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# Combinatorics of filters and ideals

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Abstract. We study the combinatorial aspects of filters and ideals on countable sets, concentrating on Borel ideals and their interaction with non-definable ones. The basic tools for this study are cardinal invariants naturally associated to ideals (filters) and the Katětov and Tukey orders.

## Introduction

This paper is part survey and part research announcement. It contains no proofs, though in some cases short hints at proofs are given. It deals with combinatorial aspects of filters and ideals on countable sets. The focus is on definable (Borel, analytic, ...) ideals and filters. The reason for this is twofold. On one hand, in a great number of cases there are "critical" ideals with respect to a given combinatorial property, which are definable. In most cases these critical ideals are even Borel of a low Borel complexity. On the other hand, definable filters and ideals allow for fewer "pathologies" and the study of these can take advantage of descriptive set-theoretic methods, such as Borel determinacy, as well as forcing and combinatorial methods combined with an absoluteness argument.

We are also interested in the interaction between definable and non-definable ideals (filters). One of the first results linking properties of non-definable filters to definable ones is A. Mathias' characterization of *selective* ultrafilters as exactly those ultrafilters which intersect every tall analytic ideal [**72**]. We will show how definable ideals can be used to naturally classify non-definable ones such as maximal ideals (or, dually, ultrafilters) and maximal almost disjoint families.

The principal tools for our considerations are cardinal invariants of the continuum and closely related partial orders on ideals (filters). Many such orderings have been successfully used in the literature. We will closely examine two of them, the Katětov order and the Tukey order, as they apparently reflect combinatorial

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properties of ideals better than the more rigid Rudin-Keisler order and Borel reducibility.

The paper is organized as follows. The first section contains basic definitions of ideals, filters and their combinatorial properties as well as the definitions of the relevant cardinal invariants.

The second section deals with the problem of destructibility of ideals by forcing. It is largely based on [**45**] where a connection is established between forcings of the type $Borel(X)/I$, where $X$ is a Polish space and $I$ is a $\sigma$-ideal of Borel subsets of $X$, and forcings of the type $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is a definable ideal on $\omega$. In particular, it is shown that for a large class of definable forcings there is a definable ideal naturally associated to the forcing critical (in the Katětov order) for ideal destructibility by the forcing. The second part of section two deals with the Mathias-Prikry and Laver-Prikry type forcings and is based mostly on [**50**]. We give a combinatorial characterization of Martin's number for these forcing notions and briefly outline a rather general scheme for analyzing preservation properties for these forcing notions. In particular, we characterize for which ideals the corresponding Mathias-Prikry forcing adds a dominating real, and state sufficient and necessary criteria for preservation of $\omega$-hitting families.

In the third section we present a list of Borel ideals critical for various combinatorial properties and calculate their cardinal invariants. These calculations are sometimes routine and sometimes nontrivial. The details of many of these can be found in [**42, 48, 78**].

The short fourth section is included mostly as a further motivation for study of the Katětov order on Borel ideals. Here it is shown how Borel ideals naturally classify non-definable objects such as ultrafilters and maximal almost disjoint families.

The fifth section is devoted to basic structural analysis of the Katětov order on Borel ideals. First we present a theorem of D. Meza showing that the structure of the order is quite complex and we briefly discuss the (open) problem of the existence of (locally) minimal tall Borel ideals and its connection to Ramsey type properties of Borel ideals. Finally, we present two dichotomies for Borel ideals and analytic P-ideals, respectively. This section is based on [**46, 49, 48, 78**].

The Tukey order, cofinal types and cofinalities of analytic ideals are considered in section 6. We review basic theory of the Tukey order on analytic ideals as developed by Todorčević, Louveau-Veličković and Fremlin in [**99, 102, 70, 37**] and introduce a new class of *fragmented* $F_\sigma$ ideals. We present a dichotomy theorem for the fragmented $F_\sigma$ ideals and prove some consistency results concerning cofinalities of Borel ideals ([**51**]).

In section 7 we propose a Wadge-like order on Borel ideals based on a natural game associated to a pair of Borel ideals (see [**47**]).

The last section (section 8) treats the quotient Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$, for definable ideals $\mathcal{I}$. We very briefly review the extensive body of work on *rigidity phenomena* and *gap structure* of the quotients done by Farah [**25, 28, 29**], Todorčević [**99, 100, 102**] and Kanovei-Reeken [**57, 58**]. We also mention some isolated results on cardinal invariants of the quotients [**1, 96, 94, 41, 31**].

We have included a rather large number of open problems. They are scattered throughout the text.

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The research presented here depends heavily (explicitly or implicitly) on previous work of many mathematicians. Several of our determinacy arguments are based on games considered by C. Laflamme [**67, 66**]. We take advantage of the works of S. Solecki [**89, 87, 88**], I. Farah [**25, 27, 28**], S. Todorčević [**98, 101, 99, 100**], A. Louveau and B. Veličković [**70**], and D. Fremlin [**36, 37**] on analytic P-ideals and Tukey order, J. Brendle's and S. Shelah's work on cardinal invariants and ultrafilters [**16, 15, 18**], and J. Zapletal's work on definable forcing [**109**].

Large parts of this text are based on the PhD thesis *Ideals and filters on countable sets* written by D. Meza under my supervision. Included are also results of joint work with B. Balcar, J. Brendle, F. Hernández, D. Meza, H. Minami, D. Rojas, E. Thümmel and J. Zapletal, some of them published, some of them in the final stages of preparation.

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## 1. Preliminaries and definitions

**1.1. Ideals and filters.** A family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of a given set $X$ is an *ideal* on $X$ if

(1) for $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$,
(2) for $A, B \subset X$, $A \subset B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and
(3) $X \notin \mathcal{I}$.

In this paper we assume that all ideals on $X$ contain all finite subsets of $X$. Dual is the notion of a *filter* on $X$, i.e. $\mathcal{F} \subset \mathcal{P}(X)$ is a *filter* on $X$ if

(1) for $F, G \in \mathcal{F}$, $F \cap G \in \mathcal{F}$,
(2) for $F, G \subset X$, $F \subset G$ and $F \in \mathcal{F}$ implies $G \in \mathcal{F}$ and
(3) $\emptyset \notin \mathcal{F}$.

Given an ideal $\mathcal{I}$ on $X$ we denote by $\mathcal{I}^*$ the *dual filter*, consisting of complements of the sets in $\mathcal{I}$. Similarly, if $\mathcal{F}$ is a filter on $X$, $\mathcal{F}^*$ denotes the dual ideal. We say an ideal $\mathcal{I}$ on $X$ is *tall*[1] if for each $Y \in [X]^\omega$ there exists $I \in \mathcal{I}$ such that $I \cap Y$ is infinite. Given an ideal $\mathcal{I}$ on a set $X$, we denote by $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets, i.e. subsets of $X$ which are not in $\mathcal{I}$. If $\mathcal{I}$ is an ideal on $X$ and $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on $Y$.

We will consider mostly ideals and filters on countable sets. In that case, we typically pretend that they are, in fact, ideals or filters on $\omega$.

We consider $\mathcal{P}(\omega)$ equipped with the natural topology induced by identifying each subset of $\omega$ with its characteristic function, where $2^\omega$ is given the product

[1] Many authors prefer the term *dense*, which is probably more descriptive.

$$\begin{tikzcd}
\mathsf{cov}\,(\mathcal{N}) \arrow[r] & \mathsf{non}\,(\mathcal{M}) \arrow[r] & \mathsf{cof}\,(\mathcal{M}) \arrow[r] & \mathsf{cof}\,(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathsf{add}\,(\mathcal{N}) \arrow[uu] \arrow[r] & \mathsf{add}\,(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{cov}\,(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{non}\,(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

Cichoń's Diagram

topology. We call an ideal $\mathcal{I}$ a Borel (analytic, co-analytic,...) ideal on $\omega$ if $\mathcal{I}$ is an ideal on $\omega$ and $\mathcal{I}$ is Borel (analytic, co-analytic,...) in this topology. The same applies to filters.

An extensively studied class of ideals is the class of analytic P-ideals. An ideal $\mathcal{I}$ on $\omega$ is a P-*ideal* if for any sequence $X_n \in \mathcal{I}$, $n \in \omega$, there is an $X \in \mathcal{I}$ such that $X_n \subseteq^* X$ for all $n \in \omega$, i.e. $X \setminus X_n$ is finite for all $n \in \omega$. An ideal $\mathcal{I}$ on $\omega$ is *countably tall (or $\omega$-hitting)* [**24**] if for any sequence $X_n \in [\omega]^\omega$, $n \in \omega$, there is an $X \in \mathcal{I}$ such that $|X_n \cap X| = \aleph_0$ for all $n \in \omega$.

Let $\mathcal{I}$ be an ideal on $\omega$. We say that $\mathcal{I}$ is a P$^+$-*ideal* if for every decreasing sequence $\{X_n : n < \omega\}$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set $X$ such that $X \subseteq^* X_n$, for all $n < \omega$. We say that $\mathcal{I}$ is a Q-*ideal* if for every partition $\{F_n : n < \omega\}$ of $\omega$ into finite sets there is an $\mathcal{I}$-positive set $Y \subseteq \omega$ such that $|Y \cap F_n| \leq 1$, for all $n < \omega$. We say that $\mathcal{I}$ is a Q$^+$-*ideal* if its restriction to every positive set is a Q-ideal, i.e. if for every $\mathcal{I}$-positive set $X$ and every partition $\{F_n : n < \omega\}$ of $X$ into finite sets there is an $\mathcal{I}$-positive set $Y \subseteq X$ such that $|Y \cap F_n| \leq 1$, for all $n < \omega$.

**1.2. Cardinal invariants.** Given an ideal $\mathcal{I}$ on a set $X$, the following are standard cardinal invariants associated with $\mathcal{I}$ :

$$\begin{aligned}
\mathsf{add}\,(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \textstyle\bigcup\mathcal{A} \notin \mathcal{I}\}, \\
\mathsf{cov}\,(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \textstyle\bigcup\mathcal{A} = X\}, \\
\mathsf{cof}\,(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})\,(\exists A \in \mathcal{A})\,(I \subseteq A)\}, \\
\mathsf{non}\,(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}.
\end{aligned}$$

We denote by $\mathcal{M}$ the ideal of meager subsets of $\mathbb{R}$ and by $\mathcal{N}$ the ideal of Lebesgue null subsets of $\mathbb{R}$ (or $2^\omega$). For $f, g \in \omega^\omega$, we consider the order by *eventual dominance* $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$. A family $F \subseteq \omega^\omega$ is *bounded* if there is $h \in \omega^\omega$ such that $f \leq^* h$ for all $f \in F$; and we say $F$ is *dominating* if for any $g \in \omega^\omega$ there is $f \in F$ such that $g \leq^* f$. The corresponding cardinal invariants are the minimal cardinality $\mathfrak{b}$ of an unbounded family, and $\mathfrak{d}$, the minimal cardinality of a dominating family. The provable inequalities between the cardinal invariants of $\mathcal{M}$ and $\mathcal{N}$ are summarized in *Cichoń's diagram*[2].

For more on cardinal invariants in general and the Cichoń's diagram in particular consult [**5**] and [**11**].

[2]As usual, the arrows in the diagram point from the smaller to the larger cardinal.

$$\begin{tikzcd}
 & & \mathsf{non}^*(\mathcal{I}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \mathsf{add}^*(\mathcal{I}) \arrow[ur] \arrow[dr] & & \mathsf{cof}^*(\mathcal{I}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathsf{cov}^*(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

When we deal with ideals on countable sets, the only one of these cardinal invariants giving any information is the cofinality, as all the others are less than or equal to $\aleph_0$.

Definition 1.1 ([**42**]). Let $\mathcal{I}$ be a tall ideal on $\omega$. Define the following cardinals associated with $\mathcal{I}$ :

$$\begin{aligned}
\mathsf{add}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in \mathcal{I})(\exists A \in \mathcal{A})(A \not\subseteq^* X)\},\\
\mathsf{cov}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^{\aleph_0})(\exists A \in \mathcal{A})(|A \cap X| = \aleph_0)\},\\
\mathsf{cof}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\},\\
\mathsf{non}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\}.
\end{aligned}$$

The $\mathsf{cof}^*$ is, of course, equal to $\mathsf{cof}$ for any uncountably generated ideal.[3] Our choice of names is somewhat justified by the following: For every tall ideal $\mathcal{I}$ on $\omega$, there is a natural ideal of Borel subsets of $\mathcal{P}(\omega)$ associated with $\mathcal{I}$ defined as

$$\widehat{\mathcal{I}} = \left\{\mathcal{X} \subseteq \mathcal{P}(\omega) : (\exists I \in \mathcal{I})(\mathcal{X} \subseteq \widehat{I})\right\},$$

where $\widehat{I} = \{X \subseteq \omega : |X \cap I| = \aleph_0\}$. One can easily check that $I \subseteq^* J$ if and only if $\widehat{I} \subseteq \widehat{J}$. Hence, $\mathcal{J} \subseteq \mathcal{P}(\omega)$ is a $P$-ideal if and only if $\widehat{\mathcal{J}}$ is a $\sigma$-ideal. Then $\mathsf{add}(\widehat{\mathcal{I}}) = \mathsf{add}^*(\mathcal{I})$, $\mathsf{cov}(\widehat{\mathcal{I}}) = \mathsf{cov}^*(\mathcal{I})$, $\mathsf{non}(\widehat{\mathcal{I}}) = \mathsf{non}^*(\mathcal{I})$ and $\mathsf{cof}(\widehat{\mathcal{I}}) = \mathsf{cof}^*(\mathcal{I})$.

The inequalities holding among these cardinals are summarized in the above diagram.

It follows directly from the definition that $\mathsf{cov}^*(\mathcal{I}) \geq \mathfrak{p}$ for any tall ideal $\mathcal{I}$. Also, $\mathsf{add}^*(\mathcal{I}) \geq \aleph_1$ if and only if $\mathcal{I}$ is a P-ideal, and $\mathsf{non}^*(\mathcal{I}) \geq \aleph_1$ if and only if $\mathcal{I}$ is $\omega$-hitting.

**1.3. Orders on ideals on $\omega$.** We consider four (pre)orders on ideals on $\omega$ and discuss their impact on cardinal invariants of the ideals. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$.

- (Katětov order) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Katětov-Blass order) $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.

[3] Some of these cardinals have been originally introduced in the dual language of filters. Brendle and Shelah in [**18**] introduced cardinal invariants $\mathfrak{p}(\mathcal{F})$ and $\pi\mathfrak{p}(\mathcal{F})$ associated with an (ultra)filter $\mathcal{F}$. For tall ideal $\mathcal{I}$, $\mathsf{add}^*(\mathcal{I}) = \mathfrak{p}(\mathcal{I}^*)$, $\mathsf{cov}^*(\mathcal{I}) = \pi\mathfrak{p}(\mathcal{I}^*)$, $\mathsf{non}^*(\mathcal{I}) = \pi\chi(\mathcal{I}^*)$ and $\mathsf{cof}^*(\mathcal{I}) = \mathsf{cof}(\mathcal{I}) = \chi(\mathcal{I}^*)$.

- (Rudin-Keisler order) $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $A \in \mathcal{I}$ if and only if $f^{-1}[I] \in \mathcal{J}$.
- (Tukey order) $\mathcal{I} \leq_T \mathcal{J}$ if there is a function $f : \mathcal{I} \to \mathcal{J}$ such that for every $\subseteq$-bounded set $X \subseteq \mathcal{J}$, $f^{-1}[X]$ is $\subseteq$-bounded in $\mathcal{I}$.

We will say $\mathcal{I}$ and $\mathcal{J}$ are *Katětov-equivalent* if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. Analogously are defined *Katětov-Blass*, *Rudin-Keisler* and *Tukey-equivalences*.

There is a close relationship between the cardinal invariants of ideals and corresponding orders. The Rudin-Blass order is the strongest; obviously, $\mathcal{I} \leq_{RB} \mathcal{J}$ implies $\mathcal{I} \leq_{RK} \mathcal{J}$, and $\mathcal{I} \leq_{RK} \mathcal{J}$ implies both $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{I} \leq_T \mathcal{J}$.

Theorem 1.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$.*

*(1) If $\mathcal{I} \leq_K \mathcal{J}$ then $\mathsf{cov}^*(\mathcal{J}) \leq \mathsf{cov}^*(\mathcal{I})$.*
*(2) If $\mathcal{I} \leq_{KB} \mathcal{J}$ then $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{non}^*(\mathcal{J})$.*
*(3) If $\mathcal{I} \leq_T \mathcal{J}$ then $\mathsf{cof}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{J})$ and $\mathsf{add}^*(\mathcal{J}) \leq \mathsf{add}^*(\mathcal{I})$.*

Shoenfield's absoluteness entails that the Katětov order among Borel ideals is absolute. When dealing with Borel (or analytic) ideals in several models of set theory, we do not consider the same set, which is unlikely to be either an ideal or Borel, but rather the ideal with the same Borel code. It should also be mentioned that the it does not matter which code we take, as codes which give the same Borel set in one model give the same Borel set in any other model containing the codes.

Proposition 1.3. *If $\mathcal{I}$ and $\mathcal{J}$ are Borel ideals on countable sets then the relation $\mathcal{I} \leq_K \mathcal{J}$ is absolute for models $M \subseteq N$ such that $\omega_1^N \subseteq M$ and $\mathcal{I}, \mathcal{J} \in M$*

The same is, of course, also true for the Rudin-Keisler order, but not necessarily for the Tukey order. T. Mátrai [**73**] has recently described two analytic ideals which are Tukey equivalent if and only if $\mathsf{CH}$ holds. As of now there are no Borel examples. On the other hand, Solecki and Todorčević [**91**] showed that among analytic P-ideals the Tukey order reduces to a Borel function and therefore is also absolute.

**1.4. Ideals and submeasures.** There is an extremely close and useful connection between $F_\sigma$ ideals and analytic P-ideals, and lower semicontinuous submeasures.

Definition 1.4. A *submeasure on* a set $X$ is a function $\varphi : \mathcal{P}(X) \to [0, \infty]$ satisfying:

- $\varphi(\emptyset) = 0$,
- If $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$ and
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

To avoid trivialities, we also require that

- $\varphi(F) < \infty$ for all finite subsets of $X$.

If $\varphi$ is a submeasure on $\omega$ and satisfies:

- $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$

then $\varphi$ is called a *lower semicontinuous submeasure*, abbreviated by lscsm. To each lscsm $\varphi$ on $\omega$ naturally correspond the following two ideals:

- $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ and
- $Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$.

It is immediate from the definition that $Exh(\varphi) \subseteq Fin(\varphi)$, $Fin(\varphi)$ is an $F_\sigma$ ideal and $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal. The following fundamental theorems of Mazur and Solecki are key to the study of both $F_\sigma$-ideals and analytic P-ideals.

THEOREM 1.5 (Mazur [**76**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then $\mathcal{I}$ is an $F_\sigma$ ideal if and only if there is a lscsm $\varphi$ such that $\mathcal{I} = Fin(\varphi)$.*

THEOREM 1.6 (Solecki [**87, 88**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then:*

- *$\mathcal{I}$ is an analytic P-ideal if and only if there is a lscsm $\varphi$ such that $\mathcal{I} = Exh(\varphi)$.*
- *$\mathcal{I}$ is an $F_\sigma$ P-ideal if and only if there is a lscsm $\varphi$ such that $\mathcal{I} = Exh(\varphi) = Fin(\varphi)$.*

In particular, all analytic P-ideals are $F_{\sigma\delta}$.

**1.5. MAD families and ultrafilters.** Given two infinite subsets $A, B$ of $\omega$ we say $A$ and $B$ are *almost disjoint* if $A \cap B$ is finite. A family $\mathcal{A}$ of infinite subsets of $\omega$ is an *almost disjoint family* if $A$ and $B$ are almost disjoint for any $A, B$ distinct elements of $\mathcal{A}$. A *MAD family* is an infinite maximal almost disjoint family, i.e. an almost disjoint family such that for every infinite set $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $A \cap X$ is infinite. Given an almost disjoint family $A$ we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. Note that $\mathcal{I}(\mathcal{A})$ is a tall ideal if and only if $\mathcal{A}$ is a MAD family. In [**72**], A. Mathias proved that ideals generated by MAD families are meager but not analytic.

Every filter can be extended to a maximal filter (*ultrafilter*) by the Kuratowski-Zorn lemma. We only consider *free* ultrafilters, i.e. ultrafilters consisting of infinite sets. Ultrafilters have been thoroughly studied by both set-theorists and topologists. The most important classes of ultrafilters are: selective ultrafilters, P-points, Q-points, rapid ultrafilters and nowhere dense ultrafilters. An ultrafilter $\mathcal{U}$ on $\omega$ is:

- *selective* if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- a *P-point* if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n|$ is finite for every $n \in \omega$.
- a *Q-point* if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into finite sets there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- *rapid* if the family of increasing enumerations of elements of $\mathcal{U}$ is dominating.
- *nowhere dense* (or a $\mathsf{nwd}$-ultrafilter) if for every map $f : \omega \to \mathbb{R}$ there is a $U \in \mathcal{U}$ such that $f[U]$ is a nowhere dense subset of $\mathbb{R}$.

It is well known that an ultrafilter $\mathcal{U}$ is selective if and only if it is both a P-point and a Q-point. Also every Q-point is rapid and every P-point is $\mathsf{nwd}$ [**5**].

## 2. Destructibility of ideals by forcing

Our interest in the Katětov order stems from the study of destructibility of ideals by forcing.

DEFINITION 2.1. Given an ideal $\mathcal{I}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $\mathcal{I}$ if there is a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of $\omega$ such that

$$\Vdash_{\mathbb{P}} \text{“}I \cap \dot{X} \text{ is finite for every } I \in \mathcal{I}\text{”}.$$

Destroying an ideal (which really means destroying *tallness* of the ideal) is, in the dual language of filters, called also *diagonalizing* or *zapping* a filter. The general question, central in combinatorial set theory of the reals, is the following:

QUESTION 2.2. *When does a given forcing destroy a given ideal?*

Many open problems boil down to instances of this question: The consistency of $\mathfrak{p} < \mathfrak{t}$, the question of Roitman as to whether the existence of a dominating family of size $\aleph_1$ implies the existence of a MAD family of size $\aleph_1$, ...

**2.1. Trace ideals.** It turns out that there is a deep connection between the proper forcings of the type $P_I$ of $I$-positive Borel subsets of a Polish space $X$, ordered by inclusion, where $I$ is a $\sigma$-ideal on $X$, studied by Zapletal in [**109**], and definable ideals on countable sets and their corresponding quotient Boolean algebras. $P_I$ is a non-separative partial order whose separative quotient is the $\sigma$-algebra $Borel(X)/I$. Zapletal [**109**] has given the following characterization of properness of these forcing notions:

$P_I$ is *proper* if and only if for every countable elementary submodel $M$ of a large enough $H(\theta)$ and every condition $B \in M \cap P_I$ the set $C = \{x \in B : x$ is $M$-generic$\}$ is $I$-positive.

Another important property of forcings of the type $P_I$ is the *Continuous Reading of Names* (CRN).

DEFINITION 2.3 (Zapletal [**109**]). If $P_I$ is a proper forcing then it *has the* CRN if for every Borel function $f : B \to 2^\omega$ with an $I$-positive Borel domain $B$ there is an $I$-positive Borel set $C \subseteq B$ such that $f \restriction C$ is continuous.

Many of the common proper forcing notions, such as Cohen, random, Sacks, Miller, Laver, ... can be naturally presented as forcings of the form $P_I$ with the CRN. In particular, every proper $\omega^\omega$-bounding poset $P_I$ has the continuous reading of names, and if the ideal $I$ is $\sigma$-generated by closed sets then the forcing $P_I$ is proper and it has the continuous reading of names (see [**45**] and [**109**]).

With Zapletal [**45**] we have studied the relationship between the forcings of type $P_I$ and quotients $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is an ideal on $\omega$. The link between these classes of posets is provided by the following definition.

DEFINITION 2.4 (Brendle [**19**]). Given a $\sigma$-ideal $I$ on $\omega^\omega$, its *trace ideal* $tr(I)$ is an ideal on $\omega^{<\omega}$ defined by $a \in tr(I)$ if and only if $\{r : \exists^\infty n \in \omega\ (r \restriction n \in a)\} \in I$.

Of course, if the $\sigma$-ideal $I$ is reasonably definable, so is the ideal $tr(I)$.

THEOREM 2.5 ([**45**]). *Let $I$ be a $\sigma$-ideal on $\omega^\omega$. If $P_I$ is a proper forcing with CRN then $\mathcal{P}(\omega^{<\omega})/tr(I)$ is a proper forcing as well and it is naturally isomorphic to a two-step iteration of $P_I$ followed by an $\aleph_0$-distributive forcing.*

In some cases, we have been able to identify the $\aleph_0$-distributive "tail" forcing as $\mathcal{P}(\omega)/\mathsf{fin}$ of the $P_I$-extension; however, we do not know what it is in many other cases, such as in the case of Cohen and random forcings.

PROPOSITION 2.6 ([**45**]). *Let $I$ be a $\sigma$-ideal on $\omega^\omega$ $\sigma$-generated by a $\sigma$-compact family of closed sets. Then the forcing $P_I$ is proper and $\omega^\omega$-bounding*[4]*, and $\mathcal{P}(\omega^{<\omega})/tr(I) = P_I * P(\omega)/\mathsf{fin}$.*

[4]Recall that a forcing is *$\omega^\omega$-bounding* if it does not add unbounded reals.

It turns out that the trace ideals are critical, in the Katětov order, with respect to $P_I$-destructibility. The following theorem was discovered independently by Kurilić [**63**] and Hrušák [**43**], for the special case of Cohen forcing, then extended by Brendle and Yatabe [**19**] to a larger class of forcings and finally took the current form in [**45**].

THEOREM 2.7 ([**45**]). *If $P_I$ is a proper forcing with CRN and $\mathcal{I}$ is an ideal on $\omega$ then the following are equivalent:*

*(1) there is a $B \in P_I$ such that $B \Vdash$ "the ideal $\mathcal{I}$ is destroyed", and*
*(2) there is a $tr(I)$-positive set $a$ such that $\mathcal{I} \leq_K tr(I) \restriction a$.*

We say that an ideal $\mathcal{I}$ on $\omega$ is *$K$-uniform* if $\mathcal{I} \restriction X \leq_K \mathcal{I}$ for every $\mathcal{I}$-positive set $X$. A forcing of the form $P_I$ where $I$ is a $\sigma$-ideal on $\omega^\omega$ is *continuously homogeneous* if for every $I$-positive Borel set $B$ there is a continuous function $F : \omega^\omega \to B$ such that $F^{-1}(A) \in I$ for all $A \in I \restriction B$. It is easy to see that if $P_I$ is continuously homogeneous, then $tr(I)$ is K-uniform and hence the theorem takes a nicer form.

THEOREM 2.8. *Let $P_I$ be a proper forcing with CRN, which is continuously homogeneous, and let $\mathcal{J}$ be an ideal on $\omega$. Then the following conditions are equivalent:*

*(1) $P_I$ destroys $\mathcal{J}$*
*(2) $\mathcal{J} \leq_K tr(I)$.*

Many of the aforementioned forcings are indeed continuously homogeneous, e.g. Cohen, random, Miller, Sacks, ...

There is a close relation between the covering number of the $\sigma$-ideal and the $\mathsf{cov}^*$-number of the corresponding trace ideal.

PROPOSITION 2.9 ([**45**]). *Suppose that $I$ is a $\sigma$-ideal on $\omega^\omega$ generated by analytic sets such that $P_I$ is a proper forcing with the CRN. Then*

$$\mathsf{cov}(I) \leq \mathsf{cov}^*(tr(I)) \leq \max\{\mathsf{cov}(I), \mathfrak{d}\}.$$

The trace ideals associated to definable forcing notions are themselves definable, though they are rarely Borel. They are Borel, in fact $F_{\sigma\delta}$, for Cohen and random forcing; however, for most other simple forcing notions they are already co-analytic (or worse). The only known Borel trace ideals come from c.c.c. forcings.

QUESTION 2.10. *Is there a non-c.c.c. forcing $P_I$ such that $tr(I)$ is Borel?*

We conjectured in [**45**] that if the trace ideal is analytic then it is even Borel. We also do not have an example of a Borel trace ideal of Borel complexity higher than $F_{\sigma\delta}$.

**2.2. Laflamme, Mathias-Prikry and Laver-Prikry type forcings.** Still considering the question 2.2, rather than fixing a forcing and investigating which ideals are being destroyed, one can fix an ideal and try to find a forcing with additional "nice" properties destroying the ideal, for instance forcing not adding unbounded or dominating reals. Laflamme [**65**] has shown that

THEOREM 2.11 (Laflamme [**65**]). *Every $F_\sigma$ ideal can be destroyed by a proper $\omega^\omega$-bounding forcing.*

A variant of *Laflamme's forcing* can be easily described using Mazur's characterization of $F_\sigma$ ideals. Let $\mathcal{I}$ be an $F_\sigma$ ideal, $\mathcal{I} = \mathrm{Fin}\,(\varphi)$ for some lower semicontinuous submeasure $\varphi$ by Theorem 1.5. Define the poset $\mathbb{P}_\varphi$ as the set of all perfect finitely branching trees $T \subseteq \omega^{<\omega}$ such that $\lim_{t\in T} \varphi\,(\mathrm{succ}_T\,(t)) = \infty$[5], ordered by inclusion.

The forcing $\mathbb{P}_\varphi$ destroys $\mathcal{I}$, is $\omega^\omega$-bounding and adds a bounded eventually different real. There are several interesting problems concerning this forcing. For instance, it is not known whether it can add random reals or even just independent reals. Also, it would be interesting to characterize those submeasures $\varphi$ such that the forcing $\mathbb{P}_\varphi$ preserves outer Lebesgue measure.

Laflamme's theorem cannot be extended even to $F_{\sigma\delta}$ ideals. However, the following seems to be an open problem:

QUESTION 2.12. *Can every $F_{\sigma\delta}$-ideal be destroyed by a proper forcing not adding a dominating real? What about the density zero ideal $\mathcal{Z}$?*

Two natural and commonly used forcing notions that destroy a given ideal $\mathcal{J}$ are the Mathias-Prikry and Laver-Prikry forcings associated to $\mathcal{J}$.

DEFINITION 2.13. Let $\mathcal{J}$ be an ideal on $\omega$.

The *Mathias-Prikry forcing* $\mathbb{M}_\mathcal{J}$ associated to the ideal $\mathcal{J}$ is defined as the set of all pairs $\langle t, a\rangle$ where $t \subset \omega$ is a finite set, $a \subset \omega$ is a set in the ideal $\mathcal{J}$, and $\langle u, b\rangle \leq \langle t, a\rangle$ if $t \subset u, a \subset b$ and $a \cap u \setminus t = 0$.

We will refer to the union of the first coordinates of conditions in the generic filter as the generic subset of $\omega$, and denote it by $\dot{a}_{gen}$.

The *Laver-Prikry forcing* $\mathbb{L}_\mathcal{J}$ associated to the ideal $\mathcal{J}$ consists of perfect subtrees $T \subseteq \omega^{<\omega}$ with stem $s_T$ such that for every $t \in T$ with $s_T \subseteq t$ the set $\mathrm{succ}_T(t) \in \mathcal{J}^*$, ordered by inclusion.

We denote by $\dot{f}_{gen}$ the name for the generic function (the union of the stems of the trees in the generic filter) and by $\dot{a}_{gen}$ the range of $\dot{f}_{gen}$.

In fact, both forcings do more than destroy the ideal $\mathcal{J}$, they *separate* $\mathcal{J}$ from $\mathcal{J}^+$, i.e. $\dot{a}_{gen}$ is forced to be almost disjoint from all ground model sets in $\mathcal{J}$ and have an infinite intersection with all $\mathcal{J}$-positive ground model sets.

It is useful to introduce the corresponding cardinal invariant, the *separating number* of an ideal $\mathcal{J}$.

$$\begin{aligned}\mathsf{sep}(\mathcal{J}) = \min\{&|\mathcal{H}| + |\mathcal{K}| : \mathcal{K} \subset \mathcal{J}, \mathcal{H} \subset \mathcal{J}^+ \text{ and}\\ &\forall A \subset \omega\,((\exists J \in \mathcal{K}(|A \cap J| = \omega) \text{ or } \exists H \in \mathcal{H}(|A \cap H| < \omega))\}.\end{aligned}$$

It is clear from the definition that $\mathsf{add}^*(\mathcal{J}) \leq \mathsf{sep}(\mathcal{J}) \leq \mathsf{cov}^*(\mathcal{J})$, and that $\mathsf{sep}(\mathcal{J}) = \mathsf{cov}^*(\mathcal{J})$ if $\mathcal{J}$ is a maximal ideal (i.e. if $\mathcal{J}^*$ is an ultrafilter).

PROPOSITION 2.14. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$. If $\mathcal{I} \leq_{RK} \mathcal{J}$ then $\mathsf{sep}(\mathcal{J}) \leq \mathsf{sep}(\mathcal{I})$.*

[5]By $\mathrm{succ}_T\,(t)$ we denote the set $\{n \in \omega : t^\frown n \in T\}$ and $\lim_{t\in T} \varphi\,(\mathrm{succ}_T\,(t)) = \infty$ means that for every $N \in \omega$ the set of those $t \in T$ such that $\varphi\,(\mathrm{succ}_T\,(t)) < N$ is finite.

Both the Mathias-Prikry and Laver-Prikry forcing notions are clearly c.c.c., in fact, $\sigma$-centered. Also, $\mathbb{L}_{\mathcal{J}}$ adds a dominating real (the generic function $\dot{f}_{gen}$ is dominating).

The *rank analysis* of names introduced by Baumgartner and Dordal [**7**] for Hechler forcing is the basic tool for analyzing forcing properties of the Laver-Prikry type forcings (see e.g. [**16, 15, 17**]). There does not seem to be a direct analogue of this for the Mathias-Prikry type forcings; however, they can be analyzed by studying the following ideal associated to an arbitrary ideal $\mathcal{I}$. Let $fin$ denote the family of all non-empty finite subsets of $\omega$.

Definition 2.15. Given $\mathcal{I}$ an ideal on $\omega$, let

$$\mathcal{I}^{<\omega} = \{A \subseteq fin : (\exists I \in \mathcal{I})(\forall a \in A)\ a \cap I \neq \emptyset\}.$$

This ideal was probably first considered implicitly by Sirota [**86**] and explicitly by Louveau [**68**] in the construction of an extremally disconnected topological group.

We are now going to characterize basic preservation properties of the forcings $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$. The first property we consider is the property of not adding a Cohen real.

Theorem 2.16 ([**14**]). *Let $\mathcal{J}$ be an ideal on $\omega$. Then*

*(1) $\mathbb{M}_{\mathcal{J}}$ does not add a Cohen real if and only if $\mathcal{J}^*$ is a selective ultrafilter.*
*(2) $\mathbb{L}_{\mathcal{J}}$ does not add a Cohen real if and only if $\mathcal{J}^*$ is a $\mathsf{nwd}$-ultrafilter.*

*Moreover, if $\mathcal{J}^*$ is a selective ultrafilter, then $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ are forcing equivalent.*

Shelah and Błaszczyk have extended (2) to prove the following:

Theorem 2.17 ([**12**]). *There is a $\sigma$-centered forcing that does not add Cohen reals if and only if there is a nowhere dense ultrafilter.*

The behavior of any forcing notion $\mathbb{P}$ can be to a large extent described by its *Martin number*

$$\mathfrak{m}(\mathbb{P}) = \min\{\kappa : \neg\mathsf{MA}_{\kappa}(\mathbb{P})\},$$

i.e. $\mathfrak{m}(\mathbb{P})$ is the minimal size of a collection of dense subsets of $\mathbb{P}$ such that no filter on $\mathbb{P}$ intersects them all.

In [**18**], Brendle and Shelah characterized the Martin numbers of the Mathias-Prikry and Laver-Prikry type forcings for ultrafilters as follows:

Theorem 2.18 ([**18**]). *Let $\mathcal{U}$ be an ultrafilter. Then:*

*(1) $\mathfrak{m}(\mathbb{M}_{\mathcal{U}^*}) = \mathsf{cov}^*(\mathcal{U}^*)$ and*
*(2) $\mathfrak{m}(\mathbb{L}_{\mathcal{U}^*}) = \min\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{U}^*)\}$.*

For arbitrary ideal the situation is similar, with three changes, first the covering number has to be replaced by the separating number, in the case of the Mathias-Prikry forcing the ideal $\mathcal{I}^{<\omega}$ has to be considered, and the fact that the forcing adds Cohen reals has to be taken into account.

Theorem 2.19 ([**50**]). *Let $\mathcal{I}$ be an ideal on $\omega$ which is not maximal. Then:*

*(1) $\mathfrak{m}(\mathbb{M}_{\mathcal{I}}) = \min\{\mathsf{sep}(\mathcal{I}^{<\omega}), \mathsf{cov}(\mathcal{M})\}$ and*
*(2) $\mathfrak{m}(\mathbb{L}_{\mathcal{I}}) = \min\{\mathsf{sep}(\mathcal{I}), \mathsf{add}(\mathcal{M})\}$.*

We have already seen that the forcing $\mathbb{L}_{\mathcal{I}}$ always adds a dominating real. The question of when the forcing $\mathbb{M}_{\mathcal{I}}$ adds a dominating real was considered by Canjar [**20**] and Brendle [**15**]. Canjar [**20**] has, assuming $\mathfrak{d} = \mathfrak{c}$, constructed an ultrafilter $\mathcal{U}$ such that the forcing $\mathbb{M}_{\mathcal{U}^*}$ does not add a dominating real and noticed that such an ultrafilter necessarily has to be a P-point without rapid Rudin-Keisler predecessors. Brendle in [**15**] has (among other things) noticed that $\mathbb{M}_{\mathcal{I}}$ does not add dominating reals for any $F_\sigma$ ideal $\mathcal{I}$.

Here we present a simple combinatorial characterization of not adding a dominating real by the Mathias-Prikry type forcings.

THEOREM 2.20 ([**50**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then $\mathbb{M}_{\mathcal{I}}$ does not add a dominating real if and only if the ideal $\mathcal{I}^{<\omega}$ is a $P^+$-ideal.*

Both Canjar's and Brendle's results follow as simple corollaries. It is not clear at the moment whether for an ultrafilter $\mathcal{U}$ not adding dominating reals is equivalent to $\mathcal{U}$ being a P-point without rapid Rudin-Keisler predecessors. We also do not know whether for a Borel ideal $\mathcal{I}$, $\mathbb{M}_{\mathcal{I}}$ does not add a dominating real if and only if $\mathcal{I}$ is $F_\sigma$. A result in this direction is the following:

THEOREM 2.21 ([**50**]). *Let $\mathcal{J}$ be a Borel ideal on $\omega$. Then the following are equivalent:*

*(1) $\mathcal{J}$ can be extended to an ideal $\mathcal{I}$ (not necessarily definable) such that $\mathbb{M}_{\mathcal{I}}$ does not add a dominating real*
*(2) $\mathcal{J}$ can be extended to an $F_\sigma$ ideal.*

The combinatorics of the Mathias-Prikry forcing and the ideal $\mathcal{I}^{<\omega}$ is closely related to the problem of Malykhin (see [**80, 39**]) in general topology: *Is there a separable non-metrizable Fréchet topological group?* To each ideal $\mathcal{I}$ on $\omega$ one can naturally associate a group topology $\tau_{\mathcal{I}}$ on the countable Boolean group $[\omega]^{<\omega}$ with the symmetric difference as the group operation. The ideal $\mathcal{I}^{<\omega}$ is the ideal of sets whose closure does not contain the neutral element $\emptyset$ in this topology. The resulting group topology $\tau_{\mathcal{I}}$ on $[\omega]^{<\omega}$ is Fréchet iff every $\mathcal{I}^{<\omega}$-positive set contains an infinite set in $(\mathcal{I}^{<\omega})^{\perp}$[6] and it is metrizable if and only if the ideal $\mathcal{I}$ is countably generated.

An ideal $\mathcal{I}$ is *Fréchet* if $\mathcal{I} = \mathcal{I}^{\perp\perp}$. In other words, $\tau_{\mathcal{I}}$ is Fréchet iff $\mathcal{I}^{<\omega}$ is a Fréchet ideal. Gruenhage and Szeptycki in [**39**] asked the following instance of Malykhin's question:

QUESTION 2.22 (Gruenhage-Szeptycki [**39**]). *Is there an uncountably generated ideal $\mathcal{I}$ such that $\mathcal{I}^{<\omega}$ is Fréchet?*

The answer is known to be positive in various models of ZFC, see [**39**]. In [**17**] we have given the following partial negative answer:

THEOREM 2.23 ([**17**]). *It is consistent with* ZFC *that $\mathcal{I}^{<\omega}$ is not Fréchet for any $\aleph_1$-generated ideal $\mathcal{I}$.*

One of the crucial elements of our proof was preservation of $\omega$-hitting families by the Laver-Prikry type forcing:

THEOREM 2.24 ([**17**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then the following are equivalent:*

[6] Recall that if $\mathcal{I}$ is an ideal on a set $X$ then $\mathcal{I}^{\perp} = \{J \subseteq X : (\forall I \in \mathcal{I}) |I \cap J| < \omega\}$.

*(1)* $\mathbb{L}_{\mathcal{I}}$ *preserves* $\omega$*-hitting families,*
*(2)* $\forall X \in \mathcal{I}^+ \ \forall \mathcal{J} \leq_K \mathcal{I} \restriction X$ *(*$\mathcal{J}$ *is not* $\omega$*-hitting).*

Note that, in particular, $\mathbb{L}_{\mathcal{I}}$ preserves $\omega$-hitting families if $\mathcal{I}$ is a Fréchet ideal, but also in many other cases, such as for $\mathcal{I}$ being the ideal $\mathsf{nwd}$ of nowhere dense subsets of the rationals. A similar result also holds for the Mathias-Prikry forcing:

Theorem 2.25 ([**50**]). *Let* $\mathcal{I}$ *be an ideal on* $\omega$*. Then the following are equivalent:*

*(1)* $\mathbb{M}_{\mathcal{I}}$ *preserves* $\omega$*-hitting families,*
*(2)* $\forall X \in (\mathcal{I}^{<\omega})^+ \ \forall \mathcal{J} \leq_K \mathcal{I}^{<\omega} \restriction X$ *(*$\mathcal{J}$ *is not* $\omega$*-hitting).*

While there is an abundance of ideals for which the forcing $\mathbb{L}_{\mathcal{I}}$ preserves $\omega$-hitting, there is no known $\mathsf{ZFC}$ example of an ideal $\mathcal{I}$ such that $\mathbb{M}_{\mathcal{I}}$ preserves $\omega$-hitting. This can be seen as a variant of the question of Gruenhage and Szeptycki.

Question 2.26. *Is there in* $\mathsf{ZFC}$ *an uncountably generated ideal* $\mathcal{I}$ *such that* $\mathbb{M}_{\mathcal{I}}$ *preserves* $\omega$*-hitting families?*

The results concerning Borel ideals contained here depend on the study of Katětov order contained in subsequent sections. We present them here as part of the motivation for the study of the Katětov order on Borel ideals.

## 3. Some critical Borel ideals and their cardinal invariants

**3.1. The nowhere dense ideal $\mathsf{nwd}$.** The nowhere dense ideal $\mathsf{nwd}$ is the ideal on the set of rational numbers $\mathbb{Q}$ whose elements are the nowhere dense subsets of $\mathbb{Q}$. $\mathsf{nwd}$ is an $F_{\sigma\delta}$ ideal. It is naturally isomorphic to the trace ideal corresponding to the Cohen forcing. In particular, for an ideal $\mathcal{I}$ on $\omega$,

$$\mathcal{I} \text{ is Cohen-destructible if and only if } \mathcal{I} \leq_K \mathsf{nwd}.$$

It is a result of Keremedis [**62**] (see also [**1**]) that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$. Fremlin [**37**] proved that $\mathsf{cof}(\mathsf{nwd}) = \mathsf{cof}(\mathcal{M})$. Finally, any countable base for open sets of $\mathbb{Q}$ is a witness for $\mathsf{non}^*(\mathsf{nwd}) = \aleph_0$.

To highlight the close relationship between the Katětov order and the $\mathsf{cov}^*$-number of an ideal, we mention the following

Proposition 3.1 ([**48**]). *Let* $\mathcal{I}$ *be a Borel ideal on* $\omega$*. Then* $\mathcal{I} \leq_K \mathsf{nwd}$ *if and only if* $\mathsf{ZFC} \vdash \mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}(\mathcal{M})$.

Proof. One implication follows directly from 1.2. To see the other, assume that $\mathsf{ZFC} \vdash \mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}(\mathcal{M})$. Add $\mathfrak{c}^+$-many Cohen reals. Then $\mathsf{cov}^*(\mathcal{I}) > \mathsf{cov}^*(\mathcal{I})^V$, so $\mathcal{I}$ is Cohen-destructible and hence $\mathcal{I} \leq_K \mathsf{nwd}$. □

**3.2. The eventually different ideals.** The *eventually different ideal* is defined by

$$\mathcal{ED} = \{A \subset \omega \times \omega : (\exists m, n \in \omega)(\forall k > n)\,(|\{l : \langle k, l\rangle \in A\}| \leq m)\}.$$

It is easily seen that the ideal $\mathcal{ED}$ is not $\omega$-hitting, so $\mathsf{add}^*(\mathcal{ED}) = \mathsf{non}^*(\mathcal{ED}) = \aleph_0$. Furthermore, $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$ and $\mathsf{cof}^*(\mathcal{ED}) = \mathfrak{c}$ (see [**48**]). The ideal $\mathcal{ED}$ is critical for *selective* ideals:

PROPOSITION 3.2. *Let $\mathcal{I}$ be an ideal on $\omega$. Then $\mathcal{ED} \leq_K \mathcal{I}$ if and only if there is a partition of $\omega$ into sets in $\mathcal{I}$ such that every selector is in $\mathcal{I}$.*

We also consider the ideal $\mathcal{ED}_{fin} = \mathcal{ED} \restriction \triangle$, where $\triangle = \{\langle m, n\rangle : n \leq m\}$. It is critical for Q-ideals, in much the same way:

PROPOSITION 3.3 ([**48**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$ if and only if there is a partition of $\omega$ into finite sets such that every selector is in $\mathcal{I}$.*

Moreover, it is the KB-least $\omega$-hitting ideal among definable ideals. We state the theorem for Borel ideals, in order to keep the determinacy arguments simple and intuitive. However, this and other similar theorems are typically true either for analytic or co-analytic ideals in ZFC, and for ideals of higher complexities assuming determinacy at corresponding levels of the projective hierarchy.

THEOREM 3.4 ([**48**]). *If $\mathcal{I}$ is a Borel ideal on $\omega$, then $\mathsf{non}^*(\mathcal{I}) = \omega$ or $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$.*

PROOF. For a Borel ideal $\mathcal{I}$, consider the following two-player game: In stage $k$, Player I chooses a finite subset $F_k$ of $\omega$ and then Player II chooses a natural number $n_k \notin F_k$.

| $I$ | $F_0 \in [\omega]^{<\omega}$ | | $F_1 \in [\omega]^{<\omega}$ | | $\dots$ | |
|---|---|---|---|---|---|---|
| $II$ | | $n_0 \notin F_0$ | | $n_1 \notin F_1$ | | $\dots$ |

Player I wins if $\{n_i : i \in \omega\} \in \mathcal{I}$ and Player II wins $\{n_i : i \in \omega\} \in \mathcal{I}^+$.
Now, by Borel determinacy, the game is determined, so it suffices to note that:

(1) If Player I has a winning strategy then $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$.
(2) If Player II has a winning strategy, then $\mathsf{non}^*(\mathcal{I}) = \omega$. $\square$

In particular, $\mathsf{non}^*(\mathcal{I}) = \omega$ or $\mathsf{non}^*(\mathcal{ED}_{fin}) \leq \mathsf{non}^*(\mathcal{I})$ for all Borel ideals $\mathcal{I}$. It also follows that every $\omega$-splitting[7] Borel ideal contains a perfect $\omega$-splitting subset, which is a special case of a theorem of Spinas [**92**].

The $\mathsf{cov}^*(\mathcal{ED}_{fin})$ and $\mathsf{non}^*(\mathcal{ED}_{fin})$ can be viewed as *bounded* versions of $\mathsf{non}(\mathcal{M})$ and $\mathsf{cov}(\mathcal{M})$, respectively, and extend in a natural way Cichoń's diagram.

PROPOSITION 3.5 ([**48**]). *The following hold:*

*(1)* $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$ *and*
*(2)* $\mathsf{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\}$.

Let us also mention that the min and max in the proposition are sharp. In the Random real model, i.e. a model obtained from a model of CH by adding at least $\aleph_2$-many random reals, $\mathsf{cov}^*(\mathcal{ED}_{fin}) > \mathfrak{d}$, and $\mathsf{cov}^*(\mathcal{ED}_{fin}) < \mathsf{add}(\mathcal{M})$ holds in the Hechler model. In a sense dually, $\mathsf{non}^*(\mathcal{ED}_{fin}) < \mathfrak{b}$ holds after adding $\aleph_1$-many random reals to a model of Martin's Axiom. The fact that $\mathsf{cof}(\mathcal{M}) < \mathsf{non}^*(\mathcal{ED}_{fin})$ holds after adding $\aleph_1$-many Hechler reals to a model of MA, is an unpublished result of J. Brendle.

[7]A family $\mathcal{S}$ of infinite subsets of $\omega$ is *$\omega$-splitting* if for every countable collection of infinite subsets of $\omega$ there is an element of $\mathcal{S}$ which splits all elements of the collection.

**3.3. Fubini products.** Given two ideals $\mathcal{I}, \mathcal{J}$ on $\omega$, the *Fubini product* $\mathcal{I} \times \mathcal{J}$ is defined by

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega : \{n : (A)_n \notin \mathcal{J}\} \in \mathcal{I}\}.$$

It is easy to see that the Fubini product of Borel ideals is also a Borel ideal. Actually, if $\mathcal{I}$ is a $\Sigma_\alpha$ ideal and $\mathcal{J}$ is a $\Sigma_\beta$ ideal then $\mathcal{I} \times \mathcal{J}$ is a $\Sigma_{\beta+\alpha}$ ideal.

Here we violate our implicit agreement, that all ideals contain all finite subsets of their underlying sets, this facilitates the definitions (and natural names) for the following two ideals.

$\mathsf{fin} \times \emptyset$ can be thought of as the ideal generated by an infinite partition of $\omega$ into infinite sets. It is countably generated, hence not tall.

$\emptyset \times \mathsf{fin}$ can be viewed as an ideal $\mathcal{I}$ for which there is a partition of $\omega$ into infinitely many infinite sets $\{P_n : n < \omega\}$, such that $I \in \mathcal{I}$ if and only if $I \cap P_n$ is finite for all $n < \omega$. It is not a tall ideal and consequently is Katětov equivalent with $\mathsf{fin}$. It is an $F_{\sigma\delta}$ $P$-ideal, with $\mathsf{cof}(\emptyset \times \mathsf{fin}) = \mathfrak{d}$ and $\mathsf{add}^*(\emptyset \times \mathsf{fin}) = \mathfrak{b}$.

$\mathsf{fin} \times \mathsf{fin}$ is an $F_{\sigma\delta\sigma}$ ideal. It is critical with respect to the following P-like property:

Proposition 3.6. *Given an ideal $\mathcal{I}$ on $\omega$, $\mathcal{I} \geq_K \mathsf{fin} \times \mathsf{fin}$ if and only if there is a partition $\{Q_n : n < \omega\}$ of $\omega$ into infinite sets in $\mathcal{I}$ such that every $A \subseteq \omega$ satisfying $|A \cap Q_n| < \aleph_0$ is in $\mathcal{I}$.* □

Its cardinal invariants are: $\mathsf{add}^*(\mathsf{fin}\times\mathsf{fin}) = \mathsf{non}^*(\mathsf{fin}\times\mathsf{fin}) = \aleph_0$, $\mathsf{cov}^*(\mathsf{fin}\times\mathsf{fin}) = \mathfrak{b}$ and $\mathsf{cof}(\mathsf{fin} \times \mathsf{fin}) = \mathfrak{d}$.

Building on an earlier work of Solecki [**89**] and a natural game introduced by Laflamme [**66**], Laczkovich and Recław [**64**] proved the following dichotomy.

Theorem 3.7 ([**64**]). *Let $\mathcal{I}$ be a Borel ideal. Then either*

*(1) $\mathcal{I} \geq_K \mathsf{fin} \times \mathsf{fin}$, or*
*(2) $\mathcal{I}$ and $\mathcal{I}^*$ can be separated by an $F_\sigma$ set, i.e there is an $F_\sigma$ set $X$ containing $\mathcal{I}$ and disjoint from $\mathcal{I}^*$.*

In particular, no ideal Katětov-above $\mathsf{fin} \times \mathsf{fin}$ can be extended to an $F_{\sigma\delta}$ ideal as any $F_{\sigma\delta}$ ideal can be separated from its dual according to a theorem of Solecki [**89**].

**3.4. $\mathsf{conv}$.** An ideal closely related to $\mathsf{fin} \times \mathsf{fin}$ is the ideal $\mathsf{conv}$, defined as the ideal on $\mathbb{Q} \cap [0,1]$ generated by sequences in $\mathbb{Q} \cap [0,1]$ convergent in $[0,1]$. $\mathsf{conv}$ is an $F_{\sigma\delta\sigma}$ ideal. Every $\mathsf{conv}$-positive set contains a positive subset $X$ such that $\mathsf{conv} \restriction X$ is naturally isomorphic to $\mathsf{fin} \times \mathsf{fin}$. The cardinal invariants of $\mathsf{conv}$ are trivial: $\mathsf{add}^*(\mathsf{conv}) = \mathsf{non}^*(\mathsf{conv}) = \aleph_0$ and $\mathsf{cov}^*(\mathsf{conv}) = \mathsf{cof}(\mathsf{conv}) = \mathfrak{c}$.

The following theorem characterizes those ideals which are Katětov above the ideal $\mathsf{conv}$.

Theorem 3.8 ([**78**]). *For any ideal $\mathcal{I}$ on $\omega$ the following are equivalent*

*(1) $\mathcal{I} \geq_K \mathsf{conv}$,*
*(2) there is a countable family $\mathcal{X} \subseteq [\omega]^\omega$ such that for every $Y \in \mathcal{I}^+$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.*

the ideal $\mathsf{conv}$ is also a lower bound for all trace ideals in the Katětov order:

Theorem 3.9 ([**78**]). *Let $\mathcal{I}$ be an ideal on $\omega$ such that the quotient $\mathcal{P}(\omega)/\mathcal{I}$ is a proper forcing adding a new real. Then there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \restriction X \geq_K \mathsf{conv}$.*

**3.5. The random graph ideal $\mathcal{R}$.** Given a graph $G$ on $\omega$, one can define a (possibly improper) ideal $\mathcal{I}_G$ as the ideal generated by the set of all the subsets of $\omega$ which are homogeneous (cliques and free sets) for $G$. We consider the ideal $\mathcal{R}$ on $\omega$ generated by the homogeneous sets in the *Rado graph* (also called the *random graph*) $E$ . The Rado graph is determined uniquely (up to an isomorphism) by the following extension property:

Given $a$ and $b$ disjoint finite subsets of $\omega$ there is $k < \omega$ such that $\{\{k,l\} : l \in a\} \subseteq E$ and $\{\{k,l\} : l \in b\} \cap E = \emptyset$.

It immediately follows that the Rado graph is *universal*, i.e. given a graph $\langle \omega, G\rangle$, there is a subset $X \subseteq \omega$ such that $\langle \omega, G\rangle \cong \langle X, E \restriction X\rangle$.

The ideal $\mathcal{R}$ is $F_\sigma$ and, by Ramseys theorem, it is tall. Its cardinal invariants are trivial: $\mathsf{add}^*(\mathcal{R}) = \mathsf{non}^*(\mathcal{R}) = \aleph_0$, $\mathsf{cov}^*(\mathcal{R}) = \mathsf{cof}(\mathcal{R}) = \mathfrak{c}$.

Consider the following Ramsey property of ideals:

Definition 3.10. Let $\mathcal{I}$ be an ideal on $\omega$. We will say that $\mathcal{I}$ satisfies

$$\omega \longrightarrow (\mathcal{I}^+)^2_2$$

if for every coloring $\varphi : [\omega]^2 \to 2$ there is an $\mathcal{I}$-positive set $X$ homogeneous with respect to $\varphi$. We will say that $\mathcal{I}$ satisfies

$$\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$$

if for every $\mathcal{I}$-positive set $X$ and every coloring $\varphi : [X]^2 \to 2$ there is an $\mathcal{I}$-positive subset $Y$ of $X$ homogeneous with respect to $\varphi$.

The ideal $\mathcal{R}$ is critical with respect to the property $\omega \longrightarrow (\mathcal{I}^+)^2_2$.

Proposition 3.11. *Let $\mathcal{I}$ be an ideal on $\omega$. Then,*

$$\omega \longrightarrow (\mathcal{I}^+)^2_2 \quad \textit{if and only if} \quad \mathcal{I} \not\geq_K \mathcal{R}.$$

In particular, the following conditions are equivalent:

- $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$,
- $\mathcal{R} \not\leq_K \mathcal{I} \restriction X$, for all $X \in \mathcal{I}^+$.

**3.6. Solecki's ideal $\mathcal{S}$.** Denote by $Clop(2^\omega)$ the (countable) family of all clopen subsets of the Cantor set $2^\omega$, and let $\lambda$ denote the standard Haar measure on $2^\omega$. *Solecki's ideal* $\mathcal{S}$ [**89**] is the ideal on the countable set

$$\Omega = \{A \in Clop(2^\omega) : \lambda(A) = \frac{1}{2}\},$$

generated by the subsets of $\Omega$ with non-empty intersection. Equivalently, a subbase for $\mathcal{S}$ is the family of all subsets of $\Omega$ of the form:

$$I_x = \{A \in \Omega : x \in A\}$$

where $x$ is an element of $2^\omega$.

The ideal $\mathcal{S}$ is a tall $F_\sigma$ ideal, whose cardinal invariants are: $\mathsf{add}^*(\mathcal{S}) = \mathsf{non}^*(\mathcal{S}) = \aleph_0$, $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{S}) = \mathfrak{c}$ [**48**].

The ideal $\mathcal{S}$ is critical for ideals which fail to satisfy the Fubini property.

DEFINITION 3.12. We will say that an ideal $\mathcal{I}$ satisfies the *Fubini property* if for any Borel subset $A$ of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda(A_n) > \varepsilon\} \in \mathcal{I}^+$ implies $\lambda^*(\{x \in 2^\omega : A^x \in \mathcal{I}^+\}) \geq \varepsilon$, where $\lambda^*$ denotes the outer Lebesgue (Haar) measure on $2^\omega$.

Of course, for ideals which are universally measurable, in particular, for Borel ideals, the outer measure in the definition can be replaced by measure. Solecki [**89**] noticed that:

THEOREM 3.13 (Solecki [**89**]). *An ideal $\mathcal{I}$ fails to satisfy the Fubini property if and only if there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.*

**3.7. Summable ideals.** Given $f : \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{n\in\omega} f(n) = \infty$, the *summable ideal* corresponding to $f$ is the ideal

$$\mathcal{I}_f = \left\{ A \subseteq \omega : \sum_{n\in A} f(n) < \infty \right\}.$$

The ideal $\mathcal{I}_f$ is tall if and only if $\lim_{n\to\infty} f(n) = 0$. The lower semicontinuous submeasure on $\omega$ corresponding to $\mathcal{I}_f$ is: $\varphi_f(A) = \sum_{n\in A} f(n)$. By definition $\mathcal{I}_f = \mathrm{Fin}(\varphi_f)$. So, summable ideals are $F_\sigma$. A typical example of a summable ideal is the ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subseteq \omega : \sum_{n\in A} \frac{1}{n} < \infty \right\}.$$

**3.8. Asymptotic density zero ideal.** The ideal $\mathcal{Z}$ of subsets of $\omega$ of *asymptotic density zero* is the ideal

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \tfrac{|A\cap n|}{n} = 0 \right\}.$$

Equivalently, $A \in \mathcal{Z}$ if and only if

$$\lim_{n\to\infty} \frac{\left|A \cap [2^n, 2^{n+1})\right|}{2^n} = 0.$$

Both the tall summable ideals and the density zero ideal $\mathcal{Z}$ are tall analytic P-ideals, hence are KB-above $\mathcal{ED}_{fin}$. Consequently, their covering numbers are below $\mathsf{cov}^*(\mathcal{ED}_{fin})$ and, dually, their uniformity numbers are above $\mathsf{non}^*(\mathcal{ED}_{fin})$. Also, the summable ideals are random-destructible; hence $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}^*(\mathcal{I})$ and $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{non}(\mathcal{N})$ for every tall summable ideal $\mathcal{I}$ [**42**].

For the density zero ideal, there are some upper and lower bounds

$$\min\{\mathfrak{b}, \mathsf{cov}(\mathcal{N})\} \leq \mathsf{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathsf{non}(\mathcal{N})\}$$

(see [**42**]) and many questions. It is not even known whether it can be destroyed by an $\omega^\omega$-bounding forcing.

For additivity and cofinality the results are optimal [**37**]: $\mathsf{add}^*(\mathcal{I}) = \mathsf{add}(\mathcal{N})$ and $\mathsf{cof}^*(\mathcal{I}) = \mathsf{cof}(\mathcal{N})$, for every tall ideal $\mathcal{I}$ which is either summable or a density ideal (see [**25**]) for the definition of a density ideal).

## 4. Katětov order, ultrafilters and MAD families

The Katětov order was introduced by Miroslav Katětov in [**59**] together with an order that became known as the Rudin-Keisler order. On ultrafilters (or equivalently maximal ideals) the two orders coincide. Whereas the Rudin-Keisler order has been extensively studied, the Katětov order has been somewhat neglected. One of the primary objectives of this survey is to show that it is both useful and intrinsically interesting, and deserves further study. For some early results on Katětov order see [**22**].

**4.1. Elementary facts about Katětov order.** Some immediate properties of Katětov order are listed here. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$.

(1) $\mathcal{I} \simeq_K \mathsf{fin}$ if and only if $\mathcal{I}$ is not tall.
(2) If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$.
(3) If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_K \mathcal{I} \restriction X$.
(4) $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{I}$ and $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{J}$.
(5) $\mathcal{I}, \mathcal{J} \leq_K \mathcal{I} \times \mathcal{J}$.

Here $\mathcal{I} \oplus \mathcal{J}$ denotes the disjoint sum of $\mathcal{I}$ and $\mathcal{J}$. Properties (4) and (5) show that Katětov order is both upward and downward directed. The following proposition lists some of the order-theoretic properties of the Katětov order.

PROPOSITION 4.1 ([**78, 44**]). *The following hold.*

*(1) Every family $\mathcal{A}$ of at most $\mathfrak{c}$ ideals has a $\leq_K$-lower bound.*
*(2) The family of maximal ideals is cofinal in Katětov order.*
*(3) Ideals generated by MAD families are coinitial among tall ideals in Katětov order.*

THEOREM 4.2 ([**78, 44**]). *Let $\mathcal{I}$ be a tall ideal on $\omega$. Then*

*(1) there is a $\leq_K$-antichain below $\mathcal{I}$ of cardinality $\mathfrak{c}$ and*
*(2) there is a $\leq_K$-decreasing chain of length $\mathfrak{c}^+$ below $\mathcal{I}$.*

**4.2. Ultrafilters and Katětov order.** In this section we study the critical ideals for well studied classes of ultrafilters: P-points, Q-points, selective ultrafilters and rapid ultrafilters. We conclude this section with the study of $\mathcal{S}$-ultrafilters, i.e. the ultrafilters which satisfy the Fubini property.

THEOREM 4.3 (Mathias [**72**]). *Let $\mathcal{U}$ be an ultrafilter on $\omega$. Then $\mathcal{U}$ is selective if and only if $\mathcal{U}$ intersects every tall analytic ideal $\mathcal{I}$.*

Zapletal [**110**] has recently found a characterization of P-points similar in spirit.

THEOREM 4.4 (Zapletal [**110**]). *Let $\mathcal{U}$ be an ultrafilter on $\omega$. Then $\mathcal{U}$ is a P-point if and only if for every Borel ideal $\mathcal{I}$ disjoint from $\mathcal{U}$ there is an $F_\sigma$-ideal $\mathcal{J}$ disjoint from $\mathcal{U}$ and containing $\mathcal{I}$.*

J. Baumgartner introduced the following definition in [**6**]. Let $I$ be a family of subsets of a set $X$ such that $I$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ is an *$I$-ultrafilter* if for every function $F : \omega \to X$ there is an $A \in \mathcal{U}$ such that $F[A] \in I$.

PROPOSITION 4.5. *Let $\mathcal{I}$ be an ideal on $\omega$. Then an ultrafilter $\mathcal{U}$ on $\omega$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \not\leq_K \mathcal{U}^*$.*

Many standard combinatorial properties of ultrafilters are easily seen to be characterized in this way by Borel ideals of a low complexity (see [**35**] for details). Let $\mathcal{U}$ be an ultrafilter and $\mathcal{U}^*$ the dual ideal. Then

- $\mathcal{U}$ is selective iff $\mathcal{ED} \not\leq_K \mathcal{U}^*$ iff $\mathcal{R} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a P-point iff $\mathsf{fin} \times \mathsf{fin} \not\leq_K \mathcal{U}^*$ iff $\mathsf{conv} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a nowhere dense ultrafilter iff $\mathsf{nwd} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a Q-point iff $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{U}^*$,
- $\mathcal{U}$ is rapid iff $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for any analytic P-ideal $\mathcal{I}$.

Another, perhaps less standard property of ultrafilters was considered by M. Benedikt [**8, 9**]. Given an ultrafilter $\mathcal{U}$ on $\omega$, and a sequence $\langle A_n : n \in \omega\rangle$ of Borel subsets of the Cantor space $2^\omega$, the *$\mathcal{U}$-limit* of the sequence $\langle A_n : n \in \omega\rangle$ is the set

$$\mathcal{U}\text{-}\lim A_n = \{x \in 2^\omega : \{n \in \omega : x \in A_n\} \in \mathcal{U}\}.$$

If $\langle x_n : n < \omega\rangle$ is a sequence of real numbers then $l \in \mathbb{R}$ is the $\mathcal{U}$-limit of $\langle x_n : n < \omega\rangle$ provided that $\{n < \omega : |x_n - l| < \varepsilon\} \in \mathcal{U}$ for all $\varepsilon > 0$.

PROPOSITION 4.6. *Let $\mathcal{U}$ be a free ultrafilter. Then the following conditions are equivalent:*

*(1) $\mathcal{S} \not\leq_K \mathcal{U}^*$,*
*(2) $\mathcal{U}^*$ satisfies the Fubini property and*
*(3) for any sequence $\langle A_n : n < \omega\rangle$ of Borel subsets of $2^\omega$,*

$$\text{if } \mathcal{U}\text{-}\lim \lambda(A_n) > 0 \text{ then } \mathcal{U}\text{-}\lim A_n \neq \emptyset.$$

It is well known that for every one of the properties considered in this section it is relatively consistent with $\mathsf{ZFC}$ that there are no ultrafilters satisfying it (see [**5**]).

QUESTION 4.7. *Is there a Borel ideal $\mathcal{I}$ such that in $\mathsf{ZFC}$ there is an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$? What about the density zero ideal $\mathcal{Z}$?*

Concerning the density zero ideal, Gryzlov [**40**] showed that in $\mathsf{ZFC}$ there is an ultrafilter $\mathcal{U}$ such that for any injective function $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ such that $f[U] \in \mathcal{Z}$. Flašková in [**34**] improved on Gryzlov's result by showing that the density ideal can be replaced by the summable ideal $\mathcal{I}_{\frac{1}{n}}$.

There is also the well-known problem:

QUESTION 4.8. *Is it consistent with $\mathsf{ZFC}$ that there are neither P-points nor Q-points?*

There are either P-points or Q-points in every model of $\mathfrak{c} \leq \aleph_2$ (see [**5**]).

**4.3. MAD families and Katětov order.** Just as we saw that the upward Katětov-cones of definable ideals stratify and classify ultrafilters, the downward cones do the same for MAD families. Given a (Borel) ideal $\mathcal{I}$ one could, dualizing Baumgartner's definition, call a MAD family $\mathcal{A}$ *$\mathcal{I}$-MAD* if $\mathcal{I}(\mathcal{A}) \not\leq_K \mathcal{I}$.

For the K-uniform trace ideals, of course, $\mathcal{A}$ is $tr(I)$-MAD if and only if it is $P_I$-indestructible. It is easy to see that $\mathcal{I}(\mathcal{A}) \leq_K \mathsf{fin} \times \mathsf{fin}$, for every MAD family $\mathcal{A}$. There are many natural questions concerning MAD families and the Katětov order. We mention three of them:

QUESTION 4.9 ([**44**]). *Is there (consistently) a MAD family $\leq_K$-maximal among MAD families?*

QUESTION 4.10 (Steprāns [**95**]). *Is there a Cohen-indestructible MAD family in* ZFC*?*

QUESTION 4.11 ([**52**]). *Is there a Sacks-indestructible MAD family in* ZFC*?*

There is a published incorrect answer to the last question. In [**43**] I showed that there is a $\overline{\mathsf{ctbl}}$-MAD family in ZFC, here $\overline{\mathsf{ctbl}}$ denotes the ideal of the subsets of the rationals with countable closure (in the reals). The mistake is that $\overline{\mathsf{ctbl}}$ is not (K-equivalent to) the trace ideal $tr(\mathsf{ctbl})$, corresponding to the Sacks forcing.

## 5. Katětov order on Borel ideals

Motivated by the results of the previous sections we now turn our attention to the study of the Katětov order restricted to Borel ideals.

**5.1. Katětov order is complex.** Of course, there are only $\mathfrak{c}$-many Borel ideals, so, for instance, the fact that there are decreasing chains of length $\mathfrak{c}^+$ in the Katětov order no longer holds when restricted to Borel ideals. However, Katětov order even when restricted to Borel ideals is complex.

THEOREM 5.1 (D. Meza [**78**]). *There is an order embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into Borel ideals ordered by the Katětov order. In fact, there is such an embedding into summable ideals, in particular, $F_\sigma$ P-ideals.*

PROOF. Fix a partition of $\omega$ into finite intervals $\langle I_n : n < \omega\rangle$ such that $\min(I_{n+1}) = \max(I_n) + 1$, and a sequence $\langle r_n : n < \omega\rangle$ of real numbers in $(0, 1]$ such that:

(1) $|I_n| \cdot r_n \geq |\bigcup_{j<n} I_j|$ and
(2) $|I_n| \cdot r_{n+1} \leq 2^{-n-1}$.

For each infinite subset $A$ of $\omega$, define a function $f_A : \omega \to (0, 1]$ such that for every $k < \omega$

$$f_A(k) = \begin{cases} r_n & \text{if } k \in I_n \text{ and } n \notin A, \\ r_{n+1} & \text{if } k \in I_n \text{ and } n \in A. \end{cases}$$

Then:

- For every infinite and co-infinite subset $A$ of $\omega$, $\mathcal{I}_{f_A}$ is a non-trivial tall summable ideal.
- Let $A, B \in [\omega]^\omega$. Then $A \subseteq^* B$ if and only if $\mathcal{I}_{f_A} \leq_K \mathcal{I}_{f_B}$. □

Note that this, in particular, shows that there are antichains of size $\mathfrak{c}$ and both increasing and decreasing chains of length $\mathfrak{b}$ in the Katětov order restricted to Borel ideals. We do not know whether there are in ZFC increasing and/or decreasing chains of length $\mathfrak{c}$ in the Katětov order on Borel ideals.

QUESTION 5.2. *Are there consistently two (or more) distinct covering numbers of summable ideals?*

**5.2. Looking for (locally) Katětov minimal Borel ideals.** We do not know the answer to the following basic question:

QUESTION 5.3. *Is there a tall Borel ideal Katětov-minimal among tall Borel ideals?*

This is, of course, equivalent to asking whether the Katětov order restricted to tall Borel ideals is $\mathfrak{c}$-downwards closed. We conjecture that the answer is negative. However:

PROPOSITION 5.4. *There is a tall projective ideal Katětov below all tall Borel ideals.*

PROOF. Let $U \subseteq \mathcal{P}(\omega) \times 2^\omega$ be a universal analytic set. Let $Y = \{x \in 2^\omega : U^x$ is a tall Borel ideal$\}$. Then, $Y$ is projective and by changing $U$ on the coordinates outside $Y$ we get a projective $V \subseteq \mathcal{P}(\omega) \times 2^\omega$ such that

(1) for every $x \in 2^\omega$ the set $V^x$ is a tall Borel ideal, and
(2) for every tall Borel ideal $\mathcal{I}$ there is an $x \in 2^\omega$ such that $\mathcal{I} = V^x$.

Having fixed such $V$ one can define an ideal $\mathcal{J}$ on $2^{<\omega}$ generated by antichains of $2^{<\omega}$ and sets of the form $\{x \restriction n : n \in I\}$, where $x \in 2^\omega$ and $I \in V^x$.

The ideal $\mathcal{J}$ is then tall and projective and such that for any tall Borel ideal $\mathcal{I}$ there is a $\mathcal{J}$-positive set $X$ such that $\mathcal{I}$ is isomorphic (hence K-equivalent) to $\mathcal{J} \restriction X$. $\square$

A reasonable weaker question is

QUESTION 5.5. *Is there a Borel tall ideal $\mathcal{J}$ such that for every Borel tall ideal $\mathcal{I}$ there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{J} \leq_K \mathcal{I} \restriction X$?*

We call such an ideal $\mathcal{J}$ *locally minimal*. There is a natural candidate, the ideal $\mathcal{R}$ introduced in section 3. Recall that for a Borel ideal $\mathcal{I}$ there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{R} \leq_K \mathcal{I} \restriction X$, if and only if $\mathcal{I}^+ \not\longrightarrow (\mathcal{I}^+)^2_2$. We have been able to give a positive answer to the question in a restricted class of Borel ideals (containing all $F_\sigma$ ideals).

THEOREM 5.6 ([**78**]). *Let $\mathcal{I}$ be a tall Borel ideal on $\omega$ such that $\mathcal{P}(\omega)/\mathcal{I}$ is proper. Then there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \restriction X \geq_K \mathcal{R}$.*

Let us remark that there are even $F_\sigma$ ideals which are not Katětov above $\mathcal{R}$, so $\mathcal{R}$ is not K-minimal. We do not even know whether $F_\sigma$ ideals are co-initial among tall Borel ideals:

QUESTION 5.7. *Does every tall Borel ideal contain a tall $F_\sigma$ subideal?*

**5.3. Ramsey and related properties.** We will take a closer look at the Ramsey property $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$ and the related P$^+$ and Q$^+$ properties here. The following is a well known fact, essentially, a reformulation of the standard proof of Ramsey's theorem.

PROPOSITION 5.8. *If $\mathcal{I}$ is an ideal which is both $P^+$ and $Q^+$ then $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$.*

It is easy to see that if $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$ then $\mathcal{I}$ has to be a Q$^+$-ideal. On the other hand, the $P^+$-property is not indispensable.

CLAIM 5.9. *There is a non-$P^+$-ideal satisfying $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$.*

PROOF. Fix for every $n<\omega$ a MAD family $\mathcal{A}_n$ such that $\mathcal{A}_{n+1}=\bigcup_{A\in\mathcal{A}_n}\mathcal{A}^A_{n+1}$, where every $\mathcal{A}^A_{n+1}$ is a MAD family in $\mathcal{P}(A)$. Let

$$\mathcal{I}=\bigcap_{n<\omega}\mathcal{I}(\mathcal{A}_n).$$

Then $\mathcal{I}^+\longrightarrow(\mathcal{I}^+)^2_2$, but $\mathcal{I}$ is not $\mathrm{P}^+$. □

Also, there are no tall Borel ideals satisfying the conditions above.

PROPOSITION 5.10. *There are no tall analytic ideals which are both* $P^+$ *and* $Q^+$.

PROOF. Let $\mathcal{I}$ be an analytic ideal on $\omega$, and suppose that $\mathcal{I}$ is a $\mathrm{P}^+$ and $\mathrm{Q}^+$-ideal. Then, by the $\mathrm{P}^+$ condition $\mathcal{I}^+$ is a $\sigma$-closed forcing, hence, it does not add new reals. Let $G$ be an $\mathcal{I}^+$-generic ultrafilter. Then, in $V[G]$, $G$ is a selective ultrafilter and $\mathcal{I}^{V[G]}=\mathcal{I}$ is an analytic ideal disjoint from $G$, contradicting Mathias' theorem 4.3 (in $V[G]$). □

We have already seen that the ideal $\mathcal{ED}_{fin}$ is critical with respect to the $\mathrm{Q}^+$-property: $\mathcal{I}$ is a $\mathrm{Q}^+$-ideal if and only if $\mathcal{I}\restriction X\not\geq_{KB}\mathcal{ED}_{fin}$ for all $\mathcal{I}$-positive sets $X$, and for Borel ideals if and only if $\mathcal{I}\restriction X$ is not $\omega$-hitting for all $\mathcal{I}$-positive sets $X$.

The $\mathrm{P}^+$-property is a lot more slippery. Let $\mathcal{I}$ be an ideal on $\omega$. We will say that $\mathcal{I}$ is *decomposable* if there is an infinite partition $\{X_n : n<\omega\}$ of $\omega$ into $\mathcal{I}$-positive sets such that for every $X\subseteq\omega$

$$X\in\mathcal{I}\ \text{ if and only if }\ (\forall n<\omega)(X\cap X_n\in\mathcal{I}).$$

We will say that $\mathcal{I}$ is *indecomposable* if it is not decomposable.

PROPOSITION 5.11. *Let* $\mathcal{I}$ *be an ideal. Then* $\mathcal{I}$ *is a* $P^+$*-ideal if and only if* $\mathcal{I}$ *is indecomposable and* $\mathsf{fin}\times\mathsf{fin}\not\leq_K\mathcal{I}\restriction X$*, for all* $X\in\mathcal{I}^+$.

There is a close relationship between the $\mathrm{P}^+$ property and $F_\sigma$ ideals. Just and Krawczyk [**56**] were probably the first to notice that every $F_\sigma$ ideal is $\mathrm{P}^+$.

THEOREM 5.12 ([**78**]). *Let* $\mathcal{I}$ *be a Borel ideal on* $\omega$*. Then the following conditions are equivalent*

1. *there is an* $F_\sigma$ *ideal* $\mathcal{J}$ *containing* $\mathcal{I}$*,*
2. *there is a* $P^+$*-ideal* $\mathcal{K}$ *containing* $\mathcal{I}$*.*

PROOF. $1\to2$ follows trivially from the above observation. Let us prove $2\to1$. Let $G$ be an $\mathcal{K}^+$-generic ultrafilter. Since $\mathcal{K}$ is $\mathrm{P}^+$, $\mathcal{K}^+$ is a $\sigma$-closed forcing and it does not add new reals, sequences of real numbers and Borel sets. Then, in $V[G]$, $G$ is a P-point, and by theorem 4.4, there is an $F_\sigma$ ideal $\mathcal{J}$ containing $\mathcal{I}$ and disjoint from $G$. Since $\mathcal{K}^+$ does not add new real numbers, the ideal $\mathcal{J}$ was already in $V$. □

$F_\sigma$ ideals can actually be combinatorially characterized among Borel ideals by a slight strengthening of the $\mathrm{P}^+$- property.

DEFINITION 5.13 (Laflamme and Leary [**67**]). Let $\mathcal{X}$ be a set of infinite subsets of $\omega$. A tree $T\subseteq([\omega]^{<\omega})^{<\omega}$ is an $\mathcal{X}$*-tree of finite sets* if for each $s\in T$ there is an $X_s\in\mathcal{X}$ such that $s^\frown a\in T$ for each $a\in[X_s]^{<\omega}$.

An ideal $\mathcal{I}$ on $\omega$ is a $\mathrm{P}^+$*(tree)-ideal* if every $\mathcal{I}^+$-tree of finite sets has a branch whose union is in $\mathcal{I}^+$.

Laflamme and Leary [**67**] have proved that an ideal $\mathcal{I}$ is not $\mathrm{P}^+$(tree) if and only if Player I has a winning strategy for the following game $G$: In step $n$, Player I chooses an $\mathcal{I}$-positive set $X_n$ and Player II chooses a finite set $F_n \subseteq X_n$. Player II wins if $\bigcup_{n<\omega} F_n \in \mathcal{I}^+$.

LEMMA 5.14. *Let $\mathcal{I}$ be a Borel ideal. Then, Player II has a winning strategy in the game $G$ if and only if $\mathcal{I}$ is an $F_\sigma$ ideal.*

So we have proved the following theorem.

THEOREM 5.15 ([**78**]). *Let $\mathcal{I}$ be a Borel ideal. Then $\mathcal{I}$ is a $P^+$(tree)-ideal if and only if $\mathcal{I}$ is an $F_\sigma$ ideal.* □

QUESTION 5.16. *Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\mathcal{I} \geq_K \mathsf{conv}$ or there is an $F_\sigma$ ideal $\mathcal{J}$ containing $\mathcal{I}$?*

An approximation to this conjecture is the following result.

THEOREM 5.17 ([**78**]). *Let $\mathcal{I}$ be a Borel ideal such that the forcing quotient $\mathcal{P}(\omega)/\mathcal{I}$ is proper. Then, either there is an $\mathcal{I}$-positive set $X$ such that $\mathsf{conv} \leq_K \mathcal{I} \restriction X$ or there is an $F_\sigma$ ideal $\mathcal{J}$ containing $\mathcal{I}$.*

A similar problem is to characterize those Borel ideals that can be extended to an $F_{\sigma\delta}$ ideal:

QUESTION 5.18. *Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\mathcal{I} \geq_K \mathsf{fin} \times \mathsf{fin}$ or there is an $F_{\sigma\delta}$ ideal $\mathcal{J}$ containing $\mathcal{I}$?*

Of course, the main open problem remains whether $\mathcal{R}$ is locally minimal:

QUESTION 5.19. *Is there a tall Borel ideal $\mathcal{I}$ such that $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2_2$?*

More results on Ramsey type properties of definable and non-definable ideals will appear in [**49**]. R. Filipów, N. Mrózek, I. Recław and P. Szuca also studied Ramsey type properties and related convergence properties in [**33, 32**]. Many of their results can be readily reformulated as results about Katětov order.

**5.4. Category dichotomy.** In this section we will prove the following structural theorem for Borel ideals.

THEOREM 5.20 (Category Dichotomy [**46, 78**]). *Let $\mathcal{I}$ be a Borel ideal. Then either $\mathcal{I} \leq_K \mathsf{nwd}$ or there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$.*

PROOF. The proof goes through a determinacy argument for the following game $G(\mathcal{I})$ associated to an ideal $\mathcal{I}$: In step $k$, Player I chooses an element $I_k$ of $\mathcal{I}$ and then Player II chooses an element $n_k$ of $\omega$ not in $I_k$. Player I wins if $\{n_k : k < \omega\} \in \mathcal{I}$.

If for every $\mathcal{I}$-positive set $X$ Player II has a winning strategy in the game $G(\mathcal{I} \restriction X)$ then $\mathcal{I} \leq_K \mathsf{nwd}$.

On the other hand, if there is an $\mathcal{I}$-positive set $Y$ such that Player I has a winning strategy for $G(\mathcal{I} \restriction Y)$ then there is an $\mathcal{I}$-positive set $X \subseteq Y$ such that $\mathcal{I} \restriction X \geq_K \mathcal{ED}$. □

One can easily turn the Category Dichotomy into a trichotomy: For every Borel ideal $\mathcal{I}$ either $\mathcal{I} \leq_K \mathsf{nwd}$ or there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \restriction X \geq_K \mathsf{fin} \times \mathsf{fin}$ or there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \restriction X \geq_{KB} \mathcal{ED}_{fin}$.

Recall that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$.

Corollary 5.21. *Let $\mathcal{I}$ be a K-uniform Borel ideal. Then* $\mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}(\mathcal{M})$ *or* $\mathsf{cov}^*(\mathcal{I}) \leq \mathsf{non}(\mathcal{M})$.

This is a particular case of a heuristically confirmed "rule of thumb" that for any simply definable (Borel) cardinal invariant $\mathfrak{j}$ either $\mathsf{ZFC} \vdash \mathfrak{j} \leq \mathsf{non}(\mathcal{M})$ or $\mathsf{ZFC} \vdash \mathfrak{j} \geq \mathsf{cov}(\mathcal{M})$. The only standard cardinal invariant I know that does not satisfy this is the *groupwise density number* $\mathfrak{g}$ [**11**], which is, of course, not Borel.

Question 5.22. *Is it true, that for any Borel cardinal invariant* $\mathfrak{j}$ *either* $\mathsf{ZFC} \vdash \mathfrak{j} \leq \mathsf{non}(\mathcal{M})$ *or* $\mathsf{ZFC} \vdash \mathfrak{j} \geq \mathsf{cov}(\mathcal{M})$*?*

**5.5. Measure dichotomy.** In this section we present a dichotomy for analytic P-ideals similar in form to the Category dichotomy. It is somewhat analogous to Christensen's result [**21**] linking the Fubini property to non-pathologicity of submeasures on atomless Boolean algebras.

Theorem 5.23 (Measure Dichotomy [**46, 78**]). *Let $\mathcal{I}$ be an analytic P-ideal. Then, either $\mathcal{I} \leq_K \mathcal{Z}$ or there is $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.*

Recall that a submeasure $\varphi$ on a set $X$ is *non-pathological* if for every $A \subseteq X$

$$\varphi(A) = \hat{\varphi}(A) =_{def} \sup\{\mu(A) : \mu \text{ is a measure on } X \text{ dominated by } \varphi\}.$$

Following Farah [**25**] we say that an analytic P-ideal $\mathcal{I}$ on $\omega$ is *non-pathological* if there is a lscsm $\varphi$ such that $\mathcal{I} = Exh(\varphi) = Exh(\hat{\varphi})$.

We define the *degree of pathology* of a submeasure $\varphi$ on $X$ such that $\varphi(X) < \infty$ by

$$P(\varphi) = \frac{\varphi(X)}{\sup\{\mu(X) : \mu \text{ is a measure dominated by } \varphi\}}.$$

*Kelley's covering number* [**61**] of a family of sets is defined as follows: Let $F$ be a set and $\mathcal{B} \subseteq \mathcal{P}(F)$. For any finite sequence $S = \langle S_0, \ldots S_n\rangle$ of (not necessarily distinct) elements of $\mathcal{B}$ let

$$m(S) = \min\left\{|\{i \leq n : x \in S_i\}| : x \in F\right\}.$$

The *covering number* $C(\mathcal{B})$ is defined as

$$C(\mathcal{B}) = \sup\left\{\frac{m(S)}{|S|} : S \in \mathcal{B}^{<\omega}\right\}.$$

The fundamental theorem of Kelley which links the covering number with submeasures is the following. Recall, that a (sub)measure $\varphi$ on a set $X$ is *normalized* if $\varphi(X) = 1$.

Theorem 5.24 (Kelley [**61**]). *For each non-empty family $\mathcal{B}$ of $\mathcal{P}(F)$ the covering number $C(\mathcal{B})$ is the minimum of the numbers $\sup\{\mu(A) : A \in \mathcal{B}\}$, where the minimum is taken over all normalized measures $\mu$ on $\mathcal{P}(F)$.* □

Using Kelley's theorem, one can deduce the following lemma, crucial in our proof of the Measure dichotomy. It can be seen as a finite (atomic) version of a theorem of Christensen [**21**] who showed that a submeasure $\varphi$ on an atomless Boolean algebra is pathological if and only if the Fubini theorem for $\varphi$ fails.

LEMMA 5.25 (Quantitative version of Christensen's lemma). *Let $F$ be a finite set, $\varepsilon > 0$, $\varphi$ a normalized submeasure on $\mathcal{P}(F)$ and $\mathcal{A}_\varepsilon = \{A \subseteq F : \varphi(A) < \varepsilon\}$. Then*

$$C(\mathcal{A}_\varepsilon) \geq 1 - \frac{1}{\varepsilon P(\varphi)}.$$

One should note that, in this context, the Kelley's covering number "measures" the failure of the Fubini theorem: $C(\mathcal{A}_\varepsilon) > \delta$ if and only if there is an $N < \omega$ and there is a set $A \subseteq F \times N$ such that all horizontal sections of $A$ have submeasure $< \varepsilon$ while all vertical sections have normalized counting measure $> \delta$. It is easily seen, that the finite $N$ can be replaced by the Cantor set and the counting measure by Lebesgue (Haar) measure. Interpreted in this way, the lemma says that "the more pathological is the submeasure, the worse the Fubini theorem for $\varphi$ fails".

COROLLARY 5.26. *If $\mathcal{I}$ is an analytic P-ideal then the following conditions are equivalent:*

*(a) $\mathcal{I} \restriction X \leq_K \mathcal{Z}$ for every $\mathcal{I}$-positive set $X$,*
*(b) $\mathcal{S} \not\leq_K \mathcal{I} \restriction X$, for every $\mathcal{I}$-positive set $X$,*
*(c) $\mathcal{I}$ has the Fubini property and*
*(d) $\mathcal{I}$ is non-pathological.*

Recall that $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ and $\mathsf{cov}^*(\mathcal{Z})$ is a close relative of $\mathsf{cov}(\mathcal{N})$. Note also that the Measure dichotomy does not hold for all Borel ideals (for instance $\mathsf{fin} \times \mathsf{fin}$ is a counterexample). We do not know whether it holds for $F_\sigma$ ideals. It is even conceivable, though unlikely, that the measure dichotomy could be extended to a trichotomy for all Borel ideals as follows: *Let $\mathcal{I}$ be a Borel ideal. Then, either $\mathcal{I} \leq_K \mathcal{Z}$ or there is $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$ or there is $X \in \mathcal{I}^+$ such that $\mathsf{fin} \times \mathsf{fin} \leq_K \mathcal{I} \restriction X$.*

## 6. Tukey order

The Tukey order on directed partial orders (i.e. partially ordered sets where any two elements have a common upper bound) was introduced by J. W. Tukey [**103**] in order to study the Moore-Smith convergence in topology.

Given two directed partial orders $P$ and $Q$ a function $f : P \longrightarrow Q$ is a *Tukey map (or Tukey reduction)* if $f$ maps unbounded sets to unbounded sets or, equivalently, if pre-images of bounded sets are bounded. We say that $P$ is *Tukey reducible* to $Q$ ($P \leq_T Q$) if there is a Tukey map $f : P \longrightarrow Q$. The existence of a Tukey map from $P$ to $Q$ is equivalent to the existence of a *convergent* map from $Q$ to $P$, i.e. a map sending cofinal subsets to cofinal subsets.

Two partially ordered sets $P, Q$ are *cofinally similar* or of the same *cofinal type* if there is another partially ordered set such that both $P$ and $Q$ are cofinal subsets of it. Tukey noticed that directed partial orders are of the same cofinal type if and only if they are Tukey-equivalent (Tukey-bi-reducible). The Tukey order was further studied by J. Isbell in a series of papers [**38, 53, 54, 55**], where basic concepts, such as the notion of a weakly bounded set, were introduced and fundamental open problems were raised. Answers to many of these were provided by Todorčević in [**98, 101**]. In particular he showed that:

THEOREM 6.1 (Todorčević [**98**]). *Assuming the Proper forcing axiom* PFA*, any directed set of size* $\aleph_1$ *is cofinally similar to one of the following:* $1$*,* $\omega$*,* $\omega_1$*,* $\omega\times\omega_1$ *and* $[\omega_1]^{<\omega}$*.*

On the other hand, he has also shown that there are $2^{\aleph_1}$ many distinct cofinal types of directed orders of size $\mathfrak{c}$. Therefore, an attempt to classify cofinal types of directed orders of size $\mathfrak{c}$ is bound to fail. There could be a better chance for a classification for definable structures.

The interest in Tukey order restricted to definable partial orders was at least partially motivated by the Bartoszyński-Raisonnier-Stern [**4, 5, 83**] results about cardinal invariants of measure and category, which can be concisely expressed in the language of Tukey order:

THEOREM 6.2 ([**5**]). $\mathcal{M}\leq_T\mathcal{N}$*.*

Another reason for such study was the fact that many ideals and directed sets naturally arising in analysis are simply definable. The study of the Tukey order on definable directed sets was initiated by D. Fremlin [**36, 37**], S. Todorčević [**101**] and P. Vojtáš [**107**].

**6.1. Tukey order on analytic ideals.** Cofinal types of Borel (or analytic) ideals were mentioned already by Isbell in [**55**], where he considered the density zero ideal $\mathcal{Z}$. Also, Fremlin [**36, 37**] and Todorčević [**101**] dealt with some ideals on a countable set, though their focus was on $\sigma$-ideals of Borel subsets of Polish spaces and structures of size $\aleph_1$, respectively.

The first papers dedicated to the study of analytic ideals on $\omega$ ([**99, 102**] and [**70**]) provided the fundamental structural theorems for the Tukey order on definable ideals.

Let $\mathcal{I}$ be an ideal. A subset $\mathcal{X}\subseteq\mathcal{I}$ is *weakly bounded*[8] if every infinite sequence of elements of $\mathcal{X}$ has a bounded subsequence (i.e. a subsequence whose union is in $\mathcal{I}$). Dually, a subset $\mathcal{X}\subseteq\mathcal{I}$ is *strongly unbounded* if no union of infinitely many members of $\mathcal{X}$ is in $\mathcal{I}$. Note that if $\mathcal{X}\subseteq\mathcal{I}$ is weakly bounded then its closure $\overline{\mathcal{X}}$ is contained in $\mathcal{I}$.

Note that any two ideals having a strongly unbounded set of size $\mathfrak{c}$ are Tukey-equivalent and are Tukey-above any ideal on $\omega$. The strongest property in this sense is for an ideal to have a perfect strongly unbounded subset. It was conjectured by Louveau and Veličković in [**70**] that any analytic ideal which has an uncountable strongly unbounded set should have a perfect one. This conjecture has recently been refuted by T. Mátrai [**74**], who showed that there is (in ZFC) an analytic ideal which has a strongly unbounded subset of size $\aleph_1$ but not a strongly unbounded subset of size $\aleph_2$ and, in particular, does not have a perfect strongly unbounded subset.

The following fundamental theorem shows that the ideal $\emptyset\times\mathsf{fin}$ is the least analytic ideal which is not $F_\sigma$.

THEOREM 6.3 (Louveau-Veličković [**70**]). *Let* $\mathcal{I}$ *be an analytic ideal. Then either* $\emptyset\times\mathsf{fin}\leq_T\mathcal{I}$ *or* $\mathcal{I}$ *is* $F_\sigma$*.*

[8] Pseudobounded according to Isbell [**55**].

Proof. Consider the following two player game: In the $n$-th inning of the game player I plays a weakly bounded set $\mathcal{X}_n \subseteq \mathcal{I}$ and player II responds by playing $a_n$ a finite subset of $\omega$ not in $\mathcal{X}_n$. Player II wins if $\bigcup_{n\in\omega} a_n \in \mathcal{I}$.

The existence of a winning strategy for player I implies that $\mathcal{I}$ can be covered by the closures of countably many weakly bounded sets, hence is $F_\sigma$.

If, on the other hand, player II has a winning strategy, then there is a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ such that

(1) for every $t \in T$ the set of $a \in [\omega]^{<\omega}$ $t^\frown a \in T$ is strongly unbounded and
(2) $\bigcup rng(f) \in \mathcal{I}$ for every branch $f \in [T]$.

This readily implies that $\omega^\omega \simeq_T \emptyset \times \mathsf{fin} \leq_T \mathcal{I}$.

The game as stated is easily seen to be determined for Borel ideals and a simple modification turns it into a closed game for any analytic ideal. □

Theorem 6.4 (Louveau-Veličković [**70**]). *Let $\mathcal{I}$ be an analytic ideal such that $\mathcal{I} \leq_T \emptyset \times \mathsf{fin}$. Then either $\emptyset \times \mathsf{fin} \simeq_T \mathcal{I}$ or $\mathcal{I}$ is countably generated.*

The focus of both the Louveau-Veličković and Todorčević papers was on analytic P-ideals. We state the results and sketch alternative proofs based on Solecki's theorem 1.6. Let us first make the following simple observation:

Claim 6.5. *Let $\mathcal{I} = Exh(\varphi)$ be an uncountably generated analytic P-ideal. Then there is a pairwise disjoint family $\{a_{m,n} : m, n \in \omega\}$ of finite subsets of $\omega$ such that $2^{-n-1} \leq \varphi(a_{m,n}) \leq 2^{-n}$.*

Theorem 6.6 (Todorčević [**99**]). *Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{I}$ is countably generated or $\emptyset \times \mathsf{fin} \leq_T \mathcal{I}$.*

Proof. Let $\mathcal{I} = Exh(\varphi)$ and $\{a_{m,n} : m, n \in \omega\}$ be as in the claim. The ideal $\emptyset \times \mathsf{fin}$ is naturally Tukey equivalent with $\omega^\omega$ ordered pointwise. For every $f \in \omega^\omega$ let $\psi(f) = \bigcup_{n\in\omega} a_{f(n),n}$. Then $\psi$ is the required Tukey map. This follows directly from the fact that for a fixed $n$ the set $\{a_{m,n} : m \in \omega\}$ is strongly unbounded. □

Theorem 6.7 (Todorčević [**102**]). *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I} \leq_T \mathcal{I}_{\frac{1}{n}}$.*

Proof. Let $\mathcal{I} = Exh(\varphi)$. Fix $\{a_{b,n} : b \in [\omega]^{<\omega}, n \in \omega\}$ a pairwise disjoint family of finite subsets of $\omega$ such that $2^{-n-1} \leq \sum_{j\in a_{b,n}} \frac{1}{j} \leq 2^{-n}$. For every $I \in \mathcal{I}$ let

$$f_I(n) = \min\{k : \varphi(I \setminus k) \leq \frac{1}{2^n}\}.$$

By exhaustivity, $f_I \in \omega^\omega$ is well defined. Let $\psi(I) = \bigcup_{n\in\omega} a_{I\cap f_I(n),n}$. Then $\psi$ is the required Tukey map. □

This result could also be attributed to Louveau and Veličković, as they were the first to notice that there is a top ideal among analytic P-ideals in the Tukey order. Louveau and Veličković in their article also showed that the Tukey order on analytic P-ideals is complex.

Theorem 6.8 (Louveau-Veličković [**70**]). *There is an embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into analytic P-ideals ordered by the Tukey order.*

They also pointed out that there are, indeed, uncountably generated $F_\sigma$ ideals which are not above $\emptyset \times \mathsf{fin}$ in the Tukey order. They introduced the *ideal of polynomial growth*

$$\mathcal{P} = \{A \subseteq \omega : (\exists k \in \omega)(\forall n \in \omega)\ |A \cap 2^n| \leq n^k\}$$

and showed that it is $\sigma$-weakly bounded, hence not Tukey-above $\emptyset \times \mathsf{fin}$.

Inspired by this example, with Zapletal and Rojas [**51**] we have introduced and studied a class of $F_\sigma$ ideals disjoint from the class of analytic P-ideals.

If $\mathcal{I}$ is an $F_\sigma$ ideal and $\varphi$ is a lscsm such that $\mathcal{I} = Fin(\varphi)$, denote by $\mathcal{I}_k^\varphi$ (or simply $\mathcal{I}_k$) the set $\{A \in \mathcal{I} : \varphi(A) \leq k\}$. Clearly, for all $k \in \omega$, $\mathcal{I}_k^\varphi$ is closed, and $\mathcal{I} = \bigcup_k \mathcal{I}_k^\varphi$.

Definition 6.9 ([**51**]). Let $\mathcal{I}$ be an $F_\sigma$ ideal on $\omega$. The ideal $\mathcal{I}$ is said to be *fragmented* if there are a lscsm $\varphi$ and a partition $\{a_i : i \in \omega\}$ of $\omega$ into finite sets, such that for every $k \in \omega$,

$$\mathcal{I}_k^\varphi = \{A \in \mathcal{I} : (\forall i \in \omega)(\varphi(A \cap a_i) \leq k)\}.$$

The ideal $\mathcal{P}$ of polynomial growth is an example of a fragmented $\sigma$-weakly bounded ideal (under $\varphi(A) = sup_{n\in\omega}\{\min\{k : |A \cap [2^n, 2^{n+1})| \leq n^k\}\}$). Other examples of fragmented ideals are: $\mathcal{ED}_{fin}$ and the ideal $\mathcal{L} = \{A \subseteq \omega : (\exists k \in \omega)(\forall n \in \omega)\ |A \cap 2^n| \leq n \cdot k\}$ of linear growth. The last two ideals both have a perfect strongly unbounded subset as opposed to the first example, which is $\sigma$-weakly bounded. The essential difference between the first and the last two ideals is that the growth of the submeasure in the fragments of the first can be controlled in the following sense:

Definition 6.10 ([**51**]). An ideal $\mathcal{I}$ is *gradually fragmented* if it is fragmented (via $\varphi$) and, moreover,

$$\forall k \exists m \forall l \forall^\infty j (\forall \mathcal{B} \in \left[P(a_j) \cap \mathcal{I}_k^\varphi\right]^l)(\cup \mathcal{B} \in \mathcal{I}_m^\varphi)$$

There is a dichotomy for fragmented $F_\sigma$ ideals.

Theorem 6.11 ([**51**]). *Let $\mathcal{I} = \mathrm{Fin}(\varphi)$ be a fragmented ideal. Then:*

- *Either $\mathcal{I}$ is gradually fragmented, or*
- *$\mathcal{I}$ contains a perfect strongly unbounded subset.*

K. Mazur in [**77**] has (essentially) shown that also the Tukey order on gradually fragmented $F_\sigma$-ideals is complex:

Theorem 6.12 (Mazur [**77**]). *There is an order embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into gradually fragmented $F_\sigma$-ideals ordered by the Tukey order.*

Solecki and Todorčević in [**91**] showed that every analytic ideal has a cofinal $G_\delta$ set, improving on a theorem of Zafrany [**108**]. They also showed that among analytic *basic* orders (a class of orders that includes all analytic P-ideals and *relative $\sigma$-ideals of compact sets*) the Tukey order reduces to the existence of a continuous cofinal map, hence is absolute. There is a large body of work on (relative) $\sigma$-ideals of compact sets and Tukey order which we do not include, and we refer the reader to [**36, 60, 71, 91**]. We only mention recent results of T. Mátrai [**74, 75**] who showed that the Tukey order on relative $\sigma$-ideals of compact sets is also complex (embeds $\mathcal{P}(\omega)/\mathsf{fin}$) and that the ideal $\mathsf{nwd}$ is not an upper bound for relative $\sigma$-ideals of compact sets.

**6.2. Cofinalities of Borel ideals.** There are very few "standard" cofinalities of Borel ideals:

- $\omega$ ... the cofinality of $\mathsf{fin}$,
- $\mathfrak{d}$ ... the cofinality of $\emptyset \times \mathsf{fin}$ and $\mathsf{fin} \times \mathsf{fin}$,
- $\mathsf{cof}(\mathcal{M})$ ... the cofinality of $\mathsf{nwd}$,
- $\mathsf{cof}(\mathcal{N})$ ... the cofinality of $\mathcal{Z}$ and $\mathcal{I}_{\frac{1}{n}}$.
- $\mathfrak{c}$ ... the cofinality of any ideal having a perfect strongly unbounded set.

PROPOSITION 6.13. *Let $\mathcal{I}$ be an uncountably generated analytic ideal. Then* $\mathsf{cof}(\mathcal{I}) \geq \mathsf{cov}(\mathcal{M})$.

Given that there does not seem to be a definable ideal of $\mathsf{cof}(\mathcal{I}) = \mathsf{cov}(\mathcal{M})$ and in light of the Louveau-Veličković theorem 6.3, one has to wonder: *Is $\mathfrak{d}$ a lower bound for cofinalities of all uncountably generated Borel ideals? Are there only finitely many distinct cofinalities of Borel ideals?* It turns out that the answer to both questions is in the negative.

A quick glance at the definition of gradually fragmented ideals reveals that their cofinalities are preserved by any proper forcing having the *Laver property* (see [**5**]) and consequently:

THEOREM 6.14 ([**51**]). *It is consistent that $\mathfrak{b} = \omega_2$ and $\mathsf{cof}(\mathcal{I}) = \omega_1$ for all gradually fragmented ideals $\mathcal{I}$.*

There is a natural forcing associated to every Borel ideal $\mathcal{I}$, which adds a new element of $\mathcal{I}$ not contained in any ground model set in $\mathcal{I}$ defined as follows: Let $\mathcal{I}$ be a Borel ideal. Let $J$ be the $\sigma$-ideal on $\mathcal{I}$ generated by the family $\{\mathcal{P}(I) : I \in \mathcal{I}\}$. Denote by $\mathbb{P}_{\mathcal{I}}$ the forcing $Borel(\mathcal{I})/J$.

We say that a forcing notion $\mathbb{P}$ *adds an unbounded element* of a Borel ideal $\mathcal{I}$ if there is a $\mathbb{P}$-name $\tau$ such that $\Vdash_{\mathbb{P}}$ "$\tau \in \mathcal{I}$ and $\tau \not\subseteq I$ for any ground model $I \in \mathcal{I}$".

General theorems of Zapletal [**109**] and simple genericity arguments give:

PROPOSITION 6.15. *Let $\mathcal{I}$ be a Borel ideal and let $\mathbb{P}_{\mathcal{I}}$ be the corresponding forcing. Then:*

- *$\mathbb{P}_{\mathcal{I}}$ is proper.*
- *$\mathbb{P}_{\mathcal{I}}$ preserves* $\mathsf{non}(\mathcal{M})$.
- *$\mathbb{P}_{\mathcal{I}}$ preserves* $\mathsf{cof}(\mathcal{M})$, *provided that $\mathcal{I}$ is $F_\sigma$.*
- *$\mathbb{P}_{\mathcal{I}}$ adds an unbounded element of $\mathcal{I}$.*

A simple consequence is:

THEOREM 6.16 ([**51**]). *It is consistent that* $\mathsf{cof}(\mathcal{M}) = \omega_1$ *and* $\mathsf{cof}(\mathcal{I}) = \omega_2$ *for all uncountably generated $F_\sigma$ ideals $\mathcal{I}$.*

Using a countable support product of the forcings of type $\mathbb{P}_{\mathcal{I}}$ for a carefully chosen family of gradually fragmented $F_\sigma$ ideals we were able to prove:

THEOREM 6.17 ([**51**]). *It is consistent with* $\mathsf{ZFC}$ *that there are uncountably many pairwise distinct cofinalities of gradually fragmented $F_\sigma$ ideals.*

It would be interesting to know how combinatorial properties of an ideal $\mathcal{I}$ impact preservation properties of the corresponding forcing $\mathbb{P}_{\mathcal{I}}$. When does $\mathbb{P}_{\mathcal{I}}$ preserve $\mathsf{cof}(\mathcal{M})$? $\mathsf{cof}(\mathcal{N})$? outer measure?

There is a close relationship between Borel ideals $\mathcal{I}$ and the corresponding forcing notions $\mathbb{P}_{\mathcal{I}}$ and a variant of the Tukey order, considered e.g. by Fremlin in [**37**]: Given two ideals $\mathcal{I}$ and $\mathcal{I}'$ we say that $\mathcal{I}$ is *$\omega$-Tukey reducible* to $\mathcal{I}'$ ($\mathcal{I} \leq_{\omega T} \mathcal{I}'$) if there is a function $f : \mathcal{I} \to \mathcal{I}'$ such that the pre-images of bounded sets are $\sigma$-bounded. It can be easily seen that the forcing $\mathbb{P}_{\mathcal{I}}$ adds an unbounded element of an ideal $\mathcal{I}$ if and only if $\mathcal{I} \leq_{\omega T} \mathcal{I}'$, where the witnessing function $f$ is sufficiently definable (*piece-wise Borel* in the following sense: Let $\mathcal{J}$ be the $\sigma$-ideal corresponding to $\mathcal{I}$. For every $\mathcal{J}$-positive set $B$ there is an $\mathcal{J}$-positive set $B \subseteq C$ such that $f \restriction C$ is Borel.)

There is a natural (trivial) characterization of the situation when a proper forcing of the type $P_J$ $(= Borel(X)/J)$ adds an unbounded element of a Borel ideal $\mathcal{I}$. It is if and only if there is a Borel function $f : X \to \mathcal{I}$ such that $f^{-1}[\mathcal{P}(I)] \in J$ for any $I \in \mathcal{I}$. This seems to be particularly interesting for the Sacks and Miller forcing, i.e. for $J$ being the $\sigma$-ideal of countable, or $\sigma$-compact sets.

Proposition 6.18. *Let $\mathcal{I}$ be a Borel ideal. Then:*

- *The Sacks forcing adds an unbounded element of $\mathcal{I}$ if and only if there is a perfect set $P \subseteq \mathcal{I}$ such that any element of $\mathcal{I}$ contains only countably many elements of $P$ (i.e. $\mathcal{I} \simeq_{\omega T} [2^\omega]^{<\omega}$).*
- *The Miller forcing adds an unbounded element of $\mathcal{I}$ if and only if $\emptyset \times \mathsf{fin} \leq_{\omega T} \mathcal{I}$ with Borel witnessing map.*

This simple observation raises the following natural questions, the second of which was asked in a stronger form (for Tukey order) in [**70**]:

Question 6.19. *Let $\mathcal{I}$ be a Borel ideal. Is it true that either $\mathcal{I} \leq_{\omega T} \mathcal{I}_{\frac{1}{n}}$ or $\mathcal{I} \simeq_{\omega T} [2^\omega]^{<\omega}$?*

Question 6.20. *Let $\mathcal{I}$ be a Borel ideal. Is it true that either $\emptyset \times \mathsf{fin} \leq_{\omega T} \mathcal{I}$ or $\mathcal{I}$ is $\sigma$-weakly bounded?*

It is also not clear whether every $\sigma$-weakly bounded ideal has cofinality consistently strictly below $\mathfrak{d}$. In fact, there does not seem to be any known example of a $\sigma$-weakly bounded ideal which is not gradually fragmented.

Another natural question concerns additivities. While we have shown that there are consistently many distinct cofinalities of Borel ideals, there are still only three ($\omega$, $\mathsf{add}(\mathcal{N})$ and $\mathfrak{b}$) known distinct additivities of Borel (analytic) ideals.

Question 6.21. *Is the additivity of every analytic P-ideal equal to either $\mathsf{add}(\mathcal{N})$ or $\mathfrak{b}$?*

Dual question is also open for cofinalities of analytic P-ideals. In particular:

Question 6.22 (Solecki-Todorčević [**91**]). *Are there two Tukey non-equivalent $F_\sigma$ P-ideals?*

There are several non-reducibility results. Some of them can be deduced from the proofs of the consistency results about distinct cofinalities, e.g. there is no uncountably generated $F_\sigma$ ideal Tukey reducible to $\mathsf{nwd}$ and $\mathcal{Z} \not\leq_T \mathsf{nwd}$ [**37**]. Other results are more involved, e.g. $\mathcal{I}_{\frac{1}{n}} \not\leq_T \mathcal{Z}$ [**70**] and some of them still open:

Question 6.23 (Fremlin [**37**]). *Is $\mathsf{nwd} \not\leq_T \mathcal{Z}$?*[9]

[9]This has been recently solved by T. Mátrai [**73**], and S. Solecki and S. Todorčević [**90**], independently.

**6.3. Tukey order on ultrafilters.** The Tukey order is also interesting when restricted to maximal ideals (or ultrafilters ordered by reverse inclusion). Isbell in [**53**] proved that there is an ultrafilter $\mathcal{U}$ of the maximal cofinal type, i.e. $\mathcal{U} \simeq_T [2^\omega]^{<\omega}$ and asked:

QUESTION 6.24 (Isbell [**53**]). *How many cofinal types of ultrafilters are there?*

In fact, it is still an open question, whether in $\mathsf{ZFC}$ there is a free ultrafilter of a cofinal type different from $[2^\omega]^{<\omega}$. There are, of course many consistency results. Any ultrafilter of character less than $\mathfrak{c}$ is an example, as is any P-point. D. Milovich in [**79**] has shown that there is consistently a non-P-point $\mathcal{U}$ such that $\mathcal{U} <_T [2^\omega]^{<\omega}$. Further research in the area is being done by Dobrinen and Todorčević [**23**].

The study of cofinal types of ultrafilters is clearly related to the following classical problem known as the *Katowice problem* [**81**]:

QUESTION 6.25. *Can the Čech-Stone remainders $\omega^*$ and $\omega_1^*$ be consistently homeomorphic?*

The question is, via Stone duality, equivalent to the question of whether the Boolean algebras $\mathcal{P}(\omega)/\mathsf{fin}$ and $\mathcal{P}(\omega_1)/\mathsf{fin}$ can be consistently isomorphic. It is, moreover, equivalent to the question whether a free (or, equivalently, any free) ultrafilter on $\omega$ can be (as partially ordered set) isomorphic to a free ultrafilter of $\omega_1$. In particular, should the answer be positive, the cofinal types of ultrafilters on $\omega$ and the cofinal types of ultrafilters on $\omega_1$ would have to coincide.

QUESTION 6.26. *What are the cofinal types of ultrafilters on $\omega_1$?*

## 7. Comparison game

In this section we propose a "rough" classification of Borel ideals based on a simple two player game. The game induces an order which is coarser than the Rudin-Keisler order, in fact coarser than the "monotone Tukey order". We hope that the order could provide insight into the structure of Borel ideals of low complexity, in particular, into the internal structure of $F_{\sigma\delta}$ ideals.

DEFINITION 7.1 ([**47**]). Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$. The *Comparison Game* for $\mathcal{I}$ and $\mathcal{J}$ denoted by $G(\mathcal{I},\mathcal{J})$ is defined as follows: In step $n$, Player I chooses an element $I_n$ of $\mathcal{I}$ and Player II chooses an element $J_n$ of $\mathcal{J}$. Player II wins if $\bigcup_n I_n \in \mathcal{I}$ if and only if $\bigcup_n J_n \in \mathcal{J}$; otherwise, Player I wins.

The comparison game defines an order on ideals on $\omega$.

DEFINITION 7.2. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$. We say $\mathcal{I} \sqsubseteq \mathcal{J}$ if Player II has a winning strategy in the comparison game $G(\mathcal{I},\mathcal{J})$. We say that $\mathcal{I} \simeq \mathcal{J}$ if $\mathcal{I} \sqsubseteq \mathcal{J}$ and $\mathcal{J} \sqsubseteq \mathcal{I}$.

Note that the relation $\sqsubseteq$ is reflexive and transitive, but not antisymmetric; and the relation $\simeq$ is an equivalence relation.

It is easy to see that the comparison game on Borel ideals is determined, in fact, it naturally reduces to *Wadge degrees*. Putting

$$\tilde{\mathcal{X}} = \{x \in \omega^\omega : rng(x) \in \mathcal{X}\}$$

for a subset $\mathcal{X}$ of $\mathcal{P}(\omega)$, one readily sees that

$$\mathcal{I} \sqsubseteq \mathcal{J} \text{ if and only if } \tilde{\mathcal{I}} \leq_{\text{Wadge}} \tilde{\mathcal{J}}.$$

Just like Wadge degrees, the comparison game order on Borel ideals is well-founded and "almost" linear.

Proposition 7.3. *(1) If $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ are Borel ideals, $\mathcal{I} \not\sqsubseteq \mathcal{J}$ and $\mathcal{J} \not\sqsubseteq \mathcal{K}$ then $\mathcal{K} \sqsubseteq \mathcal{I}$.*
*(2) Let $\mathcal{I}$ and $\mathcal{J}$ be two $\sqsubseteq$-incomparable ideals. Then, for any ideal $\mathcal{K}$ on $\omega$, $\mathcal{K} \sqsubseteq \mathcal{I}$ iff $\mathcal{K} \sqsubseteq \mathcal{J}$ or $\mathcal{I} \sqsubseteq \mathcal{K}$ iff $\mathcal{J} \sqsubseteq \mathcal{K}$.* □

We do not know whether $\sqsubseteq$ is linear. We also do not know whether it respects Borel complexity. What we do know is that it "almost" respects it.

Proposition 7.4. *If $\mathcal{I}$ is a $\Sigma^0_\alpha$ or $\Pi^0_\alpha$ ideal then $\tilde{\mathcal{I}}$ is a $\Sigma^0_{\alpha+1}$ or $\Pi^0_{\alpha+1}$ set, respectively.*

Note also that if $\mathcal{I} \leq_{RK} \mathcal{J}$ (in fact, the existence of a monotone Tukey-map $f : \mathcal{I} \longrightarrow \mathcal{J}$ suffices) then $\mathcal{I} \sqsubseteq \mathcal{J}$.

Question 7.5. *Is the order $\sqsubseteq$ linear (a well order)?*

Next we state some basic facts about the low levels of the order. Given $f \in 2^\omega$, denote by $A_f = \{f \restriction n : n < \omega\}$ the branch of the tree $2^{<\omega}$ corresponding to $f$. The ideal $\mathcal{I}_0$ is the ideal on $2^{<\omega}$ generated by the family of sets $A_f$ where $f \in 2^\omega$ is not eventually zero.

Theorem 7.6 ([**47**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then:*

- $\mathsf{fin} \sqsubseteq \mathcal{I}$,
- $\mathcal{I} \simeq \mathsf{fin}$ *if and only if* $\mathcal{I}$ *is* $F_\sigma$,
- $\mathcal{I}$ *is not an* $F_\sigma$ *ideal if and only if* $\mathcal{I}_0 \sqsubseteq \mathcal{I}$,
- $\emptyset \times \mathsf{fin} \not\sqsubseteq \mathcal{I}_0$.

Both the ideal $\mathcal{I}_0$ and $\emptyset \times \mathsf{fin}$ are $F_{\sigma\delta}$, so unlike in the case of $F_\sigma$ ideals, there are at least two classes of $F_{\sigma\delta}$ ideals.

Farah in [**25**] asked whether every $F_{\sigma\delta}$ ideal $\mathcal{I}$ is of the following canonical form: There is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that

$$\mathcal{I} = \{A \subseteq \omega : (\forall n < \omega)(\exists m < \omega)(A \setminus [0, m) \in C_n)\}.$$

We will say that $\mathcal{I}$ is a *Farah ideal* if it is of this form. Obviously, every Farah ideal $\mathcal{I}$ is an $F_{\sigma\delta}$ ideal. One can easily see that an ideal $\mathcal{I}$ is Farah if and only if there is a sequence $\{F_n : n < \omega\}$ of hereditary $F_\sigma$ sets closed under finite changes such that $\mathcal{I} = \bigcap_n F_n$. With some extra work, one can show that:

Theorem 7.7 ([**47**]). *Let $\mathcal{I}$ be an ideal on $\omega$. Then, $\mathcal{I}$ is Farah if and only if there is a sequence $\{F_n : n < \omega\}$ of $F_\sigma$ sets closed under finite changes such that $\mathcal{I} = \bigcap_n F_n$.*

We call an $F_{\sigma\delta}$ ideal $\mathcal{I}$ *weakly Farah* if there is a sequence $\langle F_n : n < \omega\rangle$ of hereditary $F_\sigma$ sets such that $\mathcal{I} = \bigcap_n F_n$.

Proposition 7.8 ([**47**]). *If $\mathcal{I}$ is a weakly Farah ideal then $\mathcal{I} \sqsubseteq \emptyset \times \mathsf{fin}$.*

Every analytic P-ideal is either equivalent with $\mathsf{fin}$ (if and only if it is $F_\sigma$) or equivalent with $\emptyset \times \mathsf{fin}$, so the class of analytic P-ideals skips the intermediate class of $\mathcal{I}_0$. Most known $F_{\sigma\delta}$ ideals are equivalent with $\emptyset \times \mathsf{fin}$, e.g. $\mathcal{Z}$, $\mathsf{nwd}$. On the other hand, the $F_{\sigma\delta\sigma}$ ideal $\mathsf{fin} \times \mathsf{fin}$ is strictly above $\emptyset \times \mathsf{fin}$. We conclude with three more open problems:

QUESTION 7.9. *(1) Is every $F_{\sigma\delta}$ ideal (weakly) Farah?*
*(2) Are there exactly two classes of $F_{\sigma\delta}$ ideals?*
*(3) How many classes of $F_{\sigma\delta\sigma}$ ideals are there?*

## 8. Quotient algebras $\mathcal{P}(\omega)/\mathcal{I}$

We now turn our attention to the study of the quotient Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ for definable ideals $\mathcal{I}$.

**8.1. Rigidity phenomena.** The starting point of any considerations in this area has to be the celebrated result of S. Shelah:

THEOREM 8.1 (Shelah [**84**]). *It is consistent that all automorphisms of $\mathcal{P}(\omega)/\mathsf{fin}$ are trivial.*

An automorphism is *trivial* if it is induced by an almost permutation of $\omega$, i.e. a bijection between two co-finite subsets of $\omega$. Shelah's original argument used the *oracle c.c.* method. Later it was shown by Shelah and Steprāns [**85**] that the result is true assuming $\mathsf{PFA}$. A careful analysis by Veličković [**106, 105**] revealed that the proof can be naturally split into two somewhat independent parts concerning *liftings* of homomorphisms between quotients.

DEFINITION 8.2. Let $\Phi : \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ be a homomorphism. A function $\varphi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a *lifting* of $\Phi$ if $[\varphi(X)]_{\mathcal{J}} = \Phi([X]_{\mathcal{I}})$ for every $X \subseteq \omega$.

Note that the function $\varphi$ is not required to be a homomorphism.

The two parts of the proof are:

(1) using forcing or some strong axiom ($\mathsf{PFA}$, $\mathsf{OCA}$, ...) show that every automorphism has a nicely definable (continuous, Baire-measurable) lifting, and
(2) an automorphism which has a definable lifting is trivial (i.e. has a completely additive lifting, see the definition below).

The *rigidity conjectures* of Farah and Todorčević, roughly speaking, assert that the same phenomenon occurs for any homomorphism (isomorphism) between quotients by definable ideals. It was quickly noticed that any homomorphism that has a Baire-measurable or a Lebesgue-measurable lifting has, in fact, a continuous lifting [**105, 100, 58**].

DEFINITION 8.3 (Farah [**25**]). An ideal $\mathcal{I}$ has the *Radon-Nikodym (RN)* property if every homomorphism $\Phi : \mathcal{P}(\omega)/\mathsf{fin} \to \mathcal{P}(\omega)/\mathcal{I}$ with a continuous lifting has a completely additive lifting.

By a *completely additive lifting* we mean a lifting of the form $\varphi(A) = h^{-1}[A]$ for some function $h$ from $\omega$ to $\omega$. Todorčević in [**100**] conjectured that every analytic P-ideal has the RN property. This has been partially confirmed by Farah

THEOREM 8.4 (Farah [**25**]). *Every non-pathological analytic P-ideal has the Radon-Nikodym property.*

and extended by Kanovei and Reeken

THEOREM 8.5 (Kanovei-Reeken [**57, 58**]). *Every analytic ideal having the Fubini property has the Radon-Nikodym property.*

However, it turned out that not all analytic P-ideals are RN [**25**], so some constraint is necessary for the positive answer to Todorčević's conjecture. Also not all RN ideals are Fubini ($\mathsf{nwd}$ is a counterexample [**28**]). It is still an open problem to find a combinatorial characterization of RN ideals. This is open even for analytic P-ideals:

QUESTION 8.6 (Farah [**25**]). *Is there a pathological analytic P-ideal with the RN property?*

Not surprisingly, Farah's *Rigidity conjecture* has two components:

(1) asks whether two quotients over Borel ideals are Baire-isomorphic (i.e. there is an isomorphism with a Baire-measurable lifting) if and only if the ideals are Rudin-Keisler equivalent,
(2) asks whether assuming $\mathsf{PFA}$ (or Martin's Maximum,...) every isomorphism between quotients over Borel ideals has a Baire-measurable lifting.

This has also been partially confirmed. Recall the definition of a Farah ideal from section 7.

THEOREM 8.7 (Farah [**28**]). *Assume* $\mathsf{PFA}$*. If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and at least one of them is Farah then every isomorphism between their quotients has a continuous lifting.*

We refer the interested reader to [**25, 28, 29, 57, 58, 100, 102**] for more information on this deep subject.

**8.2. Gap spectra of analytic quotients.** A development parallel to the study of the rigidity phenomena was the study of *gap spectra* of analytic quotients. Given an ideal $\mathcal{I}$, we call two families $\mathcal{A}, \mathcal{B}$ of subsets of $\omega$ *$\mathcal{I}$-orthogonal* if $A \cap B \in \mathcal{I}$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Two $\mathcal{I}$-orthogonal families $\mathcal{A}, \mathcal{B}$ form a *gap* if there is no $C \subseteq \omega$ such that $A \setminus C \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $C \cap B \in \mathcal{I}$ for all $B \in \mathcal{B}$. A gap $(\mathcal{A}, \mathcal{B})$ is *Hausdorff* if both $\mathcal{A}$ and $\mathcal{B}$ are $\sigma$-directed under inclusion mod $\mathcal{I}$.

Todorčević in [**100**] showed that any Baire-embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into an analytic quotient preserves Hausdorff gaps and that any Baire-embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into a quotient by an analytic P-ideal preserves all gaps. In particular, he showed that

THEOREM 8.8 (Todorčević [**100**]). *Let $\mathcal{I}$ be an analytic ideal. Then:*

*(1) $\mathcal{P}(\omega)/\mathcal{I}$ contains an $(\omega_1, \omega_1)$-gap.*
*(2) If, moreover, $\mathcal{I}$ is a P-ideal then $\mathcal{P}(\omega)/\mathcal{I}$ contains both an $(\omega_1, \omega_1)$-gap and an $(\omega, \mathfrak{b})$-gap.*

and asked to: *Determine the gap spectrum of $\mathcal{P}(\omega)/\mathcal{I}$ for every analytic ideal $\mathcal{I}$ on $\omega$.*

No Hausdorff gap in $\mathcal{P}(\omega)/\mathsf{fin}$ is analytic [**99**]. Therefore, it came as a surprise that

THEOREM 8.9 (Farah [**26**]). *There is an analytic Hausdorff gap in any quotient over an uncountably generated $F_\sigma$ P-ideal.*

Another surprising fact was proved recently by J. Brendle [**13**]:

THEOREM 8.10 (Brendle [**13**]). *There is an $(\omega_1, \omega)$-gap in $\mathcal{P}(\omega)/\mathcal{ED}_{fin}$.*

**8.3. How many quotients are there?** The title of this subsection is borrowed from [**27**]. Let us start with a problem from [**25**].

QUESTION 8.11 (Farah [**25**]). *Are there infinitely (even uncountably) many analytic P-ideals whose quotients are provably in* ZFC *pairwise non-isomorphic?*

Farah also asked the same question for arbitrary analytic and even definable ideals. He has also shown that assuming CH many classes of distinct ideals have isomorphic quotients [**27**]. The first general theorem of this kind was proved (using Parovičenko's theorem) by Just and Krawczyk:

THEOREM 8.12 (Just and Krawczyk [**56**]). *Assuming* CH *all quotients over $F_\sigma$ ideals are pairwise isomorphic.*

On the other hand, Steprāns in [**93**] produced uncountably many pairwise forcing non-equivalent (not just non-isomorphic) quotients over co-analytic ideals. In retrospect, his ideals are trace ideals, so his construction could be seen as a particular case of our theorem 2.5. The question was later answered completely by M. Oliver [**82**], who showed that:

THEOREM 8.13 (Oliver [**82**]). *There are $\mathfrak{c}$-many pairwise non-isomorphic quotients over analytic P-ideals.*

However, his method does not seem to produce quotients which are distinct as forcing notions. So we propose to reformulate the original question:

QUESTION 8.14. *Are there infinitely (uncountably) many analytic (P-)ideals whose quotients are not forcing equivalent?*

As of now, very few algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ for analytic ideal $\mathcal{I}$ are sufficiently well understood as forcing notions:

- (Farah [**30**]) $\mathcal{P}(\omega)/\mathcal{Z}$ is equivalent to the iteration $\mathcal{P}(\omega)/\mathsf{fin} * \mathbb{B}(2^\omega)$, where $\mathbb{B}(2^\omega)$ denotes the measure algebra for adding $\mathfrak{c}$- many random reals.
- ([**45**]) $\mathcal{P}(\omega)/tr(\mathcal{N})$ is proper and equivalent to the iteration $\mathbb{B}(\omega) * \mathbb{Q}$, where $\mathbb{Q}$ does not add reals.
- ([**45**]) There is an analytic P-ideal whose quotient is not proper.

So there are at least four forcing non-equivalent quotients over analytic P-ideals, the fourth being any quotient over an $F_\sigma$ P-ideal. There is also $\mathcal{P}(\mathbb{Q})/\mathsf{nwd}$, which is an iteration of Cohen forcing and a forcing not adding reals. The analysis of trace ideals gives one more candidate (or class of candidates). Let us consider again the Mathias-Prikry forcing $\mathbb{M}_\mathcal{J}$ associated to an ideal $\mathcal{J}$.

PROPOSITION 8.15 ([**45**]). *Let $\mathcal{J}$ be an ideal on $\omega$. The forcing $\mathbb{M}_\mathcal{J}$ has the continuous reading of names if and only if $\mathcal{J}$ is a P-ideal.*

PROPOSITION 8.16 ([**45**]). *Let $\mathcal{J}$ be an analytic P-ideal, and let $I$ be the $\sigma$-ideal associated with the Mathias-Prikry forcing $\mathbb{M}_\mathcal{J}$. Then the following are equivalent:*

*(1) $\mathcal{J}$ is $F_\sigma$,*
*(2) the ideal $tr(I)$ is Borel.*

So, by theorem 2.5 the quotient $\mathcal{P}(\omega^{<\omega})/tr(I)$ is a proper forcing adding a real for every $F_\sigma$ P-ideal $\mathcal{J}$, where $I$ is the natural $\sigma$-ideal associated with the forcing $\mathbb{M}_\mathcal{J}$

Question 8.17. *Are there uncountably many pairwise forcing non-equivalent quotients over the trace ideals corresponding to the Mathias-Prikry forcing with $F_\sigma$ P-ideals?*

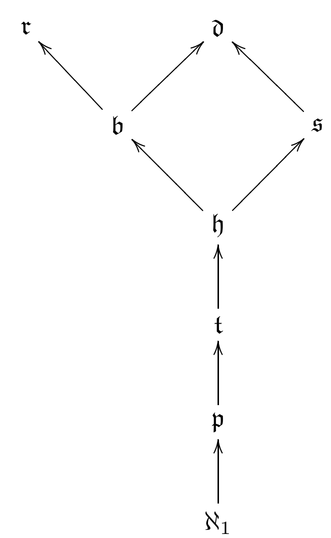
Various of our theorems about the Katětov order on Borel ideals required that the corresponding quotient be proper. We still do not know whether this assumption is essential in any of those results. However, it would be useful to have a better understanding of properness in this context. We also have only one example of a non-proper quotient and that example is over an analytic P-ideal. We need more non-proper Borel quotients against which we could test our conjectures.

**8.4. Cardinal invariants of $\mathcal{P}(\omega)/\mathcal{I}$.** Another useful way of studying the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$ is via its cardinal characteristics.

Recall some of the standard cardinal invariants corresponding to the algebra $\mathcal{P}(\omega)/\mathsf{fin}$ which form the so called *van Douwen's diagram* (see [**104, 11**] for definitions and more information).

We denote by $\mathfrak{p}_\mathcal{I}, \mathfrak{t}_\mathcal{I}, \mathfrak{h}_\mathcal{I}, \mathfrak{r}_\mathcal{I}, \mathfrak{s}_\mathcal{I}$ their direct analogues for the case of the algebra $\mathcal{P}(\omega)/\mathcal{I}$.

These cardinal invariants have so far been calculated only for a very short list of ideals. Typically, for Borel quotients that add a new real, the name for the real essentially witnesses $\mathfrak{s}_\mathcal{I} = \aleph_0$. This is the case for the ideals $\mathsf{nwd}$, $\mathcal{Z}$ and $tr(\mathcal{N})$, so in these cases, the only one of the cardinal invariants introduced of interest is the *reaping number*.




van Douwen's Diagram

Proposition 8.18. *(1) (Steprāns [**94**]) $\mathfrak{r}_{\mathcal{Z}} = \mathfrak{c}$.*
*(2) ([**1**]) $\max\{\mathfrak{r}, \mathsf{cof}(\mathcal{M})\} \leq \mathfrak{r}_{\mathsf{nwd}} \leq \mathfrak{i}$.*

Recall that $\mathfrak{i}$ is the minimal cardinality of a maximal independent family.

The situation is more interesting for ideals such that their quotients are $\sigma$-closed, in particular for $F_\sigma$ ideals. The quotient over the ideal $\mathsf{fin}\times\mathsf{fin}$ was considered by Brendle, Szymański and Zhou, and Hernández [**13, 41, 97**]:

Proposition 8.19. *(1) (Szymański and Zhou [**97**]) $\mathfrak{t}_{\mathsf{fin}\times\mathsf{fin}} = \omega_1$.*
*(2) (Brendle [**13**]) $\mathfrak{s}_{\mathsf{fin}\times\mathsf{fin}} = \mathfrak{s}$.*
*(3) (Hernández [**41**]) It is consistent with $\mathsf{ZFC}$ that $\mathfrak{h}_{\mathsf{fin}\times\mathsf{fin}} < \mathfrak{h}$.*

There are also some results concerning $\mathfrak{a}_{\mathcal{I}}$, the minimal size of an uncountable maximal antichain in $\mathcal{P}(\omega)/\mathcal{I}$. Note that uncountable is important; for many quotients there are countably infinite maximal antichains. It is well known that $\mathfrak{b} \leq \mathfrak{a}$. This has been extended to the quotients over $F_\sigma$ P-ideals by Farkas and Soukup [**31**]. However, it is not true in general, not even for $F_\sigma$ ideals:

Proposition 8.20. *(1) (Farkas and Soukup [**31**]) $\mathfrak{b} \leq \mathfrak{a}_{\mathcal{I}}$ for all $F_\sigma$ P-ideals $\mathcal{I}$.*
*(2) (Steprāns [**96**]) It is consistent with $\mathsf{ZFC}$ that $\mathfrak{a}_{\mathsf{nwd}} < \mathfrak{b}$.*
*(3) (Brendle [**13**]) It is consistent with $\mathsf{ZFC}$ that $\mathfrak{a}_{\mathcal{ED}_{fin}} < \mathfrak{b}$.*

Of course, any two Boolean algebras with distinct cardinal characteristics are non-isomorphic. Since we are interested mostly in forcing properties of the quotients $\mathcal{P}(\omega)/\mathcal{I}$, the most interesting of the cardinal invariants introduced is the *distributivity number* $\mathfrak{h}_{\mathcal{I}}$. It is particularly interesting for quotients over $F_\sigma$ ideals. While all quotients over $F_\sigma$ ideals are isomorphic under $\mathsf{CH}$ there seems to be a strong rigidity phenomenon of consistency results:

Question 8.21. *Let $\mathcal{I}$ and $\mathcal{J}$ be $F_\sigma$ ideals and suppose that there is no regular embedding of $\mathcal{P}(\omega)/\mathcal{I}$ into $\mathcal{P}(\omega)/\mathcal{J}$ with a completely additive lifting. Is it then consistent that $\mathfrak{h}_{\mathcal{I}} < \mathfrak{h}_{\mathcal{J}}$?*

If not, is the following true?

Question 8.22. *Are there infinitely (uncountably many) $F_\sigma$ ideals which have consistently pairwise different distributivity numbers?*

It is easy to see that $\mathcal{P}(\omega)/\mathsf{fin}$ regularly embeds into $\mathcal{P}(\omega)/\mathcal{I}$ and, hence, $\mathfrak{h}_{\mathcal{I}} \leq \mathfrak{h}$ for any fragmented $F_\sigma$ ideal. On the other hand Brendle [**13**] announced

Theorem 8.23 (Brendle [**13**]). *$\mathfrak{h}_{\mathcal{ED}_{fin}} < \mathfrak{h}$ is consistent.*

and asked

Question 8.24 (Brendle [**13**]). *Are $\mathfrak{h}_{\mathcal{I}} < \mathfrak{h}$ and $\mathfrak{h} < \mathfrak{h}_{\mathcal{I}}$ consistent for any summable ideal $\mathcal{I}$?*

It is also not clear whether all summable ideals have the same distributivity number.

A particular instance of a theorem of Balcar, Simon and Pelant [**2, 3**] shows that any quotient over an $F_\sigma$ ideal has a *base tree* of height $\mathfrak{h}_{\mathcal{I}}$. It follows that if the forcing $\mathcal{P}(\omega)/\mathcal{I}$ is homogeneous in density then it collapses $\mathfrak{c}$ to $\mathfrak{h}_{\mathcal{I}}$. Where does the forcing $\mathcal{P}(\omega)/\mathcal{I}$ collapse $\mathfrak{c}$ for any Borel ideal $\mathcal{I}$? In particular:

Question 8.25. *(1) Where does $\mathcal{P}(\mathbb{Q})/\mathsf{nwd}$ collapse $\mathfrak{c}$?*

*(2) Where does $\mathcal{P}(\omega)/tr(\mathcal{N})$ collapse $\mathfrak{c}$?*

*(3) Does every non-proper Borel quotient collapse $\mathfrak{c}$ to $\omega$? More precisely, is there an $\mathcal{I}$-positive set $X$ such that $\mathcal{P}(X)/\mathcal{I}\restriction X$ collapses $\mathfrak{c}$ to $\omega$?*

There is very little known about the $\mathfrak{p}_\mathcal{I}$ and $\mathfrak{t}_\mathcal{I}$.

Question 8.26. *Is there a Borel ideal $\mathcal{I}$ such that $\mathfrak{p}_\mathcal{I} < \mathfrak{t}_\mathcal{I}$ is consistent? Is $\mathfrak{p}_\mathcal{I} = \mathfrak{p}$ ($\mathfrak{t}_\mathcal{I} = \mathfrak{t}$) for every $F_\sigma$ ideal $\mathcal{I}$?*

One last question is rather ad hoc. The cardinal characteristic $\mathfrak{h}$ is equal to the minimal size of a family of tall ideals whose intersection is not tall. One can analogously define $\mathfrak{h}_{analytic}$, $\mathfrak{h}_{Borel}$,..., $\mathfrak{h}_{F_\sigma}$ as the minimal size of a family of tall analytic (Borel, ..., $F_\sigma$) ideals whose intersection is not tall. Obviously,

$$\mathfrak{h} \le \mathfrak{h}_{analytic} \le \mathfrak{h}_{Borel} \le \cdots \le \mathfrak{h}_{F_\sigma} \le \min\{\mathfrak{b}, \mathfrak{s}\}.$$

Question 8.27. *Which of the above inequalities are consistently strict?*

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