SOME REMARKS ON NON-SPECIAL COHERENT ARONSZAJN TREES

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Abstract. We introduce some guessing principles sufficient for the existence of non-special coherent Aronszajn trees and show how they relate to some of the standard principles in Set Theory (like MA_{ω_1} and $\diamondsuit$).

A variant of a question of I. Juhasz asks whether the principle $\clubsuit$ implies the existence of a non-special Aronszajn tree. Motivated by this question, we investigate when a coherent Aronszajn tree associated with the ρ_1 function of Todorčević (see [5]) is not special. To do this, we define principles $\bigstar_0$ and $\bigstar_1$, and their corresponding weak versions $w\bigstar_0$ and $w\bigstar_1$. The principles $\bigstar_0$ and $\bigstar_1$ are strong enough to construct non-special coherent Aronszajn trees. All these principles are weak in the sense that are all consistent with $MA_{\sigma-centered}$ and some of them are strong in the sense that they do not follow from $\diamondsuit$.

Our notation is mostly standard (see Kunen[4] and Jech[2]). We will use Λ to denote the collection of all countable limit ordinals. $A \sqsubseteq B$ will be used to denote that A is an initial segment of B, whenever A, B are subsets of ω_1. If A is a subset of ω_1, we will use $ot(A)$ to denote the order-type of A. The symbol $\frown$ denotes concatenation.

By a *C-sequence* (see [5]) we mean a sequence $\langle C_\alpha : \alpha \in \omega_1\rangle$ with the following properties: $C_{\alpha+1} = \{\alpha\}$, C_α is a cofinal subset of α of order-type ω, whenever α is a countable limit ordinal > 0.

Definition 1. *The principles $\bigstar_1, w\bigstar_1, \bigstar_0, w\bigstar_0$ are defined as follows:*

- $\bigstar_0$ *There is a C-sequence $\langle S_\alpha : \alpha \in \omega_1\rangle$ such that for every $\varphi : \Lambda \to \omega$ there are $\alpha, \beta \in \Lambda$ such that $\varphi(\alpha) = \varphi(\beta)$, $S_\beta \cap \alpha \sqsubseteq S_\alpha$ and $\alpha \in S_\beta$.*
- $w\bigstar_0$ *There is a C-sequence $\langle S_\alpha : \alpha \in \omega_1\rangle$ such that for every $\varphi : \Lambda \to \omega$ there are $\alpha, \beta \in \Lambda$ such that $\varphi(\alpha) = \varphi(\beta)$ and $\alpha \in S_\beta$.*
- $\bigstar_1$ *There is a C-sequence $\langle S_\alpha : \alpha \in \omega_1\rangle$ such that for every stationary set S there are $\alpha, \beta \in S$ such that $S_\beta \cap \alpha \sqsubseteq S_\alpha$ and $\alpha \in S_\beta$.*
- $w\bigstar_1$ *There is a C-sequence $\langle S_\alpha : \alpha \in \omega_1\rangle$ such that for every stationary set S there are $\alpha, \beta \in S$ such that $\alpha \in S_\beta$.*

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Following [5], to every C-sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ we associate two functions ρ_0, ρ_1. The function $\rho_0 = \rho_0(C_\alpha : \alpha < \omega_1) : [\omega_1]^2 \to \omega^{<\omega}$ is defined recursively as follows

$$\rho_0(\alpha,\beta) = \begin{cases} \langle |C_\beta \cap \alpha)|\rangle \frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)) & \text{if } \alpha < \beta \\ \varnothing & \text{if } \alpha = \beta \end{cases}.$$

Even though, ρ_0 is an important function on its own, we use it only as an auxiliar tool in some proofs of the theorems in this article.

The function $\rho_1 = \rho_1(C_\alpha : \alpha < \omega_1) : [\omega_1]^2 \to \omega$ is defined recursively by

$$\rho_1(\alpha,\beta) = \begin{cases} \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\} & \text{if } \alpha < \beta \\ 0 & \text{if } \alpha = \beta \end{cases},$$

Thus, $\rho_1(\alpha,\beta)$ is simply the maximal integer appearing in the sequence $\rho_0(\alpha,\beta)$. We will focus on the function ρ_1. Basic properties of the ρ_1 function are mentioned in the next lemma.

Lemma 2 (Todorčević [5])**.** *For all $\alpha < \beta < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leqslant \alpha : \rho_1(\xi,\alpha) \leqslant n\}$ *is finite,*
(b) $\{\xi \leqslant \alpha : \rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)\}$ *is finite.*

Let $\rho_{1\alpha} : \alpha \to \omega$ be defined by $\rho_{1\alpha}(\xi) = \rho_1(\xi,\alpha)$ for every $\xi < \alpha$. Then it follows from the previous lemma that the sequence

$$\rho_{1\alpha} : \alpha \to \omega \quad (\alpha < \omega_1)$$

of finite-to-one functions is *coherent* in the sense that $\rho_{1\alpha} =^* \rho_{1\beta} \restriction \alpha$ whenever $\alpha \leqslant \beta$. (Here $=^*$ means the fact that the functions agree on all but finitely many arguments). The corresponding tree

$$T(\rho_1) = \{\rho_{1\beta} \restriction \alpha : \alpha < \beta \leqslant \omega_1\}$$

is a coherent Aronszajn tree.

The following two theorems show the relevance of the guessing principles $\bigstar_0$ and $\bigstar_1$.

Theorem 3. $\bigstar_0$ *implies that there is a non special coherent Aronszajn tree.*

Proof. Let $T = T(\rho_1)$ be the coherent Aronszajn tree constructed from a $\bigstar_0$-sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ i.e. $\rho_1 = \rho_1(S_\alpha : \alpha < \omega_1)$. To prove the theorem it is enough to check that $A = \{\rho_{1\alpha} : \alpha \in \Lambda\} \subseteq T$ is not a countable union of antichains. Given any partion $\varphi : A \to \omega$ of A, we define a new function $\hat\varphi : \Lambda \to \omega$ by $\hat\varphi(\alpha) = \varphi(\rho_{1\alpha})$ for every $\alpha \in \Lambda$. It follows, using $\bigstar_0$, that there are $\alpha, \beta \in \Lambda$ such that $\hat\varphi(\alpha) = \hat\varphi(\beta)$, $S_\beta \cap \alpha \sqsubseteq S_\alpha$ and $\alpha \in S_\beta$. Then let us check that $\rho_{1\alpha} \subseteq \rho_{1\beta}$. Let $\{\xi_k : k \leqslant n\}$ be the increasing enumeration of $S_\beta \cap \alpha$. The proof proceeds by cases:

Case 1. If $\xi \in [0, \xi_0]$ then $\rho_0(\xi,\beta) = \langle 0\rangle \frown \rho_0(\xi,\xi_0)$. Since $S_\beta \cap \alpha \sqsubseteq S_\alpha$ the same holds for $\rho_0(\xi,\alpha)$. Then by the definition of ρ_1 we have that $\rho_1(\xi,\alpha) = \rho_1(\xi,\beta)$.

Case 2. If $\xi \in (\xi_k, \xi_{k+1}]$ then $\rho_0(\xi, \beta) = \langle |S_\beta \cap \xi| \rangle \frown \rho_0(\xi, \min(S_\beta \setminus \xi))$. However, $S_\beta \cap \alpha \sqsubseteq S_\alpha$ implies that $\xi_{k+1} = \min(S_\beta \setminus \xi) = \min(S_\alpha \setminus \xi)$ and $|S_\beta \cap \xi| = |S_\alpha \cap \xi|$ so $\rho_1(\xi, \beta) = \rho_1(\xi, \alpha)$.

Case 3.

If $\xi \in (\xi_n, \alpha)$ then $\rho_0(\xi, \beta) = \langle n \rangle \frown \rho_0(\xi, \alpha)$, and $\rho_0(\xi, \alpha) = \langle |S_\alpha \cap \xi| \rangle \frown \rho_0(\xi, \min(S_\alpha \setminus \xi))$. However, since $S_\beta \cap \alpha \sqsubseteq S_\alpha$, $n \leqslant |S_\alpha \cap \xi|$ so we have that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$.

Then $\forall \xi < \alpha\ (\rho_{1\alpha}(\xi) = \rho_{1\beta}(\xi))$. So we are done. □

Theorem 4. $\bigstar_1$ *implies that there is a coherent Aronszajn tree T which does not have stationary antichains.*

Proof. Let $T = T(\rho_1)$ be the coherent Aronszajn tree constructed from a $\bigstar_1$-sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ i.e. $\rho_1 = \rho_1(S_\alpha : \alpha < \omega_1)$. The result follows using the same argument as in the previous theorem and the following claim.

Claim. T has a stationary antichain if and only if $\{\rho_{1\alpha} : \alpha \in \omega_1\}$ has one.

Let us prove the claim. Let $A = \{t_\alpha : \alpha \in S\}$ be a stationary antichain of T, we may assume that $|T_\alpha \cap A| = 1$ and $ht(t_\alpha) = \alpha$ for every $\alpha \in S$. Note that S is a stationary set. For each $t_\alpha \in A$ there is an $F_\alpha \in [\alpha]^{<\omega}$ such that $t_\alpha(\xi) = \rho_{1\alpha}(\xi)$ for every $\xi \in (\alpha \setminus F_\alpha)$. By the pressing down lemma, we can find a stationary set $S' \subseteq S$ such that $F_\alpha = F$ for every $\alpha \in S'$. Using again the pressing down lemma we can find a stationary set $\hat{S} \subseteq S'$ such that $t_\alpha \restriction F = t_\beta \restriction F$ for every $\alpha < \beta \in \hat{S}$. Then $\forall \alpha < \beta \in \hat{S}$ there is a $\xi \in (\alpha \setminus F)$ such that $t_\alpha(\xi) \neq t_\beta(\xi)$. This implies that $t_\alpha(\xi) = \rho_{1\alpha}(\xi) \neq \rho_{1\beta}(\xi) = t_\beta(\xi)$. So $\{\rho_{1\alpha} : \alpha \in \hat{S}\}$ is a stationary antichain in $\{\rho_{1\alpha} : \alpha < \omega_1\}$, and this finishes the proof. □

As we have seen, the principles $\bigstar_0$ and $\bigstar_1$ are guessing principles which imply the existence of non-special Aronszajn trees. In order to have a better understanding of these principles we will compare them with some well known principles in set theory, summed up in the following diagram.

$$\begin{array}{ccccc}
 & & \diamondsuit^+ & & \\
 & & \downarrow & & \\
\bigstar_1 & \rightarrow & w\bigstar_1 & \not\leftarrow & \diamondsuit \\
\downarrow & & \downarrow & & \\
\bigstar_0 & \rightarrow & w\bigstar_0 & \rightarrow & \neg MA_{\omega_1} \\
\downarrow & & \uparrow & & \\
\text{There is a NSAT} & & \diamondsuit & &
\end{array}$$

Here $NSTA$ is an abbreviation for *non-special Aronszajn tree*. As the following theorem shows all the principles are relatively consistent with ZFC, even with $MA_{\sigma-centered}$.

Theorem 5. *If $V[G]$ is the generic extension obtained by adding a single Cohen real then $V[G] \models \bigstar_1$.*

Proof. From now on assume that $c : \omega \to [\omega]^{<\omega}$ is a Cohen-generic real and $e_\alpha : \alpha \to \omega \;\; (\alpha < \omega_1)$ is a coherent sequence of finite-to-one functions. Let $\langle C_\alpha : \alpha < \omega_1\rangle$ be an arbitrary C-sequence. We change this C-sequence to a C-sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ in the following way:

$$S_\alpha = \{\xi < \alpha : C_\alpha(n) \leqslant \xi < C_\alpha(n+1),\; e_\alpha(\xi) \in c(n)\},$$

where $C_\alpha(0) = 0$ and $C_\alpha(n)$ is the nth element of C_α for $0 < n < \omega$. Note that since $e'_\alpha s$ are finite-to-one $ot(S_\alpha) = \omega$. Let us check that $\langle S_\alpha : \alpha < \omega_1\rangle$ is a $\bigstar_1$-sequence. Assume that A is a stationary subset of ω_1. Note that if A is stationary in $V[G]$, then there is a stationary set $A_0 \in V$ such that $A_0 \subset A$. So without loss of generality we may assume that A is in the ground model. Fix $p \in Fn(\omega, [\omega]^{<\omega})$ with $dom(p) \in \omega$, use the pressing down lemma to find a stationary set $S \subset A$ such that S_α agree with S_β in all the places decided by p for every $\alpha, \beta \in S$. Pick an accumulation point β of S, now choose an $\alpha \in S$ in such a way that $C_\beta(n_0) < \alpha \leqslant C_\beta(n_0 + 1)$ where $dom(p) < n_0$. Let q be defined by

$$q(n) = \begin{cases} p(n) & if \;\; n \in dom(p) \\ \varnothing & if \;\; dom(p) < n < n_0 \\ \{e_\beta(\alpha)\} & if \;\; n = n_0 \end{cases}$$

then $q \Vdash$ "$\dot{S}_\beta \cap \alpha \sqsubseteq \dot{S}_\alpha \;\&\; \alpha \in \dot{S}_\beta$". □

Corollary 6. $\bigstar_1$ *(and hence also* $\bigstar_0, w\bigstar_0$ *and* $w\bigstar_1$*) are relatively consistent with* $MA_{\sigma-centered}$*.*

Proof. Let V be a model of MA and $\mathbb{P}$ a forcing which adds a single Cohen real. By the previous theorem if G is a $\mathbb{P}$-generic filter then $M[G] \models \bigstar_1$ and by the theorem of Roitman (see [1]) the extension $M[G] \models MA_{\sigma-centered}$. □

The fact that after adding a single Cohen real there is a coherent Aronszajn tree without stationary antichains was first observed by B. König in [3]. The following propositions give us some relationship beetwen $\diamondsuit$ and $\diamondsuit^+$ with our guessing principles.

Proposition 7. $\diamondsuit$ *implies* $w\bigstar_0$*.*

Proof. Let $\langle \varphi_\alpha : \alpha \in \omega_1\rangle$ be a $\diamondsuit$-sequence which guesses elements of ω_1^ω (i.e. $\varphi_\alpha \in \omega^\alpha$). Define $X_\alpha = \{n : \varphi_\alpha^{-1}(n)$ is cofinal in $\alpha\}$ for every limit α. For every $\alpha \in \Lambda$ choose $S_\alpha \subseteq \alpha$ of order type ω such that $S_\alpha \cap \varphi_\alpha^{-1}(n)$ is a cofinal in α for every $n \in X_\alpha$. This is very easy to do. Let us check that the C-sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ has the required properties. Now, let $\varphi : \Lambda \to \omega$ be given. Set $X = \{n \in \omega : \varphi^{-1}(n)$ is cofinal in $\omega_1\}$ and $C = \{\alpha : \forall n \in X \; (\varphi^{-1}(n)$ is cofinal in $\alpha)\}$. It is easy to see that C is a club in ω_1. Let be $\xi_0 = \max\{\varphi^{-1}(n) : n \notin X\} + 1$ and $S = \{\alpha : \varphi_\alpha = \varphi \restriction \alpha\}$. Pick any $\beta \in C \cap S \cap [\xi_0, \omega_1)$ then $\varphi(\beta) = n_0 \in X_\beta$. It follows from the properties of S_β that there is an $\alpha \in S_\beta$ such that $\varphi(\alpha) = n_0$. □

Proposition 8. $\diamondsuit^+$ *implies* $w\bigstar_1$.

Proof. Let $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$ be a $\diamondsuit^+$-sequence. For each α, let $S_\alpha \subset \alpha$ be a sequence of order-type ω such that $S_\alpha \cap A \neq \varnothing$ for every $A \in \mathcal{A}_\alpha$ (this can be done by an easy induction). Let us verify that $\langle S_\alpha : \alpha \in \omega_1 \rangle$ is a $w\bigstar_1$-sequence. Given a stationary set S, there is a club C such that $\forall\ \alpha \in C\ (S \cap \alpha \in \mathcal{A}_\alpha)$. Pick any $\beta \in (C \cap S)$ then $S_\beta \cap (S \cap \beta) \neq \varnothing$, now choose $\alpha \in S_\beta \cap (S \cap \beta)$. Then $\alpha, \beta \in S$ and $\alpha \in S_\beta$. So we are done. $\square$

We do not know if in the previous propositions we can replace the weak versions for the stronger ones. However, we have some limitations as the following theorem shows.

Theorem 9. $\diamondsuit$ *does not implies* $w\bigstar_1$.

To prove the theorem we need the following lemmas.

Lemma 10. *For every C-sequence $\langle S_\alpha : \alpha \in \omega_1 \rangle$ there is an α such that for every $\beta > \alpha$, $\{\gamma : (S_\gamma \setminus \alpha) \cap \beta = \varnothing\}$ is stationary.*

Proof. Suppose that this is not the case. Then for every α there is a $\beta(\alpha)$ and a club C_α such that $(S_\gamma \setminus \alpha) \cap \beta(\alpha) \neq \varnothing$, whenever $\gamma \in C_\alpha$. Pick $\alpha_0 \in \omega_1$ and define $\alpha_{n+1} = \beta(\alpha_n)$. Let $\xi \in \bigcap_{n\in\omega} C_{\alpha_n}$ be greater than $\alpha = sup\{\alpha_n : n \in \omega\}$. Since S_ξ intersects each interval $[\alpha_n, a_{n+1})$, α is an accumulation point of S_ξ, so the order-type of S_ξ is greater than ω, which is a contradiction. $\square$

The following lemma is a well known fact.

Lemma 11. (1) *Countable support iteration of σ-closed forcings is σ-closed,*
(2) *Every σ-closed forcing preserves $\diamondsuit$.*

Proof of theorem 9. For every C-sequence $\mathcal{C} = \langle C_\alpha : \alpha \in \omega_1 \rangle$, define the notion of forcing $\mathbb{P}_\mathcal{C}$ where

$$\mathbb{P}_\mathcal{C} = \{p \in 2^{<\omega_1} : \forall \alpha \in p^{-1}(1),\quad C_\alpha \cap p^{-1}(1) = \varnothing \text{ and } p \restriction \alpha_\mathcal{C} \equiv 0\}$$

Here $\alpha_\mathcal{C}$ is the α in the previous lemma which correspond to the C-sequence $\mathcal{C}$, and the order is by extension.

Claim 1. $\mathbb{P}_\mathcal{C}$ is a σ-closed forcing.

Let p_n be a decreasing sequence of conditions in $\mathbb{P}_\mathcal{C}$ and set $p_\omega = \bigcup_{n\in\omega} p_n$. Obviously, $p_\omega \in 2^{<\omega_1}$ and $p_\omega \restriction \alpha_\mathcal{C} \equiv 0$. Suppose that there are $\alpha, \beta \in p_\omega^{-1}(1)$ such that $\alpha\ \in C_\beta$ i.e. $C_\beta \cap p_\omega^{-1}(1) \neq \varnothing$, then there are $n, m \in \omega$ such that $\alpha \in dom(p_n)$ and $\beta \in dom(p_m)$ but this implies that $\alpha, \beta \in p_{m+n}^{-1}(1)$ and $C_\beta \cap p_{m+n}^{-1}(1) \neq \varnothing$ which is a contradiction.

Claim 2. $\mathbb{P}_\mathcal{C}$ forces that $\mathcal{C}$ is not a $w\bigstar_1$-sequence.

Let f_G be the $\mathbb{P}_\mathcal{C}$-generic function and $S = f_G^{-1}(1)$. To see that $\mathcal{C}$ is not a witness for $w\bigstar_1$ in $M[G]$ it suffices to prove that S is stationary in $M[G]$. Let $\dot{C}$ be a name for a club and $p \in \mathbb{P}_{\mathcal{C}}$ a condition such that $p \Vdash$ "$\dot{C}$ is a club". By Lemma 10, we can find a sequence $M_0 \subseteq M_1 \subseteq ... \subseteq M_n \subseteq ...$ of countable elementary submodels of $H(\theta)$ for θ large enough, such that $p, \langle C_\alpha : \alpha \in \omega_1\rangle, \dot{C} \in M_0$ and moreover, $(C_{\delta_n} \setminus \alpha_{\mathcal{C}}) \cap dom(p) = \varnothing$, where $\delta_n = M_n \cap \omega_1$ and we may assume that $\delta_n \in M_{n+1}$. Set $M_\omega = \bigcup_{n\in\omega} M_n$ and $\delta = M_\omega \cap \omega_1$. We will construct a sequence p_n of conditions such that $p_{n+1} \leqslant p_n$, $p_n \Vdash$ "$\delta_n \in \dot{C}$", $p_n^{-1}(1) \cap C_\delta = \varnothing$ and $p_n \in M_n$ by recursion as follows:

Let $\xi_0 = \max(C_\delta \cap \delta_0)$, and extend p to a condition $q = p \cup \{(\alpha, 0) : \alpha \in [dom(p), \xi_0]\}$. Note that $q \in M_0$. Since $M_0[G] \models$ "C is a club" there is an $\eta_0 \in \omega_1 \cap M_0$ and a $p_0 \in \mathbb{P}_{\mathcal{C}} \cap M_0, p_0 \leqslant q$ such that $p_0 \Vdash$ "$\dot{\eta}_0 \in \dot{C}$".

For the inductive step assume that we have constructed p_k for $k \leqslant n$ with the required properties. Pick $\xi_{n+1} < \delta_{n+1}$ such that $\xi_{n+1} > \max(C_\delta \cap \delta_{n+1})$.Then $q = p_n \cup \{(\alpha, 0) : \alpha \in [dom(p_n), \xi_{n+1}] \in M_{n+1}$ is a condition. As $q \Vdash$ "$\dot{C}$ is a club" there is a $\eta_{n+1} < \delta_{n+1}$ and a condition $q \geq p_{n+1} \in M_{n+1}$ such that $p_{n+1} \Vdash$ "$\eta_{n+1} \in \dot{C}$".

Finally, let

$$p_\omega = \bigcup_{n\in\omega} p_n \cup \{(\delta, 1)\}.$$

Note that p_ω is a condition as $p_\omega^{-1}(1) \cap C_\delta = \varnothing$. As $p_\omega \leqslant p_n$ for all $n \in \omega$, $p_\omega \Vdash$ "$\{\eta_n : n \in \omega\} \subseteq \dot{C}$". As $\delta = sup_{n\in\omega}\eta_n$ and since $\dot{C}$ is a name for a club $p_\omega \Vdash$ "$\delta \in \dot{S} \cap \dot{C}$. So S is stationary and Claim 2 holds.

Let $V = L$ and construct a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\rangle$ so that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{\dot{\mathcal{C}}}$ for some $C-$sequence $\dot{\mathcal{C}}$". By a standard book-keeping argument one can make sure that all C-sequences in the intermediate models are listed. Let G be a $\mathbb{P}$-generic. Since every C-sequence $\mathcal{C}$ in $M[G]$ has a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_2$, and at some stage $\beta < \omega_2$ we have that $\dot{\mathbb{Q}}_\beta = \mathbb{P}_{\dot{\mathcal{C}}}$ then $\mathcal{C}$ is not a $w\bigstar_1$-sequence. So $M[G] \models \neg w\bigstar_1$ and by the Lemma 11. $M[G] \models \diamondsuit$.

Finally we show that none of the principles is consistent with Martin's Axiom.

Theorem 12. *$MA(\omega_1)$ implies $\neg w\bigstar_0$.*

Proof. Let $\langle C_\alpha : \alpha < \omega_1\rangle$ be a C-sequence. Define

$$\mathbb{P} = \{p : A \to \omega : A \in [\Lambda]^{<\omega},\ (\forall \alpha < \beta)(p(\alpha) = p(\beta)\ \to\ \alpha \notin C_\beta)\}$$

ordered by inverse inclusion. It is easy to see that, if f_G is the generic function, then f_G is defined on Λ and forces that $\langle C_\alpha : \alpha < \omega_1\rangle$ is not a $w\bigstar_0$-sequence, to assure both we need to meet only ω_1 many dense sets. To finish the proof it suffices to check that:

Claim $\mathbb{P}$ is a c.c.c. forcing.

Suppose that $\{p_\alpha : \alpha \in \omega_1\}$ is an antichain. By a standard Δ-system type argument, we can assume that their domains form a Δ-system with root r, such that there is a $N \in \omega$ with $|dom(p_\alpha)| = N$ for each $\alpha \in \omega_1$ and all the functions agree on r. Moreover, we can assume that $dom(p_\alpha) \cap dom(p_b) = \varnothing$ for every $\alpha, \beta \in \omega_1$, and $\max(dom(p_\alpha)) < \min(dom(p_\beta))$ if $\alpha < \beta$. Now, set $dom(p_{\omega \cdot N+1}) = \{\xi_1, ..., \xi_N\}$. Since $p_{\omega \cdot N+1}$ is incompatible with p_α for every $\alpha < \omega \cdot N+1$, $(\bigcup_{i=1}^{N} C_{\xi_i}) \cap dom(p_\alpha) \neq \varnothing$ for every $\alpha < \omega \cdot N+1$. Then by the pigeon hole principle there is a i such that $ot(C_{\xi_i}) \geq \omega + 1$. However, this contradicts the fact that $\langle C_\alpha : \alpha < \omega_1 \rangle$ is a C-sequence, so we are done. $\square$

We conclude with some open problems.

Questions 13. (1) *Does* $w\bigstar_1$ *imply* $\bigstar_1$*?*
(2) *Does* $w\bigstar_0$ *imply* $\bigstar_0$*?*
(3) *Does* $\clubsuit$ *imply* $\bigstar_0$*?*

An early version of this paper contained also the following questions: (4) Does $\diamondsuit^+$ imply $\bigstar_1$? and (5) Does $\diamondsuit$ imply $\bigstar_0$?.

These questions were answered by Paul Larson. we present the proof with his kind permission.

Theorem 14 (Larson)**.** (i) $\diamondsuit$ *implies* $\bigstar_0$.
(ii) $\diamondsuit^+$ *implies* $\bigstar_1$.

Proof. Fix for every limit ordinal $\alpha < \omega_1$ a strictly increasing sequence $\{\alpha_n : n \in \omega\}$ such that $sup_{n\in\omega}\alpha_n = \alpha$ and let θ be a sufficiently large regular cardinal.

To prove (i) let $\langle \varphi_\alpha : \alpha < \omega_1 \}$ be a $\diamondsuit$-sequence which guesses elements of ω^{ω_1} (i.e. $\varphi_\alpha \in \omega^\alpha$). Construct recursively a C-sequence $\langle S_\alpha : \alpha \in \Lambda \rangle$ and a sequence $\langle e^\alpha_n : \alpha \in \Lambda, n \in \omega \rangle$ of finite subsets of α with the following properties:

(i) $S_\alpha = \bigcup_{n\in\omega} e^\alpha_n$,
(ii) $e^\alpha_n \sqsubseteq e^\alpha_{n+1}$, $\max(e^\alpha_{n+1}) > \alpha_n$,
(iii) $e^\alpha_{n+1} = e^\alpha_n \cup \{\xi\}$, where
$\xi = \min\{\eta : \eta > \max(e^\alpha_n \cup \{a_n\}) \wedge e^\alpha_n \sqsubseteq S_\eta \wedge \varphi_\alpha(\xi) = n\}$ if such ξ exists,
otherwise $\xi = \alpha_N$, where $N = \min\{k : \alpha_k > (e^\alpha_n \cup \{\alpha_n\})\}$.

Now let us check that $\langle S_\alpha : \alpha \in \Lambda \rangle$ is a $\bigstar_0$-sequence. It follows from (ii) that S_α is cofinal in α with order type ω. Let $\varphi : \omega_1 \to \omega$ be given. Set $S = \{\alpha < \Lambda : \varphi_\alpha = \varphi \restriction \alpha\}$, since $\langle \varphi_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence S is a stationary set. Since $C = \{\omega_1 \cap M : M \prec H(\theta)$ such that $\varphi, \langle S_\alpha : \alpha < \Lambda \rangle, \langle \varphi_\alpha : \alpha < \omega_1 \rangle \in M\}$ is a club there is an $M \in C$ such that $M \cap \omega_1 = \delta \in S$. Suppose that $\varphi(\delta) = n$ and let $e^\delta_{n-1} = S_\delta \restriction n-1$ (here $e^\delta_{-1} = \varnothing$) then for every $\alpha \in M$

$$H(\theta) \models \exists \beta > \alpha(\varphi(\beta) = n \ \wedge \ e^\delta_{n-1} \sqsubseteq S_\beta.$$

So, there is an $\alpha \in \delta, \alpha > (e^\delta_n \cup \{\alpha_n\})$ such that $n = \varphi(\alpha) = \varphi_\delta(\alpha)$ and $e^\delta_{n-1} \sqsubseteq S_\alpha$. It follows for the construction of e^δ_{n+1} that $e^\delta_{n+1} = e^\delta_n \cup \{\xi\}$ for some ξ with the same properties of α Then we have that $\varphi(\xi) = \varphi(\delta), \xi \in S_\delta$ and $S_\delta \cap \alpha = e^\delta_{n-1} \sqsubseteq S_\xi$.

To prove (ii) let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \}$ be a $\diamondsuit^+$-sequence. Enumerate $\mathcal{A}_\alpha$ as $\{A^\alpha_n : n \in \omega\}$. Construct recursively a C-sequence $\langle S_\alpha : \alpha \in \Lambda \rangle$ and a sequence $\langle e^\alpha_n : \alpha \in \Lambda, n \in \omega \rangle$ of finite subsets of α with the following properties:

(i) $S_\alpha = \bigcup_{n\in\omega} e^\alpha_n$,
(ii) $e^\alpha_n \sqsubseteq e^\alpha_{n+1}$, $\max(e^\alpha_{n+1}) > \alpha_n$,
(iii) $e^\alpha_{n+1} = e^\alpha_n \cup \{\xi\}$, where

$\xi = \min\{\eta : \eta > \max(e^\alpha_n \cup \{a_n\}) \wedge e^\alpha_n \sqsubseteq S_\eta \wedge (\xi) \in A^\alpha_{n+1}\}$ if such ξ exists, otherwise $\xi = \alpha_N$, where $N = \min\{k : \alpha_k > (e^\alpha_n \cup \{\alpha_n\})\}$.

Now let us check that $\langle S_\alpha : \alpha \in \Lambda \rangle$ is a $\bigstar_1$-sequence. It follows from (ii) that S_α is cofinal in α with order type ω. Let S a stationary. Set $D = \{\alpha < \Lambda : S \cap \alpha \in \mathcal{A}_\alpha\}$, since $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit^+$-sequence D is a club. Since $C = \{\omega_1 \cap M : M \prec H(\theta) \text{ such that} S, \langle S_\alpha : \alpha < \Lambda \rangle, \langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle \in M\}$ is a club there is an $M \in C$ such that $M \cap \omega_1 = \delta \in S \cap D$. Suppose that $S \cap \delta = A^\delta_n$ and let $e^\delta_{n-1} = S_\delta \restriction n-1$ then for every $\alpha \in M$

$$H(\theta) \models \exists \beta > \alpha(\beta \in S \ \wedge \ e^\delta_{n-1} \sqsubseteq S_\beta.$$

So, there is an $\alpha \in \delta, \alpha > (e^\delta_n \cup \{\alpha_n\})$ such that $\alpha \in A^\delta_{n+1}$ and $e^\delta_n \sqsubseteq S_\alpha$. It follows for the construction of e^δ_{n+1} that $e^\delta_{n+1} = e^\delta_n \cup \{\xi\}$ for some ξ with the same properties of α. Then we have that $\xi \in S, \xi \in S_\delta$ and $S_\delta \cap \xi = e^\delta_{n-1} \sqsubseteq S_\xi$. □

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