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# SCATTERED SPACES FROM WEAK DIAMONDS

BY

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ABSTRACT

Parametrized $\diamondsuit$-principles introduced in [9] associated to the cardinal invariants $\mathfrak{s}$ and $\mathfrak{b}$ are used to construct (1) a family of sequentially compact spaces whose product is not countably compact - an example for the Scarborough-Stone problem, (2) a Jakovlev space, and (3) a compact sequential space of sequential order $\omega_1$. All spaces constructed are scattered, locally compact and of size $\omega_1$.

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## 1. Introduction

We consider three classical topological problems, and construct corresponding examples, using the parametrized $\diamondsuit$-principles introduced in [9]. In particular, we construct a family of sequentially compact spaces providing an example to the Scarborough-Stone problem assuming $\diamondsuit(\mathfrak{s})$, a Jakovlev space and a compact sequential space of sequential order $\omega_1$ assuming (a variant of) $\diamondsuit(\mathfrak{b})$. All of these spaces are scattered, locally compact and of size $\omega_1$.

The main point being that the parametrized $\diamondsuit$-principles typically hold in models where the corresponding cardinal invariant is $\omega_1$ [9, 18] and, in particular, help settle the aforementioned problems in models of set theory, where the solution was up to now not known, e.g. the sequential order of compact spaces in the Cohen and Miller models, and the Scarborough-Stone problem in the Sacks model.

A secondary purpose of this note is to try to popularize the use of parametrized $\diamondsuit$-principles. These principles have been used recently to construct many interesting examples (see e.g. [9, 19, 11, 5]) and nicely complement the use of cardinal invariants of the continuum in analysis of combinatorial and topological problems.

The Scarborough-Stone problem asks whether there is a family of sequentially compact spaces whose product is not countably compact. Some consistent examples are known: van Douwen [22] constructed such a family assuming $\mathfrak{b}^* = \mathfrak{c}$, while Vaughan constructed, using $\diamondsuit$, an example consisting of $T_6$ spaces. On other hand, Weiss [25] using $\mathsf{MA} + \neg\mathsf{CH}$ proved that every countably compact $T_6$ (i.e. perfectly normal) space is compact, so, in particular, every product of countably compact $T_6$ spaces is countably compact; hence the Scarborough-Stone problem is independent of $\mathsf{ZFC}$ for the class of $T_6$ spaces. Nyikos, Soukup and Veličović, in [20], proved, assuming $\mathsf{PFA}$, that in a countably compact $T_5$

(i.e. hereditarily normal) space, every countable set has compact clousure, thus every product of countably compact $T_5$ spaces is countably compact and so the Scarborough-Stone problem is independent of $\mathsf{ZFC}$ even for $T_5$ spaces. On the other hand, Nyikos and Vaughan [21] constructed, in $\mathsf{ZFC}$, an example in which the spaces are Hausdorff but are not regular. In $\mathsf{ZFC}$, it is still open for regular spaces.

Jakovlev spaces emerged as natural tools to answer an old question of Arhangelskii [2] whether the notions of weakly first countable and first countable coincide in the class of compact spaces.

Recall that a space $X$ is *weakly first countable* if there is a family $\{C_n^x : x \in X, n \in \omega\}$ such that $x \in C_{n+1}^x \subseteq C_n^x$ for all $x \in X$ and $n \in \omega$, and such that $U \subset X$ is open if and only if for each $x \in U$ there is a $n \in \omega$ such that $C_n^x \subset U$.

A topological space $X$ is *Jakovlev* if it is uncountable and can be partitioned into $\omega$ levels $L_n$ so that

(1) for every $n \in \omega$ and every point $x \in L_n$ there is a countable compact open neighbourhood $U$ of $x$ such that $U \setminus \{x\} \subseteq \bigcup_{m<n} L_m$, and
(2) every countable sequence contained in any level has an accumulation point.

The one point compactification of a Jakovlev space answers the question of Arhangelskii negatively [14]. It is unknown if Jakovlev spaces exist in $\mathsf{ZFC}$. For more on Jakovlev spaces see [1] and [15].

Recall that a subset $A$ of a topological space $X$ is *sequentially closed* if every convergent sequence of points in $A$ has its limit point in $A$. A space $X$ is *sequential* if every sequentially closed subset of $X$ is closed. Given a subset $A$ of $X$ the *sequential clousure* of $A$ is defined as

$$seqcl(A) = \{x \in X : (\exists (a_n)_{n\in\omega} \subset A)(a_n \to x)\}.$$

Iterating the procedure one defines $seqcl^0(A) = A$,

$$seqcl^{\alpha+1}(A) = seqcl(seqcl^{\alpha}(A))$$

for $\alpha < \omega_1$, and $seqcl^{\alpha}(A) = \bigcup_{\beta<\alpha} seqcl^{\beta}(A)$ in case $\alpha \leq \omega_1$ is a limit ordinal. The *sequential order* of a topological space $X$, denoted by $so(X)$, is defined as the minimal $\alpha \leq \omega_1$ such that for every $A \subset X$ the set $seqcl^{\alpha}(A)$ is sequentially closed.

In 1974, Bashkirov [3] proved that assuming $\mathsf{CH}$ there are compact sequential spaces of any sequential order up to and including $\omega_1$. It is easy to see that the one point compactification of the Mrówka-Isbell space corresponding to a MAD family over the set of natural numbers is a compact sequential space of sequential order 2. Surprisingly, it is not known if there is a compact sequential space of sequential order bigger than 2 in ZFC alone. In [6], Dow constructed a compact space of sequential order 4 assuming $\mathfrak{b}^* = \mathfrak{c}$, and pointed out that his method cannot be generalized to get larger sequential order. Dow also proved in [7] that, under PFA, the sequential order of any compact sequential scattered space for which the sequential order and the Cantor-Bendixson rank coincide cannot be greater than $\omega$. The main question, however, remains open.

The notation we use is standard. For set-theoretic notation see [17], and for topology background consult [10]. Following [9], a function $F : 2^{<\omega_1} \to X$, where $X$ is a Polish space, is *Borel* if $F \restriction 2^\alpha$ is a Borel function for each $\alpha < \omega_1$. Recall that given $A, B \subset \omega$ we say that $A$ *splits* $B$ if both $B \cap A$ and $B \setminus A$ are infinite, and given $f, g \in \omega^\omega$ we write that $f \leq^* g$ to denote that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$.

The two principles used here are the following:

$\diamondsuit(\mathfrak{s})$ := for every Borel $F : 2^{<\omega_1} \to [\omega]^\omega$ there is a $g : \omega_1 \to [\omega]^\omega$ such that for each $f \in 2^{\omega_1}$, the set $\{\alpha < \omega_1 : g(\alpha) \text{ splits } F(f \restriction \alpha)\}$ is stationary.

$\diamondsuit(\mathfrak{b})$ := for every Borel $F : 2^{<\omega_1} \to \omega^\omega$ there is $\{g_\alpha : \alpha < \omega_1\} \subset \omega^\omega$ such that for every $f \in 2^{\omega_1}$, the set $\{\beta < \omega_1 : g_\beta \not\leq^* F(f \restriction \beta)\}$ is stationary.

For technical reasons, we shall use a different (technically stronger) version of $\diamondsuit(\mathfrak{b})$. Let us denote by $\omega^{\uparrow\omega}$ the family of all strictly increasing functions from $\omega$ to $\omega$.

$\diamondsuit(\mathfrak{b}^*)$ := For every Borel $F : 2^{<\omega_1} \to \omega^{\uparrow\omega}$ there is $\{g_\alpha : \alpha < \omega_1\} \subseteq \omega^{\uparrow\omega}$ such that for every $f \in 2^{\omega_1}$, the set of all $\beta < \omega_1$ such that there are infinitely many $n \in \omega$ for which there is $m \in \omega$ such that

$$(\blacklozenge) \qquad \Big[g_\alpha(n), g_\alpha(n+1)\Big) \supseteq \Big[F(f \restriction \beta)(m), F(f \restriction \beta)(m+1)\Big),$$

is stationary.

In fact, we do not know (it is a recurrent open problem with the parametrized $\diamondsuit$ principles) if the principles $\diamondsuit(\mathfrak{b})$ and $\diamondsuit(\mathfrak{b}^*)$ are consistently different. They hold simultaneously in all models where we can determine their validity (Sacks model, Miller model, Cohen model, ...), due to the folklore fact that the two corresponding cardinal invariants are equal, i.e.

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^{\uparrow\omega} \ \& \ \forall g \in \omega^{\uparrow\omega} \ \exists f \in \mathcal{F} \ \exists^\infty n \in \omega \ \exists m \in \omega$$
$$[g_\alpha(n), g_\alpha(n+1)) \subseteq [f(m), f(m+1))\}.$$

This can be seen as a trivial consequence of [4, Theorem 2.10]. There it is shown that $\mathfrak{b}$ is equal to the minimal size of a family $\mathcal{F}' \subseteq \omega^{\uparrow\omega}$ such that

$$\forall g \in \omega^{\uparrow\omega} \ \exists f \in \mathcal{F}' \ \exists^\infty n \in \omega \ \exists m \in \omega \ [g(n), g(n+1)) \not\supseteq [f(m), f(m+1))\}.$$

To see the non-trivial inequality, given an $f \in \mathcal{F}$ let $f_i$, $i \in \{0,1\}$, be defined by $f_i(n) = f(2n+i)$. It can be easily checked that

$$\mathcal{F} = \{f_i : f \in \mathcal{F}' \ \& \ i \in \{0,1\}\}$$

is the required family.

We usually do not apply the function $F$ from the principles directly to elements of $2^{<\omega_1}$ but rather to pairs of the form $\langle Y, \mathcal{A}\rangle$ where $Y$ is a subset of $\omega$ (or another countable set) and $\mathcal{A}$ is a countable family of subsets of $\omega$ – an approximation to the object we wish to construct. We do this by a simple coding which we sketch here. Given a countable indecomposable ordinal $\alpha > \omega$ and a function $\sigma \in 2^\alpha$ define $Y_\sigma = \{n \in \omega : \sigma(n) = 1\}$, and similarly for $\beta < \alpha$, let $A^\sigma_\beta = \{n \in \omega : \sigma(\omega\cdot(\beta+1)+n) = 1\}$. In this way, $\sigma$ codes a pair $\langle Y_\sigma, \mathcal{A}_\sigma\rangle$, $\mathcal{A}_\sigma = \{A^\sigma_\beta : \beta < \alpha\}$. Of course, sometimes the guessed pair does not satisfy some extra requirements we may impose (i.e. the set $Y$ being infinite, or the family $\mathcal{A}$ having special structure), however, in all of those cases these requirements define a Borel subset of $2^\alpha$ and the values outside this Borel set are irrelevant for our constructions, so we can define the value of $F$ outside this set arbitrarily.

Moreover, the same coding extends also to coding pairs $\langle Y, \mathcal{A}\rangle$ where $\mathcal{A}$ is an uncountable family by elements of $\mathscr{P}(\omega)$, with the important property that on a closed unbounded set the restrictions of the coded sequences are the coded initial segments, hence, as the set where we guess correctly is stationary, we can always assume that we are guessing at these ordinals.

## 2. Scarborough-Stone problem

If $\mathcal{F}$ is a filter over a Boolean algebra $\mathbb{B}$, we denote by

$$\mathcal{F}^+ = \{a \in \mathbb{B} : (\forall y \in \mathcal{F})(a \cap y \neq \emptyset)\}$$

the set of *positive* sets with respect to $\mathcal{F}$. For a Boolean algebra $\mathbb{B}$ we denote by $St(\mathbb{B})$ its Stone space, and for every $b \in \mathbb{B}$ we denote by $b^* = \{x \in St(\mathbb{B}) : b \in x\}$ the basic clopen set corresponding to $b$.

We shall use a special kind of Boolean algebras, the *minimally generated* Boolean algebras (see [16]). Here we only deal with Boolean algebras contained in $\mathscr{P}(\omega)$, in fact, every Boolean algebra considered here is a subalgebra of $\mathscr{P}(\omega)$ containing all finite sets. Given two Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ we say that $\mathbb{B}$ *is a minimal extension of* $\mathbb{A}$, we write $\mathbb{A} <_m \mathbb{B}$, if the algebra generated by $\mathbb{A} \cup \{b\}$ is equal to $\mathbb{B}$ for every $b \in \mathbb{B} \setminus \mathbb{A}$. Equivalently, $\mathbb{A} <_m \mathbb{B}$ if and only if there is a unique ultrafilter $\mathcal{U}$ on $\mathbb{A}$ that does not generate an ultrafilter on $\mathbb{B}$ and there are exactly two ultrafilters on $\mathbb{B}$ which extend $\mathcal{U}$. We say that a Boolean algebra $\mathbb{B}$ is *minimally generated* if there is a sequence $\{a_\beta : \beta < \alpha\}$ of generators of $\mathbb{B}$ and $\mathbb{B} = \bigcup_{\beta<\alpha} \mathbb{B}_\beta$ where for every $\beta$ we have that $\mathbb{B}_\beta <_m \mathbb{B}_{\beta+1}$, $\mathbb{B}_0 = \{\emptyset, \omega\}$ and $\mathbb{B}_\beta$ is the Boolean algebra generated by the set $\{a_\gamma : \gamma < \beta\}$. Following Dow and Shelah ([8]), we call a minimally generated Boolean algebra *coherently minimally generated* and its definition if, moreover, denoting by $\mathcal{U}_\alpha$ the ultrafilter on $\mathbb{B}_\alpha$ witnessing that $\mathbb{B}_\alpha <_m \mathbb{B}_{\alpha+1}$, for every $\gamma < \gamma' < \beta$ we have that $\mathcal{U}_{\gamma'} \cap \mathbb{B}_\gamma = \mathcal{U}_\gamma$.

If $\{A_\beta : \beta < \alpha\} \subset \mathcal{U}$ is a sequence of generators of a coherently minimally generated Boolean algebra and $\mathbb{B}_\beta$ is the Boolean algebra generated by the set $\{A_\gamma : \gamma < \beta\}$ denote by $X_\beta = St(\mathbb{B}_\beta) \setminus \{\mathcal{U} \cap \mathbb{B}_\beta\}$. Note that $X_\beta$ is an open subspace of $X_{\beta'}$ for every $\beta < \beta' < \alpha$, and for every $\beta < \alpha$ the set $X_{\beta+1} \setminus X_\beta$ has exactly one element $x_\beta$ (the "other" ultrafilter extending $\mathcal{U}_\beta$), and in this way we can identify every point in this space with an ordinal (just as in [8]). Then for every $\beta < \alpha$ the filter $\mathcal{U}_\beta$ is generated by $\{A_\gamma : \gamma < \beta\}$, and the point $x_\beta$ corresponds to the ultrafilter generated by $\mathcal{U}_\beta \cup \{\omega \setminus A_\beta\}$. We shall identify the point with the corresponding ultrafilter. That is

$$St(\mathbb{B}) = \{\mathcal{U} \cap \mathbb{B}\} \cup \{x_\beta : \beta < \alpha\}.$$

We shall show that under $\diamondsuit(\mathfrak{s})$ there is a family of sequentially compact spaces whose product is not countably compact, providing an example for the Scarborough-Stone problem. It is well known that this is equivalent to constructing for each $\mathcal{U} \in \omega^*$ a sequentially compact space, $X_{\mathcal{U}}$ containing $\omega$ so that $\omega$ has no $\mathcal{U}$-limit.[1] For more information concerning the Scarborough-Stone problem see [24].

The following is the combinatorial core of the construction inspired by the construction in [8].

Proposition 1: *Assume $\diamondsuit(\mathfrak{s})$, and let $\mathcal{U}$ be an ultrafilter on $\omega$. Then there is $\{A_\alpha : \alpha < \omega_1\} \subset \mathcal{U}$ such that:*

*(1) $\{A_\alpha : \alpha < \omega_1\}$ is a sequence of generators of a coherently minimally generated Boolean algebra, and*

*(2) for every infinite sequence $\{\alpha_n : n \in \omega\} \subset \omega_1$ there are an infinite $M \in [\omega]^\omega$, an $\alpha < \omega_1$ and a sequence of finite sets $\{F_n : n \in \omega\}$ such that $F_n \in [\alpha_n]^{<\omega}$ with the property that for every $n \in M$ the set $(\bigcap_{\beta\in F_n} A_\beta) \cap (\omega \setminus A_{\alpha_n}) \cap A_\alpha = \emptyset$.*

*Proof.* Fix a family of bijections $\{e_\alpha : \alpha < \omega_1\}$ such that $e_\alpha : \omega \to \alpha$ for each $\omega \le \alpha < \omega_1$. By a suitable coding[2] we may assume that the domain of $F$ consists of pairs $\langle Y, \{A_\beta : \beta < \alpha\}\rangle$, where $\alpha$ is an infinite countable ordinal, $Y \in [\alpha]^\omega$ and $\{A_\beta : \beta < \alpha\}$ is a sequence of generators of a coherently minimally generated Boolean algebra.

Given $\langle Y, \{A_\beta : \beta < \alpha\}\rangle$ as above, recursively define

$$U_0^\alpha = \omega \setminus A_{e_\alpha(0)} \text{ and } U_n^\alpha = (\omega \setminus A_{e_\alpha(n)}) \setminus \bigcup_{i<n} U_i^\alpha.$$

Finally, put

$$F(\langle Y, \{A_\beta : \beta < \alpha\}\rangle) = \{m \in \omega : (\exists \beta \in Y)(U_m \in x_\beta^+)\}$$

in case that set is infinite and put $F(\langle Y, \{A_\beta : \beta < \alpha\}\rangle) = \omega$ otherwise. It is easy to see that $F$ is a Borel function.

We shall define the algebra recursively: Let $g : \omega_1 \to [\omega]^\omega$ be the $\diamondsuit(\mathfrak{s})$-sequence for $F$. For $n \in \omega$, let $A_n = \omega \setminus n$, and for $\alpha \ge \omega$ consider the family $\{U_n^\alpha : n \in \omega\}$ defined above. As $\mathcal{U}$ is ultrafilter, either $\bigcup_{n\in g(\alpha)} U_n^\alpha$ or its complement is an element of $\mathcal{U}$, let $A_\alpha$ be that element of $\mathcal{U}$.

[1] Recall that a point $x$ in a topological space $X$ is the $\mathcal{U}$-limit of a sequence $\{x_n : n \in \omega\} \subseteq X$ if for every open neighbourhood $V$ of $x$ the set $\{n : x_n \in V\}$ is an element of $\mathcal{U}$.

[2] Here the coding differs slightly from the one outlined in the introduction: Given a countable indecomposable ordinal $\alpha > \omega$ and a function $\sigma \in 2^\alpha$ define $Y_\sigma = \{\beta \in \alpha : \sigma(\omega\cdot\beta) = 1\}$, and for $\beta < \alpha$, let $A_\beta^\sigma = \{n \in \omega : \sigma(\omega \cdot \beta + n + 1) = 1\}$. In this way, $\sigma$ codes a pair $\langle Y_\sigma, \mathcal{A}_\sigma\rangle$, where $\mathcal{A}_\sigma = \{A_\beta^\sigma : \beta < \alpha\}$.

Now, property (1) is immediate from the construction. In order to prove (2), let $Y = \{\alpha_n : n \in \omega\} \subset \omega_1$ be any sequence and let $\beta_0 < \omega_1$ be an upper bound for $Y$. Consider the branch defined by (or rather, which codes) $\{\langle Y, \{A_\beta : \beta < \alpha\}\rangle : \alpha < \omega_1\}$. [3] As $g$ is a $\diamondsuit(\mathfrak{s})$-sequence, there is a $\gamma > \beta_0$ such that $g(\gamma)$ splits $F(\langle Y, \{A_\alpha : \alpha < \gamma\rangle)$. Without loss of generality we can assume that $A_\gamma = \bigcup_{n\in g(\gamma)} U_n^\gamma$, the other case being completely analogous. Observe that $\mathcal{A} = \{U_n^\gamma : n \in \omega\}$ is a maximal antichain in $\mathbb{B}_{\gamma+1}$, such that every ultrafilter in $\mathbb{B}_{\gamma+1}$ distinct from the filter $\mathcal{U} \cap \mathbb{B}_{\gamma+1}$ generated by $\{A_\alpha : \alpha \leq \gamma\}$, contains one of the $U_n^\gamma$'s, equivalently is contained in one of the $(U_n^\gamma)^*$'s.

For each $n \in \omega$ let $F_n \in [\alpha_n]^{<\omega}$ be such that $(\bigcap_{\delta\in F_n} A_\delta) \cap (\omega \setminus A_{\alpha_n}) \subset U_m^\gamma$ where $U_m^\gamma$ is the unique element of the maximal antichain $\mathcal{A}$ that is a member of $x_{\alpha_n}$. Let

$$M = \{m \in \omega : (\forall n \in g(\gamma))(U_n^\gamma \cap (\bigcap_{\delta\in F_m} A_\delta \cap (\omega \setminus A_{\alpha_m})) = \emptyset)\}.$$

As $g(\gamma)$ splits $F(\langle Y, \{A_\alpha : \alpha < \gamma\}\rangle)$, the set $M$ is infinite as required. ∎

We recall the following lemma of Dow and Shelah:

Lemma 2 (Dow-Shelah [8]): *If $\mathbb{B}$ is a coherently minimally generated Boolean algebra, then the Stone space of $\mathbb{B}$ is scattered and sequentially compact.*

Now we are ready to state and prove the main theorem of this section.

Theorem 3: *Suppose that $\diamondsuit(\mathfrak{s})$ holds. For each $\mathcal{U} \in \omega^*$, there is a sequentially compact space $X_\mathcal{U}$ containing $\omega$ such that $\omega$ has no $\mathcal{U}$-limit in the space. Hence, the product $\prod_{\mathcal{U}\in\omega^*} X_\mathcal{U}$ is a product of sequentially compact spaces that is not countably compact.*

*Proof.* Let $\mathcal{U}$ be a free ultrafilter, and let $\{A_\alpha : \alpha < \omega_1\} \subset \mathcal{U}$ be the sequence constructed in Proposition 1. Consider the Stone space of the Boolean algebra $\mathbb{B}$ generated by $\{A_\alpha : \alpha < \omega_1\}$, and let $\mathcal{V} = \mathcal{U} \cap \mathbb{B}$. Note that $\mathcal{V}$ is the $\mathcal{U}$-limit of $\omega$ in $St(\mathbb{B})$. Let $X_\mathcal{U} = St(\mathbb{B}) \setminus \{\mathcal{V}\}$.

The space $X_\mathcal{U}$ is then a locally compact scattered space with a countable dense set of isolated points identified with $\omega$. By definition, the sequence $\omega \subset X_\mathcal{U}$ has no $\mathcal{U}$-limit. To see that $X_\mathcal{U}$ is sequentially compact, by the lemma quoted above,

[3] By this we mean that the branch $f \in 2^{\omega_1}$ is such that $Y = \{\beta \in \omega_1 : f(\omega \cdot \beta) = 1\}$ and $A_\beta = \{n \in \omega : f(\omega \cdot \beta + n + 1) = 1\}$ for every $\beta \in \omega_1$.

it suffices to show that no nontrivial sequence in $St(\mathbb{B})$ converges to $\mathcal{V}$. Recall that $St(\mathbb{B}) = \{\mathcal{V}\} \cup \{x_\beta : \beta < \omega_1\}$, hence $X_\mathcal{U} = \{x_\beta : \beta < \omega_1\}$.

Let $\{x_{\beta_n} : n \in \omega\} \subset X_\mathcal{U}$ be any sequence. To finish the proof it suffices to see that *there is a* $\gamma$ *such that* $(\omega \setminus A_\gamma)^*$ *contains infinitely many elements of* $\{x_{\beta_n} : n \in \omega\}$. Now, consider the sequence $\{\beta_n : n \in \omega\} \subset \omega_1$. By property (2) of Proposition 1 there is an infinite set $M \subseteq \omega$, an ordinal $\alpha < \omega_1$ and a sequence of finite sets $\{F_n : n \in \omega\}$ such that $F_n \in [\beta_n]^{<\omega}$ with the property that for every $n \in M$ the set $(\bigcap_{\beta \in F_n} A_\beta) \cap (\omega \setminus A_{\beta_n}) \cap A_\alpha = \emptyset$. This, however, means that the clopen set $(\omega \setminus A_\alpha)^*$ contains infinitely many of the $x_{\beta_n}$'s. ∎

## 3. Jakovlev spaces

Let us fix some notation first. We shall consider families $\mathcal{A}$ of subsets of $\omega$, written as a disjoint union of subfamilies $L_n(\mathcal{A})$, $n \in \omega$, the family $L_n(\mathcal{A})$ called the *n-th level* of $\mathcal{A}$. By $\mathscr{I}_n$ we shall denote the ideal (possibly improper) generated by $\bigcup_{i<n} L_i(\mathcal{A})$. Such a family $\mathcal{A}$ will be called a *layered family* if it has the following properties:

(a) $L_0(\mathcal{A}) = \{\{k\} : k \in \omega\}$;

(b) $A \cap B \in \mathscr{I}_n$ for every $n \in \omega$ and distinct $A, B \in L_n(\mathcal{A})$, and;

(c) given $m < n \in \omega$, $A \in L_m(\mathcal{A})$ and $B \in L_n(\mathcal{A})$, either $A \setminus B \in \mathscr{I}_m$ or $A \cap B \in \mathscr{I}_m$.

A layered family $\mathcal{A}$ is called a *Jakovlev family* if additionally it has the property:

(d) for each $n \in \omega$ and $Y \subset L_n(\mathcal{A})$ infinite, there are an $m > n$, an $A \in L_m(\mathcal{A})$ and an infinite subset $Y'$ of $Y$ such that for each $y \in Y'$ it happens that $y \setminus A \in \mathscr{I}_n$.

Given an ordinal $\alpha \leq \mathfrak{c}$ we shall say that a layered family $\mathcal{A}$ is *of length* $\alpha$ if

$$\mathcal{A} = \{A^m_\beta : \beta < \alpha, 0 < m < \omega\} \cup \{\{k\} : k \in \omega\},$$

with layers $L_m(\mathcal{A}) = \{A^m_\beta : \beta < \alpha,\}$ for $m > 0$. We shall call a countable layered family $\mathcal{B} = \{B^m_n : m, n \in \omega\}$ of length $\omega$ *canonical* if each of its layers $L_m(\mathcal{B}) = \{B^m_n : n \in \omega\}$ is pairwise disjoint, and if given $m' < m \in \omega$, $B \in L_m(\mathcal{B})$ and $B' \in L_{m'}(\mathcal{B})$ either $B' \subset B$ or $B' \cap B = \emptyset$.

The following objects will be fixed throughout this section:

- a bijection $c_\omega : \omega \to \omega \times \omega$,

- a family $\{d_\alpha : \omega \le \alpha < \omega_1\}$ of bijections $d_\alpha : \omega \to \alpha \times \omega \setminus \{0\}$.

Given a family $\mathcal{A} \subseteq \mathscr{P}(\omega)$ we denote by $\langle \mathcal{A} \rangle$ the Boolean subalgebra of $\mathscr{P}(\omega)$ generated by $\mathcal{A}$. We call a pair of layered families $\mathcal{A}$ and $\mathcal{B}$ *equivalent* if they generate the same Boolean algebra, i.e. $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$, if $\mathscr{I}_m(\mathcal{A}) = \mathscr{I}_m(\mathcal{B}) = \mathscr{I}_m$ for every $m \in \omega$, and there is a bijective function $\varphi : \mathcal{A} \to \mathcal{B}$ which preserves layers such that $A^m_\beta \triangle \varphi(A^m_\beta) \in \mathscr{I}_m$ for every $m \in \omega$ and $\beta < \alpha$. Note that, in particular, $A^m_\beta \triangle \varphi(A^m_\beta) \in \langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$.

Lemma 4: *For every $\alpha \in \omega_1$ there is a Borel map $D_\alpha$ which to each layered family $\mathcal{A}$ of length $\alpha$ assigns a canonical layered family $\mathcal{B}$ (of length $\omega$) equivalent to $\mathcal{A}$.*

*Proof.* For given $\alpha$, consider the function $d_\alpha = (d_1, d_2)$, where $d_1, d_2$ are the coordinate functions of $d_\alpha$, and let for all $n \in \omega$,

$$A_n = A^{d_2(n)}_{d_1(n)}.$$

Then recursively define $\{C_n : n \in \omega\}$ by putting $C_0 = A_0$ and

$$\begin{aligned} C_n = \Big(A_n \cup \bigcup\{C_i : i < n \ \&\ d_2(i) < d_2(n) \ \&\ A_i \subseteq_{\mathscr{I}_{d_2(i)}} A_n\}\Big) \cap \\ \cap \bigcap\{C_i : i < n \ \&\ d_2(i) > d_2(n) \ \&\ A_i \supseteq_{\mathscr{I}_{d_2(n)}} A_n\} \setminus \\ \setminus \bigcup\{C_i : i < n \ \&\ L(A_i) = L(A_n)\}. \end{aligned}$$

Finally, let $B^m_i = C_{n_i}$, where $n_i$ is the $i$-th element of the set $N_m = \{n \in \omega : d_2(n) = m\}$ in its increasing enumeration. It is easy to check that $\mathcal{B}$ is then as required. The bijection $\varphi : \mathcal{A} \to \mathcal{B}$ witnessing that $\mathcal{A}$ and $\mathcal{B}$ are equivalent is then defined by $\varphi(A_n) = C_n$ for every $n \in \omega$. ∎

Proposition 5: *Suppose that $\diamondsuit(\mathfrak{b}^*)$ holds. There is a Jakovlev family $\mathcal{A}$.*

*Proof.* We shall define a convenient Borel function $F : 2^{<\omega_1} \to \omega^\omega$. By a suitable coding we may assume that the domain of $F$ are pairs of the form $\langle Y, \mathcal{A} \rangle$, where $\mathcal{A} = \{A^n_\beta : \beta < \alpha, n \in \omega\}$ is a layered family of subsets of $\omega$ of length some countable ordinal $\alpha$, and $Y$ is an infinite subset of one of the levels of $\mathcal{A}$. Using Lemma 4, we replace the original $\mathcal{A}$ by the corresponding canonical layered family $\mathcal{B} = D_\alpha(\mathcal{A})$ and identify $Y$ with its $D_\alpha$-image as a subset of say the $r$-th level of $\mathcal{B}$.

Enumerate $\mathcal{B}$ as $\{C_k : k \in \omega\}$ according to the function $c_\omega$ fixed at the beginning, and let $F(\langle Y, \mathcal{A}\rangle)(0) = 0$, $F(\langle Y, \mathcal{A}\rangle)(n) = k_n$ and

$$F(\langle Y, \mathcal{A}\rangle)(n+1) = \min\{k : B^r_k \in Y \ \&\ B^r_k \cap \bigcup_{m<k_n} C_m = \emptyset\}$$

if the set $\{k : B^r_k \in Y \ \&\ B^r_k \cap \bigcup_{m<k_n} C_m = \emptyset\}$ is not empty, otherwise let

$$F(\langle Y, \mathcal{A}\rangle)(n+1) = k_n + 1.$$

Let $\{g_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit(\mathfrak{b}^*)$-sequence for $F$. We shall construct recursively an increasing family of countable layered families $\{\mathcal{A}_\alpha : \omega \leq \alpha < \omega_1\}$, $\mathcal{A}_\alpha$ of length $\alpha$, so that $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \omega \leq \alpha < \omega_1\}$ is a Jakovlev family.

To start, let $\mathcal{A}_\omega$ be an arbitrary layered family of length $\omega$. Having constructed $\mathcal{A}_\alpha$, for some $\omega \leq \alpha < \omega_1$, apply Lemma 4, and consider the canonical layered family

$$D_\alpha(\mathcal{A}_\alpha) = \mathcal{B} = \{B^m_n : m, n \in \omega\}.$$

Use the function $c_\omega$ to enumerate $\mathcal{B}$ as $\{C_k : k \in \omega\}$, and define for every $r \in \omega$

$$D^r_0 = \bigcup\{B^r_i : i < g_\alpha(0)\}$$

and for $n > 0$

$$D^r_n = \bigcup\{B^r_i : i < g_\alpha(n+1) \ \&\ B^r_i \cap \bigcup_{m<g_\alpha(n)} C_m = \emptyset\}.$$

Finally let $A^1_\alpha = \bigcup_{n\in\omega} D^0_n$, and $A^{r+1}_\alpha = A^r_\alpha \cup \bigcup_{n\in\omega} D^r_n$, and

$$\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{A^r_\alpha : r \in \omega \setminus \{0\}\}.$$

By possibly replacing the function $g_\alpha$ by a faster growing function one can make sure that each set $A^r_\alpha$, $r > 0$, is infinite.

To check that $\mathcal{A}_{\alpha+1}$ is a layered family of length $\alpha+1$ it suffices to note that the family $\mathcal{B} \cup \{A^r_\alpha : r \in \omega \setminus \{0\}\}$ is layered, i.e. to note that $B^m_k \cap A^m_\alpha \in \mathscr{I}_m$, and that if $m < n$ then either $B^m_k \cap A^n_\alpha \in \mathscr{I}_m$ or $B^m_k \setminus A^n_\alpha \in \mathscr{I}_m$. One can easily check this by induction on $m$ using the inductive hypothesis, and the fact that every $B^m_k$ appears as one of the $C_\ell$'s, hence all but finitely many of the $B^m_i$ (resp. $B^n_i$) whose union is considered in the definition of $A^m_\alpha$ (resp. $A^n_\alpha$) are disjoint from $B^m_k$.

Now, for limit $\alpha$ we simply let

$$\mathcal{A}_\alpha = \bigcup_{\omega\leq\beta<\alpha} \mathcal{A}_\beta$$

which, by definition, is a layered family of length $\alpha$. This concludes the construction, and all that remains to be proved is that

$$\mathcal{A} = \bigcup_{\omega \leq \beta < \omega_1} \mathcal{A}_\beta$$

is a Jakovlev family.

Let $Y$ be a countable subset of some level of $\mathcal{A}$, say level $r$. Choose a large enough $\beta$ so that $Y \subseteq L_r(\mathcal{A}_\beta)$. Consider the branch that codes $\{\langle Y, \mathcal{A}_\beta \rangle : \beta < \omega_1\}$. Since $\{g_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit(\mathfrak{b}^*)$-sequence, there are stationarily many $\alpha > \beta$ such that for infinitely many $n \in \omega$ there is $m \in \omega$ such that

$$\Big[g_\alpha(n), g_\alpha(n+1)\Big) \supseteq \Big[F(\langle Y, \mathcal{A}_\alpha \rangle)(m), F(\langle Y, \mathcal{A}_\alpha \rangle)(m+1)\Big).$$

Fix such $\alpha$. We claim that there is an $A \in \mathcal{A}_{\alpha+1}$, for which there are infinitely many $y \in Y$ such that $y \subseteq_{\mathscr{I}_r} A$.

We may assume that there is no such $A \in \mathcal{A}_\alpha$ (otherwise we are done), i.e. every element of $\mathcal{B} = D_\alpha(\mathcal{A}_\alpha)$ is (almost) disjoint from all but finitely many elements of $Y$. Let $A = A_\alpha^{r+1}$ and recall that

$$A_\alpha^{r+1} = A_\alpha^r \cup \bigcup_{n \in \omega} \bigcup \{B_i^r : i < g_\alpha(n+1) \ \&\ B_i^r \cap \bigcup_{m < g_\alpha(n)} C_m = \emptyset\}.$$

Now, as there are infinitely many $n$ and $m$ such that

$$\Big[g_\alpha(n), g_\alpha(n+1)\Big) \supseteq \Big[F(\langle Y, \mathcal{A}_\alpha \rangle)(m), F(\langle Y, \mathcal{A}_\alpha \rangle)(m+1)\Big).$$

and (by the above assumption) for every $m \in \omega$, if $j_m = F(\langle Y, \mathcal{A}_\alpha \rangle)(m)$,

$$F(\langle Y, \mathcal{A}_\alpha \rangle)(m+1) = \min\{k : B_k^r \in Y \ \&\ B_k^r \cap \bigcup_{i < j_m} C_i = \emptyset\},$$

i.e., there are infinitely many $m$ and $k_m$ such that $B_{k_m}^r \in Y$ and $B_{k_m}^r \cap \bigcup_{i<j_m} C_i = \emptyset$, hence $B_{k_m}^r \subseteq A$ for each of those $m$. ∎

THEOREM 6: *The principle $\diamondsuit(\mathfrak{b}^*)$ implies the existence of a separable Jakovlev space of size $\omega_1$.*

*Proof.* The space is the Stone space of the Boolean subalgebra $\mathbb{B}_\mathcal{A}$ of $\mathscr{P}(\omega)$ generated by the family $\mathcal{A}$ with the top point (ultrafilter) $\{\omega\}$ removed. The set $\{\{n\} : n \in \omega\}$ is then a dense set of isolated points of the space. Note that for any ultrafilter $\mathcal{U}$ over $\mathbb{B}_\mathcal{A}$ distinct from the top, there is a unique element $A$ of $\mathcal{A}$ (say on level $n$) such that $\mathcal{U} = \{B \in \mathbb{B}_\mathcal{A} : A \setminus B \in \mathscr{I}_n\}$, hence $St(\mathbb{B}_\mathcal{A})$ can be naturally identified with $\mathcal{A}$ equipped with the topology defined as follows:

Each element $A$ of $\mathcal{A}$ ($A \in L_n(\mathcal{A})$) has a countable compact neighbourhood

$$A^* = \{A\} \cup \{B \in \bigcup_{m<n} L_m(\mathcal{A}) : B \setminus A \in \mathscr{I}_m \ \&\ B \in L_m(\mathcal{A})\}.$$

It follows directly from the properties of $\mathcal{A}$ that the space is a separable Jakovlev space. ∎

Abraham, Gorelic, Juhász in [1] proved that there is a Jakovlev space assuming $\mathfrak{b} = \mathfrak{c}$. There is a Jakovlev space of cardinality $\omega_1$ in any model obtained by adding $\aleph_1$-many Cohen reals or $\aleph_1$-many dominating reals. As $\diamondsuit(\mathfrak{b})$ holds in the last two models, Theorem 6 generalizes and provides uniform proof for these results. It also follows that there are Jakovlev spaces in any *canonical model* of $\mathsf{ZFC}$, i.e. a model produced as a countable support iteration of a single sufficiently definable and sufficiently homogeneous proper partial order $\mathbb{P}$, as in any such model either $\mathfrak{b} = \mathfrak{c}$ or $\diamondsuit(\mathfrak{b})$ holds by [9, Theorem 6.6].

## 4. Sequential order

In this section our goal is to construct, assuming $\diamondsuit(\mathfrak{b}^*)$, a compact scattered sequential space of sequential order $\omega_1$. Recall the following simple facts:

LEMMA 7: *Suppose that there is a family $\{X_\alpha : \alpha < \omega_1\}$ of compact scattered sequential spaces such that each $X_\alpha$ has $so(X_\alpha) < \omega_1$. If $\{so(X_\alpha) : \alpha < \omega_1\}$ is cofinal in $\omega_1$, then the one point compactification of $\bigoplus_{\alpha<\omega_1} X_\alpha$ is a compact scattered space of sequential order $\omega_1$.*

Thus, to define the space of sequential order $\omega_1$ it is enough to define, for each ordinal $\eta < \omega_1$, a compact sequential scattered space of sequential order $\eta + 1$.

The following is probably well known but we did not find a reference for it.

LEMMA 8 (Folklore): *Let $X$ be a compact scattered space of countable height (Cantor-Bendixson rank, scattered index). Then $X$ is sequential.*

*Proof.* For an ordinal $\gamma$, denote by $X^{(\gamma)}$ the set of all points of $X$ of (scattered) height $\gamma$. Given $G \subseteq X$ not closed, pick $x \in \overline{G} \setminus G$ of minimum height $\gamma$. Consider $U$ a compact neighbourhood of $x$ "looking down", i.e. $U \subseteq \{x\} \cup \bigcup_{\beta<\gamma} X^{(\beta)}$. Observe that $\overline{G \cap U} \setminus (G \cap U) = \{x\}$.

Since being a scattered space is hereditary, $K = \{x\} \cup (G \cap U)$ is also a compact scattered space (with likely very different scattered levels). Again, there is compact neighbourhood $W$ of $x$ in $K$ "looking down". Let $\xi$ be the height of $x$ in $K$. Then either $\xi$ is a successor ordinal or a limit ordinal less than $\omega_1$. If it is a successor ordinal, the last infinite level of $W$ is a closed discrete subset of $G \cap U$, any sequence of points in it converges to the point $x$. If $\xi < \omega_1$ is limit, choose an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ of ordinals less than $\xi$ converging to $\xi$ and choose a point from each level $x_n \in W \cap K^{(\alpha_n)}$. Then the sequence $\langle x_n : n \in \omega \rangle$ necessarily converges to the point $x$. ∎

In many ways, the construction follows the general pattern of the construction given in the previous section, with several important differences: First of all our space can not be locally countable, so we have to be "expanding" the locally compact neighbourhoods of every point cofinally along the construction. Also, the sequences we need to take care of are not only those contained in a fixed level. To begin we define notions analogous to the ones introduced at the beginning of last section:

*Definition 9:* Let $\eta$ be an infinite countable ordinal. A family of subsets of $\omega$ will be called *$\eta$-layered* if $\mathcal{A} = \bigcup_{\xi \leq \eta} L_\xi(\mathcal{A})$, where

(1) $L_0(\mathcal{A}) = \{\{n\} : n \in \omega\}$, $L_\xi(\mathcal{A})$ is a countable family of proper infinite subsets of $\omega$ for $0 < \xi < \eta$ and $L_\eta(\mathcal{A}) = \{\omega\}$,
(2) for each $\xi \leq \eta$ and $A, B \in L_\xi(\mathcal{A})$ , $A \cap B \in \mathscr{I}_\xi$, where $\mathscr{I}_\xi$ denotes the ideal generated by the family $\bigcup_{\gamma < \xi} L_\gamma(\mathcal{A})$,
(3) for every $\xi < \zeta < \eta$, $A \in L_\xi(\mathcal{A})$ and $B \in L_\zeta(\mathcal{A})$, either $A \setminus B \in \mathscr{I}_\xi$ or $A \cap B \in \mathscr{I}_\xi$.

Given $A \in \mathcal{A}$, we say that $A$ is on the *level* $\xi$ of $\mathcal{A}$ and write $L(A) = \xi$ if $A \in L_\xi(\mathcal{A})$. We say that an $\eta$-layered family $\mathcal{A} \subseteq \mathscr{P}(\omega)$ is a *canonical $\eta$-layered* family if given $A, B \in \mathcal{A}$, either $A \subseteq B$, $A \supseteq B$ or $A \cap B = \emptyset$. We also write $A \subseteq_{\mathscr{I}} B$ to mean that $A \setminus B$ belongs to the ideal $\mathscr{I}$ and we say that $A$ is contained in $B$ mod $\mathscr{I}$. [4] We consider the notion of equivalence of $\eta$-layered families defined in a way analogous to the previous section. Note that if two

[4] Let us remark that unlike in the previous section, we do not require the $\eta$-layered families to be enumerated in type $\eta \times \omega$ but rather to be enumerated arbitrarily in a countable transfinite sequence by the "coding mechanism".

$\eta$-layered families are equivalent then, in particular, they generate the same Boolean subalgebra of $\mathscr{P}(\omega)$, and the same ideals $\mathscr{I}_\xi$, for $\xi \le \eta$.

For the rest of the section, given an infinite ordinal $\alpha < \omega_1$ we fix bijections $a_\alpha : \alpha \to \eta \times \omega$ and $c_\alpha : \omega \to \alpha$. The function $a_\alpha$ will be used to canonically enumerate a given countable $\eta$-layered family $\mathcal{A}$ of length $\alpha$ in such a way that $L_\gamma(\mathcal{A}) = \{A^\gamma_m : m \in \omega\}$ for every $\gamma \le \eta$. We shall also fix for every $0 < \xi < \eta$ a bijection $e_\xi : \omega \to \xi \times \omega$.

We have a result similar to Lemma 4.

Lemma 10: *For each $\alpha \in \omega_1$ there is a Borel map $D_\alpha$ which to each countable $\eta$-layered family $\mathcal{A}$ indexed by $\alpha$ assigns a canonical $\eta$-layered family $\mathcal{B}$ equivalent to $\mathcal{A}$.*

*Proof.* For given $\alpha$, consider the function $a_\alpha = (a_1, a_2)$, where $a_1, a_2$ are the coordinate functions of $a_\alpha$, and let, for every $n \in \omega$

$$A_n = A^{a_1(c_\alpha(n))}_{a_2(c_\alpha(n))}.$$

Then recursively define $\{C_n : n \in \omega\}$ by putting $C_0 = A_0$ and

$$\begin{aligned} C_n = \Big(A_n \cup \bigcup\{C_i : i < n \ \&\ L(A_i) < L(A_n) \ \&\ A_i \subseteq_{\mathscr{I}_{L(A_i)}} A_n\}\Big) \cap \\ \cap \bigcap\{C_i : i < n \ \&\ L(A_i) > L(A_n) \ \&\ A_i \supseteq_{\mathscr{I}_{L(A_n)}} A_n\} \setminus \\ \setminus \bigcup\{C_i : i < n \ \&\ L(A_i) = L(A_n)\}. \end{aligned}$$

Observe that $A_n \triangle C_n \in \mathscr{I}_{L(A_n)}$. Finally, let $B^\gamma_i = C_{n_i}$, where $n_i$ is the $i$-the element of the set

$$N_\gamma = \{n \in \omega : a_1(c_\alpha(n)) = \gamma\}$$

in its increasing enumeration. It is easy to check that $\mathcal{B} = \{B^\gamma_i : \gamma < \eta \ \&\ i \in \omega\}$ is a canonical $\eta$-layered family. The bijective function $\varphi : \mathcal{A} \to \mathcal{B}$ witnessing that $\mathcal{A}$ and $\mathcal{B}$ are equivalent is defined by $\varphi(A_n) = C_n$. ∎

Given an $\eta$-layered family $\mathcal{A}$, call an infinite $Y \subseteq \mathcal{A}$ *slim* if there is a $\xi \le \eta$ such that $\langle L(y) : y \in Y\rangle$ increasingly converges to $\xi$, if $\xi$ is limit, or $Y \subseteq L_\zeta(\mathcal{A})$, if $\xi = \zeta + 1$, and there is a $C \in \mathcal{A}$ such that

- $(\forall^\infty y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} C)$,
- the set $\{y \in Y : y \subseteq_{\mathscr{I}_{L(y)}} D\}$ is finite for every $D \in \mathcal{A}$ such that $L(D) < L(C)$.

Note that such a $C \in \mathcal{A}$ is uniquely determined by $Y$.

Proposition 11: *The principle $\diamondsuit(\mathfrak{b}^*)$ implies that for each ordinal $\eta < \omega_1$ there is an $\eta$-layered family $\mathcal{A}$ such that for every slim $Y \in [\mathcal{A}]^\omega$ there is a $C \in \mathcal{A}$ such that*

*(1)* $(\forall^\infty y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} C)$,
*(2)* $|\{y \in Y : y \subseteq_{\mathscr{I}_{L(y)}} D\}| < \omega$ *for every* $D \in \mathcal{A}$ *with* $L(D) < L(C)$, *and*
*(3)* $L(C) = \sup\{L(y) + 1 : y \in Y\}$.

*Proof.* Fix $\eta < \omega_1$. We shall define a Borel function $F : 2^{<\omega_1} \to \omega^\omega$ as follows. By a suitable coding we may assume that the domain of $F$ are pairs of the form $\langle \overline{Y}, \mathcal{B}\rangle$, where $\mathcal{B}$ is an $\eta$-layered family of subsets of $\omega$ ordered in a sequence of length $\alpha$ for some $\alpha < \omega_1$, and $\overline{Y}$ is a slim subset of $\mathcal{B}$.

Using Lemma 10, we consider the canonical $\eta$-layered family $\mathcal{A} = D_\alpha(\mathcal{B})$ equivalent to $\mathcal{B}$, as witnessed by the (unique) bijection $\varphi : \mathcal{B} \to \mathcal{A}$ preserving levels, and let $Y = \varphi[\overline{Y}]$. Note that $Y$ is then slim with respect to $\mathcal{A}$, and if $C$ is the (unique) element of $\mathcal{A}$ witnessing that $Y$ is slim then

(1) $(\forall^\infty y \in Y)(y \subseteq C)$ and
(2) $|\{y \in Y : y \subseteq D\}| < \omega$ for every $D \in \mathcal{A}$ such that $L(D) < L(C)$.

If also the clause (3) is satisfied we do not have to do anything, i.e we can define $F(\langle \overline{Y}, \mathcal{B}\rangle)$ to be arbitrary, e.g. constant 0. If (3) is not satisfied, i.e. if $\gamma = L(C) > \sup\{L(y) + 1 : y \in Y\}$, use the function $e_\gamma$ to provide a canonical enumeration $\{C_n : n \in \omega\}$ of all proper subsets of $C$ which are members of $\mathcal{A}$ defined recursively as follows: If $e_\gamma(0) = (\zeta, k)$ then let $C_0 = A^\zeta_\ell$, where

$$\ell = \min\{j \in \omega : A^\zeta_j \subseteq C\}$$

and similarly, for $n > 0$, if $e_\gamma(n) = (\zeta, k)$ then let $C_n = A^\zeta_\ell$, where

$$\ell = \min\{j \in \omega : \ A^\zeta_j \subseteq C \text{ and } A^\zeta_j \notin \{C_i : i < n\}.$$

Then define $F(\langle \overline{Y}, \mathcal{B}\rangle) = \langle k_n : n \in \omega\rangle$ by putting

$$k_0 = \min\{k \in \omega : (\exists y \in Y)(y \subseteq C_k)\},$$

and

$$(\maltese) \qquad k_{n+1} = \min\{k > k_n : (\exists y \in Y)(y \subseteq C_k \ \&\ y \nsubseteq \bigcup_{i \le k_n} C_i)\}.$$

Note that as $\mathcal{A}$ is canonical, the set $y \in Y$ whose existence is required in the definition of $k_{n+1}$ has the property that $y \cap C_i = \emptyset$, or $C_i \subseteq y$ and hence $C_i \in \mathscr{I}_{L(y)}$, for every $i \le k_n$.

Let $\{g_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit(\mathfrak{b}^*)$-sequence for $F$ satisfying ($\blacklozenge$). Assume that each $g_\alpha$ is an increasing function. We shall recursively construct an increasing family of countable $\eta$-layered families $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ so that $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ will have the properties stated in the proposition.

To begin with, let $\mathcal{A}_0$ be an arbitrary canonical countable $\eta$-layered family.

Assume now that we have already constructed $\mathcal{A}_\alpha$, for some $\alpha < \omega_1$. Applying Lemma 10, we may assume that $\mathcal{A}_\alpha$ is canonical. We shall construct a family $\{D(A,\xi) : A \in \mathcal{A}_\alpha \;\&\; \xi < L(A)\}$ of subsets of $\omega$ whose members will be candidates for elements of the next layered family $\mathcal{A}_{\alpha+1}$. To define $D(A,\xi)$, use again the function $e_\xi$ to enumerate

$$\{A' \in \mathcal{A}_\alpha : A' \subset A\} = \{C_n : n \in \omega\}$$

then let

$$D_n(A,\xi) = \bigcup\{C_i : i < g_\alpha(n+1) \;\&\; L(C_i) < \xi\} \setminus \bigcup\{C_j : j \leq g_\alpha(n)\}$$

and

$$(❄) \qquad D(A,\xi) = \bigcup\{D_n(A,\xi) : n \in \omega\}.$$

Consider the family $\mathcal{D}_{\mathcal{A}_\alpha} = \{D(A,\xi) : A \in \mathcal{A}_\alpha \;\&\; \xi < L(A)\}$ and enumerate it as $\{E_n : n \in \omega\}$ (note that here we do not require the enumeration to be canonical). In order to suitably modify the sets $E_n$'s to add them to the next $\eta$-layered family, define recursively finite families $\mathcal{D}_n \subseteq \mathscr{P}(E_n)$ of pairwise disjoint $\mathscr{I}_{L(E_i)}$-positive subsets of $E_n$ as follows: First, let $\mathcal{D}_0 = \{E_0\}$. Now, if $n > 0$ assume that $\mathcal{D}_i$, for $i < n$ have already been defined. If $E_n = D(A,\xi)$, for some $A \in \mathcal{A}_\alpha$ and some $\xi < L(A)$, let $\mathcal{D}_n$ be the set of those atoms of the finite Boolean algebra generated by

$$(✪) \qquad \{E_n\} \cup \bigcup_{i<n} \mathcal{D}_i$$

which are subsets of $E_n$ and are $\mathscr{I}_\xi$-positive.

Extend the family $\mathcal{A}_\alpha$ to a family $\mathcal{A}_{\alpha+1}$ by putting

$$L_\xi(\mathcal{A}_{\alpha+1}) = L_\xi(\mathcal{A}_\alpha) \cup \bigcup\{\mathcal{D}_i : L(E_i) = \xi\}.$$

Observe that instead of adding $D(A,\xi)$ to $\mathcal{A}_{\alpha+1}$ we have added a finite $\mathscr{I}_\xi$-almost partition of it. Note also that if $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{A}_{\alpha+1} \setminus \mathcal{A}_\alpha$ then either $A \subseteq_{\mathscr{I}_{L(A)}} B$ or $A \cap B \in \mathscr{I}_{L(A)}$.

To verify that $\mathcal{A}_{\alpha+1}$ is an $\eta$-layered family, we have to check clauses (2) and (3) of Definition 9.

For (2), let $\xi < \eta$ and $A, B \in L_\xi(\mathcal{A}_{\alpha+1})$. There are several cases.

*(i)* If $A, B \in \mathcal{A}_\alpha$, it is clear that $A \cap B \in \mathscr{I}_\xi$.

*(ii)* If $A \in \mathcal{A}_\alpha$, $B \notin \mathcal{A}_\alpha$, then there is $H \in \mathcal{A}_\alpha$ such that $B \subseteq D(H, \xi)$. Since we are assuming that $\mathcal{A}_\alpha$ is canonical, we have $A \cap H = \emptyset$ or $A \subseteq H$. It follows that $A \cap D(H, \xi) \in \mathscr{I}_\xi$ either because is empty, or because by (❋) $D(H, \xi)$ is $\mathscr{I}_\xi$-almost disjoint from every element of $\mathcal{A}_\alpha$ below $H$. Hence $A \cap B \in \mathscr{I}_\xi$.

*(iii)* Now, assume that $A, B \notin \mathcal{A}_\alpha$ such that there are distinct $G, H \in \mathcal{A}_\alpha$ with $A \subseteq D(G, \xi)$ and $B \subseteq D(H, \xi)$. Now, as $\mathcal{A}_\alpha$ is canonical, either $G \cap H = \emptyset$ or one is contained in the other, say $G \subseteq H$. Then $A \cap B \in \mathscr{I}_\xi$ as either it is empty, or as $D(H, \xi)$ is $\mathscr{I}_\xi$-almost disjoint from every element of $\mathcal{A}_\alpha$ below $H$, $D(H, \xi)$ is $\mathscr{I}_\xi$-almost disjoint from $G \supseteq D(G, \xi) \supseteq A$.

*(iv)* The remaining case deals with $A, B \notin \mathcal{A}_\alpha$ which are contained in the same $D(H, \xi)$. They are then $\mathscr{I}_\xi$-almost disjoint by definition.

For clause (3) of Definition 9, let $\xi < \zeta \leq \eta$ and $A \in L_\xi(\mathcal{A}_{\alpha+1})$, $B \in L_\zeta(\mathcal{A}_{\alpha+1})$. Once again, we have to consider a few cases.

*(i)* If $A, B \in \mathcal{A}_\alpha$, it is clear that $A \cap B \in \mathscr{I}_\xi$.

*(ii)* If $A \in \mathcal{A}_\alpha$, $B \notin \mathcal{A}_\alpha$, then there is $H \in \mathcal{A}_\alpha$ such that $B \subseteq D(H, \xi)$ and $L(H) \geq \zeta + 1$. Then $A \cap H = \emptyset$, or $A \subseteq H$ and then $A \cap D(H, \zeta) \in \mathscr{I}_\xi$ by (❋).

*(iii)* If $A \notin \mathcal{A}_\alpha$ and $B \in \mathcal{A}_\alpha$. Then $A \subseteq D(G, \xi)$ for some $G \in \mathcal{A}_\alpha$ and $G \cap B = \emptyset$, $G \subseteq B$ or $B \subsetneq G$. Hence $A \cap B = \emptyset$ or $A \subseteq B$ in the first two cases. In the remaining case $B \subsetneq G$, the definition of $D(G, \xi)$ in (❋) implies that $A \cap B \in \mathscr{I}_\xi$. To see this, observe that to define $D(G, \xi)$ we are using the enumeration $\{C_n : n \in \omega\}$ of all elements of $\mathcal{A}_\alpha$ which are subsets of $G$, in particular $B$ appears as some $C_{n_0}$. So,

$$D(G, \xi) \cap B \subseteq \bigcup \{B \cap C_i : i \leq n_0 \ \& \ L(C_i) < \xi\} \in \mathscr{I}_\xi$$

by (❋). Thus $D(G, \xi) \cap B \in \mathscr{I}_\xi$.

*(iv)* If $A, B \notin \mathcal{A}_\alpha$, there are $G, H \in \mathscr{A}_\alpha$ such that $A \subseteq D(G, \xi)$ and $B \subseteq D(H, \zeta)$. Then there are four subcases:

(a) $G \cap H = \emptyset$, in which case $A \cap B = \emptyset$.

(b) $H \subsetneq G$ in which case $\zeta \leq L(G)$, then $A \cap H \in \mathscr{I}_\xi$. This is analogous to case *(iii)* using $H$ instead of $B$.

(c) $G \subsetneq H$, there are two options here: $\xi < L(G) \leq \zeta < L(H)$ or $\xi < \zeta < L(G)$. Both of these are analogous, let us assume that $\xi < L(G) \leq \zeta < L(H)$.

Like in the case *(iii)*, to define $D(H,\zeta)$ we are using the enumeration $\{C_n^H : n \in \omega\}$ of the family of all elements of $\mathcal{A}_\alpha$ which are subsets of $H$, thus for some $n_0 \in \omega$, we have that,

$$D(H,\zeta) \cap G \subseteq \bigcup_{i \leq n_0} G \cap C_i^H \subseteq \bigcup_{i \leq n_0} C_i^H.$$

To define $D(G,\xi)$ we are using the enumeration $\{C_n^G : n \in \omega\}$ of all elements of $\mathcal{A}_\alpha$ which are subsets of $G$. There is $k \in \omega$ such that $C_i^H \cap C_j^G = \emptyset$ for all $i \leq n_0$ and all $j \geq k$. Therefore

$$D(G,\xi) \cap D(H,\zeta) \subseteq \bigcup_{j \leq k} \bigcup \{C_j^G \cap C_i^H : i \leq n_0 \ \& \ L(C_j^G) < \xi\} \in \mathscr{I}_\xi$$

by (❄).

(d) $G = H$, then the definition in (❄) implies that $D(G,\xi) \subseteq D(H,\zeta)$. Moreover $A$ and $B$ are subsets of $D(G,\xi)$ and $D(H,\zeta)$, respectively, and they are atoms of finite Boolean algebras according to their definition in (❂). So, depending to which set $D(G,\xi)$ or $D(H,\zeta)$ comes first in the enumeration $\{E_n : n \in \omega\}$ we have that $A \subseteq B$ or $A \cap B = \emptyset$.

For $\alpha \leq \omega_1$, a limit ordinal, if $\mathcal{A}_\xi$ is defined for all $\xi < \alpha$, define $\mathcal{A}_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. It is clear that an increasing union of $\eta$-layered families is an $\eta$-layered family. Finally we let

$$\mathcal{A} = \bigcup_{\beta<\omega_1} \mathcal{A}_\beta.$$

We are left with showing that the $\eta$-layered family $\mathcal{A}$ has the required properties. To that end let $\overline{Y} \subseteq \mathcal{A}$ be slim with respect to $\mathcal{A}$. Note that $\overline{Y}$ is then slim with respect to $\mathcal{A}_\beta$ for every $\beta < \omega_1$ such that $\overline{Y} \subseteq \mathcal{A}_\beta$.

As $\{g_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit(\mathfrak{b}^*)$-sequence for $F$ satisfying (♦), there is an $\alpha < \omega_1$ such that for infinitely many $n \in \omega$ there is $m \in \omega$ such that

$$\Big[g_\alpha(n), g_\alpha(n+1)\Big) \supseteq \Big[F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)(m), F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)(m+1)\Big)$$

and $\overline{Y} \subseteq \mathcal{A}_\alpha$ is slim with respect to $\mathcal{A}_\alpha$. We apply Lemma 10 to transform $\mathcal{A}_\alpha$ into an equivalent canonical $\eta$-layered family $\mathcal{B}$, and denote by $\varphi$ the bijection which witnesses that $\mathcal{A}_\alpha$ and $\mathcal{B}$ are equivalent. Finally we consider $Y = \varphi[\overline{Y}]$ a slim subset $Y$ of $\mathcal{B}$. Recall that the set $C \in \mathcal{B}$ witnessing that $Y$ is slim has the property that

(1) $(\forall^\infty y \in Y)(y \subseteq C)$,
(2) $|\{y \in Y : y \subseteq D\}| < \omega$ for every $D \in \mathcal{B}$ with $L(D) < L(C)$.

Now, if (3) $L(C) = \sup\{L(y)+1 : y \in Y\}$, the same will hold in $\mathcal{A}$, i.e.

(1) $(\forall^\infty y \in \overline{Y})(y \subseteq_{\mathscr{I}_{L(y)}} \overline{C} = \varphi^{-1}(C))$,
(2) $\{y \in \overline{Y} : y \subseteq_{\mathscr{I}_{L(y)}} D\}$ is finite for every $D \in \mathcal{A}$ with $L(D) < L(C) = L(\overline{C})$, and
(3) $L(\overline{C}) = \sup\{L(y)+1 : y \in Y\}$,

and we are done.

Now, if $\xi = \sup\{L(y)+1 : y \in Y\} < L(C)$, then by the definitions of the function $F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)$ and $D(C,\xi)$

$$(\exists^\infty y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} D(C,\xi)).$$

To see this note that if $F(\overline{Y}, \mathcal{A}_\alpha)(m) = k_m$ then

$$(\exists y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} C_{k_m} \setminus \bigcup_{i<k_m} C_i)$$

where $\{C_i : i \in \omega\}$ is the same canonical enumeration of all elements of $\mathcal{B}$ which are proper subsets of $C$ as the one used in the definitions of $F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)$ and $D(C,\xi)$. For infinitely many $n \in \omega$ there is $m \in \omega$ such that

$$\Big[g_\alpha(n), g_\alpha(n+1)\Big) \supseteq \Big[F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)(m), F(\langle \overline{Y}, \mathcal{A}_\alpha\rangle)(m+1)\Big)$$

and as

$$D(C,\xi) = \bigcup_{n\in\omega} \Big(\bigcup\{C_i : i < g_\alpha(n+1) \ \&\ L(C_i) < \xi\} \setminus \bigcup\{C_j : j \le g_\alpha(n)\}\Big),$$

it follows that there are infinitely many $y \in Y$ such that $y \subseteq_{\mathscr{I}_{L(y)}} D(C,\xi)$, and we are done.

Now, recall that $D(C,\xi)$ is almost partitioned into a finite family of $\mathscr{I}_\xi$-positive subsets $D \in \mathcal{A}_{\alpha+1}$ all of them with the property that for all $B \in \mathcal{A}_\alpha$ then either $B \subseteq_{\mathscr{I}_{L(B)}} D$ or $D \cap B \in \mathscr{I}_{L(B)}$. In particular, $y \subseteq_{\mathscr{I}_{L(y)}} D$ or $D \cap y \in \mathscr{I}_{L(y)}$ for every $y \in Y$. Therefore for one of these $D$'s there are infinitely many $y \in Y$ such that $y \subseteq_{\mathscr{I}_{L(y)}} D$. Finally, since $y \triangle \varphi(y) \in \mathscr{I}_\xi$ for every $y \in \overline{Y}$, there are infinitely many $y \in \overline{Y}$ such that $y \subseteq_{\mathscr{I}_{L(y)}} D$. ∎

THEOREM 12: *The principle $\diamondsuit(\mathfrak{b}^*)$ implies that for each ordinal $\eta < \omega_1$, there is a separable compact sequential scattered space of sequential order $\eta + 1$.*

*Proof.* Let $\mathcal{A}$ be a $\eta$-layered family from the previous proposition. Let $\mathbb{B}_{\mathcal{A}} \subseteq \mathscr{P}(\omega)$ be the Boolean algebra generated by $\mathcal{A}$. Observe that every $A \in \mathcal{A}$ produces an ultrafilter $x_A$ on $\mathbb{B}_{\mathcal{A}}$ defined by

$$x_A = \{B \in \mathbb{B}_{\mathcal{A}} : A \setminus B \in \mathscr{I}_{L(A)}\}.$$

Let $X = St(\mathbb{B}_{\mathcal{A}}) = \{x_A : A \in \mathcal{A}\}$ be the Stone space of $\mathbb{B}_{\mathcal{A}}$. Note that a slim subset $Y$ of the $\eta$-layered family $\mathcal{A}$, corresponds to a convergent sequence in $X$ and that, if $C \in \mathcal{A}$ is the witness to $Y$ being slim in $\mathcal{A}$, then the sequence $\{x_y : y \in Y\}$ converges to $x_C$. Also observe that the scattered levels, $X^{(\gamma)}$, of the space $X$ correspond to the levels of the $\eta$-layered family $\mathcal{A}$. Thus $X$ is a compact scattered space of height $\eta + 1$ and, by Lemma 8, $X$ is also sequential.

To prove that $X$ is of sequential order $\eta + 1$, it is enough to consider the level 0 of our space and show that we need to iterate the operator *seqcl* $\eta$ times to get the whole space $X$. Moreover, by the properties of the $\eta$-layered family $\mathcal{A}$ in the previous proposition, we have that if $\{y_n : n \in \omega\}$ is contained in $\bigcup_{\xi<\gamma} L_\xi(\mathcal{A})$, then every slim subset of $\{y_n : n \in \omega\}$ is witnessed by some element of $\bigcup_{\xi<\gamma+1} L_\xi(\mathcal{A})$. In other words, every convergent subsequence of a sequence contained in $X^{(\leq\gamma)}$, the first $\gamma$ levels of $X$, has a its limit in $X^{(\leq\gamma+1)}$. This shows that $seqcl^\alpha(X^{(0)}) \subseteq \bigcup_{\beta\leq\alpha} X^{(\beta)}$ for all $\alpha < \eta$. ∎

COROLLARY 13: *In any model where $\diamondsuit(\mathfrak{b}^*)$ is valid there are compact sequential scattered spaces of any sequential order $\alpha \leq \omega_1$.*

As in the previous section, our result combined with the result of [6] shows that there is a compact sequential scattered spaces of sequential order 4 in any canonical model of ZFC.

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