SPACES IN WHICH EVERY DENSE SUBSET IS BAIRE

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Abstract. We deal with several types of spaces in which every dense subspace is Baire (D-Baire spaces). Baire almost P-spaces and open-hereditarily irresolvable Baire spaces are example of D- spaces. We give a characterization of D-Baire spaces and characterize a particular class of them. We give an example of a D-Baire space whose square is not Baire.

1. Introduction

There is a wide variety of topological spaces in which every dense subset is Baire. The most simple of them are those spaces which have a discrete open dense subset, like locally compact Hausdorff extensions of discrete spaces. Baire almost P-spaces are also D-Baire and the open-hereditarily irresolvable Baire spaces form a a class of D-spaces (it is known that irresolvable D-Baire spaces without isolates points exist only in some models of set theory (see [17] and [18])). Our purpose of this paper is to give several characterizations of D-Baire spaces and open-hereditarily irresolvable Baire spaces. We find some sufficient conditions on D-Baire spaces to be metrizable or to have a discrete dense subspace. We finally explore some invariance properties under finite products or under continuous open images.

2. Definitions and preliminary results

Our spaces will be T_3. We recall the reader some basic definitions and after that we list five equivalent known definitions of Baire spaces (for the proofs we referred the reader to [12] which offers a complete survey on Baire spaces).

$A \subseteq X$ is *nowhere dense* (respect to X) if $\operatorname{int} A^{-} = \emptyset$. A subset $A \subseteq X$ is *a meager set* (or of *the first category*) in X if A is a countable union of nowhere dense sets. A space is called *Baire* if the intersection of countably many open dense subsets of the space is dense.

Proposition 2.1. *The following properties of a topological space X are equivalent:*

1. *X is a Baire space.*
2. *For every countable closed cover $\{H_n : n \in \mathbb{N}\}$ of X, the set $\bigcup_{n=1}^{\infty} \operatorname{int} H_n$ is dense in X.*

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(3) *For every sequence* $V_1, V_2, \ldots$ *of open sets with the same closure* K*, we have* $K = \left(\bigcap_{n=1}^{\infty} V_n\right)^-$.
(4) *Every meager* G_δ*-set in* X *is nowhere dense.*
(5) *Every meager set has empty interior.*

Every topological space which has a dense Baire subspace is evidently a Baire space. The converse is not true: for instance, the real line is a Baire space but the subspace of rationals is not. A useful necessary and sufficiente condition for a dense subset A of a Baire space to be Baire is given in the next theorem of J. M. Aarts and D. J. Lutzer [1] (for a proof see [12, Th. 1.24]):

Theorem 2.2. *Let* X *be a Baire space and let* $A \subseteq X$ *be dense. Then* A *is a Baire space if and only if every* G_δ*-set in* X *contained in* $X \setminus A$ *is nowhere dense.*

For the sake of completeness, we are going to give a proof of this theorem:

Lemma 2.3. *Let* X *be a Baire space. If* $G = \bigcap_{n\in\mathbb{N}} V_n$ *is a nonempty nowhere dense* G_δ*-set of* X*, where* V_n *is an open subset of* X *for all* $n \in \mathbb{N}$*, then for every nonempty open subset* V *of* X *there is* $n \in \mathbb{N}$ *such that* $int[(X \setminus V_n) \cap (V \setminus G^-)] \neq \emptyset$.

Proof. Let V be a nonempty open subset of X. Then, $V \setminus G^-$ is a nonempty open subset of X; hence, $V \setminus G^-$ is also Baire. Since $V \setminus G^- \subseteq \bigcup_{n\in\mathbb{N}} X \setminus V_n$ and each $X \setminus V_n$ is a closed subset of X, by the third clause of Proposition 2.1, there is $n \in \mathbb{N}$ such that $int[(X \setminus V_n) \cap (V \setminus G^-)] \neq \emptyset$. □

Proof of Theorem 2.2. Necessity. Let $G = \bigcap_{n\in\mathbb{N}} V_n$, where V_n is an open subset of X for each $n \in \mathbb{N}$, that is contained in $X \setminus A$. Then, $A \subseteq \bigcup_{n\in\mathbb{N}} X \setminus V_n$. In virtue of Proposition 2.1, $\bigcup_{n\in\mathbb{N}} int_A(A \cap (X \setminus V_n))$ is dense in A. Suppose that $intG^- \neq \emptyset$. Then, there is $m \in \mathbb{N}$ such that $\emptyset \neq intG^- \cap int_A(A \cap (X \setminus V_m))$. On the other hand, we know that $G^- \subseteq V_m^- = (V_m \cap A)^-$. Hence,

$$\emptyset \neq intG^- \cap int_A(A \cap (X \setminus V_m)) \subseteq (V_m \cap A)^- \cap A = cl_A(V_m \cap A)$$

which implies that $intG^- \cap int_A(A \cap (X \setminus V_m)) \cap V_m \cap A \neq \emptyset$, but this is impossible.

Sufficiency. Assume that A is no Baire. According to Proposition 2.1, there is a countable closed cover $\{H_n : n \in \mathbb{N}\}$ of A such that $\bigcup_{n\in\mathbb{N}} int_A H_n$ is not dense in A. For each $n \in \mathbb{N}$, choose a closed subset C_n of X such that $H_n = A \cap C_n$ for each $n \in \mathbb{N}$. Let $G = \bigcap_{n\in\mathbb{N}}(X \setminus C_n)$ which is a G_δ-set of X contained in $X \setminus A$. If $G = \emptyset$, then $\{C_n : n \in \mathbb{N}\}$ would be a closed cover of X and, by Proposition 2.1, then $\bigcup_{n\in\mathbb{N}} intC_n$ would be dense in X which is not possible. So, $G \neq \emptyset$. Choose an nonempty open subset V of X such that $V \cap A \cap int_A H_n = \emptyset$, for all $n \in \mathbb{N}$. By Lemma 2.3, we can find $n \in \mathbb{N}$ such that $int[C_n \cap (V \setminus G^-)] \neq \emptyset$. Hence, $\emptyset \neq int[C_n \cap (V \setminus G^-)] \cap A \subseteq int_A(C_n \cap A) \cap V \cap A \subseteq int_A H_n \cap V \cap A$, but this is a contradiction. Thus, A is Baire. □

In this paper, we shall study the following class of Baire spaces inspired in Theorem 2.2.

Definition 2.4. We say a space X is *D-Baire* if every dense subspace of X is Baire.

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.5. *Let X be a Baire space. Then, X is D-Baire if and only if every G_δ-set in X with empty interior is nowhere dense.*

A further corollary will be obtained after the next definition.

Following R. Levy [20], we say that a topological space X is an *almost P-space* if every non-empty G_δ-set in X has a non-empty interior.

Corollary 2.6. *Every Baire almost P-space is D-Baire.*

The following results are taken from [20]:

Theorem 2.7. a) *If X is locally compact and realcompact, then $\beta X \setminus X$ is almost P-space.*

b) *A Tychonoff space X is almost P-space if and only if its Hewitt realcompactification υX is almost P-space.*

c) *If X is a Tychonoff space, then βX is almost P-space if and only if X is pseudocompact and almost P-space.*

Corollary 2.8. *If X is an infinite discrete space whose cardinality is not Ulam measurable, then $\beta X \setminus X$ is D-Baire and βX is not almost P-space.*

3. D-Baire spaces

To start this section we give several characterizations of D-Baire spaces. First, we need to recall the definition of the σ-algebra PB.

Given a space X, the *class $PB(X)$* is the σ-algebra in X generated by all open sets and all nowhere dense sets. In [19] it is proved that $A \subseteq X$ belongs to the class $PB(X)$ if and only if A may be expressed in the form $A = L \cup D$, where L is a G_δ-set and D is meager. Obviously, the σ-algebra of Borel sets is contained in the class $PB(X)$.

Theorem 3.1. *The following seven conditions on a space X are equivalent:*

(1) X is D-Baire.
(2) X is Baire and every G_δ-set with empty interior is nowhere dense.
(3) Every meager subset $A \subseteq X$ is nowhere dense.
(4) X is Baire and every dense G_δ-set has dense interior.
(5) X is Baire and every set in the class $PB(X)$ with empty interior is nowhere dense.
(6) X is Baire and every Borel set with empty interior is nowhere dense.
(7) X is Baire and the union of a G_δ-set with empty interior and a meager set of X is nowhere dense.

Proof. (1) $\Longleftrightarrow$ (2). This es Corollary 2.5.

(2) $\Longrightarrow$ (3). Let $A \subseteq X$ be a meager set. Assume $A = \bigcup_{n=1}^{\infty} H_n$ where H_n is nowhere dense for all $n \in \mathbb{N}$. Therefore, $L = X \setminus \bigcup_{n=1}^{\infty} H_n^- = \bigcap_{n=1}^{\infty} X \setminus H_n^-$ is a G_δ-set in X and L is dense in X because its complemet is a meager set and X is Baire. Let $V = \operatorname{int} L$. The set $L - V$ clearly has empty interior. Hence, $L - V^-$ is a G_δ-set with empty interior, by hypothesis, $L - V^-$ is nowhere dense. Also $L \cap \operatorname{Fr} V$

is a nowhere dense set. Therefore, $L - V = (L - V^-) \cup (L \cap \operatorname{Fr} V)$ is a nowhere dense set as well. On the other hand,

$$X \setminus V = (L \setminus V) \cup (X \setminus L) = (L \setminus V) \cup \bigcup_{n=1}^{\infty} H_n^-$$

is a meager set. Since X is Baire,

$$\emptyset = \operatorname{int}(X \setminus V) = X \setminus V^- .$$

Therefore, $V^- = X$ and $A \subseteq X \setminus V = \operatorname{Fr} V$ is nowhere dense.

(3) $\Longrightarrow$ (4). It follows from Proposition 2.1 that X is a Baire space. Let $L \subseteq X$ be a dense G_δ-set of X. Since $X \setminus L$ is a meager set, the hypothesis implies that $X \setminus L$ is nowhere dense, i.e. $(X \setminus L)^-$ has empty interior. Therefore, $V = X \setminus (X \setminus L)^- = \operatorname{int} L$ is an open dense subspace of X.

(4) $\Longrightarrow$ (2). Let G be a G_δ-set with empty. First observe that $\operatorname{int} G^- \subseteq (G^- \setminus G)^-$. Since $G^- \setminus G$ is an F_σ-set with empty interior, $X \setminus (G^- \setminus G)$ is a dense G_δ-set of X. By assumption, $\operatorname{int}\,(X \setminus (G^- \setminus G))$ is also dense in X. That is, $X \setminus (G^- \setminus G)^-$ is dense in X. Hence, $\operatorname{int}\,(G^- \setminus G)^- = \emptyset$ and so $\operatorname{int} G^- = \emptyset$.

(4) $\Longrightarrow$ (5). We have already established above the equivalence among the clauses (1), (2), (3) and (4). The fifth clause follows directly from the properties of the class $PB(X)$ (see [19]) and the clauses (2) and (3).

(5) $\Longrightarrow$ (6). This implication is obvious because the σ-algebra of Borel sets is contained in the class $PB(X)$.

(6) $\Longrightarrow$ (1). It is enough to observe that (6) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (7). We know the first six statements are equivalent on to each other. Thus clause (7) follows directly from clauses (2) and (3).

(7) $\Longrightarrow$ (1). This is a consequence of Theorem 2.1 and Corollary 2.5. □

Corollary 3.2. *Every open subset of a D-Baire space is also D-Baire.*

Let us state some particular classes of D-Baire spaces.

Definition 3.3. Let X be a Baire space.

(1) X is said to be D'-*Baire* if every set with empty interior is nowhere dense.
(2) We say that X is D''-*Baire* if X has a dense discrete subspace.

As our spaces are T_1, it is evident that in the definition of D''-space we may say that the space contains a dense subset of isolated points, and also we can removed the condition Baire in the definition of D''-Baire space. Thus, we have directly that every D''-Baire space is D'-Baire and every D'-Baire space is D-Baire. The simplest examples of non-discrete D''-space are those whose have only one non-isolated point, and the Stone-Čech compactifications of discrete spaces are also D''-Baire.

We give now equivalent formulations for D'-Baire spaces.

For brevity, we say that $A \subseteq X$ is a *boundary set* if $\operatorname{int} A = \emptyset$ and let us consider the following subsets of a space X.

$$B_1 = \{A \subseteq X \mid \operatorname{int} A^- = \emptyset\}, B_2 = \{A \subseteq X \mid A \text{ is a meager set in } X\}$$

and

$$B_3 = \{A \subseteq X \mid \operatorname{int} A = \emptyset\}.$$

Obviously $B_1 \subseteq B_2$ and we also have that X is a Baire space iff $B_2 \subseteq B_3$.

Theorem 3.4. *In a topological space X, the following properties are equivalent:*

1. *X is D'-Baire.*
2. *X is Baire and for each dense set $A \subseteq X$, $\operatorname{int} A$ is also dense in X.*
3. *X is Baire and each dense set contains a dense G_δ-set.*
4. *The concepts boundary set, nowhere dense set and meager set are equivalent.*
5. *If $\{H_n : n \in \mathbb{N}\}$ is a countable family of dense subsets of X, then $\bigcap_{n=1}^{\infty} H_n$ is also dense in X.*
6. *X is Baire and each finite intersection of dense sets in X is also dense in X.*
7. *The family of all boundary subsets of X is a σ-ideal.*

Proof. The implications (2) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (6) and the equivalence (4) $\Longrightarrow$ (4) are obvious.

(1) $\Longrightarrow$ (2). Let $A \subseteq X$ be dense. As $X \setminus A$ has empty interior, by hypothesis, $X \setminus A$ is nowhere dense. Therefore, $\operatorname{int} A = X \setminus (X \setminus A)^-$ is an open dense subset of X.

(3) $\Longrightarrow$ (4). We only have to prove $B_3 \subseteq B_1$. Indeed, if $L \subseteq X$ has empty interior, $X \setminus L$ is dense in X and, by hypothesis, there exists a dense G_δ-set $H \subseteq X \setminus L$. Therefore, $X \setminus H$ is an F_δ-set with empty interior containing L. $X \setminus H$ and L are then meager sets and $B_2 = B_3$. The hypothesis implies also that X is D-Baire, since if $D \subseteq X$ is dense in X and C is a G_δ-set of X disjoint from D, then $C \subseteq X \setminus A$ where A is a dense G_δ-set of X contained in D. Therefore, $X \setminus A$ and C are meager sets. Being a meager G_δ-set in a Baire space, C is nowhere dense. Therefore, by Theorem 2.2, D is a Baire subspace of X and X is D-Baire. By Theorem 3.1, $B_1 = B_2$ and hence conclude that $B_3 = B_2 = B_1$.

(4) $\Longrightarrow$ (5). Let $\{H_n : n \in \mathbb{N}\}$ be a countable family of dense subsets of X. Then, for every $n \in \mathbb{N}$, $X \setminus H_n$ has empty interior and, by hypothesis, $X \setminus H_n$ is a meager set. Then

$$\bigcup_{n=1}^{\infty} (X \setminus H_n) = X \setminus \bigcap_{n=1}^{\infty} H_n$$

is also a meager set and, by hypothesis, it has empty interior. Therefore, $\bigcap_{n=1}^{\infty} H_n$ is dense in X.

(6) $\Longrightarrow$ (1). Let $A \subseteq X$ be a set with empty interior and let $L = X \setminus A$. Define $V = \operatorname{int} L$. Since $L \setminus V$ is a set with empty interior, the set $X \setminus (L \setminus V) = V \cup A$ is dense in X. $L = X \setminus A$ is also dense in X. Therefore, by hypothesis, $(V \cup A) \cap (X \setminus A) = V$ is dense in X. Hence, $X \setminus V$ is a nowhere dense set. Since $A \subseteq X \setminus V$, we deduce that A is also a nowhere dense set and the proof is complete. □

The condition stated in clause (7) of Theorem 3.4 was considered in [21].

Corollary 3.5. *Every D'-Baire space is D-Baire.*

Proof. Use condition (3) from 3.1 and condition (4) in 3.4. □

By using the equality between two of the the sets B_1, B_2 and B_3, we obtain that X is a D-space iff $B_1 = B_2$ (Theorem 3.1) and:

Corollary 3.6. *For a space X the following conditions are equivalent:*

(1) X is D'-Baire.
(2) Every boundary set is meager.
(3) Every boundary set is nowhere dense.

Corollary 3.7. *Every nonempty open subset of a D'-Baire space is also D'-Baire.*

Proof. Assume that X is a D'-Baire space and let $U \subseteq X$ be open and nonempty. It is known that U is also a Baire space (see [7, Ex. 3.9.J (a)]). Suppose that A is a dense subset of U. Since $A \cup (X \setminus U)$ is dense in X, by Theorem 3.4, int $(A \cup (X \setminus U))$ is also dense in X. Let $\emptyset \neq V \subseteq U$. Then, $\emptyset \neq V \cap$ int $(A \cup (X \setminus U))$ and it is clear that $V \cap$ int $(A \cup (X \setminus U)) \subseteq A$. This implies that $V \cap$ int$A \neq \emptyset$. This shows that int A is a dense subset of U. According to Theorem 3.4, U is D'-Baire. □

Following E. Hewitt [13] we say that a topological crowded[1] space X is *resolvable* if X has a dense subspace D whose complement $X \setminus D$ is also dense in X. A space that cannot be split in two disjoint dense subsets is called *irresolvable*[2]. Most of the spaces which we handle are resolvable. For example, it is shown in [13] that all metric crowded spaces and all compact crowded spaces are resolvable (maximally resolvable). In a more general setting, E. G. Pytke'ev [22] showed that every crowded k-space is resolvable. For more examples of resolvable spaces the reader is referred to [5]. However, we may find multiple examples of irresolvable spaces in the literature (see for instance [6], [8] and [13]).

A space X is called *open-hereditarily irresolvable* if every open subset of X is irresolvable. In the following corollary, we shall prove that the D'-spaces are precisely the open-hereditarily irresolvable Baire spaces. The proof of the next lemma is left to the reader.

Lemma 3.8. *In a topological space X, the following properties are equivalent:*

(1) Every subset of X with empty interior is nowhere dense.
(2) X is open-hereditarily irresolvable.

The following statement is a direct application of Theorem 3.4 and the previous lemma.

Corollary 3.9. *A space X is D'-Baire iff X is Baire and open-hereditarily irresolvable.*

Thus, we have that every D'-space must be irresolvable. Hence, by Corollary 2.8, $\beta\mathbb{N} - \mathbb{N}$ is D-Baire and, by Pytke'ev's Theorem, we obtain that $\beta\mathbb{N} - \mathbb{N}$ cannot be D'-Baire. Corollary 3.9 is a particular case of Proposition 1.2 from [17] and the implication (1) $\Longrightarrow$ (2) of Theorem 3.4 lies, in a more general form, in [13].

It is shown in [18] (see also [17]) that if there is a Baire irresolvable crowded space, then there is a measurable cardinal in the inner model. Hence, if $V = L$, then every Baire space without isolated points is resolvable. Using this assertion and Corollary 3.9, we can prove that every D'-Baire space is D''-Baire in a model of ZFC where $V = L$.

[1] A space without isolated points is called *crowded*.

[2] A space with at least one isolated points cannot be divided in two disjoint dense subset; hence, we may omit the condition crowded in the definition of irresolvable space.

Theorem 3.10. *Under the assumption of $V = L$, the set of isolated points of a D'-Baire space is dense in the space. Thus, $V = L$ implies that a space is D'-Baire iff it is D''-Baire.*

Proof. Assume $V = L$. As we pointed above every Baire space without isolated points must be resolvable. Hence and from Corollary 3.9 we must have that every nonempty open subset of X has an isolated point. Therefore, X contains a dense discrete subset. □

S. Shelah [23] showed the consistency (modulo reasonably large cardinals) of the existence of a topological Baire irresolvable space with no isolated points of size ω_1. It is not hard to see that every Baire irresolvable crowded space must contain a nonempty open subset open-hereditarily irresolvable. Since every open subset of a Baire space is also Baire, Shelah's example contains a D'-Baire crowded subspace which cannot be D''-Baire. So, the existence of a D'-Baire space which is not D''-Baire is undecidable in ZFC.

Next, we state a sufficient condition on a D-Baire space to be D''-Baire.

Lemma 3.11. *Let X be a crowded space. If X has a σ-locally finite π-base*[3]*, then X has a dense meager subset.*

Proof. Let $\mathcal{B} = \bigcup_{n\in\mathbb{B}} \mathcal{B}_n$ be a π-base of X such that each family $\mathcal{B}_n$ is locally finite. For each $n \in \mathbb{N}$, enumerate $\mathcal{B}_n$ as $\{B_i^n : i \in I_n\}$ and choose $x_i^n \in B_i^n$ for each $n \in \mathbb{N}$ and for each $i \in I_n$. Now, we define $N_n = \{x_i^n : i \in I_n\}$ for every $n \in \mathbb{N}$. Clearly, N_n is discrete for every $n \in \mathbb{N}$. Since X is crowded, we must have that N_n is nowhere dense in X for all $n \in \mathbb{N}$. Thus, $N = \bigcup_{n\in\mathbb{N}} N_n$ is meager and dense. □

The following results follows directly from the previous lemma.

Theorem 3.12. *Every D-Baire space with a σ-locally finite π-base is a D''-Baire space.*

Proof. Suppose that X has a nonempty set U without isolated points. It is evident that U also has a σ-locally finite π-base and, by Corollary , U is a D-Baire crowded space. So, by Lemma 3.11, U has a dense meager subset which contradicts Theorem 3.1. □

Corollary 3.13. *Every metric D-Baire space is D''-Baire.*

Proof. Suppose that the set of isolated points of X is not dense. For each $n \in \mathbb{N}$, let $\{B(d, \frac{1}{n+1}) : d \in D_n\}$[4] be a maximal pairwise disjoint family whose elements do not contain any isolated point of X. Put $U = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in D_n} B(d, \frac{1}{n+1})$. Since U is an open subset of X, by Corollary 3, U is a metric D-Baire crowded space. Clearly U has a σ-locally finite π-base. By Theorem 3.12, U is a D''-Baire space and so contains a dense subset of isolated points which is a contradiction to the fact that U does not contain any isolated point of X. □

Theorem 3.14. *Let X be a D-Baire space. If there exists a dense set $L \subseteq X$ having a σ-discrete network, then X is D''-Baire. In particular, a D-Baire, separable space is D''-Baire.*

[3] A family $\mathcal{B}$ of nonempty open subsets of a space X is a π-*base* if every nonempty open subset of X contains an element of $\mathcal{B}$.

[4] $B(x, \epsilon)$ denotes the ball with center x and radio ϵ in a metric space.

Proof. We may suppose, without loss of generality, that $L = X$. Let $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ be a network of X, where each family $\mathcal{H}_n$ is discrete (with respect to X). Choosing a point in each member of $\mathcal{H}$, we may find a dense set $D \subseteq X$ which is a countable union of closed discrete sets $\{D_n : n \in \mathbb{N}\}$. Let $E_n = D_n - X^a$ and $F_n = D_n \cap X^a$. The set $E = \bigcup_{n=1}^{\infty} E_n$ is open and discrete in X and each set F_n is nowhere dense. Hence $F = \bigcup_{n=1}^{\infty} F_n$ is a meager set. Since X is D-Baire, the set F is nowhere dense (see condition 3) in 3.1). Since $D = E \cup F$, we deduce $X = E^- \cup F^-$. Necessarily $E^- = X$, because if $V = X \setminus E^- \neq \emptyset$, the open set V would be contained in F^-, contradicting the fact that F is nowhere dense. Therefore, E is an open discrete dense subspace of X and X is D''-Baire. □

We give below a sufficient condition on a D'-Baire space to be D''-Baire. We give first a definition:

The *derived sets* of a space X are defined as follows:

$$X^{(0)} = X$$
$$X^{(1)} = X^a$$

Assuming $X^{(\alpha)}$ is already defined for an ordinal number α, we define $X^{(\alpha+1)}$ as the set of limit points of $X^{(\alpha)}$. If α is an infinite limit ordinal and if $X^{(\gamma)}$ is already defined for each $\gamma < \alpha$, we set:

$$X^{(\alpha)} = \bigcap_{\gamma<\alpha} X^{(\gamma)}$$

Therefore, there exists a minimum ordinal number β such that $X^{(\beta)}$ is crowded or empty, i.e., such that $X^{(\beta)} = X^{(\beta+1)}$. This set $X^{(\beta)}$ is called the *last derived set* of X.

Theorem 3.15. *Let X be a D'-Baire space whose last derived set $X^{(\beta)}$ is resolvable. Then X is D''-Baire. In fact, $X \setminus X^a$ is dense in X.*

Proof. Let $L \subseteq X^{(\beta)}$ be such that $L^- = (X^{(\beta)} - L)^- = X^{(\beta)}$. Clearly

$$D = \bigcup_{0\leq\alpha<\beta} \left(X^{(\alpha)} - X^{(\alpha+1)}\right) \cup L$$

is dense in X. Because X is D'-Baire, $\operatorname{int} D$ is also dense in X.

But $(X \setminus \operatorname{int} D)^- = (X^{(\beta)} - L)^- = X^{(\beta)}$. Therefore, $\operatorname{int} D = X \setminus X^{(\beta)} = \bigcup_{0\leq\alpha<\beta} \left(X^{(\alpha)} - X^{(\alpha+1)}\right)$. We prove $X \setminus X^{(1)}$ is dense in $\operatorname{int} D$ and, hence, it is dense in X. Suppose on the contrary, there exists a point

$$p \in (\operatorname{int} D) \cap \left[X \setminus (X \setminus X^{(1)})^-\right] .$$

Therefore, there exists an open set $W \subseteq X$ such that

$$p \in W \subseteq \operatorname{int} D \text{ and } W \cap (X \setminus X^{(1)})^- = \emptyset .$$

Let α be the minimum ordinal such that

$$W \cap \left(X^{(\alpha)} - X^{(\alpha+1)}\right) \neq \emptyset$$

and select a point $q \in W \cap \left(X^{(\alpha)} - X^{(\alpha+1)}\right)$. Let T be an open set in X such that $T \cap X^{(\alpha)} = \{q\}$. Therefore, $W \cap T = \{q\}$ and $q \in X \setminus X^{(1)}$, a contradiction. Hence, the discrete set $X \setminus X^{(1)}$ is dense in X and X is D''-Baire. □

Next, let us introduce a property that a D-space needs to be a D'-space.

Definition 3.16. A space X is called *PB* if $PB(X) = \mathcal{P}(X)$[5].

Theorem 3.17. *A space X is D'-Baire iff X is PB and D-Baire.*

Proof. Necessity. Assume that X is D'-Baire. By Corollary 3.5, X is D-Baire. Let $A \subseteq X$. We know that $A = \text{int}A \cup (A \setminus \text{ int } A)$. Clearly int A is a G_δ-set and since $(A \setminus \text{ int } A)$ has empty interior, $(A \setminus \text{ int } A)$ is nowhere dense. Then, $A \in PB(X)$.

Sufficiency. Assume that $A \subseteq X$ has empty interior. By assumption, $A = G \cup M$, where G is a G_δ-set and M is meager. Since X is a D-space, by Theorem 3.1, we have that G is nowhere dense. So, A is nowhere dense. Therefore, X is a D'-space. □

We now consider the following class of Baire spaces.

Definition 3.18. A space X is called *extremally Baire* if the union of a boundary G_δ-set and a meager set is boundary.

It follows from Theorem 2.1 that every extremally Baire space is Baire and from Theorem 3.1 that every D-Baire space is extremally Baire.

Theorem 3.19. *If X is PB and extremally Baire, then X is D-Baire.*

Proof. Let D be a dense subset of X. By hypothesis, $D = G_0 \cup M_0$ and $X \setminus D = G_1 \cup M_1$, where G_0 and G_1 are a G_δ-sets and M_0 and M_1 are meager. Without loss of generality, we may assume that $G_0 \cap M_0 = G_1 \cap M_1 = \emptyset$. We claim that G_0 is a dense subset of $G_0 \cup G_1$. Indeed, suppose that there is a nonempty open subset V of X such that $(G_0 \cup G_1) \cap V \subseteq G_1$. It is clear that $V \cap G_1$ is a boundary G_δ-set and since X is extremally Baire, we must have that $(V \cap G_1) \cup (V \cap M_0) \cup (V \cap M_1) = V$ is a boundary set which is a contradiction. Thus, G_0 is a dense subset of $G_0 \cup G_1$. But, it is not hard to see that $G_0 \cup G_1$ is a dense subset of X. So, G_0 is a dense G_δ-set of X and since $X$is a Baire space, we have that G_0 is also Baire. Therefore, D is also a Baire space. □

4. Real-valued functions

Problem 109 of the Scottish Book posed by M. Katětov is the following: Is there a crowded space on which every real-valued function is continuous at some point ? In a very nice paper, V. I. Malykhin [21] prove that there is a irresolvable Baire crowded space iff there is a space on which every real-valued function is continuous at some point. Years later, it was shown in [10] that a Baire space X is open-hereditarily irresolvable iff every real-valued function on X has a dense set of points of continuity. In connection with these results, R. Bolstein [3] introduced the notion of almost-resolvability: A space is called *almost resolvable* if it is the

[5] $\mathcal{P}(X)$ denotes the family of all subsets of a set X

countable union of boundary sets (it is clear that every resolvable space is almost resolvable). It is shown in [3] and [10] that a space X is almost resolvable iff one of the following equivalent conditions holds:

(1) X admits an everywhere discontinuous real-valued function with countable range.
(2) X admits an everywhere discontinuous real-valued function.

V. I. Malykhin [21] found a model of ZFC in which every topological space is almost resolvable, and in the model of Shelah [23] there is a crowded Baire space which is not almost resolvable. As a particular case of Proposition 1.2 from [17] is the next result.

Proposition 4.1. *For every space X the following conditions are equivalent.*

(1) X is D'-Baire.
(2) For every space Y of countable weight and for every function $f : X \to Y$ the set of points of continuity of f contains a dense open set.

From Corollary 3.7 and the previous proposition we have:

Corollary 4.2. *A crowded space X is D'-Baire iff X does not contain a nonempty almost resolvable open subset.*

It is pointed out in [2] that a space X is D''-Baire iff there is a dense subset D of X such that every function $f : X \to \mathbb{R}$, $f|_D$ is continuous. In the paper, [2], the D''-Baire spaces are called UB-spaces.

5. Invariance properties

It is a well known fact that the Baire property is invariant under open continuous maps. As far as the invariance under continuous maps is concerned, we can prove:

Theorem 5.1. *Let $\varphi\colon X \to Y$ be open, continuous and onto. Then*

i) If X is D-Baire, Y is also D-Baire.
ii) If X is almost P-space, Y is also almost P-space.
iii) If X is D'-Baire, Y is also D'-Baire.
iv) If X is D''-Baire, Y is also D''-Baire.

Proof. *i*). Let $E \subseteq Y$ be a dense subset. Then $D = \varphi^{-1}(E)$ is dense in X. By hypothesis, D is a Baire space. Since $\varphi\,|\,D\colon D \to E$ is continuous, open and surjective, we deduce that E is also a Baire space.

ii). Let $L \supseteq Y$ be a non-empty G_δ-set. Since $\varphi^{-1}(L)$ is also a non-empty G_δ-set, we have $\operatorname{int}\varphi^{-1}(L) \neq \emptyset$. Hence $\operatorname{int} L \supseteq \varphi(\operatorname{int}\varphi^{-1}(L)) \neq \emptyset$ and Y is an almost P-space.

iii). Let $E \subseteq Y$ be a dense subset of Y. Then $D = \varphi^{-1}(E)$ is dense in X. Since X is D'-Baire, $\operatorname{int} D$ is dense in X. Therefore, $\varphi(\operatorname{int} D)$ is dense in Y. But $\varphi(\operatorname{int} D) \subseteq E$. Hence, $\operatorname{int} E$ is dense in Y and Y is a D'-Baire space.

iv). Let $D \subseteq X$ be open, discrete and dense in X. To prove $\varphi(D)$ is discrete, select a point $x \in D$. Then $\{x\}$ is an open set in X contained in D. Therefore $\{\varphi(x)\}$ is an open set in Y contained in $\varphi(D)$ and $\varphi(D)$ is discrete. Therefore, Y is a D''-Baire space. □

It is also obvious that the almost P-space and the D''-Baire properties are preserved under finite products. There are many examples in the literature of Baire spaces (even metrizable Baire spaces) whose square is not Baire (see [4] and [9]). On the other hand, there exist several topological properties P which imply the Baire property and are invariant under arbitrary products: For instance, either $P =$ pseudocompleteness (see [1]) or $P =$ weak pseudocompactness (see [11]).

We exhibit next a D-Baire space X whose square is not Baire. Obviously X cannot be an almost P-space.

Example 5.2. To construct our example we need some basic notions from Set Theory that the reader may find them in text books like [16] and [14]. We consider the first uncountable ordinal number ω_1 equipped with the order topology. Let S be a stationary subset of ω_1. Then, we define X_S as the set of all compact subsets of S. For $A \in X_S$, we let let $con(A) = \{B \in X_S : A \subseteq B \text{ and } max(A) < min(B \setminus A)\}$, that is $con(A)$ is the set of all end-extensions of A. It is clear that if $A, B \in X_S$ and $con(A) \cap con(B) \neq \emptyset$, then either A is an end-extension of B or B is an end-extension of A. The topology on X_S is the topology generated by all cones and their complements. Obviously, by definition, X_S is zero dimensional and Hausdorff.

Claim 1. Let $D \subseteq X_S$. Then, D contains dense open subset of X_S if and only if for every A in X_S there is $B \in con(A)$ such that $con(B) \subseteq D$.

Proof of Claim 1. It suffices to show that $\mathcal{C} = \{con(A) : A \in X_S\}$ is a base for the topology of X_S. Indeed, suppose that $A \in X_S \setminus con(B)$. Without loss of generality, we may assume that $con(A) \cap con(B) \neq \emptyset$. Then, we must have that B is a proper end-extension of A and so $con(B) \subseteq con(A)$. Let $\gamma = min(S \setminus max(B))$. Then, $A \in con(A \cup \{\gamma\}) \subseteq X_S \setminus con(B)$. This shows that $\mathcal{C}$ is a base for X_S.

Claim 2. The intersection of countably many dense open sets contains a dense open set.

Proof of Claim 2. For each $n \in \mathbb{N}$ take a dense open subset D_n of X_S. Fix $A \in X_S$. We need to find an end-extension B of A so that $con(B) \subseteq \bigcap_{n\in\mathbb{N}} D_n$. In fact, let M be a countable elementary submodel such that $S, D_i \in M$ and $\gamma = M \cap \omega_1$ is in S. Choose an increasing sequence of ordinals γ_n converging to γ. Recursively construct an increasing sequence B_n of elements of $X_S \cap M$ so that:

1) $B_0 \in con(A)$,
2) $B_{n+1} \in con(B_n)$, for each $n \in \mathbb{N}$,
3) $\gamma_n \leq max(B_n)$, for each $n \in \mathbb{N}$, and
4) $con(B_n) \subseteq D_n$, for each $n \in \mathbb{N}$.

The construction of the B_n's follows directly from Claim 1 using the fact that each D_n is dense open in X_S and M knows it. Put $B = (\bigcup_{n\in\mathbb{N}} B_n) \cup \{\gamma\}$. As $\gamma \in S$ and $\gamma_n \nearrow \gamma$, $B \in X_S$ and $con(B) \subseteq con(B_n)$, for all $n \in \mathbb{N}$. Another proof without using elementary submodels can be achieved by proving that the set

$$C = \{\gamma < \omega_1 : \text{ there is a sequence } (B_n)_{n\in\mathbb{N}} \text{ in } X_S \text{ such that:}$$

1) $A \in con(B_0)$,
2) $B_{n+1} \in con(B_n)$, for each $n \in \mathbb{N}$,
3) $con(B_n) \subseteq D_n$, for each $n \in \mathbb{N}$, and
4) $(\bigcup_{n\in\mathbb{N}} B_n) \cup \{\gamma\} \in X_S\}$

is closed and unbounded in ω_1.

Thus, according to Proposition 2.1, Theorem 3.1 and Claim 2, the space X_S is D-Baire.

Claim 3. If S and T are disjoint stationary subsets of ω_1, then $X_S \times X_T$ is not Baire.

Proof of Claim 3. For $A \in X_S$ and $B \in X_T$ we let

$$osc(A,B) = |\{\alpha \in A \cup B : \alpha \in A \text{ iff } min((A \cup B) \setminus \alpha) \in B\}|;$$

That is, $osc(A,B)$ is the number of "changes" from A to B and vice-versa. Given $A \in X_S$ and $B \in X_T$, $osc(A,B)$ is finite. Indeed, if α_n is an alternating element of A and B, for each $n \in \mathbb{N}$, then

$$\alpha = sup\{\alpha_n : n \in \mathbb{N}\} = sup\{min((A \cup B) \setminus \alpha_n) : n \in \mathbb{N}\} \in A \cap B$$

since both sets are compact, but this is impossible since S and T are disjoint. For each $n \in \mathbb{N}$, we define $E_n = \{(A,B) \in X_S \times X_T : osc(A,B) \geq n\}$. Is clear that each pair of elements $A \in X_S$ and $B \in X_T$ can extended to $A' \in X_S$ and $B' \in X_T$ by alternating members of S and T, respectively, making the $osc(A',B')$ as large as desired. Thus, for each $n \in \mathbb{N}$ and for each $(A,B) \in X_S \times X_T$, we can find $(A',B') \in X_S \times X_T$ so that $A' \in con(A)$, $B' \in con(B)$ and $cone(A') \times cone(B') \subseteq E_n$. Therefore, E_n is a dense open subset of $X_S \times X_T$ for all $n \in \mathbb{N}$. Since for every $A \in X_S$ and $B \in X_T$ $osc(A,B)$ is finite, we must have that $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. This shows that the product $X_S \times X_T$ cannot be Baire. Thus, our space is the topological sum $X = X_S \sqcup X_T$ where S and T are disjoint stationary subsets of ω_1. We have that X is D-Baire but $X \times X$ is not Baire.

We end this section with the following question.

Question 5.3. Is there a D'-Baire space whose square is not Baire in some model of ZFC ?

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