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Another $\diamondsuit$-like principle

by

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Abstract. A new $\diamondsuit$-like principle $\diamondsuit_{\mathfrak{d}}$ consistent with the negation of the Continuum Hypothesis is introduced and studied. It is shown that $\neg\diamondsuit_{\mathfrak{d}}$ is consistent with CH and that in many models of $\mathfrak{d} = \omega_1$ the principle $\diamondsuit_{\mathfrak{d}}$ holds. As $\diamondsuit_{\mathfrak{d}}$ implies that there is a MAD family of size $\aleph_1$ this provides a partial answer to a question of J. Roitman who asked whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$. It is proved that $\diamondsuit_{\mathfrak{d}}$ holds in any model obtained by adding a single Laver real, answering a question of J. Brendle who asked whether $\mathfrak{a} = \omega_1$ in such models.

The motivation for this paper is twofold. One motivating factor is the question (according to A. Miller due to J. Roitman) as to whether the existence of a dominating subset of ω^ω of size $\aleph_1$ implies the existence of a maximal almost disjoint (MAD) family of subsets of ω of size $\aleph_1$.

The other source of motivation comes from recent investigations of $\diamondsuit$-like principles and their impact on the values of the standard cardinal invariants of the continuum. The principle $\diamondsuit$ itself is known to imply CH. Other, similar, principles are consistent with $\neg$CH. The best known of these principles is the $\clubsuit$ principle which states that there exists $\{A_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that for all $\alpha \in \mathrm{Lim}(\omega_1)$, $A_\alpha \subseteq \alpha$, $\sup(A_\alpha) = \alpha$ and for every $X \in [\omega_1]^{\omega_1}$ there exists $\alpha \in \mathrm{Lim}(\omega_1)$ such that $A_\alpha \subseteq X$. The $\clubsuit$ principle has been used by Ostaszewski (see [Os]) to construct the famous Ostaszewski space, a countably compact non-compact S-space with closed sets either countable or co-countable. In the presence of CH, $\clubsuit$ is equivalent to $\diamondsuit$. The original proof that $\clubsuit + \neg$CH is relatively consistent with ZFC can be found in [Sh].

The relationship between $\clubsuit$, its variants, and cardinal invariants has been extensively studied recently. We will give a brief account of the situation. It is folklore that $\clubsuit$ implies that $\mathrm{cov}(\mathcal{N}_{\omega_1}) = \mathrm{cov}(\mathcal{M}_{\omega_1}) = \mathfrak{p} = \mathfrak{t} = \omega_1$, and $\min\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} = \mathrm{add}(\mathcal{N}) = \omega_1$ follows easily from a result of

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Truss (see [Tr]). On the other hand, it has been observed by many (including I. Juhász and P. Komjáth) that $\clubsuit$ is consistent with Martin's Axiom for countable posets (see e.g. [FSS]). J. Brendle (to appear in [Br2]) announced that $\clubsuit + \mathrm{cov}(\mathcal{N}) = \omega_2$ is also consistent. A paper [DžSh] claims to construct a model where $\clubsuit$ holds and every Aronszajn tree is special. According to J. Brendle there seems to be some problem with their argument but the construction still shows that $\mathrm{add}(\mathcal{M}) = \omega_2$ is consistent with $\clubsuit$. J. Baumgartner (in an unpublished note) showed that $\clubsuit$ holds in a model obtained from a model of $V = L$ by adding many Sacks reals side-by-side (for a proof see [Hr]). According to the author's best knowledge the following question remains unanswered: *Is* $\clubsuit + \mathfrak{h} > \omega_1$ *consistent?*

Here a new $\diamondsuit$-like principle, denoted by $\diamondsuit_{\mathfrak{d}}$, is introduced and studied. A sequence $\bar{d} = \{d_\alpha : \alpha < \omega_1\}$ is called a $\diamondsuit_{\mathfrak{d}}$-*sequence* if $d_\alpha : \alpha \to \omega$ and

$$\forall f : \omega_1 \to \omega \ \exists \alpha \geq \omega \quad f{\restriction}\alpha \leq^* d_\alpha.$$

We shall say that $\diamondsuit_{\mathfrak{d}}$ holds if there is a $\diamondsuit_{\mathfrak{d}}$-sequence. The impact of $\diamondsuit_{\mathfrak{d}}$ on many of the standard cardinal invariants is settled in this paper. It is shown that $\neg\diamondsuit_{\mathfrak{d}}$ is consistent with CH and that $\diamondsuit_{\mathfrak{d}}$ implies that there is a MAD family of size $\aleph_1$, providing a partial answer to a question of Roitman. It is also proved that $\diamondsuit_{\mathfrak{d}}$ holds in many models of $\mathfrak{d} = \omega_1$ including any model obtained by adding a single Laver real, answering a question of J. Brendle, who asked whether $\mathfrak{a} = \omega_1$ in such models. The set-theoretic notation is mostly standard, following [Ku]. For definitions of cardinal invariants of the continuum consult [Bl], or [vD] and [BJ].

I. Principle $\diamondsuit_{\mathfrak{d}}$: Not all dominating families are the same. If there is a $\diamondsuit_{\mathfrak{d}}$-sequence then there is one satisfying an additional monotonicity property

$$\forall \alpha < \beta < \omega_1 \quad d_\alpha \leq^* d_\beta{\restriction}\alpha.$$

Even though this condition is superfluous we will assume it as it makes many an argument more transparent.

It is obvious that $\diamondsuit \Rightarrow \diamondsuit_{\mathfrak{d}}$ and that the functions $d_\alpha{\restriction}\omega$ form a dominating family, hence $\diamondsuit_{\mathfrak{d}} \Rightarrow \mathfrak{d} = \omega_1$. Notice that for every f the set of those α such that $f{\restriction}\alpha \leq^* d_\alpha$ is unbounded. The principle $\diamondsuit_{\mathfrak{d}}$ is seemingly just a slight strengthening of the assumption that $\mathfrak{d} = \omega_1$ as indicated by the following proposition.

PROPOSITION I.1. $\mathfrak{d} = \omega_1$ *if and only if there exists a sequence* $\{d_\alpha : \alpha < \omega_1\}$, $d_\alpha : \alpha \to \omega$, *such that*

$$\forall f : \omega_1 \to \omega \ \forall \alpha < \omega_1 \ \exists \beta \geq \alpha \quad f{\restriction}\alpha \leq d_\beta{\restriction}\alpha.$$

Proof. Having a sequence $\{d_\alpha : \alpha < \omega_1\}$ as above it is immediate that the family $\{d_\alpha{\restriction}\omega : \alpha < \omega_1\}$ is dominating.

For the other direction fix for every $\beta < \omega_1$ a strictly (not mod *fin*) dominating family $\mathcal{F}_\beta$ on ω^β of size $\aleph_1$. Enumerate $\bigcup\{\mathcal{F}_\beta : \beta < \omega_1\}$ as $\{f_\alpha : \alpha < \omega_1\}$. Choose $d_\alpha : \alpha \to \omega$ such that $f_\alpha \leq d_\alpha \restriction \mathrm{dom}(f_\alpha)$. In order to check that this works let $f : \omega_1 \to \omega$ and $\beta < \omega_1$ be given. As $\mathcal{F}_\beta$ is a dominating family (and uncountable) there is an $\alpha \geq \beta$ such that $f_\alpha \in \mathcal{F}_\beta$ and f_α strictly dominates $f\restriction\beta$. Then $d_\alpha\restriction\beta \geq f\restriction\beta$. ∎

The following proposition can be viewed as a partial solution to Roitman's problem. As $\diamondsuit_{\mathfrak{d}}$ holds in many models where $\mathfrak{d} = \omega_1$ it can be perceived as a "machine" for constructing small MAD families.

PROPOSITION I.2. $\diamondsuit_{\mathfrak{d}} \Rightarrow \mathfrak{a} = \omega_1$.

Proof. Let $\{A_i : i \in \omega\}$ be a fixed infinite partition of ω into infinite pieces. Let $\{d_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit_{\mathfrak{d}}$-sequence such that

$$\forall \alpha \geq \omega \quad A_\alpha = \omega \setminus \bigcup_{\beta<\alpha} [d_\alpha(\beta), \rightarrow) \cap A_\beta$$

is infinite for every α (by $[n, \rightarrow)$ we denote the set of all integers greater than or equal to n). It is easy to make sure that A_α is infinite by possibly inductively replacing the d_α with a larger function.

• $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ *is a MAD family*: Assume it is not the case. Let X be a witness to that and let $f_X : \omega_1 \to \omega$ be defined by $f_X(\alpha) = \max(A_\alpha \cap X) + 1$. Let $\beta \geq \omega$ be such that $f_X\restriction\beta \leq^* d_\beta$. We can assume that $f_X\restriction\beta \leq d_\beta$. This can be accomplished by changing X by a finite set. Let $x \in X \setminus A_\beta$. Then $x \in \bigcup_{\gamma<\beta}[d_\beta(\gamma), \rightarrow) \cap A_\gamma$, hence there is a $\gamma < \beta$ such that $x \in [d_\beta(\gamma), \rightarrow) \cap A_\gamma$ and in particular $x \in [f_X(\gamma), \rightarrow) \cap A_\gamma$, which contradicts the definition of f_X. ∎

PROPOSITION I.3. $\clubsuit + \mathfrak{d} = \omega_1$ *implies* $\diamondsuit_{\mathfrak{d}}$.

Proof. Let $\{g_\alpha : \alpha < \omega_1\}$, $g_\alpha : \alpha \to \omega$, be a sequence such that

$$\forall f : \omega_1 \to \omega \ \forall \alpha < \omega_1 \ \exists \beta \geq \alpha \quad f\restriction\alpha \leq g_\beta\restriction\alpha.$$

The existence of such a sequence follows from $d = \omega_1$ by Proposition I.1. Fix also a $\clubsuit$-sequence $\{A_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$. Without loss of generality, the order type of A_α is ω for every α. Let $\{a_n^\alpha : n \in \omega\}$ be an increasing enumeration of A_α and let $a_{-1}^\alpha = 0$. Put

$$d_\alpha = \bigcup_{n\in\omega} g_{a_{n+1}^\alpha}\restriction[a_{n-1}^\alpha, a_n^\alpha)$$

for every α limit. For isolated α let d_α be your favorite function from α to ω.

To verify that the sequence $\{d_\alpha : \alpha < \omega_1\}$ is, indeed, a $\diamondsuit_{\mathfrak{d}}$-sequence let f be a function from ω_1 to ω. Construct inductively an uncountable set $X \subseteq \omega_1$ such that for every $\alpha, \beta \in X$, $\alpha < \beta \Rightarrow f\restriction\alpha \leq g_\beta\restriction\alpha$. This is easy to do using the property of $\{g_\alpha : \alpha < \omega_1\}$. Now, as $\{A_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ is a

$\clubsuit$-sequence, there is an α limit such that $A_\alpha \subseteq X$. It is immediate from the definitions of d_α and X that $f{\restriction}\alpha \le d_\alpha$. ∎

It follows from the proof of the proposition that, in fact, $\clubsuit + \mathfrak{d} = \omega_1$ implies a stronger version of $\diamondsuit_{\mathfrak{d}}$, namely:

$$\exists\{d_\alpha : \alpha \in \omega_1\}\ \forall f : \omega_1 \to \omega \quad \{\alpha \in \omega_1 : f{\restriction}\alpha \le d_\alpha\} \text{ is stationary.}$$

We do not know whether this strengthening of $\diamondsuit_{\mathfrak{d}}$ is really stronger.

QUESTION I.4. *Does $\diamondsuit_{\mathfrak{d}}$ imply that there is a sequence $\{d_\alpha : \alpha \in \omega_1\}$ such that $\{\alpha \in \omega_1 : f{\restriction}\alpha \le d_\alpha\}$ is stationary for every $f : \omega_1 \to \omega$?*

However, it was pointed out by J. Brendle that if there is a $\diamondsuit_{\mathfrak{d}}$-sequence then there is one such that $\{\alpha \in \omega_1 : f{\restriction}\alpha \le^* d_\alpha\}$ is stationary. J. Brendle (see [Br2]) (independently) proved Proposition I.3 and observed that $\clubsuit + \mathfrak{d} = \omega_1$ implies that there is a MAD family of size $\aleph_1$.

II. $\diamondsuit_{\mathfrak{d}}$ in the presence of CH. With every $\diamondsuit_{\mathfrak{d}}$-sequence $\bar{d}$ comes a natural forcing notion destroying it. We shall denote it by $\mathbb{P}_{\bar{d}}$. A function p is a condition if and only if $p : \alpha \to \omega$ for some $\alpha < \omega_1$ and

$$\forall \beta < \alpha + \omega \quad |\{\gamma < \alpha : d_\beta(\gamma) < p(\gamma)\}| = \aleph_0.$$

As usual the ordering is reverse inclusion.

In what follows we would like to show that it is possible to iterate these forcings without adding reals and then show that CH is consistent with $\neg\diamondsuit_{\mathfrak{d}}$. Let us remind the reader of the following notation.

A forcing notion $\mathbb{P}$ is said to be *totally proper* if for every countable elementary submodel N of $H(\theta)$ (for θ large enough) such that $\mathbb{P} \in N$ and for every $p \in N \cap \mathbb{P}$ there is a lower bound $q \le p$ for a $\mathbb{P}$-generic filter over N containing p. Every such q will be called *totally $(N,\mathbb{P})$-generic*. $\mathbb{P}$ is *$<\omega_1$-proper* if for every $\alpha < \omega_1$, every increasing $\in$-sequence $\{N_\beta : \beta \le \alpha\}$ of elementary submodels of large enough $H(\theta)$ such that $\mathbb{P} \in N_0$, and every $p \in N_0 \cap \mathbb{P}$, there is a $q \le p$ which is $(N_\beta, \mathbb{P})$-generic for every $\beta \le \alpha$. It is easy to see that $\mathbb{P}$ is proper not adding reals if and only if it is totally proper. However, the property of being totally proper is, in general, not preserved under countable support iteration.

LEMMA II.1. *Let $\bar{d}$ be a $\diamondsuit_{\mathfrak{d}}$-sequence. Then*:

(1) *If $p \in \mathbb{P}_{\bar{d}}$ and $q : \mathrm{dom}(p) \to \omega$ is such that $|\{\gamma : p(\gamma) \ne q(\gamma)\}| < \aleph_0$ then $q \in \mathbb{P}_{\bar{d}}$.*

(2) *q is $(N,\mathbb{P}_{\bar{d}})$-generic if and only if q is totally $(N,\mathbb{P}_{\bar{d}})$-generic.*

(3) $\Vdash_{\mathbb{P}_{\bar{d}}}$ "*$\bar{d}$ is not a $\diamondsuit_{\mathfrak{d}}$-sequence*".

Proof. Clause (1) is obvious. For (2) notice that p and q are compatible if and only if $p \le q$ or $q \le p$, and (3) follows from the fact that the set of conditions p with $\alpha \subseteq \mathrm{dom}(p)$ is dense for every $\alpha < \omega_1$. ∎

LEMMA II.2. *$\mathbb{P}_{\bar{d}}$ is totally proper.*

Proof. Let $\{D_n : n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}_{\bar{d}}$ in a suitable elementary submodel N and let $p \in \mathbb{P}_{\bar{d}} \cap N$. Let further $\alpha = N \cap \omega_1$ and let f be a function from α to ω almost dominating d_β for every $\beta < \alpha + \omega$ (for instance d_γ for some $\gamma \ge \alpha + \omega$). Fix also $\beta_n \nearrow \alpha$. Recursively choose $p_n \in N$ so that

(1) $p \ge p_0$ and $p_n \ge p_{n+1}$,
(2) $p_n \in D_n$,
(3) $\mathrm{dom}(p_n) = \gamma_n + 1$ and $p_n(\gamma_n) > f(\gamma_n)$, where $\gamma_n \ge \beta_n$.

Let $q = \bigcup p_n$. The only thing left to verify is that $q \in \mathbb{P}_{\bar{d}}$. To that end we have to show that $|\{\gamma : d_\beta(\gamma) < q(\gamma)\}| = \aleph_0$ for all $\beta < \alpha + \omega$.

• $\beta < \alpha$: There is an n such that $\beta \subseteq \mathrm{dom}(p_n)$ and since the above holds for p_n and $q \restriction \beta = p_n \restriction \beta$ it also holds for q.

• $\alpha \le \beta < \alpha + \omega$: $\{\gamma_n : n \in \omega\} \subseteq^* \{\gamma < \beta : d_\beta(\gamma) < q(\gamma)\}$ by our choice of f, hence the latter set is infinite. ∎

LEMMA II.3. *$\mathbb{P}_{\bar{d}}$ is $<\omega_1$-proper.*

Proof. Given $\alpha < \omega_1$, $\{N_\beta : \beta \le \alpha\}$ and $p \in \mathbb{P}_{\bar{d}} \cap N_0$ let $\delta_\beta = N_\beta \cap \omega_1$. We shall prove the lemma by induction on α assuming the following induction hypothesis:

• For all $\gamma < \beta < \alpha$ if $f \ge^* d_{\delta_\beta + \omega}$ and $q \in \mathbb{P}_{\bar{d}} \cap N_\gamma$ then there exists $q' \le q$ which is $(N_\varrho, \mathbb{P}_{\bar{d}})$-generic for all $\gamma \le \varrho \le \beta$ and $q' \not\le^* f$.

Assume $f \ge^* d_{\delta_\alpha + \omega}$.

• $\alpha = \beta + 1$: Let $q \in N_\alpha$ be as in the induction hypothesis, q generic over all N_ϱ, $\varrho \le \beta$, such that $q \not\le^* d_{\delta_\beta + \omega}$. As in Lemma II.2, extend q to q' which is generic over N_α and $q' \not\le^* f$.

• α *limit*: We mimic the proof of Lemma II.2. Let $\{D_n : n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}_{\bar{d}}$ in N_α and let $\beta_n \nearrow \alpha$ so that $D_n \in N_{\beta_n}$. Construct a decreasing sequence $p_n \in N_\alpha$ so that

(1) $p \ge p_0$ and $p_n \ge p_{n+1}$,
(2) $p_n \in D_n$,
(3) p_n is N_β-generic for every $\beta \le \beta_n$,
(4) $\mathrm{dom}(p_n) = \gamma + 1$ and $p_n(\gamma_n) > f(\gamma_n)$, where again $\gamma_n \ge \delta_{\beta_n}$.

Let $q = \bigcup p_n$. As in Lemma II.2, it is easy to verify that this construction works. ∎

In order to prove that we can iterate these forcings without adding reals we shall appeal to a theorem of T. Eisworth. First we have to introduce another definition.

Let $\mathbb{P}$ be totally proper and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion. We say that $\dot{\mathbb{Q}}$ *is* 2*-complete for* $\mathbb{P}$ if whenever

(1) $N_0 \in N_1 \in N_2$ are countable elementary models of large enough $H(\theta)$,
(2) $\mathbb{P}, \dot{\mathbb{Q}} \in N_0$,
(3) $G \in N_1$ is $\mathbb{P}$-generic over N_0 having a lower bound,
(4) $\dot{q} \in N_0$ is a $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$,

then there is a $G' \in V$, $\dot{\mathbb{Q}}$-generic over $N_0[G]$ with $\dot{q}[G] \in G'$ so that if $r \in \mathbb{P}$ is a lower bound for G and r is $(N_i, \mathbb{P})$-generic for $i = 0, 1, 2$, then there is a $\mathbb{P}$-name $\dot{s}$ so that $r \Vdash$ "$\dot{s}$ is a lower bound for G'".

THEOREM II.4 ([ER]). *If $\mathbb{P}_\alpha$ is a countable support iteration of $<\omega_1$-proper forcings such that $\dot{\mathbb{Q}}_\beta$ is 2-complete for $\mathbb{P}_\beta$ for every $\beta < \alpha$ then $\mathbb{P}_\alpha$ is totally proper and $<\omega_1$-proper.*

LEMMA II.5. *Let $\mathbb{P}$ be a totally proper $<\omega_1$-proper poset and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{P}_{\bar{d}}$ for some $\diamondsuit_{\mathfrak{d}}$-sequence $\bar{d}$. Then $\dot{\mathbb{Q}}$ is 2-complete for $\mathbb{P}$.*

Proof. Let $N_0 \in N_1 \in N_2$ be countable elementary submodels of large enough $H(\theta)$, $\mathbb{P}, \dot{\mathbb{Q}} \in N_0$. Let $G \in N_1$ be a filter $\mathbb{P}$-generic over N_0 having a lower bound and let $\dot{p} \in N_0$ be a $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$. We have to find a $\dot{\mathbb{Q}}[G]$-generic filter G' over $N_0[G]$ such that whenever $r \in \mathbb{P}$ is a lower bound for G which is also $\mathbb{P}$-generic over N_i, $i = 1, 2$, then there is a $\mathbb{P}$-name $\dot{s}$ such that $r \Vdash$ "$\dot{s}$ is a lower bound for G'".

Let $\delta = \omega_1 \cap N_0$ and $\mathcal{D} = \{D \in N_0[G] : N_0[G] \models D$ is dense open in $\dot{Q}[G]\}$. Since N_0, $\dot{Q}$ and G are all elements of N_1, we have $\mathcal{D} \in N_1$ by elementarity. N_1 also knows that $\mathcal{D}$ is countable so (working in N_1) we can enumerate $\mathcal{D}$ as $\{D_n : n \in \omega\}$. If N_1 knew what $\dot{d}_{\delta+\omega}$ evaluates to, we could simulate the proof of Lemma II.2 in N_1 and that would give us the G'. This is simply not the case. What is the case, however, is that N_2 knows that no matter how the generic filter evaluates the sequence $\bar{d}$, any r as above forces $d_{\delta+\omega} \in N_1$. To express this more formally let $\dot{G}$ denote the canonical $\mathbb{P}$-name for the $(V, \mathbb{P})$-generic filter extending G. The previous remark can then be expressed (in a slight abuse of notation) as

$$N_2 \models \dot{d}_{\delta+\omega}[\dot{G}] \in N_1.$$

Recall that by the convention we adopted, $d_{\delta+\omega}$ restricted to α dominates d_α. Let $f : \delta \to \omega$ be an element of N_2 such that that $g \leq^* f$ for every $g : \delta \to \omega$, $g \in N_1$. Now we are all set to describe the G'. Construct a decreasing sequence $p_n \in N_0[G]$ so that

(1) $\dot{p}[G] \geq p_0$ and $p_n \geq p_{n+1}$,
(2) $p_n \in D_n$,
(3) $\mathrm{dom}(p_n) = \gamma_n + 1$ and $p_n(\gamma_n) > f(\gamma_n)$, for some $\gamma_n \geq \delta_n$,

where $\delta_n \nearrow \delta$. This is done exactly the same way as in Lemma II.2. Let

$$G' = \{q \in N_0[G] : (\exists n \in \omega)\ p_n \leq q\}.$$

G' is then obviously $(N_0[G], \dot{Q}[G])$-generic and $\dot{p}[G] \in G'$. Let $r \in \mathbb{P}$ be a lower bound for G, $(N_i, \mathbb{P})$-generic for $i = 1, 2$. All that is left to prove is the following:

Claim. $r \Vdash$ "$\{p_n : n \in \omega\}$ *has a lower bound*".

To prove the Claim it is enough to realize that $s = \bigcup\{p_n : n \in \omega\} \in \dot{Q}[\overline{G}]$ for every generic filter $\overline{G}$ extending G. Note that by the construction $r \Vdash$ "$d_\delta \leq^* f$". ∎

Theorem II.6. $\mathrm{Con}(\mathrm{CH} + \neg\diamondsuit_{\mathfrak{d}})$.

Proof. Let $V \models$ GCH and construct a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < 2^{\omega_1} \rangle$ so that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{\bar{d}}$ for some $\diamondsuit_{\mathfrak{d}}$-sequence $\bar{d}$". By a standard bookkeeping argument one can make sure that all $\diamondsuit_{\mathfrak{d}}$-sequences in the intermediate models are listed. To be more careful one should consider the case where some $\diamondsuit_{\mathfrak{d}}$-sequence could have been killed by accident before it was listed but in that case we can just force with a σ-closed forcing adding many new such sequences and obviously not violating the total properness of the iteration.

By Theorem II.4, $\mathbb{P}$ is totally proper, hence CH holds in the extension. All the $\diamondsuit_{\mathfrak{d}}$-sequences that appeared in the intermediate models have been killed so the only thing that could possibly cause there being one would be if some cardinals were collapsed to ω_1. To finish the proof all we have to show is

Claim. $\mathbb{P}$ *does not collapse cardinals.*

To prove the Claim it is sufficient (since we know that $\mathbb{P}$ is proper and hence does not collapse ω_1) to show that $\mathbb{P}$ has the $\aleph_2$-chain condition. This is standard, as $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| = \aleph_1$". ∎

The next natural questions are:

(1) *Does* $\diamondsuit_{\mathfrak{d}} + \mathrm{CH}$ *imply* $\diamondsuit$*?*
(2) *Does* $\diamondsuit_{\mathfrak{d}}$ *imply* CH*?*

We shall answer both questions in the negative. In particular, we will show that if we add $\aleph_1$-many Hechler reals by iteration with finite support to any ground model we also add a $\diamondsuit_{\mathfrak{d}}$-sequence. Then we show that forcing with Hechler forcing does not add $\diamondsuit$-sequences.

Recall that the elements of the Hechler forcing (denoted by $\mathbb{H}$) are pairs (s,f), where $s \in \omega^n$ for some $n \in \omega$ and $f \in \omega^\omega$, ordered by $(s,f) < (t,g)$ if $t \subseteq s$, $g \leq f$ and $g(i) < s(i)$ for every $i \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$. As usual $\mathbb{H}_{\omega_1}$ denotes the finite support iteration of Hechler forcing. Let $\langle r_\alpha : \alpha < \omega_1 \rangle$ be the $\mathbb{H}_{\omega_1}$-generic sequence of reals.

LEMMA II.7. *The Hechler forcing $\mathbb{H}_{\omega_1}$ adds a $\diamondsuit_{\mathfrak{d}}$-sequence.*

Proof. In V fix for every $\alpha < \omega_1$ a bijection $b_\alpha : \alpha \to \omega$. In $V[\langle r_\beta : \beta \leq \alpha \rangle]$ let $d_\alpha = r_\alpha \circ b_\alpha$.

CLAIM 1. *For every $f \in \omega^\alpha \cap V[\langle r_\beta : \beta < \alpha \rangle]$, $f \leq^* d_\alpha$.*

This follows immediately from the fact that $\mathbb{H}$ adds a dominating real.

CLAIM 2. *The sequence $\{d_\alpha : \alpha < \omega_1\}$ forms a $\diamondsuit_{\mathfrak{d}}$-sequence in $V[\langle r_\alpha : \alpha < \omega_1 \rangle]$.*

Let $f \in V[\langle r_\alpha : \alpha < \omega_1 \rangle]$ be a function from ω_1 to ω. All we have to show is that there is an $\alpha < \omega_1$ such that $f{\restriction}\alpha \in V[\langle r_\beta : \beta < \alpha \rangle]$ since then by Claim 1, $f{\restriction}\alpha \leq^* d_\alpha$. Notice that for every γ there is a β such that $f{\restriction}\alpha \in V[\langle r_\delta : \delta < \beta \rangle]$. So just let $\alpha = \sup\{\alpha_n : n \in \omega\}$, where $\alpha_0 = \omega$ and $f{\restriction}\alpha_n \in V[\langle r_\delta : \delta < \alpha_{n+1} \rangle]$. ■

It is worth mentioning that the use of Hechler forcing is not essential here. The proof would go through (basically unchanged) for any iteration of length ω_1 of forcings adding a dominating real.

THEOREM II.8. Con($\diamondsuit_{\mathfrak{d}} + \neg$CH).

Proof. Let V be a model of $\neg$CH and let G be an $\mathbb{H}_{\omega_1}$-generic over V. Since $\mathbb{H}_{\omega_1}$ is a c.c.c. forcing, $\neg$CH holds in $V[G]$ and $V[G] \models \diamondsuit_{\mathfrak{d}}$ by Lemma II.7. ■

THEOREM II.9. Con(CH $+ \diamondsuit_{\mathfrak{d}} + \neg\diamondsuit$).

Proof. Let $V \models \mathrm{CH}+\neg\diamondsuit$ and let again G be an $\mathbb{H}_{\omega_1}$-generic over V. Then $V[G] \models \mathrm{CH} + \diamondsuit_{\mathfrak{d}}$ by Lemma II.7 and since $\mathbb{H}_{\omega_1}$ adds only $\aleph_1$-many reals. So the only thing we have to show is that $\mathbb{H}_{\omega_1}$ does not add a $\diamondsuit$-sequence. Assume on the contrary that $\{S_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit$-sequence in $V[G]$. Put (in V)

$$A_\alpha = \{S \subset \alpha : (\exists p \in \mathbb{H}_{\omega_1})\ p \Vdash \text{“}S = \dot{S}_\alpha\text{”}\}.$$

• *$\{A_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit^-$-sequence*: Since $\mathbb{H}_{\omega_1}$ is c.c.c., $|A_\alpha| \leq \aleph_0$ for every α. Let $A \subset \omega_1$ be a set in V. Since $\{S_\alpha : \alpha < \omega_1\}$ is forced to be a $\diamondsuit$-sequence, there is a $q \in G$ and an $\alpha < \omega_1$ such that $q \Vdash$ “$A \cap \alpha = S_\alpha$”, which implies that $A \cap \alpha \in A_\alpha$.

So if there were a $\diamondsuit$-sequence in $V[G]$ there would have to be a $\diamondsuit^-$-sequence in V, which is a contradiction since $\diamondsuit \Leftrightarrow \diamondsuit^-$ (see [Ku]). ■

III. $\diamondsuit_{\mathfrak{d}}$ in the Random real extension. In this section it will be shown that a measure algebra is essentially completely oblivious to the validity of the principle $\diamondsuit_{\mathfrak{d}}$. Let $\mathbb{B}$ denote any measure algebra (i.e. any atomless Boolean algebra carrying a strictly positive probability measure) and let G be a $\mathbb{B}$-generic filter. The measure on $\mathbb{B}$ will be denoted by μ.

THEOREM III.1. *If $V \models \neg\diamondsuit_{\mathfrak{d}}$ then $V[G] \models \neg\diamondsuit_{\mathfrak{d}}$.*

Proof. Assume on the contrary that there is a $\diamondsuit_{\mathfrak{d}}$-sequence $\bar{d} = \{d_\alpha : \alpha < \omega_1\}$ in $V[G]$. Put

$$S_\alpha = \{f \in \omega^\alpha : (\exists q \in \mathbb{B})\ q \Vdash \text{“}f \le^* d_\alpha\text{”}\}.$$

Since $\mathbb{B}$ is ω^ω-bounding and satisfies the countable chain condition there is a function $d'_\alpha \in \omega^\alpha$ which dominates S_α in V. Let $f : \omega_1 \to \omega$. There is an α such that $f{\restriction}\alpha \le^* d_\alpha$ (in $V[G]$). By the definition of S_α we have $f{\restriction}\alpha \in S_\alpha$, hence $f{\restriction}\alpha \le^* d'_\alpha$. Therefore $\{d'_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit_{\mathfrak{d}}$-sequence in V, which is a contradiction. ∎

The model obtained by forcing with a large measure algebra over a ground model produced as in Theorem II.6 is the only model of ¬CH we know where $\mathfrak{d} = \omega_1$ and $\diamondsuit_{\mathfrak{d}}$ fails. However, if one is not careful, $\diamondsuit_{\mathfrak{d}}$ can hold even in the Random real model.

THEOREM III.2. *If $V \models \diamondsuit$ then $V[G] \models \diamondsuit_{\mathfrak{d}}$.*

Proof. Using $\diamondsuit$ we will construct a $\diamondsuit_{\mathfrak{d}}$-sequence indestructible by any measure algebra.

CLAIM. ($\diamondsuit$) *There is a sequence $\{d_\alpha : \alpha < \omega_1\}$ in V such that*

(1) *$d_\alpha : \alpha \to \omega$ for every $\alpha < \omega_1$ and*

(2) *for every $F : \omega_1 \times \omega \to [0,1]$ such that $\sum_{n\in\omega} F(\xi, n) = 1$ for every $\xi < \omega_1$, there exists $\omega \le \alpha < \omega_1$ with*

$$\sum_{\xi<\alpha} \Big(\sum_{k>d_\alpha(\xi)} F(\xi, k) \Big) < 1.$$

A standard coding argument provides a sequence $\{a_\alpha : \alpha < \omega_1\}$ such that for every $F : \omega_1 \times \omega \to [0,1]$ there is a stationary set S such that $F{\restriction}(\alpha \times \omega) = a_\alpha$ for every $\alpha \in S$. Having this, construct d_α as follows: If a_α is not a function from $\alpha \times \omega$ to $[0,1]$ or if there is a $\xi < \alpha$ such that $\sum_{n\in\omega} a_\alpha(\xi, n) \ne 1$ let d_α be any function from α to ω. If $a_\alpha : \alpha \times \omega \to [0,1]$ and $\sum_{n\in\omega} a_\alpha(\xi, n) = 1$ for every $\xi < \alpha$, then enumerate α as $\{\xi_n : n \in \omega\}$ and let

$$d_\alpha(\xi_n) = \min\Big\{k : \sum_{i>k} a_\alpha(\xi_n, i) < 2^{-(n+2)}\Big\}.$$

It is easy to verify that this construction works.

To finish the proof of the theorem let $\dot f$ be a $\mathbb{B}$-name for a function from ω_1 to ω such that $\Vdash_{\mathbb{B}}$ "$(\forall\omega \le \alpha < \omega_1)\ \dot f \not\le^* d_\alpha$" and put

$$F(\alpha, n) = \mu([\![\dot f(\alpha) = n]\!]).$$

By the Claim there is an infinite α such that

$$\sum_{\xi<\alpha}\Big(\sum_{k>d_\alpha(\xi)} F(\xi,k)\Big) < 1.$$

Then, however,

$$\begin{aligned}\mu([\![(\forall\xi<\alpha)\ \dot f < d_\alpha(\xi)]\!]) &= \mu([\![(\forall\xi<\alpha)(\exists k \le d_\alpha(\xi))\ \dot f(\xi) = k]\!])\\ &= \mu\Big(\bigwedge_{\xi<\alpha}\bigvee_{k\le d_\alpha(\xi)} [\![\dot f(\xi) = k]\!]\Big)\\ &= \mu\Big(\Big(\bigvee_{\xi<\alpha}\bigvee_{k>d_\alpha(\xi)} [\![\dot f(\xi) = k]\!]\Big)^c\Big)\\ &\ge 1 - \sum_{\xi<\alpha}\Big(\sum_{k>d_\alpha(\xi)} F(\xi,k)\Big) > 0,\end{aligned}$$

which is a contradiction. ■

QUESTION III.3. *Does $V \models \diamondsuit_{\mathfrak{d}}$ imply that $V[G] \models \diamondsuit_{\mathfrak{d}}$?*

IV. $\diamondsuit_{\mathfrak{d}}$ holds after adding a single Laver real. This section is devoted to showing that adding a single Laver real adds a $\diamondsuit_{\mathfrak{d}}$-sequence. This answers a question of J. Brendle (see [Br1]) who asked whether adding one Laver real adds a MAD family of size $\aleph_1$. The proof is an extension of Brendle's result that adding a Laver real adds a dominating family of size ω_1, contained in [Br1].

Recall that a tree $T \subseteq \omega^{<\omega}$ is called a *Laver tree* if there is a $t \in T$ (called the *stem* of T) such that for all $s \in T$, $s \subseteq t$ or $t \subseteq s$, and $t \subseteq s \Rightarrow |\{n \in \omega : s^\frown n \in T\}| = \aleph_0$. If $T \subseteq \omega^{<\omega}$ is a tree and $s \in T$ we let $T_s = \{t \in T : t \subseteq s \text{ or } s \subseteq t\}$ and $[T] = \{f \in \omega^\omega : f{\restriction}n \in T \text{ for all } n \in \omega\}$.

The *Laver forcing* $\mathbb{L}$ is the set of all Laver trees ordered by inclusion. It is well known that the Laver forcing satisfies Axiom A for some sequence of orderings $\le_n$. There is no need to specify the orderings here. For a Laver tree T we say that $A \subseteq T$ is a *front* if for every $f \in [T]$ there is exactly one $n \in \omega$ such that $f{\restriction}n \in A$. The following can be found in [JS] or [Br1].

LEMMA IV.1 (The Strong Fusion Lemma, [JS]). *Given a family $\{D_n : n \in \omega\}$ of open dense subsets of $\mathbb{L}$ and a $T \in \mathbb{L}$ there is a $T' \le T$ such that $\{t \in T' : T'_t \in D_n\}$ contains a front for every n.*

We shall actually use a fact from the proof of the above lemma, namely: *Given $T \in \mathbb{L}$, an open dense subset D of $\mathbb{L}$ and an $n \in \omega$ there is a $T' \le_n T$*

such that $\{t \in T' : T'_t \in D\}$ *contains a front*. In this situation define a rank function on T' as follows:

$$\mathrm{rk}_D(t) = \begin{cases} 0 & \text{if } T'_t \in D, \\ \sup\{\mathrm{rk}_D(t^\frown n) : t^\frown n \in T'\} & \text{otherwise.} \end{cases}$$

Since $\{t \in T' : T'_t \in D\}$ contains a front $\mathrm{rk}_D(t)$ is defined for every $t \in T'$. Let $\alpha_D(T) = \mathrm{rk}_D(\emptyset)$.

THEOREM IV.2. *Let G be an $\mathbb{L}$-generic filter over V. Then $V[G] \models \diamondsuit_{\mathfrak{d}}$.*

Proof. We shall start by constructing names for the functions d_α. The construction is virtually identical to the one in [Br1]. For fixed $\alpha < \omega_1$ let T_α be a well-founded tree of rank α such that if $t^\frown n \in T_\alpha$ for some n then $t^\frown n \in T_\alpha$ for every n and let ϱ_α be the standard rank function on T_α. Define ϱ^n_α on a subset of $\omega^{<\omega}$ recursively by putting

$$\varrho^0_\alpha(t) = \begin{cases} \varrho_\alpha(t) & \text{if } t \in T_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

let $\mathrm{dom}(\varrho^{n+1}_\alpha) = \{t : \varrho^n_\alpha = 0\}$ and for $t \in \mathrm{dom}(\varrho^{n+1}_\alpha)$, let m be minimal such that $\varrho^n_\alpha(t{\restriction}m) = 0$ and let s be such that $t = t{\restriction}m^\frown s$. Then put

$$\varrho^{n+1}_\alpha(t) = \begin{cases} \varrho_\alpha(s) & \text{if } s \in T_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Fix also a bijection $b_\alpha : \alpha \to \omega$ for every α. Then define a name $\dot{d}_\alpha$ so that

$$T \Vdash \text{“}\dot{d}_\alpha(\beta) = m\text{” if and only if } \varrho^{b_\alpha(\beta)}_\alpha(s) = 0 \text{ and } s(k-1) = m,$$

where s is the stem of T and k is minimal such that $\varrho^{b_\alpha(\beta)}_\alpha(s{\restriction}k) = 0$.

• $V[G] \models \{\dot{d}_\alpha[G] : \alpha < \omega_1\}$ *is a* $\diamondsuit_{\mathfrak{d}}$*-sequence*: Let τ be an $\mathbb{L}$-name for a function from ω_1 to ω and let $T \in \mathbb{L}$. Fix a countable elementary submodel N of large enough $H(\theta)$ containing $\mathbb{L}, \tau$ and T and let $\delta = N \cap \omega_1$. We will construct an $S \le T$ such that $S \Vdash$ “$\tau{\restriction}\delta \le^* \dot{d}_\delta$”. Let $D_n = \{S' : S'$ decides $\tau(b^{-1}_\delta(n))\}$. Notice that $D_n \in N$ for every n even though N does not know b_δ. Recursively choose $T_n \in N$ so that

(1) $T_0 \le T$,
(2) $T_{n+1} \le_n T_n$,
(3) $\{t \in T_n : (T_n)_t \in D_n\}$ contains a front.

Let $T' = \bigcap\{T_n : n \in \omega\}$. Then $\{t \in T' : T'_t \in D_n\}$ contains a front for every n (since T_n does) and $\alpha_{D_n}(T') < \delta$. The latter may require a little bit of an argument. Note that $\alpha_{D_n}(T_n)$ can be evaluated without leaving N and that $\alpha_{D_n}(T') \le \alpha_{D_n}(T_n)$ as $T' \le T_n$.

Let t be the stem of T'. Let n be minimal such that $\varrho^n_\delta(t) > 0$ or $t \notin \mathrm{dom}(\varrho^n_\delta)$. By, possibly, extending t we can assume the former. Construct a tree $S \leq T'$ so that $t \in S$ and

$$\begin{aligned} s \in S \;\Rightarrow\; (\forall m \geq n)\; & 0 \leq \mathrm{rk}_{D_m}(s) < \varrho^m_\delta(s) \text{ or} \\ & 0 = \mathrm{rk}_{D_m}(s) = \varrho^m_\delta(s) \text{ and } \exists l_1 < l_2 < |s| \text{ minimal} \\ & \text{such that } \mathrm{rk}_{D_m}(s{\restriction}l_1) = 0 \text{ and } \varrho^m_\delta(s{\restriction}l_2) = 0 \text{ and} \\ & s(l_2 - 1) > l_3, \text{ where } T'_{s\restriction l_1} \Vdash \text{“}\tau(b^{-1}_\delta(m)) = l_3\text{”}. \end{aligned}$$

Intuitively, we make sure that the value of τ is decided prior to the value of d_δ which is then decided to be at least as large as the value of τ.

To do this assume $s \in S$ and that $m \geq n$ is minimal such that $\varrho^m_\delta(s) > 0$ (and also $\mathrm{rk}_{D_m}(s) < \varrho^m_\delta(s)$). If $\varrho^m_\delta(s) = 1$ and $\mathrm{rk}_{D_m}(s) = 0$ then let $l_1 \leq |s|$ be minimal such that $\mathrm{rk}_{D_m}(s{\restriction}l_1) = 0$ and let l_3 be such that $T'_{s\restriction l_1} \Vdash$ “$\tau(b^{-1}_\delta(m)) = l_3$”. Put $s^\frown l$ into S if and only if $s^\frown l \in T'$ and $l > l_3$. In all the other cases $\varrho^m_\delta(s^\frown l) > \mathrm{rk}_{D_m}(s^\frown l)$ for all but finitely many l such that $s^\frown l \in T'$ and we put these in S. We do not have to worry about m's such that $\varrho^m_\delta(s) = 0$ since then the condition is satisfied automatically, and about $\overline{m}$'s such that $\overline{m} > m$ since then $s \notin \mathrm{dom}(\varrho^{\overline{m}}_\delta)$. That finishes the construction of S.

In order to verify that, indeed, $S \Vdash$ “$\tau{\restriction}\delta \leq^* \dot{d}_\delta$” let β be such that $m = b_\delta(\beta) > n$. All that has to be checked is that whenever S_s decides both $\dot{d}_\delta(\beta)$ and $\tau(\beta)$ then $S_s \Vdash$ “$\dot{d}_\delta(\beta) > \tau(\beta)$”. By the definition of $\dot{d}_\delta$, $S_s \Vdash$ “$\dot{d}_\delta(\beta) = l$” if and only if $\varrho^m_\delta(s) = 0$ and $s(l_2 - 1) = l$ where l_2 is minimal such that $\varrho^m_\delta(s{\restriction}l_2) = 0$ and by the construction of S in that case $S_s \Vdash$ “$l > l_3 = \tau(\beta)$”. ∎

V. Concluding remarks. Now we can summarize the impact of $\diamondsuit_\mathfrak{d}$ on the values of the cardinal invariants of the continuum. As mentioned in Section I, $\diamondsuit_\mathfrak{d} \Rightarrow \mathfrak{d} = \mathfrak{a} = \omega_1$. An immediate consequence of this is that all of the following cardinal invariants are equal to ω_1 assuming $\diamondsuit_\mathfrak{d}$: $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{h}$, $\mathfrak{s}$, $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{a}$, $\mathrm{add}(\mathcal{N})$, $\mathrm{add}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$.

It has been shown in [Br1] that it is consistent that adding a single Laver real to a model of MA can leave $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c} > \omega_1$. By Theorem IV.2 that model satisfies $\diamondsuit_\mathfrak{d}$. An alternative proof of $\mathrm{Con}(\diamondsuit_\mathfrak{d} + \mathrm{non}(\mathcal{N}) > \omega_1)$ is provided by Lemma II.7. It is well known that adding $\aleph_1$ many Hechler reals to a model of MA keeps $\mathrm{non}(\mathcal{N})$ big.

As a corollary of Theorem III.2 one finds that $\diamondsuit_\mathfrak{d} + \mathrm{cov}(\mathcal{N}) = \mathfrak{r} = \mathfrak{i} = \mathfrak{u} > \omega_1$ is consistent with ZFC.

A question we do not know the answer to is whether $\diamondsuit_\mathfrak{d}$ implies that the irrationals can be partitioned into $\aleph_1$-many compact sets.

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