n-th element of u_α in its increasing enumeration. Moreover, fix $\{U_\alpha : \alpha < \mathfrak{c}\}$ a partition of $\mathfrak{c}$ such that

1. $|U_0| = \mathfrak{c}$ and $\omega \subseteq U_0$,
2. for $\alpha > 0$, $|U_\alpha| = \mathfrak{b}$ and $\alpha \leq \min U_\alpha < \sup U_\alpha \leq \alpha + \mathfrak{b}$.

Finally, enumerate $[\omega]^\omega$ as $\{x_\alpha : \alpha \in U_0\}$.

Having fixed another enumeration of all infinite subsets of ω as $\{B_\alpha : \alpha < \mathfrak{c}\}$, we will recursively construct an AD family $\{A_\alpha : \alpha < \mathfrak{c}\}$ and a one-to-one sequence $\{\sigma_\alpha : \alpha < \mathfrak{c}\}$ of elements of $2^{<\mathfrak{c}}$ together with functions $C_\alpha : 2^{<\mathfrak{c}} \to \mathcal{P}(\omega)$ so that

1. $A_\alpha \subseteq^* C_\alpha(\sigma_\alpha \restriction \xi)^{\sigma_\alpha(\xi)}$ for every $\alpha < \mathfrak{c}$ and every $\xi \in dom(\sigma_\alpha)$,
2. $A_\alpha \subseteq B_\alpha$ or B_α is almost covered by finitely many A_β, $\beta < \alpha$,
3. $\sigma_\alpha \notin \sigma_\beta$ for every $\beta < \alpha < \mathfrak{c}$.

The function C_α is definable from $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ and $\{\sigma_\beta : \beta < \alpha\}$ in the following way:

1. If $\eta \in 2^\xi$ and $\xi \in U_0$, let $C_\alpha(\eta) = x_\xi$.
2. If $\eta \in 2^\xi$ and $\xi \in U_\delta$ for some $\delta > 0$ then define

   $$B_n^{\alpha,\eta} = \bigcap_{i \le n} \left[C_\alpha(\eta \restriction u_\delta(i))^{\eta(u_\delta(i))} \setminus A_{\eta \restriction u_\delta(i-1)} \right],$$

   here $A_{\eta \restriction u_\delta(i-1)} = A_\beta$ (for some $\beta < \alpha$) if $\eta \restriction u_\delta(i-1) = \sigma_\beta$ or $A_{\eta \restriction u_\delta(i-1)} = \emptyset$. For the decreasing sequence $\{B_n^{\alpha,\eta} : n \in \omega\}$ fix a family $\mathcal{P}^\eta$ of size $\mathfrak{b}$ as in lemma 1. Enumerate $\mathcal{P}^\eta$ (once and for all) as $\{c_\zeta^\eta : \zeta \in U_\delta\}$ and let $C_\alpha(\eta) = c_\xi^\eta$.

An important feature of the sequence of functions $\{C_\alpha : \alpha < \mathfrak{c}\}$ is that they are coherent in the following sense:

$$C_\beta(\sigma_\beta \restriction \xi) = C_\alpha(\sigma_\beta \restriction \xi) \text{ for every } \beta < \alpha < \mathfrak{c} \text{ and every } \xi \in dom(\sigma_\beta).$$

This follows directly from the definition of C_α as the only way the values would change is if there would appear a new $A_{\eta \restriction u_\delta(i-1)}$ in the definition of the $B_n^{\alpha,\eta}$ for one of the u_δ. However, this is impossible by the third clause. In particular,

$$A_\alpha \subseteq^* C_\beta(\sigma_\alpha \restriction \xi)^{\sigma_\alpha(\xi)} \text{ for every } \alpha < \beta < \mathfrak{c} \text{ and every } \xi \in dom(\sigma_\alpha).$$

Having constructed $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ and $\{\sigma_\beta : \beta < \alpha\}$ and having defined C_α as above, we show now how to find A_α and σ_α.

First, for any $X \in \mathcal{I}^+(\mathcal{A}_\alpha)$ recursively define τ_X as follows

$$\tau_X(\xi) = i \text{ if and only if } X \setminus C_\alpha(\tau_X \restriction \xi)^i \in \mathcal{I}(\mathcal{A}_\alpha).$$

If ξ is the first ordinal for which there is no such i (i.e. $C_\alpha(\tau_X \restriction \xi)$ splits X into two $\mathcal{I}(\mathcal{A}_\alpha)$-positive sets), let $\tau_X = \tau_X \restriction \xi$. Note, that τ_X is well-defined as each set gets eventually split (definitely on U_0 if not elsewhere).

We will recursively construct/define ordinals α_s,$\tau_s \in 2^{\alpha_s}$ and $X_s \in \mathcal{I}^+(\mathcal{A}_\alpha)$ for $s \in 2^{<\omega}$ so that

- Let $X_\emptyset = B_\alpha$ if $B_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, and let $X_\emptyset$ be another element of $\mathcal{I}^+(\mathcal{A}_\alpha)$ otherwise,
- $\tau_\emptyset = \tau_{X_\emptyset}$ and $\alpha_\emptyset = dom(\tau_{X_\emptyset})$,
- $X_{s^\frown i} = X_s \cap C_\alpha(\tau_{X_s})^i \in \mathcal{I}^+(\mathcal{A}_\alpha)$ for $i \in \{0,1\}$,
- $\tau_{s^\frown i} = \tau_{X_{s^\frown i}}$ and $\alpha_{s^\frown i} = dom(\tau_{X_{s^\frown i}})$.

Note that these objects are uniquely defined. For $f \in 2^\omega$, let $\tau_f = \bigcup_{n \in \omega} \tau_{f \restriction n}$, let α_f be the domain of τ_f and use lemma 2 to find a set $X_f \in \mathcal{I}^+(\mathcal{A}_\alpha)$ such that $X_f \subseteq^* X_{f \restriction n}$ for all n. As $\alpha < \mathfrak{c}$ there is an $f \in 2^\omega$ such that $\tau_f \notin \sigma_\beta$ for all $\beta < \alpha$. We let $\sigma_\alpha = \tau_f$ and assume for a moment that we can choose $A_\alpha \subseteq X_f \cap X_\emptyset$ almost disjoint from all A_β, $\beta < \alpha$. Then

$$A_\alpha \subseteq^* C_\alpha(\tau_f \restriction \xi)^{\tau_f(\xi)} \text{ for every } \xi < \alpha_f.$$

To see this recall, that X_f has not been split into two $\mathcal{I}(\mathcal{A}_\alpha)$-positive sets by any $C_\alpha(\tau_f \restriction \xi)$, $\xi < \alpha_f$. That is for every $\xi < \alpha_f$ there is a finite set $\mathcal{B}_\xi \subseteq \mathcal{A}_\alpha$ such that

$$X_f \setminus C_\alpha(\tau_f \restriction \xi)^{\tau_f(\xi)} \subseteq^* \bigcup \mathcal{B}_\xi.$$

This, however, implies that $A_\alpha \cap \bigcup \mathcal{B}_\xi$ is finite and so $A_\alpha \subseteq^* C_\alpha(\tau_f \restriction \xi)^{\tau_f(\xi)}$ for every $\xi < \alpha_f$.

So the only thing missing is proving that $\mathcal{A}_\alpha \restriction X_f$ is not MAD.

Claim: $\mathcal{A}_\alpha \restriction X_f$ is not maximal.

Let

$$W = \{\xi < \alpha_f : \ (\exists \beta < \alpha)\ (|A_\beta \cap X_f| = \omega \text{ and}$$
$$(\tau_f \restriction \xi = \sigma_\beta \text{ or } \xi = \min\{\zeta : \tau_f(\zeta) \neq \sigma_\beta(\zeta)\}))\}.$$

Now, if $|A_\beta \cap X_f| = \omega$ for $\beta < \alpha$ then there is a $\xi \in W$ such that

$$\tau_f \restriction \xi = \sigma_\beta \text{ or } \xi = \min\{\zeta : \tau_f(\zeta) \neq \sigma_\beta(\zeta)\}$$

and for each $\xi \in W$ there is at most one $\beta < \alpha$ such that $\tau_f \restriction \xi = \sigma_\beta$ and there are only finitely many $\beta < \alpha$ such that $\xi = \min\{\zeta : \tau_f(\zeta) \neq \sigma_\beta(\zeta)\}$, each such $A_\beta \in \mathcal{B}_\xi$. So, it suffices to show that $|W| < \mathfrak{b} \leq \mathfrak{a}$.

Aiming toward a contradiction assume that $|W| \geq \mathfrak{b}$ and let W_0 be the set of the first $\mathfrak{b}$-many elements of W. By lemma 2 there is a $\delta < \sup W_0$ such that $u_\delta \cap W$ is infinite. This means that X_f satisfies the assumption of lemma 1. Hence, there is a $\xi \in U_\delta$ such that $C_\alpha(\tau_f \restriction \xi)$ splits X_f into two $\mathcal{I}(\mathcal{A}_\alpha)$-positive sets, i.e. $\alpha_f \leq \xi$. This, however, leads to a contradiction as this $\xi < \delta + \mathfrak{b}$ and below δ there are fewer than $\mathfrak{b}$ elements of W and, obviously, there are fewer than $\mathfrak{b}$ elements of W between δ and ξ. □

An important feature of completely separable MAD families is that they can be used as tools for recursively constructing other AD families, or related objects much in the way ultrafilters can be constructed along independent families. One of the known applications of the existence of a completely separable MAD family is the existence of a *Čech function* - a function $cl : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that (i) $cl(\emptyset) = \emptyset$, (ii) $A \subseteq cl(A)$ and (iii) $cl(A \cup B) = cl(A) \cup cl(B)$ which is surjective yet not the identity.

Čech's 1947 question as to whether such function exists was first solved using CH by Price [132], then Galvin noticed that the existence of a completely separable MAD family is sufficient. Recently Galvin and Simon realized [64] that maximality of the AD family was not necessary and that an almost disjoint family constructed in [8] is sufficient.

The main problem, however, remains open:

Problem 4 ([55]). Is there a completely separable MAD family in ZFC?

5 Almost disjoint families of graphs of functions

Almost disjoint families on $\omega\times\omega$ consisting of graphs of functions and partial functions received a lot of attention recently. A longstanding problem of van Douwen (see [121]) as to whether there is an almost disjoint family of total functions which is a MAD family when augmented by vertical sections was recently answered by Raghavan in [134].

We say that two functions are *eventually different (ED)* if their graphs are almost disjoint. We call a family of functions eventually different if any two functions in the family are eventually different.

Theorem 14 ([134]). *There is a van Douwen MAD family, i.e. a family of eventually different functions maximal with respect to partial functions.*

Another basic problem that attracted much attention is relatively recent and somewhat surprisingly hard:

Problem 5 ([100]). Is there an analytic (or even closed) maximal family of eventually different functions?

It is an old result of Mathias [116] that no MAD family is analytic (a co-analytic MAD family was constructed by Miller [120] assuming $V = L$) and there seems to be a consensus that the same should hold true here. However, somehow the problem is still open. Steprāns (see [100]) introduced the notion of a strongly maximal family of eventually different functions as follows:

Definition 5. An eventually different family $\mathcal{E}$ is *strongly maximal* if given a family $\{f_i : i\in\omega\}$ of functions each not covered by finitely many elements of $\mathcal{E}$ there is a $g\in\mathcal{E}$ such that $|g\cap f_i| = \omega$ for every $i\in\omega$.

Steprāns showed that no strongly maximal eventually different family is analytic ([100]). Zhang and Kastermans [99] introduced a strengthening of this notion:

Definition 6. An eventually different family $\mathcal{E}$ is *very maximal* if given a cardinal $\lambda < |\mathcal{E}|$ and a family $\{f_\xi : \xi\in\lambda\}$ of functions each not covered by finitely many elements of $\mathcal{E}$ there is a $g\in\mathcal{E}$ such that $|g\cap f_\xi| = \omega$ for every $\xi\in\lambda$.

Kastermans [99] showed that very maximal ED families exist assuming Martin's axiom, while Raghavan [133] showed that assuming $\mathfrak{b} = \mathfrak{c}$ strongly maximal ED families exist. He has also shown that it is relatively consistent with $\mathfrak{b} = \mathfrak{c}$ that very maximal ED families do not exist.

Theorem 15 ([133]). *Assuming* $\mathsf{cov}(\mathcal{M}) < \mathsf{non}(\mathcal{M})$ *there are no very maximal ED families.*

In particular, it is consistent with ZFC that there are strongly maximal ED families while there are no very maximal ones.

Problem 6 ([133]). Is it consistent that there are no strongly maximal ED families?

Several papers, e.g. [33, 78, 99, 165], studied families of eventually different functions with special properties. Among these the most prominent is the notion of a cofinitary group:

Definition 7. A permutation $\pi \in Sym(\omega)$ is *cofinitary* if it has only finitely many fixed points. A group $\mathbb{G} \leq Sym(\omega)$ is *cofinitary* if all of its elements, other than the identity are cofinitary.

Obviously, a permutation group $\mathbb{G}$ is cofinitary if and only if its elements are mutually eventually different, i.e. it is a permutation group which is also an ED family. For a nice survey of algebraic aspects of cofinitary groups consult Cameron's [37].

Truss [160] and Adeleke [2] showed that no countable cofinitary group is maximal, while Cameron [37] showed that there is a (maximal) cofinitary group of size $\mathfrak{c}$ and asked what is the minimal size of a maximal cofinitary group. In particular, he asked whether it is possible to have a maximal cofinitary group of size strictly less than $\mathfrak{c}$. Zhang answered the question in [165] by showing that (1) assuming Martin Axiom every maximal cofinitary group has size $\mathfrak{c}$ and (2) it is consistent to have a maximal cofinitary group of size strictly less than $\mathfrak{c}$. Brendle, Spinas and Zhang [33] showed that $\mathsf{non}(\mathcal{M})$ is a lower bound on the minimal size of a maximal cofinitary group.

Similar to the case of ED families, it is an open problem whether there can be an analytic maximal cofinitary group. On the other hand, Gao and Zhang in [65] constructed, assuming $V = L$ a maximal cofinitary group with a co-analytic set of generators. Friedman and Zdomskyy [63] proved that it is consistent with $\mathfrak{b} = \mathfrak{c} = \omega_2$ that there is a Π^1_2 strongly maximal ED family.

6 Compactifications and partitioners

An important notion for studying topological and combinatorial properties of almost disjoint families is the notion of a partitioner.

Definition 8. Given an almost disjoint family $\mathcal{A}$ a set $X \subseteq \omega$ is a *partitioner* of $\mathcal{A}$ if $A \cap X =^* \emptyset$ or $A \subseteq^* X$ for every $A \in \mathcal{A}$.

Note that if X is a partitioner for $\mathcal{A}$ than so is $\omega \setminus X$. A partitioner is *trivial* if $X \in \mathcal{I}(\mathcal{A})$ or $\omega \setminus X \in \mathcal{I}(\mathcal{A})$.

6.1 Čech-Stone remainders of Ψ-spaces

Non-trivial partitioners correspond to clopen subsets of $\beta(\Psi(\mathcal{A})) \setminus \Psi(\mathcal{A})$.

Proposition 7 ([18]). *$\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is connected if and only if $\mathcal{A}$ has no non-trivial partitioners.*

Proof. If $\mathcal{A}$ has a non-trivial partitioner then it has a compactification with exactly two points. Hence $\beta\Psi(\mathcal{A})\setminus\Psi(\mathcal{A})$ is disconnected.

On the other hand, if $\beta\Psi(\mathcal{A})\setminus\Psi(\mathcal{A})$ has a non-trivial clopen subset C, then there are open (in $\Psi(\mathcal{A})$) sets U,V which separate C and $(\beta\Psi(\mathcal{A})\setminus\Psi(\mathcal{A}))\setminus C$. Note that $U\cup V$ covers $\beta\Psi(\mathcal{A})\setminus\Psi(\mathcal{A})$ so, $F=\mathcal{A}\cap(\beta\Psi(\mathcal{A})\setminus(U\cup V))$ is finite. So $P=(U\cap\omega)\setminus\bigcup F$ is a non-trivial partitioner of $\mathcal{A}$. $\square$

Mrówka in [126] constructed a MAD family, such that $\Psi(\mathcal{A})$ has a unique compactification, i.e. its Čech-Stone compactification and its one-point compactifications coincide.

Definition 9. A MAD family $\mathcal{A}$ is called *Mrówka* if $|\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})|=1$.

Theorem 16 ([126]). *There is a Mrówka family.*

Proof. Extend the AD family $\mathcal{A}_{2^\omega}$ to a MAD family $\mathcal{B}_0$. List $\mathcal{B}_0\setminus\mathcal{A}_{2^\omega}$ as $\{B_f: f\in Y\}$ for some $Y\subseteq 2^\omega$. Let

$$\mathcal{B}_1=\{A_f: f\in 2^\omega\setminus Y\}\cup\{A_f\cup B_f: f\in Y\}$$

Claim. Every non-trivial partitioner of $\mathcal{B}_1$ contains $\mathfrak{c}$ elements of $\mathcal{B}_1$.

Let $X\subseteq 2^{<\omega}$ be a non-trivial partitioner. Consider $Z=\{f\in 2^\omega: A_f\subseteq^* X\}$ then Z is an uncountable set, as $\mathcal{B}_1\restriction X$ is MAD, and Z is F_σ (hence of size $\mathfrak{c}$) as $Z=\{f\in 2^\omega:\exists n\in\omega\ \forall m\geq n\ f\restriction m\in X\}$.

Enumerate all non-trivial partitioners of $\mathcal{B}_1$ as $\{P_\alpha:\alpha<\kappa\}$ for some $\kappa\leq\mathfrak{c}$. Recursively choose $A_\alpha\neq B_\alpha\in\mathcal{B}_1\setminus\{A_\beta,B_\beta:\beta<\alpha\}$ so that $A_\alpha\subseteq^* P_\alpha$ and $B_\alpha\cap P_\alpha=^*\emptyset$. (There is no problem choosing as there are always $\mathfrak{c}$-many possibilities and less than $\mathfrak{c}$-many already chosen.) Then let

$$\mathcal{B}_2=\{A_\alpha\cup B_\alpha:\alpha<\kappa\}\cup(\mathcal{A}\setminus\{A_\alpha,B_\alpha:\alpha<\kappa\}).$$

$\mathcal{B}_2$ is then a MAD family with no non-trivial partitioners.

Since $\mathcal{B}_2$ has no nontrivial partitioners, the remainder $\beta(\Psi(\mathcal{B}_2))\setminus\Psi(\mathcal{B}_2)$ is connected. If it is a singleton, we already have the desired Mrówka family. If not, enumerate as $\{f_\alpha:\alpha<\mathfrak{c}\}$ all maps in $[0,1]^{2^{<\omega}}$ which extend to a continuous function from $\Psi(\mathcal{B}_2)$ onto $[0,1]$. Then recursively choose $A_\alpha\neq B_\alpha\in\mathcal{B}_2\setminus\{A_\beta,B_\beta:\beta<\alpha\}$ so that if f_α extends to a continuous onto function $F:\Psi(\mathcal{B}_2)\to[0,1]$ then $F(A_\alpha)\neq F(B_\alpha)$. Then let

$$\mathcal{A}=\{A_\alpha\cup B_\alpha:\alpha<\mathfrak{c}\}\cup\mathcal{B}_2\setminus\{A_\alpha,B_\alpha:\alpha<\mathfrak{c}\}.$$

$\mathcal{A}$ is then a MAD family and (1) $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is connected since $\mathcal{A}$ has no non-trivial partitioners (every non-trivial partitioner of $\mathcal{A}$ would be a non-trivial partitioner of $\mathcal{B}_2$), and (2) $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is zero-dimensional (if not, there would be a continuous surjection $F:\Psi(\mathcal{A})\to[0,1]$, however, there is no such F as $F\restriction 2^{<\omega}$ was enumerated as f_α and by the construction F is not continuous at $A_\alpha\cup B_\alpha\in\mathcal{A}$). Being both connected and zero-dimensional, $|\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})|=1$. □

Aside from presenting a fundamental and surprising example, the theorem and its proof gave both a novel method for constructing special MAD families and the starting point for investigation as to which spaces can appear as the Čech-Stone remainders of $\Psi(\mathcal{A})$ for MAD $\mathcal{A}$. Terasawa showed in [156]

Theorem 17 ([156]). *For every compact metric space X without isolated points there is a MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to X.*

A stronger result was proved by Bashkirov in [19].

Theorem 18 ([19]). *For every separable Fréchet compact space X there is a MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to X.*

As noticed by Levy and Kulesza in [103], the strongest possible result actually follows from CH and the results of Baumgartner and Weese [21] mentioned in the next section:

Theorem 19 ([103]). *Assuming CH, for every continuous image X of $\beta\omega\setminus\omega$ there is a MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to X.*

On the other hand Dow [42] proved that it is consistent that every $\beta(\Psi(\mathcal{A}))$ has size at most $\mathfrak{c}$.

Dow and Vaughan recently considered the problem of which ordinals can be homeomorphic to the Čech-Stone remainders of Ψ-spaces in [53]. According to Terasawa, the space ω_1+1 is a Čech-Stone remainder of a $\Psi(\mathcal{A})$ for MAD family $\mathcal{A}$, a fact he attributes to Mrówka. Dow and Vaughan extended the result by proving that

Theorem 20 ([53]). *Assume that there is a $\subseteq^*$-increasing chain of order type α. Then there is a MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to $\alpha+1$.*

In particular:

Theorem 21 ([53]). *There is a MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to $\alpha+1$ for every $\alpha<\mathfrak{t}^+$.*

Dow and Vaughan also pointed out that Baumgartner and Weese [21] showed that after adding more than $\aleph_2$ many Cohen reals to a model of CH there is no MAD family $\mathcal{A}$ such that $\beta(\Psi(\mathcal{A}))\setminus\Psi(\mathcal{A})$ is homeomorphic to ω_2+1

6.2 Partitioner algebras

Baumgartner and Weese in [21] introduced the notion of a *partition (or partitioner) algebra* of a MAD family $\mathcal{A}$ as the quotient Boolean algebra of the subalgebra of $\mathcal{P}(\omega)$ consisting of all partitioners modulo the ideal of partitioners in $\mathcal{I}(\mathcal{A})$ as an attempt to classify MAD families. Using the method of Mrówka they showed that:

Theorem 22 ([21]). *Every countable Boolean algebra is isomorphic to a partition algebra.*

and more generally:

Theorem 23 ([21]). *Assuming the Continuum Hypothesis, every Boolean algebra of size at most $\mathfrak{c}$ is isomorphic to a partition algebra.*

On the other hand, they also showed that it is consistent with the negation of the Continuum Hypothesis that any Boolean algebra which contains the free algebra on $\aleph_2$-generators is not representable as a partition algebra hence, in particular, no infinite complete Boolean algebra is representable. They conclude the paper with several interesting questions, many of which have since been solved by Dow and co-authors:

Theorem 24 ([49]). *The free algebra on $\aleph_1$-generators is representable as a partition algebra.*

Theorem 25 ([48]). *It is consistent with* ZFC *that there is a Boolean algebra of size $\aleph_1$ which is not representable as a partition algebra.*

The model for this is any model where $\omega_1 = \mathfrak{s} < \mathfrak{b}$ and the algebra is the subalgebra of $\mathcal{P}(\omega)$ generated by the small splitting family and finite sets.

The last two facts together show that:

Theorem 26 ([48]). *It is consistent with* ZFC *that the class of Boolean algebras representable as partition algebras is not closed under homomorphic images.*

Frankiewicz and Zbierski [62] showed that it is relatively consistent with MA(σ-linked) that $\mathfrak{c}$ is arbitrarily large and every Boolean algebra of size at most $\mathfrak{c}$ is representable as a partition algebra. The last result on partition algebras that we will mention is deep and difficult to prove.

Theorem 27 ([54]). *It is consistent, assuming a version of* PFA*, that there exists a partitioner algebra which contains a subalgebra which is not representable as a partitioner algebra.*

7 Almost disjoint families and Katětov order

Another attempt at classifying MAD families uses Katětov order.

Definition 10. Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$ ($\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$. We say that $\mathcal{I}$ is *Katětov-Blass below* $\mathcal{J}$ ($\mathcal{I} \leq_{KB} \mathcal{J}$) if the function f is finite-to-one.

Given two almost disjoint families $\mathcal{A}$ and $\mathcal{B}$ we shall write $\mathcal{A} \leq_K \mathcal{B}$ (resp. $\mathcal{A} \leq_{KB} \mathcal{B}$) if $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}(\mathcal{B})$ (resp. $\mathcal{I}(\mathcal{A}) \leq_{KB} \mathcal{I}(\mathcal{B})$). We shall write $\mathcal{I} \simeq_K \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{I} \geq_K \mathcal{J}$.

Some elementary properties of Katětov order are listed here. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

1. $\mathcal{I} \simeq_K \mathsf{fin}$ if and only if $\mathcal{I}$ is not tall.
2. If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$.
3. If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_K \mathcal{I} \restriction X$.
4. $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{I}$ and $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{J}$.
5. $\mathcal{I}, \mathcal{J} \leq_K \mathcal{I} \times \mathcal{J}$.

Here $\mathcal{I} \oplus \mathcal{J}$ denotes the disjoint sum and $\mathcal{I} \times \mathcal{J}$ the Fubini product of $\mathcal{I}$ and $\mathcal{J}$ (see section 2). Properties (4) and (5) show that Katětov order is both upward and downward directed. It is easily seen that the family of maximal ideals is cofinal in Katětov order, while the ideals generated by MAD families are coinitial among tall ideals in Katětov order. The definable ideals sit somewehere in the middle and can be used to classify both maximal ideals (dually, ultrafilters) and MAD families.

For more information about Katětov order in general and Katětov order on Borel ideals in particular consult the recent survey [87]. Before, we turn our attention to MAD families let us briefly summarize an analogous classification of ultrafilters.

7.1 Ultrafilters and Katětov order

We only consider *free* ultrafilters, i.e. ultrafilters consisting of infinite sets. The most important classes of ultrafilters, arguably, are: selective ultrafilters, P-points, Q-points, rapid ultrafilters and nowhere dense ultrafilters. An ultrafilter $\mathcal{U}$ on ω is:

- *selective* if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- a *P-point* if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n|$ is finite for every $n \in \omega$.

- a *Q-point* if for every partition $\{I_n : n \in \omega\}$ of ω into finite sets there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- *rapid* if the family of increasing enumerations of elements of $\mathcal{U}$ is dominating.
- *nowhere dense* (or a nwd-ultrafilter) if for every map $f : \omega \to \mathbb{R}$ there is a $U \in \mathcal{U}$ such that $f[U]$ is a nowhere dense subset of $\mathbb{R}$.

Baumgartner introduced the following definition in [20]. Let I be a family of subsets of a set X such that I contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ is an *I-ultrafilter* if for every $F : \omega \to X$ there is an $A \in \mathcal{U}$ such that $F[A] \in I$.

Proposition 8. *Let $\mathcal{I}$ be an ideal on ω. Then an ultrafilter $\mathcal{U}$ on ω is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \not\leq_K \mathcal{U}^*$.*

Most standard combinatorial properties of ultrafilters are characterized in this way by Borel ideals of a low complexity (see e.g. [61] for details). Let $\mathcal{U}$ be an ultrafilter and $\mathcal{U}^*$ the dual ideal. Then

- $\mathcal{U}$ is selective iff $\mathcal{ED} \not\leq_K \mathcal{U}^*$ iff $\mathcal{R} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a P-point iff $\mathsf{fin} \times \mathsf{fin} \not\leq_K \mathcal{U}^*$ iff $\mathsf{conv} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a nowhere dense ultrafilter iff $\mathsf{nwd} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a Q-point iff $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{U}^*$,
- $\mathcal{U}$ is rapid iff $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for any analytic P-ideal $\mathcal{I}$.

The moral of the story is that upward cones of Borel ideals nicely stratify/classify non-definable objects e.g. ultrafilters. We will see that the same happens for downward cones and MAD families.

7.2 Destructibility of ideals by forcing and Katětov order

Definition 11. Given an ideal $\mathcal{I}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $\mathcal{I}$ if there is a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω such that $\Vdash_{\mathbb{P}}$ "$I \cap \dot{X}$ is finite for every $I \in \mathcal{I}$".

It turns out that Katětov order is an indispensable tool for studying destructibility of ideals by forcing. In [92], Katětov order is used to fully characterize destructibility of ideals for a large class of forcing notions: the proper forcings of the type P_I of I-positive Borel subsets of the Baire space ω^ω ordered by inclusion, where I is a σ-ideal on ω^ω, (see [164]) which have the *Continuous Reading of Names* (CRN) [3], see [92, 164].

[3] If P_I is a proper forcing then it *has the CRN* if for every Borel function $f : B \to 2^\omega$ with an I-positive Borel domain B there is an I-positive Borel set $C \subseteq B$ such that $f \upharpoonright C$ is continuous.

Many of the commonly used proper forcing notions can be naturally presented as forcings of the form P_I with the CRN which are, moreover, *continuously homogeneous* i.e. for every I-positive Borel set B there is a continuous function $F:\omega^\omega \to B$ such that $F^{-1}(A)\in I$ for all $A\in\mathcal{I}\restriction B$. Now, given a σ-ideal I on ω^ω, its *trace ideal* $tr(I)$ is an ideal on $\omega^{<\omega}$ defined by

$$tr(I)=\{a\subseteq\omega^{<\omega}:\{r\in\omega^\omega:\exists^\infty n\in\omega\ r\restriction n\in a)\}\in I\}$$

.

It turns out that the trace ideals are critical, in the Katětov order, with respect to P_I-destructibility.

Theorem 28 ([92]). *Let P_I be a proper forcing with CRN, which is continuously homogeneous, and let $\mathcal{J}$ be an ideal on ω. Then the following conditions are equivalent:*

1. *P_I destroys $\mathcal{J}$*
2. *$\mathcal{J}\leq_K tr(I)$.*

We now return to the study of Katětov order restricted to MAD families.

7.3 Katětov order on MAD families

Particular cases of the results of the preceding section can be expressed as follows.

Theorem 29 ([34, 86, 92, 105]). *Let $\mathcal{A}$ be a MAD family. Then:*

1. *$\mathcal{A}$ is Sacks-indestructible if and only if $\mathcal{A}\nleq_K tr(\mathsf{ctbl})$,*
2. *$\mathcal{A}$ is Miller-indestructible if and only if $\mathcal{A}\nleq_K \mathsf{scattered}$,*
3. *$\mathcal{A}$ is Cohen-indestructible if and only if $\mathcal{A}\nleq_K \mathsf{nwd}$.*

It is easy to see that $\mathcal{A}\leq_K \mathsf{fin}\times\mathsf{fin}$ for every MAD family $\mathcal{A}$. Moreover, the Katětov order on MAD families is sufficiently complex, hence worth studying.

Theorem 30 ([88]). *Let $\mathcal{A}$ be a MAD family. Then*

(1) there is a $\leq_K$-antichain below $\mathcal{A}$ of cardinality $\mathfrak{c}$ and
(2) there is a $\leq_K$-decreasing chain of length $\mathfrak{c}^+$ below $\mathcal{A}$.

It follows that there are no K-minimal MAD families. Recently, in [4] it has been proved that consistently there are MAD families, which are K-maximal among MAD families. Call a MAD family $\mathcal{A}$ *weakly tight* provided that $(\forall\langle I_n:n\in\omega\rangle\subseteq\mathcal{I}^+(\mathcal{A}))(\exists A\in\mathcal{I}(\mathcal{A}))(\exists^\infty n\in\omega)\ |A\cap I_n|=\aleph_0$. The weakly tight families are "almost" maximal:

Proposition 9 ([88]). *Let $\mathcal{A}$ be a weakly tight MAD family and let $\mathcal{B}$ be a MAD family. If $\mathcal{A} \leq_K \mathcal{B}$ then there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{B} \leq_K \mathcal{A} \restriction X$.*

As a corollary, it was noted in [88] that a MAD family which is both weakly tight and *K-uniform* (K-equivalent to all of its restrictions to positive sets) is, in fact, K-maximal. Assuming a version of Martin's Axiom both weakly tight and K-uniform MAD families were constructed in [88]. In [4] it was shown that it can be done simultaneously:

Theorem 31 ([4]). *Assuming $\mathfrak{t} = \mathfrak{c}$ there is a MAD family $\mathcal{A}$ which is both weakly tight and K-uniform, hence K-maximal.*

It is not known, whether weakly tight, or even K-maximal MAD families exist in ZFC. Raghavan and Steprāns [136] have modified Shelah's argument for constructing a completely separable MAD family (see section 4) to prove that

Theorem 32 ([136]). *Assuming $\mathfrak{s} \leq \mathfrak{b}$ there is a weakly tight MAD family.*

Surprisingly, we do not even know, in ZFC, whether there is no K-largest MAD family. However, at least consistently, there is not.

Theorem 33 ([88]). *Assuming $\mathfrak{b} = \mathfrak{c}$, for every MAD family $\mathcal{A}$ there is a MAD family $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are K-incomparable.*

The notion of a weakly tight MAD family is a weakening of a more natural notion of tight MAD family closely related to Cohen-indestructibility. A MAD family $\mathcal{A}$ is *tight* (or $\aleph_0$-MAD [105, 111]) if

$$(\forall \langle I_n : n \in \omega\rangle \subseteq \mathcal{I}^+(\mathcal{A}))(\exists A \in \mathcal{I}(\mathcal{A}))(\forall n \in \omega)\ |A \cap I_n| = \aleph_0.$$

The next proposition provides a useful characterization of tight MAD families.

Proposition 10 ([88]). *A MAD family $\mathcal{A}$ is tight if and only if $\forall f : \mathbb{Q} \to \omega$ $\exists A \in \mathcal{I}(\mathcal{A})$ $f^{-1}[A]$ is either dense or has non-empty interior.*

It immediately follows that each tight MAD family is Cohen-indestructible, and that being tight is upward closed in the Katětov order. The relation with Cohen-idestructibility is extremely close:

Proposition 11 ([105]). *For every Cohen-indestructible MAD family $\mathcal{A}$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{A} \restriction X$ is tight.*

The existence of a tight (or, equivalently, Cohen-indestructible) MAD family is known to follow from $\mathfrak{b} = \mathfrak{c}$, $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$ and $\diamond(\mathfrak{d})$ (see [124]). It is an open question whether Cohen-indestructible MAD families exists in ZFC alone:

Problem 7 ([150]). Is there a Cohen-indestructible MAD family?

In fact, it is not even known, whether there is a Sacks-indestructible MAD family. In [86] it is shown that

Proposition 12 ([86]). *There is a MAD family $\mathcal{A}$ such that $\mathcal{A} \nleq_K \overline{\mathsf{ctbl}}$.*

which is there (in [86]) falsely identified as Sacks-indestructible. Interesting open questions concern the notion of K-maximality:

Problem 8. 1. Is there a K-maximal MAD family in ZFC?
2. Is there is a K-uniform MAD family in ZFC?
3. Is every K-maximal MAD family (weakly) tight?
4. Is it consistent that there a K-maximal MAD family of size less than $\mathfrak{c}$?
5. Is it true that there is no K-largest MAD family?

8 Hyperspaces and selections

The hyperspace of a topological space X (denoted by 2^X) consists of all nonempty closed subsets of X and is equipped with the *Vietoris topology*, i.e. the topology generated by sets of the form:

$$\langle U; V_0, \dots, V_n \rangle = \left\{ F \in 2^X : F \subseteq U \text{ and } F \cap V_i \neq \varnothing \text{ for every } i \leq n \right\},$$

where $U, V_0, \dots, V_n$ are nonempty open subsets of X.

8.1 Hyperspaces of Mrówka-Isbell spaces

Mrówka-Isbell spaces have been succesfully used in the study of continuous selections and psedocompactness of hyperspaces. Concerning countable compactness and pseudocompactness of 2^X Ginsburg [72] proved the following

Theorem 34 ([72]). *(a) If every power of a space X is countably compact then so is 2^X.*

(b) If 2^X is countably compact (pseudocompact) then so is every finite power of X.

and asked: *Is there any relationship between the pseudocompactness of X^ω and that of the hyperspace 2^X?* Cao, Nogura and Tomita [38] gave the following partial answer:

Theorem 35 ([38]). *If X is a homogeneous Tychonoff space such that 2^X is pseudocompact then X^ω is pseudocompact.*

Cao and Nogura then explicitly asked whether 2^X is pseudocompact for some/every Mrówka-Isbell space X. This problem was considered in [89] where it was shown that

Proposition 13 ([89]). *The space* $(\Psi(\mathcal{A}))^\omega$ *is pseudocompact for every MAD family* $\mathcal{A}$*.*

while

Theorem 36 ([89]).

1. $(\mathfrak{p} = \mathfrak{c})$ $2^{\Psi(\mathcal{A})}$ *is pseudocompact for every MAD family* $\mathcal{A}$*, and*
2. $(\mathfrak{h} < \mathfrak{c})$ *There is a MAD family* $\mathcal{A}$ *such that* $2^{\Psi(\mathcal{A})}$ *is not pseudocompact.*

The assumption in (2) can actually be weakened to the existence of a base tree for $\mathcal{P}(\omega)/\mathsf{fin}$ (see section 2) without branches of length $\mathfrak{c}$. Rather surprisingly, the following problem is open:

Problem 9. Is there in ZFC a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is pseudocompact?

Inspired by theorem 36 the authors answered Ginsburg's question as follows:

Theorem 37 ([89]). *There is a subspace* X *of* $\beta\omega$ *such that* X^ω *is pseudocompact yet* 2^X *is not.*

8.2 Ψ-spaces and selections

A function φ defined on 2^X (or some subspace of 2^X) is a *selection* if $\varphi(F) \in F$ for every $F \in dom(\varphi)$. A selection is *continuous* if it is continuous with respect to the Vietoris topology. In particular, a *weak selection* is a selection defined on $[X]^2$, the set of all two-element subsets of the space X.

The study of continuous selections was initiated by Michael in [118] in 1951. The general question studied by Michael is: *When does a space admit a continuous (weak) selection?* Michael himself showed that a sufficient condition for a space X to admit a continuous weak selection is that it admits a weaker topology generated by a linear order, i.e. that the space is *weakly orderable*. The natural question, whether this characterizes spaces which admit continuous weak selections, implicit in Michael's paper, was stated explicitly in a paper by van Mill and Wattel [162].

The answer is positive for (a) connected spaces [118], (b) compact spaces [162] and even (c) pseudocompact spaces (see [6,67]). However, in [90] it was shown that

Theorem 38 ([90]). *There is an almost disjoint family* $\mathcal{A}$ *such that the space* $\Psi(\mathcal{A})$ *admits a continuous weak selection but is not weakly orderable.*

In fact, the Ψ-space constructed in [90] admits a continuous selection for all compact sets, yet is not weakly orderable. It is also shown there that a separable space which admits a continuous weak selection admits, in fact, a continuous selection for all finite sets. On the other hand:

Theorem 39 ([90]). *There is an almost disjoint family $\mathcal{A}$ such that the space $\Psi(\mathcal{A})$ admits a continuous selection for triples but no continuous weak selection.*

Selections on Ψ-spaces have also been considered in [85]. There it is shown that $\Psi(\mathcal{A})$ does not admit a selection for closed sets for any uncountable AD family $\mathcal{A}$. For more on continuous selections see Gutev's article in this volume.

9 Spaces of continuous functions

9.1 Ψ-spaces and Lindelöf C_p

Recall that $C_p(X,Y)$ denotes the space of all continuous functions from X to Y with the *topology of pointwise convergence*, i.e the topology inherited from the Tychonoff product Y^X. The space $C_p(X,\mathbb{R})$ is denoted simply by $C_p(X)$. The first result on $C_p(\Psi(\mathcal{A}))$ we are aware of is due to Just, Sipacheva and Szeptycki:

Theorem 40 ([96]). *Assuming $\diamond$ there is an almost disjoint family $\mathcal{A}$ such that the space $C_p(\Psi(\mathcal{A}))$ has countable extent but is not normal.*

The main interest in the study of spaces of continuous functions over Ψ-spaces stems from trying to understand the Lindelöf property in spaces of the form $C_p(X)$. Buzyakova [36] showed that $C_p(X)$ is Lindelöf where X is the set of successor ordinals and ordinals of countable cofinality below any ordinal α. This result inspired a study of the Lindelöf and related properties in function spaces over Mrówka-Isbell spaces:

Theorem 41 ([51]). *The space $C_p(\Psi(\mathcal{A}))$ is not Lindelöf for any MAD family $\mathcal{A}$.*

The situation for $C_p(X,2)$ is somewhat more interesting:

Theorem 42 ([51, 84]).

1. ($\mathfrak{b} > \omega_1$) *The space $C_p(\Psi(\mathcal{A},2))$ is not Lindelöf for any MAD family $\mathcal{A}$.*
2. (CH) *There is a maximal almost disjoint family $\mathcal{A}$ such that the space $C_p(\Psi(\mathcal{A},2))$ is Lindelöf.*

The family constructed is a *Mrówka* family, i.e. MAD family with unique compactification (see section 6.1). In particular, assuming CH one can construct two Mrówka MAD families $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $C_p(\Psi(\mathcal{A}_0, 2))$ is Lindelöf while $C_p(\Psi(\mathcal{A}_1, 2))$ is not. This clearly demonstrates the complexity of the problem of characterizing Lindelöf property in function spaces.

For a compact scattered space K the topology of pointwise convergence and the weak topology on $C(K)$ coincide. We were informed by Koszmider that several of the above results were obtained already by Pol in [131]. They were also used in [147] in relation to problems posed in [102].

9.2 Almost disjoint families in functional analysis

According to [94]: "*almost disjoint families of* $\mathbb{N}$ *have found many applications in the theory of Banach spaces, some classical cases include Whitley's short proof of Philips' theorem in [163], Haydon's constructions of Grothendieck spaces in [76], or Johnson and Lindenstrauss' counterexamples concerning weakly compactly generated Banach spaces in [95].*" The most common (see [76, 163]) use of almost disjoint families in the theory of Banach spaces is as follows: Given a bounded sequence $\{x_n : n \in \omega\}$ of elements of a Banach space, one wishes to find an $A \subseteq \omega$ such that the sequence $\{x_n : n \in A\}$ is in some sense good. The argument is that if $\{A_\xi : \xi < \omega_1\}$ is an AD family then not all A_ξ can be bad. Banach spaces of the form $C(\Psi(\mathcal{A}))$ were probably first considered by Johnson and Lindenstrauss in [95] and Moltó in [123].

It is well known that if K is an infinite compact Hausdorff and scattered space, then the Banach space $C(K)$ of continuous functions on K has complemented copies of c_0, i.e., $C(K) \sim c_0 \oplus X \sim c_0 \oplus c_0 \oplus X \sim c_0 \oplus C(K)$. Koszmider in [101] addresses the question if this could be the only type of decompositions of $C(K) \not\sim c_0$ into infinite-dimensional summands for an infinite scattered space K.

Theorem 43 ([101]). *Assuming* $\mathfrak{p} = \mathfrak{c}$ *there is an almost disjoint family* $\mathcal{A}$ *such that the Banach space* $C(\Psi(\mathcal{A}))$ *has the following properties:*

1. *Every operator* $T : C(\Psi(\mathcal{A})) \to C(\Psi(\mathcal{A}))$ *is of the from* $cI + S$*, where* c *is a real number and* S *has range included in a subspace isomorphic to* c_0*.*
2. *The only decompositions* $C(\Psi(\mathcal{A})) = A \oplus B$ *into two infinite dimensional complemented subspaces are such that* $A \sim c_0$ *and* $B \sim C(\Psi(\mathcal{A}))$ *or* $B \sim c_0$ *and* $A \sim C(\Psi(\mathcal{A}))$*.*

Koszmider posed the obvious problem:

Problem 10 ([101]). Is the assumption $\mathfrak{p} = \mathfrak{c}$ necessary?

as well as the following (we denote by $F(\mathcal{A})$ the one point compactification of $\Psi(\mathcal{A})$, see section 10.2):

Problem 11 ([101]).

1. Suppose MA. Is it true that if $|\mathcal{A}| = |\mathcal{A}'| < \mathfrak{c}$, then $C(F(\mathcal{A}))$ is isomorphic to $C(F(\mathcal{A}'))$?
2. Suppose MA. Is it true that if $|\mathcal{A}| < \mathfrak{c}$, then $C(F(\mathcal{A}))$ is isomorphic to its square?
3. Are there two AD families $\mathcal{A}$ and $\mathcal{A}'$ of the same cardinality such that $C(F(\mathcal{A}))$ is not isomorphic to $C(F(\mathcal{A}'))$?

For more on $C(\Psi(\mathcal{A}))$ consult [59] and [60].

Marciszewski and Pol [113,114] considered the spaces $C(F(\mathcal{A}))$ equipped with the *weak topology*. Marciszewski [113] answered a question of Corson by proving:

Theorem 44 ([113]). *There is an almost disjoint family $\mathcal{A}$ such that the Banach space $C(F(\mathcal{A}))$ with the weak topology is not homeomorphic to $C(F(\mathcal{A})) \times C(F(\mathcal{A}))$.*

In [114] Marciszewski and Pol also studied the Banach spaces $C(F(\mathcal{A}_{2^\omega}))$ and $C(F(\mathcal{A}_{\omega^\omega}))$ and showed that $C(F(\mathcal{A}_{2^\omega}))$ is weak-norm-perfect while $C(F(\mathcal{A}_{\omega^\omega}))$ is not[4].

A different application of almost disjoint families to functional analysis was presented by Shelah and Steprāns in [142]. It deals with the notion of *masa* i.e. maximal abelian self-adjoint subalgebras of C^*-algebras. It was known that if the Continuum Hypothesis is assumed, then there is a non-liftable masa in the Calkin algebra generated by its projections (consult [56] for non-defined notions). Shelah and Steprāns in [142] introduced the notion of a strongly separable MAD family as follows

Definition 12 ([142]). Given an almost disjoint family $\mathcal{A}$ let

$$\mathcal{I}^+_*(\mathcal{A}) = \{H \subseteq [\omega]^{<\omega} \setminus \{\varnothing\} : \forall I \in \mathcal{I}(\mathcal{A}) \ \exists a \in H \ a \cap I = \varnothing\}.$$

An almost disjoint family $\mathcal{A}$ is *strongly separable* if for every $H \in \mathcal{I}^+_*(\mathcal{A})$ there are $\mathfrak{c}$-many $A \in \mathcal{A}$ such that for every $n \in \omega$ there is an $a \in H$ such that $a \subseteq A \setminus n$.

They showed that

Theorem 45 ([142]). *If there is a strongly separable almost disjoint family then there is a masa in the Calkin algebra generated by its projections which does not lift to a masa of $\mathfrak{B}(\mathbb{H})$.*

Raghavan in [135] showed that it is consistent with ZFC that strongly separable MAD families do not exist (see section 10.2).

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[4] A Banach space E is *weak-norm-perfect* if every norm-open subset of E is F_σ in the weak topology.

10 Fréchet and sequential spaces

Recall that a topological space X is *sequential* if any set which is not closed, contains a sequence converging to a point outside of the set. Given A subset of a sequential space X, let $\hat{A} = \{x \in X : x$ is a limit of a sequence of elements of $A\}$. For an ordinal $\alpha < \omega_1$ we define $A^{(\alpha)}$ recursively: $A^{(0)} = A$, $A^{(\alpha+1)} = \widehat{A^{(\alpha)}}$ and $A^{(\lambda)} = \bigcup_{\alpha<\lambda} A^{(\alpha)}$ for λ limit. In a sequential space X for any $A \subseteq X$ the closure of A is equal to $A^{(\alpha)}$ for some $\alpha \le \omega_1$. We denote by $\sigma(A)$ the minimal such α. The *sequential order* of a sequential space X is defined as the supremum of $\sigma(A)$ for $A \subseteq X$.

A space X is *Fréchet-Urysohn* or just *Fréchet* if it is sequential of sequential order 1, i.e. every point in the closure of a subset of X is a limit point of convergent sequence from the set.

10.1 Compact Fréchet and sequential spaces

Let $\mathcal{A}$ be an AD family. The one-point compactification of $\Psi(\mathcal{A})$ is often referred to as a *Franklin compactum* and is denoted by $F(\mathcal{A})$. It is a sequential space of sequential order at most two.

We call an AD family $\mathcal{A}$ *nowhere maximal* or *nowhere MAD* if for every $X \in \mathcal{I}^+(\mathcal{A})$ there is a $B \in [X]^\omega$ which is almost disjoint from every element of $\mathcal{A}$. The following simple observation characterizes when $\Psi(\mathcal{A})$ is Fréchet:

Proposition 14. *$F(\mathcal{A})$ is Fréchet if and only if $\mathcal{A}$ is nowhere MAD.*

Michael in [117] asked whether the product of compact Fréchet spaces is Fréchet. Malykhin [110], Olson [129], and Boehme and Rosenfeld [24] all independently constructed counter-examples of the same kind - Franklin compacta, assuming various set-theoretic axioms. The main point of the proofs was another rather simple observation:

Proposition 15. *Assume that $\mathcal{A}$ is a MAD family which is partitioned as $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ so that each $\mathcal{A}_i$, $i \in 2$, is nowhere MAD. Then $F(\mathcal{A}_0) \times F(\mathcal{A}_1)$ is not Fréchet.*

It is straightforward to show that if $F(\mathcal{A}_i) = \Psi(\mathcal{A}_i) \cup \{\infty\}$ then the set $\Delta = \{(n,n) : n \in \omega\} \subseteq F(\mathcal{A}_0) \times F(\mathcal{A}_1)$ has the point (∞,∞) in its closure, but no sequence from Δ converges to it.

Hence, the problem reduces to finding a "partitionable" MAD family. Now, doing this is easy asssuming, for instance, the Continuum Hypothesis or even just the existence of a completely separable MAD family. However, Simon [145] found an ingenious and simple way of proving the existence of such a family in ZFC:

Theorem 46 ([145]). *For every MAD family $\mathcal{A}$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{A} \restriction X = \mathcal{A}_0 \cup \mathcal{A}_1$ so that each $\mathcal{A}_i$, $i \in 2$, is nowhere MAD.*

Proof. The proof proceeds by contradiction. Assume that $\mathcal{A}$ is such that:

$(*)$ for every $X \in \mathcal{I}^+(\mathcal{A})$ and every partition $\mathcal{A} \restriction X = \mathcal{A}_0 \cup \mathcal{A}_1$ there is a $Y \subseteq X$, $Y \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{A}_i \restriction Y$ is maximal for some $i \in 2$.

List such $\mathcal{A}$ as $\{A_f : f \in Z\}$ for some $Z \subseteq 2^\omega$. Recursively, construct a function $g \in 2^\omega$ and a decreasing sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ so that for every $n \in \omega$

$$\mathcal{A}_n^{g(n)} \restriction X_n \text{ is maximal,}$$

where $\mathcal{A}_n^i = \{A_f : f \in Z \text{ such that } f(n) = i\}$. This can be done, as a direct consequence of $(*)$.

Now, use proposition 2 to find $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \subseteq^* X_n$ for every $n \in \omega$. As $X \in \mathcal{I}^+(\mathcal{A})$ there are infinitely (even uncountably) many $f \in Z$ such that $A_f \cap X$ is infinite, in particular, there is an $f \neq g$ such that $A_f \cap X$ is infinite. Then, however, there is an $n \in \omega$ such that $g(n) \neq f(n)$, hence $A_f \notin \mathcal{A}_n^{g(n)}$, hence, $A_f \cap X_n$ is finite and, consequently, $A_f \cap X$ is finite, which is a contradiction. □

It was not clear for a long time whether the positive restriction in Simon's theorem is, in fact, necessary. Recently, Dow [46] showed that it is necessary indeed.

Theorem 47 ([46]). *It is consistent with* ZFC *that there is a MAD family $\mathcal{A}$ of size ω_2 such that every $\mathcal{B} \subseteq \mathcal{A}$ of size ω_2 is somewhere MAD, i.e. not nowhere MAD.*

That is, every partitionable restriction of $\mathcal{A}$ has size ω_1 and, in particular $\mathcal{A}$ itself is not partitionable.

Berner in [22] constructed, assuming $\mathfrak{a} = \mathfrak{c}$, a non-compact space X such that βX is Fréchet. Apparently a ZFC example of such a space has since been constructed by Reznichenko (unpublished).

If $\mathcal{A}$ is not nowhere MAD then $F(\mathcal{A})$ is a compact sequential space of sequential order 2. Bashkirov in [17] constructed a whole scale of compact sequential spaces assuming the Continuum Hypothesis:

Theorem 48 ([17]). *Assuming* CH*, there are compact sequential spaces of sequential order α for every $\alpha \leq \omega_1$.*

Amazingly, 2 is the highest sequential order of a compact sequential space known in ZFC.

Problem 12 ([5]). Is there, in ZFC, a compact sequential space of sequential order 3? of infinite sequential order?

After much labour Dow in [45] constructed a compact sequential space of sequential order 4, assuming Martin's Axiom. We quote from [45]:"*There is a compelling motivation for determining the maximum possible sequential order in the presence of the Proper Forcing Axiom,* PFA*, which comes from the Moore-Mrówka problem (see [14]]). First of all, it is quite remarkable that the best known lower bounds are 2 in* ZFC *and now 4 under* PFA*. Secondly, Balogh proved that each compact space of countable tightness is sequential if* PFA *is assumed (which is known to imply Martin's Axiom and* $\mathfrak{c} = \omega_2$*). If there is some finite bound on the sequential order of compact sequential spaces in models of* PFA*, it would mean that compact spaces of countable tightness are literally but a few steps away from being Fréchet-Urysohn.*"

The question whether there is consistently some non-trivial upper bound on the sequential order of compact sequential spaces remains open. Some impressive progress has been made by Dow [47] though:

Theorem 49 ([47]). *Assuming* PFA *there is no compact scattered space of height greater than* ω *in which the sequential order and the scattering heights coincide.*

10.2 Fréchet groups and strongly separable MAD families

There are many problems in set theory and set-theoretic topology which require that certain ideals be destroyed while the tallness of others is being preserved in forcing extensions of the universe. This section discusses instances of this phenomenon related to the study of Fréchet groups and almost disjoint families.

A classical theorem of Kakutani and Birkhoff states that a T_1 topological group is metrizable if and only if it is first countable. The natural question as to what extent can the first countability be weakened was addressed by Malykhin, who asked if there is a separable (equivalently, countable) Fréchet-Urysohn group that is not metrizable?

There are known consistent positive solutions under either of the following assumptions: $\mathfrak{p} > \omega_1$ (Malykhin), the existence of an uncountable γ-set[5] (Gerlits-Nagy [69]) and $\mathfrak{p} = \mathfrak{b}$ (Nyikos [128]). In fact, by a recent result of Tsaban and Ohrenstein [130] the existence of an uncountable γ-sets is the weakest of the assumptions.

The problem was very recently proved idependent in [79]:

[5] A separable metric space X is called a γ-*set* if every ω-cover of X has a γ-subcover. An open cover $\mathcal{U} = \{U_n : n \in \omega\}$ is an ω-*cover* if for every finite $F \subseteq X$ there is a $n \in \omega$ such that $F \subseteq U_n$; $\mathcal{U} = \{U_n : n \in \omega\}$ is a γ-*cover* if for every $x \in X$ and for all but finitely many $n \in \omega$, $x \in U_n$.

Theorem 50 ([79]). *It is relatively consistent with* ZFC *that every separable Fréchet topological group is metrizable.*

The method of the proof is based on an earlier result form [31]:

Theorem 51 ([31]). *It is consistent with the continuum arbitrarily large that no uncountably generated filter of character less than $\mathfrak{c}$ is a FUF-filter.*[6]

The question as to what extent is algebra involved in the problem was raised by I. Juhasz:

Problem 13 (Juhász). Is there in ZFC a countable Fréchet space of uncountable π-weight?

A consistency result here seemed probable given that

Theorem 52 ([79]). *It is consistent with the continuum arbitrarily large that every countable Fréchet space of π-weight less than $\mathfrak{c}$ is metrizable.*

complemented by a results of Barman and Dow [15]

Theorem 53 ([15]). *It is consistent with* ZFC *that every countable Fréchet space has π-weight at most $\aleph_1$.*

However, Dow [41] has recently shown that

Theorem 54 ([41]). *There is a countable Fréchet space of uncountable π-weight, assuming $\mathfrak{b} = \mathfrak{c}$.*

Which combined with an earlier result of Nyikos shows that there is a countable Fréchet space of uncountable π-weight, assuming $\mathfrak{c} \leq \aleph_2$.

The method of proof of theorem 51, together with a clever Ramsey theoretic argument, was used by Raghavan in [135] to answer a question of Shelah and Steprāns [142] (see section 9):

Theorem 55 ([135]). *It is consistent with* ZFC *that there are no strongly separable MAD families.*

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[6] A filter $\mathcal{F}$ on ω is a *FUF-filter* if given a family $\mathcal{H} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that every element of $\mathcal{F}$ contains an element of $\mathcal{H}$ there is a sequence $\{a_n : n \in \omega\} \subseteq \mathcal{H}$ such that every element of $\mathcal{F}$ contains all but finitely many a_n's. Reznichenko and Sipacheva in [137] noted that a FUF-filter (short for Fréchet-Urysohn for finite sets) produces a group topology on the Boolean group $[\omega]^{<\omega}$ which is always Fréchet and is metrizable if and only if the filter has countable character.

11 Concluding remarks

There are many important topics on almost disjoint families that have been omitted. For instance continuous functions on Ψ-spaces were considered by Bashkirov, García-Ferreira, Kulesza, Levy, Malykhin and Tamariz-Mascarúa in [18, 66, 103, 112].

In several cases the reason for omission was that the topic was too set-theoretic without immediate connection to topology. Such is the case of the study of cardinal invariants of the continuum related to almost disjoint families. We will mention the main results rather telegraphically here.

It is well known that the almost disjointness number $\mathfrak{a}$ is larger or equal to the boundedness number $\mathfrak{b}$. The fact that the strict inequality $\mathfrak{b} < \mathfrak{a}$ is consistent was proved by Shelah in [139] by a countable support iteration of proper forcings. It was later re-proved by Brendle in [26] by finite support iteration of σ-centered posets, with the added advantage that the continuum and the values of $\mathfrak{b}$ and $\mathfrak{a}$ can be made arbitrarily large (though the proof gives that $\mathfrak{a}$ is the successor of $\mathfrak{b}$). The longstanding open problem whether $\mathfrak{d}$ can be strictly smaller than $\mathfrak{a}$ was quite recently solved also by Shelah in [140] by introducing a new type of non-linear iteration technique called *iteration along templates*. The method was used and further developed by Brendle in [28–30] and used to show, among other results, that the cardinal invariant $\mathfrak{a}$ can have countable cofinality. The surprising feature of the method is that it works only for cardinals above ω_1. So the following problem attributed to Roitman remains open:

Problem 14. Is it consistent that $\mathfrak{d} = \omega_1$ yet $\mathfrak{a} > \omega_1$?

In [81] a combinatorial principle called $\diamondsuit_{\mathfrak{d}}$ was introduced as a slight strengthening of the assumption $\mathfrak{d} = \omega_1$ and it is showed there that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{a} = \omega_1$. It was noticed by Brendle that the template method producing a model of $\mathfrak{d} = \omega_1$ would produce a model of $\diamondsuit_{\mathfrak{d}}$ and hence is not suitable for answering Roitman's question. The idea of associating a guessing principle to a cardinal invariant was further explored in [124] where a whole scheme of $\diamondsuit$-like principles was introduced and studied.

Another class of cardinal invariants related to almost disjoint families involves extensions of AD families to MAD families. Probably the first of these was considered by Leathrum in [108]. The cardinal invariant $\mathfrak{o}$ is defined as the minimal size of an almost disjoint family $\mathcal{A}$ on $2^{<\omega}$ such that $\mathcal{A}_{2^\omega} \cup \mathcal{A}$ is a MAD family (see [26, 108]). A whole variety of similar cardinal invariants was introduced by Fuchino, Geschke and Soukup in [70]. Fixing an AD family $\mathcal{A}$ they define $\mathfrak{a}^+(\mathcal{A})$ as the minimal size of an almost disjoint family $\mathcal{B}$ such that $\mathcal{B} \cup \mathcal{A}$ is a MAD family. In this notation $\mathfrak{o} = \mathfrak{a}^+(\mathcal{A}_{2^\omega})$. They also introduce the cardinal invariant $\mathfrak{a}^+(\kappa)$ as the supremum of $\mathfrak{a}^+(\mathcal{A})$ for $\mathcal{A}$ an AD family of size κ and show that both $\mathfrak{a} < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ and $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ are consistent with ZFC. However, in their second model $\mathfrak{a}^+(\aleph_1) = \aleph_2 < \mathfrak{c}$ so they ask the natural question:

Problem 15. [70] Is it consistent that $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$?

Another subject we did not consider is treating almost disjoint families themselves as topological spaces, as subspaces of $\mathcal{P}(\omega)$. It is easy to check that every AD family is both meager and of Lebesgue measure zero. There are no analytic MAD families [116] though assuming $V = L$ there is a co-analytic one as shown by Miller [120]. Recently, Brendle and Khomskii [25] showed that it is consistent with $\mathfrak{b} > \aleph_1$ that there is a co-analytic MAD family, answering a question of Friedman and Zdomskyy. Törnquist, unpublished, (see [25]) showed that the existence of a Σ^1_2 MAD family is equivalent to the existence of a Π^1_1 MAD family.

Miller in [122] has constructed a consistent example of a MAD family which is a Q-set. Brendle and Piper [32], assuming the Continuum Hypothesis, constructed a MAD family which is a σ-set and another MAD family which is concentrated on a countable set.[7]

We have also not considered the combinatorics of almost disjoint families on uncountable cardinals. The area is too vast and too detached from topology to venture even briefly into it. We will mention only a few recent topological constructions involving *strongly almost disjoint* (i.e. almost disjoint mod finite) families on uncountable cardinals. The main point being, that to any strongly almost disjoint family $\mathcal{A}$ of subsets of an uncountable cardinal κ one can naturaly associate a generalized Mrówka-Isbell space $\Psi(\kappa, \mathcal{A})$ defined on $\kappa \cup \mathcal{A}$ in the obvious way. The first such space was probably considered by Solomon in [148], where the author gave an example of a scattered Tychonoff space which is not zero-dimensional (see [156] for separable examples).

Szeptycki in [151] proved, assuming $\mathfrak{c} = \mathfrak{b}^+$, that for every MAD family $\mathcal{A}$ of countable subsets of ω_1 the space $\Psi(\omega_1, \mathcal{A})$ is not countably metacompact while Burke [35] did the same assuming $\mathfrak{a} = \mathfrak{c}$.

Dow and Vaughan [52, 53] considered ordinal remainders of spaces of the form $\Psi(\kappa, \mathcal{A})$. First, in [52], they showed, extending results of Mrówka [126], that for every $\kappa \leq \mathfrak{c}$ there is a MAD family $\mathcal{A}$ of countable subsets of κ such that the space $\Psi(\kappa, \mathcal{A})$ has unique compactification and then in [53] they proved that for every infinite cardinal $\kappa \leq \mathfrak{c}^+$ there is a maximal almost disjoint family $\mathcal{A}$ of countable sets such that the Čech-Stone remainder of $\Psi(\kappa, \mathcal{A})$ is homeomorphic to $\kappa + 1$. In [83] another version of Mrówka family is constructed: It is shown there that there is a maximal strongly almost disjoint family $\mathcal{A}$ of subsets of $\kappa = (2^{\omega_1})^+$ such that for every $\mathcal{B} \subseteq \mathcal{A}$ of size $\aleph_1$ and every $D \subseteq \kappa$ of size $\aleph_1$ there is an $A \in \mathcal{A}$ such that $(B \setminus D) \cap A \neq \varnothing$ for all $B \in \mathcal{B}$. This implies that not only does $\Psi(\kappa, \mathcal{A})$ have a unique compactification, but also that every function $f : \Psi(\kappa, \mathcal{A}) \to \mathbb{R}$ continuous with respect to the G_δ-topology on $\Psi(\kappa, \mathcal{A})$ is constant on a co-countable subset of $\mathcal{A}$.

[7] Recall that a set of reals X is a *σ-set* if every G_δ subset of X is F_σ, and X is *concentrated on a countable set* $D \subseteq X$ if every open set containing D contains all but countably many elements of X.

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