FUNCTIONAL ANALYSIS – Semester 2024-2

Contents

1 Topological and metric spaces

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U. U is called *dense* in S iff $\overline{U} = S$.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- *$\emptyset \in \mathcal{B}$.*
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exists a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6 (Filter)**.** Let S be a set. A subset $\mathcal{F}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *filter* iff it has the following properties:

- $\emptyset \notin \mathcal{F}$ and $S \in \mathcal{F}$.
- Let $U, V \in \mathcal{F}$. Then $U \cap V \in \mathcal{F}$.
- Let $U \in \mathcal{F}$ and $U \subseteq V \subseteq S$. Then $V \in \mathcal{F}$.

Definition 1.7. Let $\mathcal{F}$ be a filter. A subset $\mathcal{B}$ of $\mathcal{F}$ is called a *base* of $\mathcal{F}$ iff every element of $\mathcal{F}$ contains an element of $\mathcal{B}$.

Proposition 1.8. *Let S be a set and $\mathcal{B} \subseteq \mathfrak{P}(S)$. Then $\mathcal{B}$ is the base of a filter on S iff it satisfies the following properties:*

- $\emptyset \notin \mathcal{B}$ *and* $\mathcal{B} \neq \emptyset$.
- *Let $U, V \in \mathcal{B}$. Then there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.*

Proof. **Exercise.** □

Let S be a topological space and $x \in S$. It is easy to see that the set of neighborhoods of x forms a filter. It is called the *filter of neighborhoods* of x and denoted by $\mathcal{N}_x$. The family of filters of neighborhoods in turn encodes the topology:

Proposition 1.9. *Let S be a topological space and $\{\mathcal{N}_x\}_{x\in S}$ the family of filters of neighborhoods. Then a subset U of S is open iff for every $x \in U$, there is a set $W_x \in \mathcal{N}_x$ such that $W_x \subseteq U$.*

Proof. **Exercise.** □

Proposition 1.10. *Let S be a set and $\{\mathcal{F}_x\}_{x\in S}$ an assignment of a filter to every point in S. Then this family of filters are the filters of neighborhoods of a topology on S iff they satisfy the following properties:*

1. *For all $x \in S$, every element of $\mathcal{F}_x$ contains x.*
2. *For all $x \in S$ and $U \in \mathcal{F}_x$, there exists $W \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in W$.*

Proof. If $\{\mathcal{F}_x\}_{x\in S}$ are the filters of neighborhoods of a topology it is clear that the properties are satisfied: 1. Every neighborhood of a point contains the point itself. 2. For a neighborhood U of x take W to be an open neighborhood of x contained in U. Then W is a neighborhood for each point in W.

Conversely, suppose $\{\mathcal{F}_x\}_{x\in S}$ satisfies Properties 1 and 2. Given x we define a provisional open neighborhood of x to be an element $U \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in U$. This definition is not empty since at least S itself is a provisional open neighborhood of every point x in this way. Moreover, for any $y \in U$, by the same definition, U is a provisional open neighborhood of y. Now take $y \notin U$. Then, by Property 1, U is not a provisional open neighborhood of y. We define a provisional open set as a set that is a provisional open neighborhood for one (and thus any) of its points. We also declare the empty set to be a provisional open set. Let $\mathcal{T}$ be the set of provisional open sets.

We proceed to verify that $\mathcal{T}$ satisfies the axioms of a topology. Property 1 of Definition 1.1 holds since $S \in \mathcal{T}$, and we have declared $\emptyset \in \mathcal{T}$. Let $\{U_\alpha\}_{\alpha\in I}$ be a family in $\mathcal{T}$ and consider their union $U = \bigcup_{\alpha\in I} U_\alpha$. Assume U is not empty (otherwise $U \in \mathcal{T}$ trivially) and pick $x \in U$. Thus, there is $\alpha \in I$ such that $x \in U_\alpha$. But then $U_\alpha \in \mathcal{F}_x$ and also $U \in \mathcal{F}_x$. This is true for any $x \in U$. Hence, $U \in \mathcal{T}$. Consider now $U, V \in \mathcal{T}$. Assume the intersection $U \cap V$ to be non-empty (otherwise $U \cap V \in \mathcal{T}$ trivially) and pick a point x in it. Then $U \in \mathcal{F}_x$ and $V \in \mathcal{F}_x$ and therefore $U \cap V \in \mathcal{F}_x$. The same is true for any point in $U \cap V$, hence $U \cap V \in \mathcal{T}$. We thus drop the adjective "provisional".

It remains to show that $\{\mathcal{F}_x\}_{x \in S}$ are the filters of neighborhoods for the topology just defined. It is already clear that any open neighborhood of a point x is contained in $\mathcal{F}_x$. We need to show that every element of $\mathcal{F}_x$ contains an open neighborhood of x. Take $U \in \mathcal{F}_x$. We define V to be the set of points y such that $U \in \mathcal{F}_y$. This cannot be empty as $x \in V$. Moreover, Property 1 implies $V \subseteq U$. Let $y \in V$, then $U \in \mathcal{F}_y$ and we can apply Property 2 to obtain a subset $W \subseteq V$ with $W \in \mathcal{F}_y$. But this implies $V \in \mathcal{F}_y$. Since the same is true for any $y \in V$ we find that V is an open neighborhood of x. This completes the proof. □

Definition 1.11 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous at* $p \in S$ iff $f^{-1}(\mathcal{N}_{f(p)}) \subseteq \mathcal{N}_p$. f is called *continuous* iff it is continuous at every $p \in S$. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.12. *Let S, T be topological spaces and $f : S \to T$ a map. Then, f is continuous iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open.*

Proof. **Exercise.** □

Proposition 1.13. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S, T)$ and $g \in \mathrm{C}(T, U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.14. Let S, T be topological spaces. A bijection $f : S \to T$ is called a *homeomorphism* iff f and f^{-1} are both continuous. If such a homeomorphism exists S and T are called *homeomorphic*.

Definition 1.15. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

Definition 1.16 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.17 (Product Topology)**.** Let S be the Cartesian product $S = \prod_{\alpha \in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha \in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

<u>Exercise</u> 1. Show that alternatively, the product topology can be characterized as the coarsest topology on $S = \prod_{\alpha \in I} S_\alpha$ such that all projections $S \twoheadrightarrow S_\alpha$ are continuous.

Proposition 1.18. *Let S, T, X be topological spaces and $f \in \mathrm{C}(S \times T, X)$, where $S \times T$ carries the product topology. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x, y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. □

Definition 1.19 (Quotient Topology)**.** Let S be a topological space and $\sim$ an equivalence relation on S. Then, the *quotient topology* on $S/{\sim}$ is the finest topology such that the quotient map $S \twoheadrightarrow S/{\sim}$ is continuous.

Definition 1.20. Let S, T be topological spaces and $f : S \to T$. For $a \in S$ we say that f is *open at a* iff for every neighborhood U of a the image $f(U)$ is a neighborhood of $f(a)$. We say that f is *open* iff it is open at every $a \in S$.

Proposition 1.21. *Let S, T be topological spaces and $f : S \to T$. f is open iff it maps any open set to an open set.*

Proof. Straightforward. □

Definition 1.22 (Ultrafilter)**.** Let $\mathcal{F}$ be a filter. We call $\mathcal{F}$ an *ultrafilter* iff $\mathcal{F}$ cannot be enlarged as a filter. That is, given a filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ we have $\mathcal{F}' = \mathcal{F}$.

Lemma 1.23. *Let S be a set, $\mathcal{F}$ an ultrafilter on S and $U \subseteq S$ such that $U \cap V \neq \emptyset$ for all $V \in \mathcal{F}$. Then $U \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be an ultrafilter on S and $U \subseteq S$ such that $U \cap V \neq \emptyset$ for all $V \in \mathcal{F}$. Then, $\mathcal{B} := \{U \cap V : V \in \mathcal{F}\}$ forms the base of a filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $U \in \mathcal{F}'$. But since $\mathcal{F}$ is ultrafilter we have $\mathcal{F} = \mathcal{F}'$ and hence $U \in \mathcal{F}$. □

Proposition 1.24 (Ultrafilter lemma)**.** *Let $\mathcal{F}$ be a filter. Then there exists an ultrafilter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$.*

Proof. **<u>Exercise</u>.**Use Zorn's Lemma. □

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.25 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property*. We also say that S is a *Hausdorff space*.

Definition 1.26. A topological space S is called *completely regular* iff given a closed subset $C \subseteq S$ and a point $p \in S \setminus C$ there exists a continuous function $f : S \to [0, 1]$ such that $f(C) = \{0\}$ and $f(p) = 1$.

Definition 1.27. A topological space is called *normal* iff it is Hausdorff and if given two disjoint closed sets A and B there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 1.28. *Let S be a normal topological space, U an open subset and C a closed subset such that $C \subseteq U$. Then, there exists an open subset U' and a closed subset C' such that $C \subseteq U' \subseteq C' \subseteq U$.*

Proof. **Exercise.** □

Theorem 1.29 (Uryson's Lemma)**.** *Let S be a normal topological space and A, B disjoint closed subsets. Then, there exists a continuous function $f : S \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Let $C_0 := A$ and $U_1 := S \setminus B$. Applying Lemma 1.28 we find an open subset $U_{1/2}$ and a closed subset $C_{1/2}$ such that

$$C_0 \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_1.$$

Performing the same operation on the pairs $C_0 \subseteq U_{1/2}$ and $C_{1/2} \subseteq U_1$ we obtain

$$C_0 \subseteq U_{1/4} \subseteq C_{1/4} \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_{3/4} \subseteq C_{3/4} \subseteq U_1.$$

We iterate this process, at step n replacing the pairs $C_{(k-1)/2^n} \subseteq U_{k/2^n}$ by $C_{(k-1)/2^n} \subseteq U_{(2k-1)/2^{n+1}} \subseteq C_{(2k-1)/2^{n+1}} \subseteq U_{k/2^n}$ for all $k \in \{1, \ldots, n\}$.

Now define

$$f(p) := \begin{cases} 1 & \text{if}\, p \in B \\ \inf\{x \in (0, 1] : p \in U_x\} & \text{if}\, p \notin B \end{cases}$$

Obviously $f(B) = \{1\}$ and also $f(A) = \{0\}$. To show that f is continuous it suffices to show that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are open for $0 < a \leq 1$ and $0 \leq b < 1$. But,

$$f^{-1}([0, a)) = \bigcup_{x<a} U_x, \quad f^{-1}((b, 1]) = \bigcup_{x>b} (S \setminus C_x).$$

□

Corollary 1.30. *Every normal space is completely regular.*

Definition 1.31. Let S be a topological space. S is called *first-countable* iff for each point in S there exists a countable base of its filter of neighborhoods. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.32. Let S be a topological space and $U, V \subseteq S$ subsets. U is called *dense* in V iff $V \subseteq \overline{U}$.

Definition 1.33 (separable)**.** A topological space is called *separable* iff it contains a countable dense subset.

Proposition 1.34. *A topological space that is second-countable is separable.*

Proof. **Exercise.** □

Definition 1.35 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha\in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha\in A} U_\alpha$.

Proposition 1.36. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **Exercise.** □

Definition 1.37 (compact)**.** Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Definition 1.38. Let S be a topological space and $U \subseteq S$ a subset. Then, U is called *relatively compact* in S iff the closure of U in S is compact.

Proposition 1.39. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise.** □

Proposition 1.40. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise.** □

Lemma 1.41. *Let T_1 be a compact Hausdorff space, T_2 be a Hausdorff space and $f : T_1 \to T_2$ a continuous bijective map. Then, f is a homeomorphism.*

Proof. The image of a compact set under f is compact and hence closed in T_2. But every closed set in T_1 is compact, so f is open and hence a homeomorphism. □

Lemma 1.42. *Let T be a Hausdorff topological space and C_1, C_2 disjoint compact subsets of T. Then, there are disjoint open subsets U_1, U_2 of T such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. In particular, if T is compact, then it is normal.*

Proof. We first show a weaker statement: Let C be a compact subset of T and $p \notin C$. Then there exist disjoint open sets U and V such that $p \in U$ and $C \subseteq V$. Since T is Hausdorff, for each point $q \in C$ there exist disjoint open sets U_q and V_q such that $p \in U_q$ and $q \in V_q$. The family of sets $\{V_q\}_{q\in C}$ defines an open covering of C. Since C is compact, there is a finite subset $S \subseteq C$ such that the family $\{V_q\}_{q\in S}$ already covers C. Define $U := \bigcap_{q\in S} U_q$ and $V := \bigcup_{q\in S} V_q$. These are open sets with the desired properties.

We proceed to the prove the first statement of the lemma. By the previous demonstration, for each point $p \in C_1$ there are disjoint open sets U_p and V_p such that $p \in U_p$ and $C_2 \subseteq V_p$. The family of sets $\{U_p\}_{p\in C_1}$ defines an open covering of C_1. Since C_1 is compact, there is a finite subset $S \subseteq C_1$ such that the family $\{U_p\}_{p\in S}$ already covers C_1. Define $U_1 := \bigcup_{p\in S} U_p$ and $U_2 := \bigcap_{p\in S} V_p$.

For the second statement of the lemma observe that if T is compact, then every closed subset is compact. □

Definition 1.43. A topological space is called *locally compact* iff every point has a compact neighborhood.

Definition 1.44. A topological space is called *σ-compact* iff it is locally compact and admits a covering by countably many compact subsets.

Definition 1.45. Let T be a topological space. A *compact exhaustion* of T is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open and relatively compact subsets such that $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} U_i = T$.

Proposition 1.46. *A topological space admits a compact exhaustion iff it is σ-compact.*

Proof. Suppose the topological space T is σ-compact. Then there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets such that $\bigcup_{n\in N} K_n = T$. Since T is locally compact, every point possesses an open and relatively compact neighborhood. (Take an open subneighborhood of a compact neighborhood.) We cover K_1 by such open and relatively compact neighborhoods around every point. By compactness a finite subset of those already covers K_1. Their union, which we call U_1, is open and relatively compact. We proceed inductively. Suppose we have constructed the open and relatively compact set U_n. Consider the compact set $\overline{U_n} \cup K_{n+1}$. Covering it with open and relatively compact neighborhoods and taking the union of a finite subcover we obtain the open and relatively compact set U_{n+1}. It is then clear that the sequence $\{U_n\}_{n\in\mathbb{N}}$ obtained in this way provides a compact exhaustion of T since $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $T = \bigcup_{n\in\mathbb{N}} K_n \subseteq \bigcup_{n\in\mathbb{N}} U_n$.

Conversely, suppose T is a topological space and $\{U_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of T. Then, the sequence $\{\overline{U_n}\}_{n\in\mathbb{N}}$ provides a countable covering of T by compact sets. Also, given $p \in T$ there exists $n \in \mathbb{N}$ such that $p \in U_n$. Then, the compact set $\overline{U_n}$ is a neighborhood of p. That is, T is locally compact. □

Proposition 1.47. *Let T be a topological space, $K \subseteq T$ a compact subset and $\{U_n\}_{n\in\mathbb{N}}$ a compact exhaustion of T. Then, there exists $n \in \mathbb{N}$ such that $K \subseteq U_n$.*

Proof. **Exercise.** □

Exercise 2 (One-point compactification)**.** Let S be a locally compact Hausdorff space. Let $\tilde{S} := S \cup \{\infty\}$ to be the set S with an extra element ∞ adjoint. Define a subset U of $\tilde{S}$ to be open iff either U is an open subset of S or U is the complement of a compact subset of S. Show that this makes $\tilde{S}$ into a compact Hausdorff space.

1.3 Sequences and convergence

Definition 1.48 (Convergence of sequences)**.** Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for every neighborhood U of p we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n : x_k \in U$.

Proposition 1.49. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$. Conversely, if S is first-countable and for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.50. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $p \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Definition 1.51. Let S be a topological space and $U \subseteq S$ a subset. Consider the set B_U of sequences of elements of U. Then the set $\overline{U}^s$ consisting of the points to which some element of B_U converges is called the *sequential closure* of U.

Proposition 1.52. *Let S be a topological space and $U \subseteq S$ a subset. Let x be a sequence of points in U which has an accumulation point $p \in S$. Then, $p \in \overline{U}$.*

Proof. Suppose $p \notin \overline{U}$. Since $\overline{U}$ is closed $S \setminus \overline{U}$ is an open neighborhood of p. But $S \setminus \overline{U}$ does not contain any point of x, so p cannot be accumulation point of x. This is a contradiction. □

Corollary 1.53. *Let S be a topological space and U a subset. Then, $U \subseteq \overline{U}^s \subseteq \overline{U}$.*

Proof. Immediate. □

Proposition 1.54. *Let S be a first-countable topological space and U a subset. Then, $\overline{U}^s = \overline{U}$.*

Proof. **Exercise.** □

Definition 1.55. Let S be a topological space and $U \subseteq S$ a subset. U is said to be *limit point compact* iff every sequence in U has an accumulation point (limit point) in U. U is called *sequentially compact* iff every sequence of elements of U contains a subsequence converging to a point in U.

Proposition 1.56. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable base $\{U_n\}_{n\in\mathbb{N}}$ of the filter of neighborhoods at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^{n} U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Proposition 1.57. *Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.*

Proof. **Exercise.** □

Proposition 1.58. *A compact set is limit point compact.*

Proof. Consider a sequence x in a compact set S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

1.4 Filters and convergence

Definition 1.59 (convergence of filters)**.** Let S be a topological space and $\mathcal{F}$ a filter on S. $\mathcal{F}$ is said to *converge* to $p \in S$ iff every neighborhood of p is contained in $\mathcal{F}$, i.e., $\mathcal{N}_p \subseteq \mathcal{F}$. Then, x is said to be a *limit* of x. Also, $p \in S$ is called *accumulation point* of $\mathcal{F}$ iff $p \in \bigcap_{U\in\mathcal{F}} \overline{U}$.

Proposition 1.60. *Let S be a topological space and $\mathcal{F}$ a filter on S converging to $p \in S$. Then, p is accumulation point of $\mathcal{F}$.*

Proof. **Exercise.** □

Proposition 1.61. *Set S be a topological space and $\mathcal{F}, \mathcal{F}'$ filters on S such that $\mathcal{F} \subseteq \mathcal{F}'$. If $p \in S$ is accumulation point of $\mathcal{F}'$, then it is also accumulation point of $\mathcal{F}$. If $\mathcal{F}$ converges to $p \in S$, then so does $\mathcal{F}'$.*

Proof. Immediate. □

Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We define the filter $\mathcal{F}_x$ associated with this sequence as follows: $\mathcal{F}_x$ contains all the subsets U of S such that U contains all x_n, except possibly finitely many.

Proposition 1.62. *Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. Then x converges to a point $p \in S$ iff the associated filter $\mathcal{F}_x$ converges to p. Also, $p \in S$ is accumulation point of x iff it is accumulation point of $\mathcal{F}_x$.*

Proof. **Exercise.** □

Proposition 1.63. *Let S be a topological space and $U \subseteq S$ a subset. Consider the set A_U of filters containing U. Then, the closure $\overline{U}$ of U coincides with the set of points to which some element in A_U converges.*

Proof. If $U = \emptyset$, then A_U is empty and the proof is trivial. Assume the contrary. If $x \in \overline{U}$, then the intersection of U with any neighbourhood of x is non-empty and thus generates a filter that contains U as well as all neighborhoods of x and thus converges to x. If $x \notin \overline{U}$, then there exists a neighborhood V of x such that $U \cap V = \emptyset$. So no filter containing U can contain V. □

Proposition 1.64. *Let S,T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and filter $\mathcal{F}$ converging to p, the filter generated by $f(\mathcal{F})$ in T converges to $f(p)$. Conversely, if for any $p \in S$ and filter $\mathcal{F}$ converging to p, the filter generated by $f(\mathcal{F})$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.65. *Let S be a Hausdorff topological space, $\mathcal{F}$ a filter on S converging to a point $p \in S$. Then $\mathcal{F}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Proposition 1.66. *Let S be a topological space and $K \subseteq S$ a subset. Then, K is compact iff every filter containing K has at least one accumulation point in K.*

Proof. Let $K \subseteq S$ be compact. We suppose that there is a filter $\mathcal{F}$ containing K that has no accumulation point in K. For each $U \in \mathcal{F}$ consider the open set $O_U := S \setminus \overline{U}$. By assumption, these open sets cover K. Since K is compact, there must be a finite subset $\{U_1, \ldots, U_n\}$ of elements of $\mathcal{F}$ such that $\{O_{U_1}, \ldots, O_{U_n}\}$ covers K. But this implies

$K \cap \bigcap_{i=1}^n \overline{U_i} = \emptyset$ and thus, in particular, also $K \cap \bigcap_{i=1}^n U_i = \emptyset$, contradicting the fact that $\mathcal{F}$ is a filter. Thus, any filter containing K must have an accumulation point in K.

Now suppose that $K \subseteq S$ is not compact. Then, there exists a cover of K by open sets $\{U_\alpha\}_{\alpha \in A}$ which does not admit any finite subcover. Now consider finite intersections of the sets $C_\alpha := K \setminus U_\alpha$. These are non-empty and form the base of a filter containing K. But this filter clearly has no accumulation point in K. Thus, if every filter containing K is to possess an accumulation point, K must be compact. □

1.5 Metric and pseudometric spaces

Definition 1.67. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x, y) = d(y, x) \quad \forall x, y \in S$. (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in S$. (triangle inequality)
- $d(x, x) = 0 \quad \forall x \in S$.

Then d is called a *pseudometric* on S. S is also called a *pseudometric space*. Suppose d also satisfies

- $d(x, y) = 0 \implies x = y \quad \forall x, y \in S$. (definiteness)

Then d is called a *metric* on S and S is called a *metric space*.

Definition 1.68. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x, y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x, y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.69. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.70. A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

Proposition 1.71. *In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.*

Proof. **Exercise.** □

Proposition 1.72. *Let S be a set equipped with two pseudometrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise.** □

Proposition 1.73 (epsilon-delta criterion)**.** *Let S, T be pseudometric spaces and $f : S \to T$ a map. Then, f is continuous at $x \in S$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise.** □

1.6 Elementary properties of pseudometric spaces

Proposition 1.74. *Every metric space is normal.*

Proof. Let A, B be disjoint closed sets in the metric space S. For each $x \in A$ choose $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap B = \emptyset$ and for each $y \in B$ choose $\epsilon_y > 0$ such that $B_{\epsilon_y}(y) \cap A = \emptyset$. Then, for any pair (x, y) with $x \in A$ and $y \in B$ we have $B_{\epsilon_x/2}(x) \cap B_{\epsilon_y/2}(y) = \emptyset$. Consider the open sets $U := \bigcup_{x \in A} B_{\epsilon_x/2}(x)$ and $V := \bigcup_{y \in B} B_{\epsilon_y/2}(y)$. Then, $U \cap V = \emptyset$, but $A \subseteq U$ and $B \subseteq V$. So S is normal. □

Proposition 1.75. *Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq n_0$.*

Proof. Immediate. □

Definition 1.76. Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Exercise 3. Give an example of a set S, a sequence x in S and two metrics d^1 and d^2 on S that generate the same topology, but such that x is Cauchy with respect to d^1, but not with respect to d^2.

Proposition 1.77. *Any converging sequence in a pseudometric space is a Cauchy sequence.*

Proof. **Exercise.** □

Proposition 1.78. *Suppose x is a Cauchy sequence in a pseudometric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise.** □

Definition 1.79. Let S be a pseudometric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U, then U is called *complete*.

Proposition 1.80. *A complete subset of a metric space is closed. A closed subset of a complete pseudometric space is complete.*

Proof. **Exercise.** □

Exercise 4. Give an example of a complete subset of a pseudometric space that is not closed.

Definition 1.81 (Totally boundedness)**.** Let S be a pseudometric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 1.82. *A subset of a pseudometric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.58) and hence (Proposition 1.78) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha \in A}$ of U that does not admit a finite subcover. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k \in \mathbb{N}}$. Since U is complete, the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open, there exists $r > 0$ such that $B_r(p) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 1.83. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Proposition 1.84. *A totally bounded pseudometric space is second-countable.*

Proof. **Exercise.** □

Proposition 1.85. *The notions of separability and second-countability are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Theorem 1.86 (Baire's Theorem)**.** *Let S be a complete metric space and $\{U_n\}_{n\in\mathbb{N}}$ a sequence of open and dense subsets of S. Then, the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in S.*

Proof. Set $U := \bigcap_{n\in\mathbb{N}} U_n$. Let V be an arbitrary open set in S. It suffices to show that $V \cap U \neq \emptyset$. To this end we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of S and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive numbers. Choose $x_1 \in U_1 \cap V$ and then $0 < \epsilon_1 \leq 1$ such that $\overline{B_{\epsilon_1}(x_1)} \subseteq U_1 \cap V$. Now, consecutively choose $x_{n+1} \in U_{n+1} \cap B_{\epsilon_n/2}(x_n)$ and $0 < \epsilon_{n+1} < 2^{-n}$ such that $\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq U_{n+1} \cap B_{\epsilon_n}(x_n)$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy since by construction $d(x_n, x_{n+1}) < 2^{-n}$ for all $n \in \mathbb{N}$. So by completeness it converges to some point $x \in S$. Indeed, $x \in \overline{B_{\epsilon_1}(x_1)} \subseteq V$. On the other hand, $x \in \overline{B_{\epsilon_n}(x_n)} \subseteq U_n$ for all $n \in \mathbb{N}$ and hence $x \in U$. This completes the proof. □

Proposition 1.87. *Let S be equipped with a pseudometric d. Then $p \sim q \iff d(p,q) = 0$ for $p, q \in S$ defines an equivalence relation on S. The prescription $\tilde{d}([p],[q]) := d(p,q)$ for $p, q \in S$ is well-defined and yields a metric $\tilde{d}$ on the quotient space $S/\sim$. The topology induced by this metric on $S/\sim$ is the quotient topology with respect to that induced by d on S. Moreover, $S/\sim$ is complete iff S is complete.*

Proof. **Exercise.** □

1.7 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

Exercise 5. Let S be a metric space.

- Let $x := \{x_n\}_{n\in\mathbb{N}}$ and $y := \{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists.
- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}_0^+$ by $\tilde{d}(x,y) := \lim_{n\to\infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.
- Show that T is complete.
- Define $\overline{S}$ as the metric quotient $T/\sim$ as in Proposition 1.87. Then, $\overline{S}$ is complete.

- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 1.88. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 1.89 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric map. Then, there is a unique isometric map $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **Exercise.** □

2 Vector spaces with additional structure

In the following $\mathbb{K}$ denotes a field which might be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let V be a vector space over $\mathbb{K}$. A subset A of V is called *balanced* iff for all $v \in A$ and all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ the vector λv is contained in A. A subset A of V is called *convex* iff for all $x, y \in A$ and $t \in [0,1]$ the vector $(1-t)x + ty$ is in A. Let A be a subset of V. Consider the smallest subset of V which is convex and which contains A. This is called the *convex hull* of A, denoted $\operatorname{conv}(A)$.

Proposition 2.2. *(a) Intersections of balanced sets are balanced. (b) The sum of two balanced sets is balanced. (c) A scalar multiple of a balanced set is balanced.*

Proof. **Exercise.** □

Proposition 2.3. *Let V be vector space and A a subset. Then*

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, \lambda_i \in [0,1], x_i \in A, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Proof. **Exercise.** □

We denote the space of linear maps between a vector space V and a vector space W by $\mathrm{L}(V, W)$.

2.1 Topological vector spaces

Definition 2.4. A set V that is equipped both with a vector space structure over $\mathbb{K}$ and a topology is called a *topological vector space (tvs)* iff the vector addition $+ : V \times V \to V$ and the scalar multiplication $\cdot : \mathbb{K} \times V \to V$ are both continuous. (Here the topology on $\mathbb{K}$ is the standard one.)

Proposition 2.5. *Let V be a tvs, $\lambda \in \mathbb{K} \setminus 0$, $w \in V$. The map $V \to V : v \mapsto \lambda v$ is an automorphism of V as a tvs. In particular, the topology $\mathcal{T}$ of V is invariant under rescalings: $\lambda \mathcal{T} = \mathcal{T}$. What is more, it is invariant under translations: $\mathcal{T} + w = \mathcal{T}$. In terms of filters of neighborhoods, $\lambda \mathcal{N}_v = \mathcal{N}_{\lambda v}$ and $\mathcal{N}_v + w = \mathcal{N}_{v+w}$ for all $v \in V$.*

Proof. It is clear that non-zero scalar multiplication is a vector space automorphism. To see that scalar multiplication with λ and translation by w are continuous use Proposition 1.18. The inverse maps are of the same type hence also continuous. Thus, we have homeomorphisms. The scale- and translation invariance of the topology follows. □

Note that this implies that the topology of a tvs is completely determined by the filter of neighborhoods of one of its points, say 0.

Definition 2.6. Let V be a tvs and U a subset. U is called *bounded* iff for every neighborhood W of 0 there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$.

Remark: Changing the allowed range of λ in the definition of boundedness from $\mathbb{R}^+$ to $\mathbb{K}$ leads to an equivalent definition, i.e., is not weaker. However, the choice of $\mathbb{R}^+$ over $\mathbb{K}$ is more convenient in certain applications.

Proposition 2.7. *Let V be a tvs. Then:*

1. *Every point set is bounded.*
2. *Every neighborhood of 0 contains a balanced subneighborhood of 0.*
3. *Let U be a neighborhood of 0. Then there exists a subneighborhood W of 0 such that $W + W \subseteq U$.*

Proof. We start by demonstrating Property 1. Let $x \in V$ and U some open neighborhood of 0. Then $Z := \{(\lambda, y) \in \mathbb{K} \times V : \lambda y \in U\}$ is open by continuity of multiplication. Also $(0, x) \in Z$ so that by the product topology there exists an $\epsilon > 0$ and an open neighborhood W of x in V such that $B_\epsilon(0) \times W \subseteq Z$. In particular, there exists $\mu > 0$ such that $\mu x \in U$, i.e., $\{x\} \subseteq \mu^{-1}U$ as desired.

We proceed to Property 2. Let U be an open neighborhood of 0. By continuity $Z := \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ is open. By the product topology, there are open neighborhoods X of $0 \in \mathbb{K}$ and W of $0 \in V$ such that $X \times W \subseteq Z$. Thus, $X \cdot W \subseteq U$. Now X contains an open ball of some radius $\epsilon > 0$ around 0 in $\mathbb{K}$. Set $Y := B_\epsilon(0) \cdot W$. This is an (open) neighborhood of 0 in V, it is contained in U and it is balanced.

We end with Property 3. Let U be an open neighborhood of 0. By continuity $Z := \{(x, y) \in V \times V : x + y \in U\}$ is open. By the product topology, there are open neighborhoods W_1 and W_2 of 0 such that $W_1 \times W_2 \subseteq Z$. This means $W_1 + W_2 \subseteq U$. Now define $W := W_1 \cap W_2$. □

Proposition 2.8. *Let V be a vector space and $\mathcal{F}$ a filter on V. Then $\mathcal{F}$ is the filter of neighborhoods of 0 for a compatible topology on V iff 0 is contained in every element of $\mathcal{F}$ and $\lambda\mathcal{F} = \mathcal{F}$ for all $\lambda \in \mathbb{K} \setminus \{0\}$ and $\mathcal{F}$ satisfies the properties of Proposition 2.7.*

Proof. It is already clear that the properties in question are necessary for $\mathcal{F}$ to be the filter of neighborhoods of 0 of V. It remains to show that they are sufficient. If $\mathcal{F}$ is to be the filter of neighborhoods of 0 then, by translation invariance, $\mathcal{F}_x := \mathcal{F} + x$ must be the filter of neighborhoods of the point x. We show that the family of filters $\{\mathcal{F}_x\}_{x \in V}$ does indeed define a topology on V. To this end we will use Proposition 1.10. Property 1 is satisfied by assumption. It remains to show Property 2. By translation invariance it will be enough to consider $x = 0$. Suppose $U \in \mathcal{F}$. Using Property 3 of Proposition 2.7 there is $W \in \mathcal{F}$ such that $W + W \subseteq U$. We claim that Property 2 of Proposition 1.10 is now satisfied with this choice of W. Indeed, let $y \in W$ then $y + W \in \mathcal{F}_y$ and $y + W \subseteq U$ so $U \in \mathcal{F}_y$ as required.

We proceed to show that the topology defined in this way is compatible with the vector space structure. Take an open set $U \subseteq V$ and consider its preimage $Z = \{(x, y) \in V \times V : x + y \in U\}$ under vector addition. Take some point $(x, y) \in Z$. $U - x - y$ is an open neighborhood of 0. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W \subseteq U - x - y$, i.e., $(x + W) + (y + W) \subseteq U$. But $x + W$ is an open neighborhood of x and $y + W$ is an open neighborhood of y so $(x + W) \times (y + W)$ is an open neighborhood of (x, y) in $V \times V$ contained in Z. Hence vector addition is continuous.

We proceed to show continuity of scalar multiplication. Consider an open set $U \subseteq V$ and consider its preimage $Z = \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ under scalar multiplication. Take some point $(\lambda, x) \in Z$. $U - \lambda x$ is an open neighborhood of 0 in V. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W = U - \lambda x$. By Property 2 of Proposition 2.7 there exists a balanced subneighborhood X of W. By Property 1 of Proposition 2.7 (boundedness of points) there exists $\epsilon > 0$ such that $\epsilon x \in X$. Now define $Y := (\epsilon + |\lambda|)^{-1} X$. Note that scalar multiples of (open) neighborhoods of 0 are (open) neighborhoods of 0 by assumption. Hence Y is open since X is. Thus $B_\epsilon(\lambda) \times (x + Y)$ an open neighborhood of (λ, x) in $\mathbb{K} \times V$. We claim that it is contained in Z. First observe that since X is balanced, $B_\epsilon(0) \cdot x \subseteq X$. Similarly, we have $B_\epsilon(\lambda) \cdot Y \subseteq B_{\epsilon+|\lambda|}(0) \cdot Y = B_1(0) \cdot X \subseteq X$. Thus we have $B_\epsilon(0) \cdot x + B_\epsilon(\lambda) \cdot Y \subseteq X + X \subseteq W + W \subseteq U - \lambda x$. But this implies $B_\epsilon(\lambda) \cdot (x + Y) \subseteq U$ as required. □

Proposition 2.9. *(a) The interior of a balanced set is balanced. (b) The closure of a balanced set is balanced.*

Proof. Let U be balanced and let $\lambda \in \mathbb{K}$ with $0 < |\lambda| \le 1$. It is then enough to observe that for (a) $\lambda \mathring{U} = \mathring{\overbrace{\lambda U}} \subseteq \mathring{U}$ and for (b) $\lambda \overline{U} = \overline{\lambda U} \subseteq \overline{U}$. □

Proposition 2.10. *In a tvs every neighborhood of* 0 *contains a closed and balanced subneighborhood.*

Proof. Let U be a neighborhood of 0. By Proposition 2.7.3 there exists a subneighborhood $W \subseteq U$ such that $W + W \subset U$. By Proposition 2.7.2 there exists a balanced subneighborhood $X \subseteq W$. Let $Y := \overline{X}$. Then, Y is obviously a closed neighborhood of 0. Also Y is balanced by Proposition 2.9. Finally, let $y \in Y = \overline{X}$. Any neighborhood of y must intersect X. In particular, $y + X$ is such a neighborhood. Thus, there exist $x \in X$, $z \in X$ such that $x = y + z$, i.e., $y = x - z \in X - X = X + X \subseteq U$. So, $Y \subseteq U$. □

Proposition 2.11. *(a) Subsets of bounded sets are bounded. (b) Finite unions of bounded sets are bounded. (c) The closure of a bounded set is bounded. (d) The sum of two bounded sets is bounded. (e) A scalar multiple of a bounded set is bounded.*

Proof. **Exercise.** □

Definition 2.12. Let V be a tvs and $C \subseteq V$ a subset. Then, C is called *totally bounded* iff for each neighborhood U of 0 in V there exists a finite subset $F \subseteq C$ such that $C \subseteq F + U$.

Proposition 2.13. *(a) Subsets of totally bounded sets are totally bounded. (b) Finite unions of totally bounded sets are totally bounded. (c) The closure of a totally bounded set is totally bounded. (d) The sum of two totally bounded sets is totally bounded. (e) A scalar multiple of a totally bounded set is totally bounded.*

Proof. **Exercise.** □

Proposition 2.14. *Compact sets are totally bounded. Totally bounded sets are bounded.*

Proof. **Exercise.** □

Let A, B be topological vector spaces. We denote the space of maps from A to B that are linear and continuous by $\mathrm{CL}(A, B)$.

Definition 2.15. Let A, B be tvs. A linear map $f : A \to B$ is called *bounded* iff there exists a neighborhood U of 0 in A such that $f(U)$ is bounded. A linear map $f : A \to B$ is called *compact* iff there exists a neighborhood U of 0 in A such that $\overline{f(U)}$ is compact.

Let A, B be tvs. We denote the space of maps from A to B that are linear and bounded by $\mathrm{BL}(A, B)$. We denote the space of maps from A to B that are linear and compact by $\mathrm{KL}(A, B)$.

Proposition 2.16. *Let A, B be tvs and $f \in \mathrm{L}(A, B)$. (a) f is continuous iff the preimage of any neighborhood of 0 in B is a neighborhood of 0 in A. (b) If f is continuous it maps bounded sets to bounded sets. (c) If f is bounded then f is continuous, i.e.,* $\mathrm{BL}(A, B) \subseteq \mathrm{CL}(A, B)$. *(d) If f is compact then f is bounded.*

Proof. **Exercise.** □

A useful property for a topological space is the Hausdorff property, i.e., the possibility to separate points by open sets. It is not the case that a tvs is automatically Hausdorff. However, the way in which a tvs may be non-Hausdorff is severely restricted. Indeed, we shall see int the following that a tvs may be split into a part that is Hausdorff and another one that is maximally non-Hausdorff in the sense of carrying the trivial topology.

Proposition 2.17. *Let V be a tvs and $C \subseteq V$ a vector subspace. Then, the closure $\overline{C}$ of C is also a vector subspace of V.*

Proof. **Exercise.**[Hint: Use Proposition 1.63.] □

Proposition 2.18. *Let V be a tvs. The closure of $\{0\}$ in V coincides with the intersection of all neighborhoods of 0. Moreover, V is Hausdorff iff $\overline{\{0\}} = \{0\}$.*

Proof. **Exercise.** □

Proposition 2.19. *Let V be a tvs and $C \subseteq V$ a vector subspace.*

1. *The quotient space V/C is a tvs.*

2. *V/C is Hausdorff iff C is closed in V.*

3. *The quotient map $q : V \to V/C$ is linear, continuous and open. Moreover, the quotient topology on V/C is the only topology such that q is continuous and open.*

4. *The image of a base of the filter of neighborhoods of 0 in V is a base of the filter of neighborhoods of 0 in V/C.*

Proof. **Exercise.** □

Thus, for a tvs V the exact sequence

$$0 \to \overline{\{0\}} \to V \to V/\overline{\{0\}} \to 0$$

describes how V is composed of a Hausdorff piece $V/\overline{\{0\}}$ and a piece $\overline{\{0\}}$ with trivial topology. We can express this decomposition also in terms of a direct sum, as we shall see in the following.

A (vector) subspace of a tvs is a tvs with the subset topology. Let A and B be tvs. Then the direct sum $A \oplus B$ is a tvs with the product topology. Note that as subsets of $A \oplus B$, both A and B are closed.

Definition 2.20. Let V be a tvs and A a subspace. Then another subspace B of A in V is called a *topological complement* iff $V = A \oplus B$ as tvs (i.e., as vector spaces and as topological spaces). A is called *topologically complemented* if such a topological complement B exists.

Note that algebraic complements (i.e., complements merely with respect to the vector space structure) always exist (using the Axiom of Choice). However, an algebraic complement is not necessarily a topological one. Indeed, there are examples of subspaces of tvs that have no topological complement.

Proposition 2.21 (Structure Theorem for tvs)**.** *Let V be a tvs and B an algebraic complement of $\overline{\{0\}}$ in V. Then B is also a topological complement of $\overline{\{0\}}$ in V. Moreover, B is canonically isomorphic to $V/\overline{\{0\}}$ as a tvs.*

Proof. **Exercise.** □

We conclude that every tvs is a direct sum of a Hausdorff tvs and a tvs with the trivial topology.

2.2 Metrizable and pseudometrizable vector spaces

In this section we consider *(pseudo)metrizable vector spaces* (mvs), i.e., tvs that admit a (pseudo)metric compatible with the topology.

Definition 2.22. A pseudometric on a vector space V is called *translation-invariant* iff $d(x+a, y+a) = d(x, y)$ for all $x, y, a \in V$. A translation-invariant pseudometric on a vector space V is called *balanced* iff its open balls around the origin are balanced.

As we shall see it will be possible to limit ourselves to balanced translation-invariant pseudometrics on mvs. Moreover, these can be conveniently described by pseudo-seminorms.

Definition 2.23. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}_0^+ : x \mapsto \|x\|$ is called a *pseudo-seminorm* iff it satisfies the following properties:

1. $\|0\| = 0$.
2. For all $\lambda \in \mathbb{K}$, $|\lambda| \leq 1$ implies $\|\lambda x\| \leq \|x\|$ for all $x \in V$.
3. For all $x, y \in V : \|x+y\| \leq \|x\| + \|y\|$.

$\|\cdot\|$ is called a *pseudo-norm* iff it satisfies in addition the following property.

4. $\|x\| = 0$ implies $x = 0$.

Proposition 2.24. *There is a one-to-one correspondence between pseudo-seminorms and balanced translation invariant pseudometrics on a vector space via $d(x, y) := \|x - y\|$. This specializes to a correspondence between pseudo-norms and balanced translation invariant metrics.*

Proof. **Exercise.** □

Proposition 2.25. *Let V be a vector space. The topology generated by a pseudo-seminorm on V is compatible with the vector space structure iff for every $x \in V$ and $\epsilon > 0$ there exists $\lambda \in \mathbb{R}^+$ such that $x \in \lambda B_\epsilon(0)$.*

Proof. Assume we are given a pseudo-seminorm on V that induces a compatible topology. It is easy to see that the stated property of the pseudo-seminorm then follows from Property 1 in Proposition 2.7 (boundedness of points).

Conversely, suppose we are given a pseudo-seminorm on V with the stated property. We show that the filter $\mathcal{N}_0$ of neighborhoods of 0 defined by the pseudo-seminorm has the properties required by Proposition 2.8 and hence defines a compatible topology on V. Firstly, it is already clear that every $U \in \mathcal{N}_0$ contains 0. We proceed to show that $\mathcal{N}_0$ is scale invariant. It is enough to show that for $\epsilon > 0$ and $\lambda \in \mathbb{K} \setminus \{0\}$ the scaled ball $\lambda B_\epsilon(0)$ is open. Choose a point $\lambda x \in \lambda B_\epsilon(0)$. Take $\delta > 0$ such that $\|x\| < \epsilon - \delta$. Then $B_\delta(0) + x \subseteq B_\epsilon(0)$. Choose $n \in \mathbb{N}$ such that $2^{-n} \leq |\lambda|$. Observe that the triangle inequality implies $B_{2^{-n}\delta}(0) \subseteq 2^{-n}B_\delta(0)$ (for arbitrary δ and n in fact). Hence $B_{2^{-n}\delta}(\lambda x) = B_{2^{-n}\delta}(0) + \lambda x \subseteq \lambda B_\delta(0) + \lambda x \subseteq \lambda B_\epsilon(0)$ showing that $\lambda B_\epsilon(0)$ is open.

It now remains to show the properties of $\mathcal{N}_0$ listed in Proposition 2.7. As for Property 3, we may take U to be an open ball of radius ϵ around 0 for some $\epsilon > 0$. Define $W := B_{\epsilon/2}(0)$ Then $W + W \subseteq U$ follows from the triangle inequality. Concerning Property 2 we simple notice that open balls are balanced by construction. The only property that is not automatic for a pseudo-seminorm and does require the stated condition is Property 1 (boundedness of points). The equivalence of the two is easy to see. □

Theorem 2.26. *A tvs V is pseudometrizable iff it is first-countable, i.e., iff there exists a countable base for the filter of neighborhoods of* 0*. Moreover, if V is pseudometrizable it admits a compatible pseudo-seminorm.*

Proof. It is clear that pseudometrizability implies the existence of a countable base of $\mathcal{N}_0$. For example, the sequence of balls $\{B_{1/n}(0)\}_{n\in\mathbb{N}}$ provides such a base. Conversely, suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a base of the filter of neighborhoods of 0 such that all U_n are balanced and $U_{n+1} + U_{n+1} \subseteq U_n$. (Given an arbitrary countable base of $\mathcal{N}_0$ we can always produce another one with the desired properties.) Now for each finite subset H of $\mathbb{N}$ define $U_H := \sum_{n\in H} U_n$ and $\lambda_H := \sum_{n\in H} 2^{-n}$. Note that each U_H is a balanced neighborhood of 0. Define now the function $V \to \mathbb{R}_0^+ : x \mapsto \|x\|$ by

$$\|x\| := \inf_H \{\lambda_H | x \in U_H\}$$

if $x \in U_H$ for some H and $\|x\| = 1$ otherwise. We proceed to show that $\|\cdot\|$ defines a pseudo-seminorm and generates the topology of V.

Fix $x \in V$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. Since U_H is balanced for each H, λx is contained at least in the same sets U_H as x. Because the definition of $\|\cdot\|$ uses an infimum, $\|\lambda x\| \leq \|x\|$. This confirms Property 1 of Definition 2.23.

To show the triangle inequality (Property 3 of Definition 2.23) we first note that for finite subsets H, K of $\mathbb{N}$ with the property $\lambda_H + \lambda_K < 1$ there is another unique finite subset L of $\mathbb{N}$ such that $\lambda_L = \lambda_H + \lambda_K$. Furthermore, $U_H + U_K \subseteq U_L$ in this situation. Now, fix $x, y \in V$. If $\|x\| + \|y\| \geq 1$ the triangle inequality is trivial. Otherwise, we can find $\epsilon > 0$ such that $\|x\| + \|y\| + 2\epsilon < 1$. We now fix finite subsets H, K of $\mathbb{N}$ such that $x \in U_H$, $y \in U_K$ while $\lambda_H < \|x\| + \epsilon$ and $\lambda_K < \|y\| + \epsilon$. Let L be the finite subset of $\mathbb{N}$ such that $\lambda_L = \lambda_H + \lambda_K$. Then $x + y \in U_L$ and hence $\|x + y\| \leq \lambda_L = \lambda_H + \lambda_K < \|x\| + \|y\| + 2\epsilon$. Since the resulting inequality holds for any $\epsilon > 0$ we must have $\|x + y\| \leq \|x\| + \|y\|$ as desired.

It remains to show that the pseudo-seminorm generates the topology of the tvs. Since the topology generated by the pseudo-seminorm as well as that of the tvs are translation invariant, it is enough to show that the open balls around 0 of the pseudo-seminorm form a base of the filter of neighborhoods of 0 in the topology of the tvs. Let $n \in \mathbb{N}$. Clearly $B_{2^{-n}}(0) \subseteq U_n \subseteq B_{2^{-(n-1)}}(0)$. But this shows that $\{B_{2^{-n}}(0)\}_{n\in\mathbb{N}}$ generates the same filter as $\{U_n\}_{n\in\mathbb{N}}$. This completes the proof. □

Exercise 6. Show that for a tvs with a balanced translation-invariant pseudometric the concepts of totally boundedness of Definitions 1.81 and 2.12 coincide.

Proposition 2.27. *Let V be a mvs with pseudo-seminorm. Let $r > 0$ and $0 < \mu \leq 1$. Then, $B_{\mu r}(0) \subseteq \mu B_r(0)$.*

Proof. **Exercise.** □

Proposition 2.28. *Let V, W be mvs with compatible metrics and $f \in \mathrm{L}(V, W)$. (a) f is continuous iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $f(B^V_\delta(0)) \subseteq B^W_\epsilon(0)$. (b) f is bounded iff there exists $\delta > 0$ such that for all $\epsilon > 0$ there is $\mu > 0$ such that $f(\mu B^V_\delta(0)) \subseteq B^W_\epsilon(0)$.*

Proof. **Exercise.** □

Proposition 2.29. *Let V be a mvs and C a subspace. Then, the quotient space V/C is a mvs.*

Proof. **Exercise.** □

2.3 Locally convex tvs

Definition 2.30. A tvs is called *locally convex* iff every neighborhood of 0 contains a convex neighborhood of 0.

Definition 2.31. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}^+_0 : x \mapsto \|x\|$ is called a *seminorm* iff it satisfies the following properties:

1. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
2. For all $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

A seminorm is called a *norm* iff it satisfies in addition the following property:

3. $\|x\| = 0 \implies x = 0$.

Proposition 2.32. *A seminorm induces a balanced translation-invariant pseudometric via $d(x, y) := \|x - y\|$. Moreover, the open balls of this metric are convex.*

Proof. **Exercise.** □

Proposition 2.33. *Let V be a vector space and $\{\|\cdot\|_\alpha\}_{\alpha \in A}$ a set of seminorms on V. For any finite subset $I \subseteq A$ and any $\epsilon > 0$ define*

$$U_{I,\epsilon} := \{x \in V : \|x\|_\alpha < \epsilon \; \forall \alpha \in I\}.$$

Then, the sets $U_{I,\epsilon}$ form the base of the filter of neighborhoods of 0 in a topology on V that makes it into a locally convex tvs. If A is countable, then V is pseudometrizable. Moreover, the topology is Hausdorff iff for any $x \in V \setminus \{0\}$ there exists $\alpha \in A$ such that $\|x\|_\alpha > 0$.

Proof. Let $I, I' \subseteq A$ be finite and $\epsilon, \epsilon' > 0$. Set $I'' := I \cup I'$ and $\epsilon'' := \min(\epsilon, \epsilon')$. Then, $U_{I'',\epsilon''} \subseteq U_{I,\epsilon} \cap U_{I',\epsilon'}$. So the $U_{I,\epsilon}$ really form the basis of a filter $\mathcal{F}$. We proceed to verify that $\mathcal{F}$ satisfies the properties required by Proposition 2.8. Clearly, $0 \in U$ for all $U \in \mathcal{F}$ since $\|0\|_\alpha = 0$ and so $0 \in U_{I,\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$. Also $\lambda\mathcal{F} = \mathcal{F}$ since $\lambda U_{I,\epsilon} = U_{I,|\lambda|\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$ by linearity of seminorms. As for property 1 of Proposition 2.7 consider $x \in V$, $I \subseteq A$ finite and $\epsilon > 0$ arbitrary. Set $\mu := \max_{\alpha\in I}\{\|x\|_\alpha\}$. Then, $x \in \frac{\mu+1}{\epsilon} U_{I,\epsilon}$. Property 2 of Proposition 2.7 is satisfied since open balls of a seminorm are balanced and the sets $U_{I,\epsilon}$ are finite intersections of such open balls and hence also balanced. Property 3 of Proposition 2.7 is sufficient to satisfy for a base. Observe then, $U_{I,\epsilon/2} + U_{I,\epsilon/2} \subseteq U_{I,\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$ due to the triangle inequality. Thus, the so defined topology makes V into a tvs.

Observe that the sets $U_{I,\epsilon}$ are convex, being finite intersections of open balls which are convex by Proposition 2.32. Thus, V is locally convex. If A is countable, then there is an enumeration $I_1, I_2, \ldots$ of the finite subsets of A. It is easy to see that $U_{I_j,1/n}$ with $j \in \{1, \ldots\}$ and $n \in \mathbb{N}$ provides then a countable basis of the filter of neighborhoods of 0. That is, V is pseudometrizable. Concerning the Hausdorff property suppose that for any $x \in V \setminus \{0\}$ there exists $\alpha \in A$ such that $\|x\|_\alpha > 0$. Then, for this x we have $x \notin U_{\{\alpha\},\|x\|_\alpha}$. So V is Hausdorff. Conversely, suppose V is Hausdorff. Given $x \in V \setminus \{0\}$ there exist thus $I \subseteq A$ finite and $\epsilon > 0$ such that $x \notin U_{I,\epsilon}$. In particular, there exists $\alpha \in I$ such that $\|x\|_\alpha \geq \epsilon > 0$. □

Exercise 7. In the context of Proposition 2.33 show that the topology is the coarsest such that all seminorms $\|\cdot\|_\alpha$ are continuous.

Definition 2.34. Let V be a tvs and $W \subseteq V$ a neighborhood of 0. The map $\|\cdot\|_W : V \to \mathbb{R}_0^+$ defined as

$$\|x\|_W := \inf\{\lambda \in \mathbb{R}_0^+ : x \in \lambda W\}$$

is called the *Minkowski functional* associated to W.

Proposition 2.35. *Let V be a tvs and $W \subseteq V$ a neighborhood of* 0*.*

1. $\|\mu x\|_W = \mu\|x\|_W$ *for all* $\mu \in \mathbb{R}_0^+$ *and* $x \in V$*.*
2. *If W is balanced, then* $\|cx\|_W = |c|\|x\|_W$ *for all* $c \in \mathbb{K}$ *and* $x \in V$*.*
3. *If W is convex, then* $\|x + y\|_W \leq \|x\|_W + \|y\|_W$ *for all* $x, y \in V$*.*
4. *If V is Hausdorff and W is bounded, then* $\|x\|_W = 0$ *implies* $x = 0$*.*

Proof. **Exercise.** □

Theorem 2.36. *Let V be a tvs. Then, V is locally convex iff there exists a set of seminorms inducing its topology as in Proposition 2.33. Also, V is locally convex and pseudometrizable iff there exists a countable such set.*

Proof. Given a locally convex tvs V, let $\{U_\alpha\}_{\alpha\in A}$ be a base of the filter of neighborhoods such that U_α is balanced and convex for all $\alpha \in A$. (**Exercise.**How can this be achieved?) In case that V is pseudometrizable we choose the base such that A is countable. Let $\|\cdot\|_\alpha$ be the Minkowski functional associated to U_α. Then, by Proposition 2.35, $\|\cdot\|_\alpha$ is a seminorm for each $\alpha \in A$. We claim that the topology generated by the seminorms is precisely the topology of V. **Exercise.**Complete the proof. □

Exercise 8. Let V be a locally convex tvs and W a balanced and convex neighborhood of 0. Show that the Minkowski functional associated to W is continuous on V.

Exercise 9. Let V be a vector space and $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ a sequence of seminorms on V. Define the function $q : V \to \mathbb{R}^+_0$ via

$$q(x) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|x\|_n}{\|x\|_n + 1}.$$

(a) Show that q is a pseudo-seminorm on V. (b) Show that the topology generated on V by q is the same as that generated by the sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

2.4 Normed and seminormed vector spaces

Definition 2.37. A tvs is called *locally bounded* iff it contains a bounded neighborhood of 0.

Proposition 2.38. *A locally bounded tvs is pseudometrizable.*

Proof. Let V be a locally bounded tvs and U a bounded and balanced neighborhood of 0 in V. The sequence $\{U_n\}_{n\in N}$ with $U_n := \frac{1}{n}U$ is the base of a filter $\mathcal{F}$ on V. Take a neighborhood W of 0. By boundedness of U there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$. Choosing $n \in \mathbb{N}$ with $n \geq \lambda$ we find $U_n \subseteq W$, i.e., $W \in \mathcal{F}$. Hence $\mathcal{F}$ is the filter of neighborhoods of 0 and we have presented a countable base for it. By Theorem 2.26, V is pseudometrizable. □

Proposition 2.39. *Let A, B be a tvs and $f \in \mathrm{CL}(A, B)$. If A or B is locally bounded then f is bounded. Hence, $\mathrm{CL}(A, B) = \mathrm{BL}(A, B)$ in this case.*

Proof. **Exercise.** □

Definition 2.40. A tvs V is called *(semi)normable* iff the topology of V is induced by a (semi)norm.

Theorem 2.41. *A tvs V is seminormable iff V is locally bounded and locally convex.*

Proof. Suppose V is a seminormed vector space. Then, every ball is bounded and also convex, so in particular, V is locally bounded and locally convex.

Conversely, suppose V is a tvs that is locally bounded and locally convex. Take a bounded neighborhood U_1 of 0 and a convex subneighborhood U_2 of U_1. Now take a balanced subneighborhood U_3 of U_2 and its convex hull $W = \mathrm{conv}(U_3)$. Then W is a balanced, convex and bounded (since $W \subseteq U_2 \subseteq U_1$) neighborhood of 0 in V. Thus, by Proposition 2.35 the Minkowski functional $\|\cdot\|_W$ defines a seminorm on V. It remains to show that the topology generated by this seminorm coincides with the topology of V. Let U be an open set in the topology of V and $x \in U$. The ball $B_1(0)$ defined by the seminorm is bounded since $B_1(0) \subseteq W$ and W is bounded. Hence there exists $\lambda \in \mathbb{R}^+$ such that $B_1(0) \subseteq \lambda(U - x)$, i.e., $\lambda^{-1}B_1(0) \subseteq U - x$. But $\lambda^{-1}B_1(0) = B_{\lambda^{-1}}(0)$ by linearity and thus $B_{\lambda^{-1}}(x) \subseteq U$. Hence, U is open in the seminorm topology as well. Conversely, consider a ball $B_\epsilon(0)$ defined by the seminorm for some $\epsilon > 0$ and take $x \in B_\epsilon(0)$. Choose $\delta > 0$ such that $\|x\|_W < \epsilon - \delta$. Observe that $\frac{1}{2}W \subseteq B_1(0)$ and thus by linearity $\frac{\delta}{2}W \subseteq B_\delta(0)$. It follows that $\frac{\delta}{2}W + x \subseteq B_\epsilon(0)$. But $\frac{\delta}{2}W + x$ is a neighborhood of x so it follows that $B_\epsilon(0)$ is open. This completes the proof. □

Exercise 10. Let V be locally convex tvs with its topology generated by a finite family of seminorms. Show that V is seminormable.

Proposition 2.42. *Let V be a seminormed vector space and $U \subseteq V$ a subset. Then, U is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|x\| \le c$ for all $x \in U$.*

Proof. **Exercise.** □

Proposition 2.43. *Let A, B be seminormed vector spaces and $f \in \mathrm{L}(A, B)$. f is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|f(x)\| \le c\,\|x\|$ for all $x \in A$.*

Proof. **Exercise.** □

Proposition 2.44. *Let V be a tvs and C a vector subspace. If V is locally convex, then so is V/C. If V is locally bounded, then so is V/C.*

Proof. **Exercise.** □

2.5 Inner product spaces

As before $\mathbb{K}$ stands for a field that is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.45. Let V be a vector space over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ a map. $\langle\cdot,\cdot\rangle$ is called a *bilinear* (if $\mathbb{K} = \mathbb{R}$) or *sesquilinear* (if $\mathbb{K} = \mathbb{C}$) form iff it satisfies the following properties:

- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and
  $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

- $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ and $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbb{K}$ and $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *symmetric* (if $\mathbb{K} = \mathbb{R}$) or *hermitian* (if $\mathbb{K} = \mathbb{C}$) iff it satisfies in addition the following property:

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

$\langle\cdot,\cdot\rangle$ is called *positive* iff it satisfies in addition the following property:

- $\langle v, v\rangle \geq 0$ for all $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *definite* iff it satisfies in addition the following property:

- If $\langle v, v\rangle = 0$ then $v = 0$ for all $v \in V$.

A map with all these properties is also called a *scalar product* or an *inner product*. V equipped with such a structure is called an *inner product space* or a *pre-Hilbert space*.

Theorem 2.46 (Schwarz Inequality)**.** *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, the following inequality is satisfied:*

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle \quad \forall v, w \in V.$$

Proof. By definiteness $\alpha := \langle v, v\rangle \neq 0$ and we set $\beta := -\langle w, v\rangle$. By positivity we have,

$$0 \leq \langle \beta v + \alpha w, \beta v + \alpha w\rangle.$$

Using bilinearity and symmetry (if $\mathbb{K} = \mathbb{R}$) or sesquilinearity and hermiticity (if $\mathbb{K} = \mathbb{C}$) on the right hand side this yields,

$$0 \leq |\langle v, v\rangle|^2\langle w, w\rangle - \langle v, v\rangle|\langle v, w\rangle|^2.$$

(**Exercise.**Show this.) Since $\langle v, v\rangle \neq 0$ we can divide by it and arrive at the required inequality. □

Proposition 2.47. *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, V is a normed vector space with norm given by $\|v\| := \sqrt{\langle v, v\rangle}$.*

Proof. **Exercise.**Hint: To prove the triangle inequality, show that $\|v+w\|^2 \leq (\|v\|+\|w\|)^2$ can be derived from the Schwarz inequality (Theorem 2.46). □

Proposition 2.48. *Let V be an inner product space. Then, $\forall v, w \in V$,*

$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) \quad \textit{if} \quad \mathbb{K} = \mathbb{R},$$
$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 + \mathrm{i}\|v + \mathrm{i}w\|^2 - \mathrm{i}\|v - \mathrm{i}w\|^2\right) \quad \textit{if} \quad \mathbb{K} = \mathbb{C}$$

Proof. **Exercise.** □

Proposition 2.49. *Let V be an inner product space. Then, its scalar product $V \times V \to \mathbb{K}$ is continuous.*

Proof. **Exercise.** □

Theorem 2.50. *Let V be a normed vector space. Then, there exists a scalar product on V inducing the norm iff the parallelogram equality holds,*

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \forall v, w \in V.$$

Proof. **Exercise.** □

Example 2.51. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner product spaces via

$$\langle v, w \rangle := \sum_{i=1}^{n} v_i \overline{w_i},$$

where v_i, w_i are the coefficients with respect to the standard basis.

3 First examples and properties

3.1 Elementary topologies on function spaces

If V is a vector space over $\mathbb{K}$ and S is some set, then the set of maps $S \to V$ naturally forms a vector space over $\mathbb{K}$. This is probably the most important source of topological vector spaces in functional analysis. Usually, the spaces S and V carry additional structure (e.g. topologies) and the maps in question may be restricted, e.g. to be continuous etc. The topology given to this vector space of maps usually depends on these additional structures.

Example 3.1. Let S be a set and $F(S, \mathbb{K})$ be the set of functions on S with values in $\mathbb{K}$. Consider the set of seminorms $\{p_x\}_{x\in S}$ on $F(S, \mathbb{K})$ defined by $p_x(f) := |f(x)|$. This gives $F(S, \mathbb{K})$ the structure of a locally convex tvs. The topology defined in this way is also called the *topology of pointwise convergence.*

Exercise 11. Show that this topology is the coarsest topology making all evaluation maps, i.e., maps of the type $f \mapsto f(x)$, continuous. Show also that a sequence in $F(S, \mathbb{K})$ converges with respect to this topology iff it converges pointwise.

Example 3.2. Let S be a set and $B(S, \mathbb{K})$ be the set of bounded functions on S with values in $\mathbb{K}$. Then, $B(S, \mathbb{K})$ is a normed vector space with the supremum norm:

$$\|f\| := \sup_{x\in S} |f(x)| \quad \forall f \in B(S, \mathbb{K}).$$

The topology defined in this way is also called the *topology of uniform convergence.*

Exercise 12. Show that a sequence in $B(S, \mathbb{K})$ converges with respect to this topology iff it converges uniformly on all of S.

Exercise 13. (a) Show that on $B(S, \mathbb{K})$ the topology of uniform convergence is finer than the topology of pointwise convergence. (b) Under which circumstances are both topologies equal?

Example 3.3. Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. For $K \in \mathfrak{K}$ define on $\mathrm{C}(S, \mathbb{K})$ the seminorm

$$\|f\|_K := \sup_{x\in K} |f(x)| \quad \forall f \in \mathrm{C}(S, \mathbb{K}).$$

The topology defined in this way on $\mathrm{C}(S, \mathbb{K})$ is called the *topology of compact convergence.*

Exercise 14. Show that a sequence in $\mathrm{C}(S, \mathbb{K})$ converges with respect to this topology iff it converges compactly, i.e., uniformly in any compact subset.

Exercise 15. (a) Show that on $\mathrm{C}(S, \mathbb{K})$ the topology of compact convergence is finer than the topology of pointwise convergence. (b) Show that on the space $\mathrm{C}_\mathrm{b}(S, \mathbb{K})$ of bounded continuous maps the topology of uniform convergence is finer than the topology of compact convergence. (c) Give a sufficient condition for them to be equal.

Definition 3.4. Let S be a set, V a tvs. Let $\mathfrak{S}$ a non-empty set of non-empty subsets of S with the property that for X, Y in $\mathfrak{S}$ there exists $Z \in \mathfrak{S}$ such that $X \cup Y \subseteq Z$. Let $\mathcal{B}$ be a base of the filter of neighborhoods of 0 in V. Then, for $X \in \mathfrak{S}$ and $U \in \mathcal{B}$ the sets

$$M(X,U) := \{f \in F(S,V) : f(X) \subseteq U\}$$

define a base of the filter of neighborhoods of 0 for a translation invariant topology on $F(S,V)$. This is called the $\mathfrak{S}$-topology on $F(S,V)$.

Proposition 3.5. *Let S be a set, V a tvs and $\mathfrak{S} \subseteq \mathfrak{P}(S)$ as in Definition 3.4. Let $A \subseteq F(S,V)$ be a vector subspace. Then, A is a tvs with the the $\mathfrak{S}$-topology iff $f(X)$ is bounded for all $f \in A$ and $X \in \mathfrak{S}$.*

Proof. **Exercise.** □

Exercise 16. (a) Let S be a set and $\mathfrak{S}$ be the set of finite subsets of S. Show that the $\mathfrak{S}$-topology on $F(S,\mathbb{K})$ is the topology of pointwise convergence. (b) Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. Show that the $\mathfrak{K}$-topology on $\mathrm{C}(S,\mathbb{K})$ is the topology of compact convergence. (c) Let S be a set and $\mathfrak{S}$ a set of subsets of S such that $S \in \mathfrak{S}$. Show that the $\mathfrak{S}$-topology on $B(S,\mathbb{K})$ is the topology of uniform convergence.

3.2 Completeness

In the absence of a pseudometric we can use the vector space structure of a tvs to complement the information contained in the topology in order to define a Cauchy property which in turn will be used to define an associated notion of completeness.

Definition 3.6. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a tvs V is called a *Cauchy sequence* iff for every neighborhood U of 0 in V there is a number $N > 0$ such that $x_n - x_m \in U$ for all $n, m \geq N$.

Proposition 3.7. *Let V be a mvs with translation-invariant pseudometric. Then, the Cauchy property for sequences in tvs coincide with the previuosly defined one in pseudometric spaces. That is, Definition 3.6 coincides then with Definition 1.76.*

Proof. Straightforward. □

This Proposition implies that there is no conflict with our previous definition of a Cauchy sequence in pseudometric spaces if we restrict ourselves to translation-invariant pseudometrics. Moreover, it implies that for this purpose it does not matter which pseudometric we use, as long as it is translation-invariant. This latter condition is indeed essential.

Exercise 17. Give an example of an mvs with two compatible metrics d^1, d^2 and a sequence x, such that x is Cauchy with respect to d^1, but not with respect to d^2.

In the following, whenever we talk about a Cauchy sequence in a tvs (possibly with additional) structure, we mean a Cauchy sequence according to Definition 3.6.

For a topologically sensible notion of completeness, we need something more general than Cauchy sequences: Cauchy filters.

Definition 3.8. A filter $\mathcal{F}$ on a tvs V is called a *Cauchy filter* iff for every neighborhood U of 0 in V there is an element $W \in \mathcal{F}$ such that $W - W \subseteq U$.

Proposition 3.9. *A sequence is Cauchy iff the associated filter is Cauchy.*

Proof. **Exercise.** □

Proposition 3.10. *Let V be a tvs, $\mathcal{F}$ a Cauchy filter on V. If $p \in V$ is accumulation point of $\mathcal{F}$, then $\mathcal{F}$ converges to p.*

Proof. Let U be a neighborhood of 0 in V. Then, there exists a neighborhood W of 0 in U such that $W + W \subseteq U$. Since $\mathcal{F}$ is a Cauchy filter there exists $F \in \mathcal{F}$ such that $F - F \subseteq W$. On the other hand, p is accumulation point of $\mathcal{F}$ so there exists $q \in F \cap (p + W)$. Then, we have $F - q \subseteq W$ and thus $F \subseteq q + W \subseteq p + W + W \subseteq p + U$. This shows that every neighborhood of p is contained in $\mathcal{F}$, i.e., $\mathcal{F}$ converges to p. □

Proposition 3.11. *A converging filter is Cauchy.*

Proof. **Exercise.** □

Definition 3.12. A subset U of a tvs is called *complete* iff every Cauchy filter containing U converges to a point in U. It is called *sequentially complete* iff every Cauchy sequence in U converges to a point in U.

Since completeness is an important and convenient concept in functional analysis, the complete versions of Hausdorff tvs have special names. In particular, a complete metrizable locally convex tvs is called a *Fréchet space*, a complete normable tvs is called a *Banach space*, and a complete inner product space is called a *Hilbert space*.

Obviously, completeness implies sequential completeness, but not necessarily the other way round. Note that for a mvs with translation-invariant pseudometric, completeness in the sense of metric spaces (Definition 1.79) is now called sequential completeness. However, we will see that in this context it is equivalent to completeness in the sense of the above definition.

Proposition 3.13. *Let V be a mvs. Then, V is complete (in the sense of tvs) iff it is sequentially complete.*

Proof. We have to show that sequential completeness implies completeness. (The opposite direction is obvious.) We use a translation-invariant pseudometric on V. Suppose $\mathcal{F}$ is a Cauchy filter on V. That is, for any $\epsilon > 0$ there exists $U \in \mathcal{F}$ such that $U - U \subseteq B_\epsilon(0)$.

Now, for each $n \in \mathbb{N}$ choose consecutively $U_n \in \mathcal{F}$ such that $U_n - U_n \subseteq B_{1/n}(0)$ and $U_n \subseteq U_{n-1}$ if $n > 1$ (possibly by using the intersection property). Thus, for every $N \in \mathbb{N}$ we have that for all $n, m \geq N : U_n - U_m \subseteq B_{1/N}(0)$. Now for each $n \in \mathbb{N}$ choose an element $x_n \in U_n$. These form a Cauchy sequence and by sequential completeness converge to a point $x \in V$. Given n observe that for all $y \in U_n : d(y,x) \leq d(y,x_n) + d(x_n,x) < \frac{1}{n} + \frac{1}{n}$, hence $U_n \subseteq B_{2/n}(x)$ and thus $B_{2/n}(x) \in \mathcal{F}$. Since this is true for all $n \in \mathbb{N}$, $\mathcal{F}$ contains arbitrarily small neighborhoods of x and hence all of them, i.e., converges to x. □

Proposition 3.14. *(a) Let V be a Hausdorff tvs and A be a complete subset. Then A is closed. (b) Let V be a tvs and A be a closed subset of a complete subset B. Then A is complete.*

Proof. **Exercise.** □

We proceed to show the analogue of Proposition 1.82.

Lemma 3.15. *Let V be a tvs, $C \subseteq V$ totally bounded and $\mathcal{F}$ an ultrafilter containing C. Then $\mathcal{F}$ is Cauchy.*

Proof. Let U be a neighborhood of 0 in V. Choose another neighborhood W of 0 such that W is balanced and $W + W \subseteq U$. Since C is totally bounded there is a finite subset $F = \{x_1, \dots, x_n\}$ of V such that $C \subseteq F + W$. This implies in turn that there is $k \in \{1, \dots, n\}$ such that $(x_k + W) \cap X \neq \emptyset$ for all $X \in \mathcal{F}$. To see that this is true suppose the contrary. Then for each $i \in \{1, \dots, n\}$ there is $X_i \in \mathcal{F}$ such that $(x_i + W) \cap X_i = \emptyset$. But, then $\emptyset = C \cap \bigcap_{i=1}^n X_i \in \mathcal{F}$, a contradiction. Thus, since $\mathcal{F}$ is ultrafilter we must have $x_k + W \in \mathcal{F}$ by Lemma 1.23. But $(x_k + W) - (x_k + W) = W - W = W + W \subseteq U$ by construction. So $\mathcal{F}$ is a Cauchy filter. □

Proposition 3.16. *Let V be a tvs and $C \subseteq V$ a compact subset. Then, C is complete and totally bounded.*

Proof. **Exercise.** □

Proposition 3.17. *Let V be a tvs and $C \subseteq V$ a subset. If C is totally bounded and complete then it is compact.*

Proof. Let $\mathcal{F}$ be a filter containing C. By Proposition 1.24 there exists an ultrafilter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$. Since C is totally bounded, Lemma 3.15 implies that $\mathcal{F}'$ is Cauchy. Since C is complete, $\mathcal{F}'$ must converge to some point $p \in C$. By Proposition 1.60, this means that p is accumulation point of $\mathcal{F}'$. By Proposition 1.61 this implies that p is accumulation point of $\mathcal{F}$. Since $\mathcal{F}$ was arbitrary, Proposition 1.66 implies that C is compact. □

Proposition 3.18. *Let V be a complete mvs and C a vector subspace. Then V/C is complete.*

Proof. **Exercise.** □

Exercise 18. Which of the topologies defined above are complete? Which become complete under additional assumptions on the space S?

3.3 Finite dimensional tvs

Theorem 3.19. *Let V be a Hausdorff tvs of dimension $n \in \mathbb{N}$. Then, any isomorphism of vector spaces from $\mathbb{K}^n$ to V is also an isomorphism of tvs. Moreover, any linear map from V to any tvs is continuous.*

Proof. We first show that any linear map from $\mathbb{K}^n$ to any tvs W is continuous. Define the map $g : \mathbb{K}^n \times W^n \to W$ given by

$$g((\lambda_1, \ldots, \lambda_n), (v_1, \ldots, v_n)) := \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

This map can be obtained by taking products and compositions of vector addition and scalar multiplication, which are continuous. Hence it is continuous. On the other hand, any linear map $f : \mathbb{K}^n \to W$ takes the form $f(\lambda_1, \ldots, \lambda_n) = g((\lambda_1, \ldots, \lambda_n), (v_1, \ldots, v_n))$ for some fixed set of vectors $\{v_1, \ldots, v_n\}$ in W and is thus continuous by Proposition 1.18.

We proceed to show that any linear map $V \to \mathbb{K}^n$ is continuous. We do this by induction in n starting with $n = 1$. For $n = 1$ any such non-zero map takes the form $g : \lambda e_1 \to \lambda$ for some $e_1 \in V \setminus \{0\}$. (If $g = 0$ continuity is trivial.) For $r > 0$ consider the element $re_1 \in V$. Since V is Hausdorff there exists an open neighborhood U of 0 in V that does not contain re_1. Moreover, we can choose U to be balanced. But then it is clear that $U \subseteq g^{-1}(B_r(0))$. That is, $g^{-1}(B_r(0))$ is a neighborhood of 0 in V. Since open balls centered at 0 form a base of neighborhoods of 0 in $\mathbb{K}$ this implies that the preimage of any neighborhood of 0 in $\mathbb{K}$ is a neighborhood of 0 in V. By Proposition 2.16.a this implies that g is continuous.

We now assume that we have proofed the statement in dimension $n - 1$. Let V be a Hausdorff tvs of dimension n. Consider now some non-zero linear map $h : V \to \mathbb{K}$. We factorize h as $h = \tilde{h} \circ p$ into the projection $p : V \to V/\ker h$ and the linear map $\tilde{h} : V/\ker h \to \mathbb{K}$. $\ker h$ is a vector subspace of V of dimension $n - 1$. In particular, it is a Hausdorff tvs and hence by assumption of the induction isomorphic as a tvs to $\mathbb{K}^{n-1}$. Thus, it is complete and by Proposition 3.14.a closed as a subspace of V. Therefore by Proposition 2.19 the quotient tvs $V/\ker h$ is Hausdorff. Since $V/\ker h$ is also one-dimensional it is isomorphic as a tvs to $\mathbb{K}$ as we have shown above. Thus, $\tilde{h}$ is continuous. Since the projection p is continuous by definition, the composition $h = \tilde{h} \circ p$ must be continuous. Hence, any linear map $V \to \mathbb{K}$ is continuous. But a linear map $V \to \mathbb{K}^n$ can be written as a composition of the continuous map $V \to V^n$ given by $v \mapsto (v, \ldots, v)$ with the product of n linear (and hence continuous) maps $V \to \mathbb{K}$. Thus, it must be continuous.

We have thus shown that for any n a Hausdorff tvs V of dimension n is isomorphic to $\mathbb{K}^n$ as a tvs via any vector space isomorphism. Thus, by the first part of the proof any linear map $V \to W$, where W is an arbitrary tvs must be continuous. □

Proposition 3.20. *Let X be a Hausdorff tvs. Then, any finite dimensional subspace of X is complete and closed.*

Proof. Let $A \subseteq X$ be a subspace of dimension n. By Theorem 3.19, A as a tvs is isomorphic to $\mathbb{K}^n$. In particular, A is complete and thus closed in X by Proposition 3.14. □

Proposition 3.21. *Let X be a Hausdorff tvs, C a closed subspace of X and F a finite-dimensional subspace of X. Then, $F + C$ is closed in X.*

Proof. Since C is closed X/C is a Hausdorff tvs. Let $p : X \to X/C$ be the continuous projection. Then, $p(F)$ is finite-dimensional, hence complete and closed in X/C by Proposition 3.20. Thus, $F + C = p^{-1}(p(F))$ is closed. □

Proposition 3.22. *Let C be a bounded subset of $\mathbb{K}^n$ with the standard topology. Then C is totally bounded.*

Proof. **Exercise.** □

Theorem 3.23 (Riesz)**.** *Let V be a Hausdorff tvs. Then, V is locally compact iff it is finite dimensional.*

Proof. If V is a finite dimensional Hausdorff tvs, then its is isomorphic to $\mathbb{K}^n$ for some n by Theorem 3.19. But closed balls around 0 are compact neighborhoods of 0 in $\mathbb{K}^n$, i.e., $\mathbb{K}^n$ is locally compact.

Now assume that V is a locally compact Hausdorff tvs. Let K be a compact and balanced neighborhood of 0. We can always find this since given a compact neighborhood by Proposition 2.10 we can find a balanced and closed subneighborhood which by Proposition 1.39 must then also be compact. Now let U be an open subneighborhood of $\frac{1}{2}K$. By compactness of K, there exists a finite set of points $\{x_1, \ldots, x_n\}$ such that $K \subseteq \bigcup_{i=1}^n (x_i + U)$. Let W be the finite dimensional subspace of V spanned by $\{x_1, \ldots, x_n\}$. By Theorem 3.19 W is isomorphic to $\mathbb{K}^m$ for some $m \in \mathbb{N}$ and hence complete and closed in V by Proposition 3.14. So by Proposition 2.19 the quotient space V/W is a Hausdorff tvs. Let $\pi : V \to V/W$ be the projection. Observe that, $K \subseteq W + U \subseteq W + \frac{1}{2}K$. Thus, $\pi(K) \subseteq \pi(\frac{1}{2}K)$, or equivalently $\pi(2K) \subseteq \pi(K)$. Iterating, we find $\pi(2^k K) \subseteq \pi(K)$ for all $k \in \mathbb{N}$ and hence $\pi(V) = \pi(K)$ since $V = \bigcup_{k=1}^\infty 2^k K$ as K is balanced. Since π is continuous $\pi(K) = \pi(V) = V/W$ is compact. But since V/W is Hausdorff any one dimensional subspace of it is isomorphic to $\mathbb{K}$ by Theorem 3.19 and hence complete and closed and would have to be compact. But $\mathbb{K}$ is not compact, so V/W cannot have any one-dimensional subspace, i.e., must have dimension zero. Thus, $W = V$ and V is finite dimensional. □

Exercise 19. (a) Show that a finite dimensional tvs is always locally compact, even if it is not Hausdorff. (b) Give an example of an infinite dimensional tvs that is locally compact.

3.4 Equicontinuity

Definition 3.24. Let S be a topological space, T a tvs and $F \subseteq \mathrm{C}(S,T)$. Then, F is called *equicontinuous* at $a \in S$ iff for all neighborhoods W of 0 in T there exists a neighborhood U of a in S such that $f(U) \subseteq f(a)+W$ for all $f \in F$. Moreover, F is called *equicontinuous* iff F is equicontinuous for all $a \in S$.

Exercise 20. Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$. (a) Show that F is bounded in $\mathrm{C}(S,\mathbb{K})$ with the topology of pointwise convergence iff for each $x \in S$ there exists $c > 0$ such that $|f(x)| < c$ for all $f \in F$. (b) Show that F is bounded in $\mathrm{C}(S,\mathbb{K})$ with the topology of compact convergence iff for each $K \subseteq S$ compact there exists $c > 0$ such that $|f(x)| < c$ for all $x \in K$ and for all $f \in F$.

Lemma 3.25. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. Then, F is bounded with respect to the topology of pointwise convergence iff it is bounded with respect to the topology of compact convergence.*

Proof. **Exercise.** □

Lemma 3.26. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. Then, the closures of F in the topology of pointwise convergence and in the topology of compact convergence are equicontinuous.*

Proof. **Exercise.** □

Proposition 3.27. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. If F is closed then it is complete, both in the topology of pointwise convergence and in the topology of compact convergence.*

Proof. We first consider the topology of pointwise convergence. Let $\mathcal{F}$ be a Cauchy filter containing F. For each $x \in S$ induce a filter $\mathcal{F}_x$ generated by $e_x(\mathcal{F})$ on $\mathbb{K}$ through the evaluation map $e_x : \mathrm{C}(S,\mathbb{K}) \to \mathbb{K}$ given by $e_x(f) := f(x)$. Then each $\mathcal{F}_x$ is a Cauchy filter on $\mathbb{K}$ and thus convergent to a uniquely defined $g(x) \in \mathbb{K}$. This defines a function $g : S \to \mathbb{K}$. We proceed to show that g is continuous. Fix $a \in S$ and $\epsilon > 0$. By equicontinuity, there exists a neighborhood U of a such that $f(U) \subseteq B_\epsilon(f(a))$ for all $f \in F$ and hence $|f(x) - f(y)| < 2\epsilon$ for all $x, y \in U$ and $f \in F$. Fix $x, y \in U$. Then, there exists $f \in F$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$. Hence,

$$|g(x) - g(y)| \le |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| < 4\epsilon,$$

showing that g is continuous. Thus, $\mathcal{F}$ converges to g and $g \in F$ if F is closed.

We proceed to consider the topology of compact convergence. Let $\mathcal{F}$ be a Cauchy filter containing F (now with respect to compact convergence). Then, $\mathcal{F}$ is also a Cauchy filter with respect to pointwise convergence and the previous part of the proof shows that there exists a function $g \in \mathrm{C}(S,\mathbb{K})$ to which $\mathcal{F}$ converges pointwise. But since $\mathcal{F}$ is Cauchy with respect to compact convergence it must converge to g also compactly. Then, if F is closed we have $g \in F$ and F is complete. □

Theorem 3.28 (generalized Arzela-Ascoli)**.** *Let S be a topological space. Let $F \subseteq \mathrm{C}(S,\mathbb{K})$ be equicontinuous and bounded in the topology of pointwise convergence. Then, F is relatively compact in $\mathrm{C}(S,\mathbb{K})$ with the topology of compact convergence.*

Proof. We consider the topology of compact convergence on $\mathrm{C}(S,\mathbb{K})$. By Lemma 3.25, F is bounded in this topology. The closure $\overline{F}$ of F is bounded by Proposition 2.11.c, equicontinuous by Lemma 3.26 and complete by Proposition 3.27. Due to Proposition 3.17 it suffices to show that $\overline{F}$ is totally bounded. Let U be a neighborhood of 0 in $\mathrm{C}(S,\mathbb{K})$. Then, there exists $K \subseteq S$ compact and $\epsilon > 0$ such that $U_{K,3\epsilon} \subseteq U$, where

$$U_{K,\delta} := \{f \in \mathrm{C}(S,\mathbb{K}) : |f(x)| < \delta \ \forall x \in K\}.$$

By equicontinuity we can choose for each $a \in K$ a neighborhood W of a such that $|f(x) - f(a)| < \epsilon$ for all $x \in W$ and all $f \in \overline{F}$. By compactness of K there is a finite set of points $\{a_1, \dots, a_n\}$ such that the associated neighborhoods $\{W_1, \dots, W_n\}$ cover S. Now consider the continuous linear map $p : \mathrm{C}(S,\mathbb{K}) \to \mathbb{K}^n$ given by $p(f) := (f(a_1), \dots, f(a_n))$. Since $\overline{F}$ is bounded, $p(\overline{F})$ is bounded in $\mathbb{K}^n$ (due to Proposition 2.16.b) and hence totally bounded (Proposition 3.22). Thus, there exists a finite subset $\{f_1, \dots, f_m\} \subseteq \overline{F}$ such that $p(\overline{F})$ is covered by balls of radius ϵ centered at the points $p(f_1), \dots, p(f_m)$. In particular, for any $f \in \overline{F}$ there is then $k \in \{1, \dots, m\}$ such that $|f(a_i) - f_k(a_i)| < \epsilon$ for all $i \in \{1, \dots, n\}$. Specifying also $x \in K$ there is $i \in \{1, \dots, n\}$ such that $x \in W_i$. We obtain the estimate

$$|f(x) - f_k(x)| \le |f(x) - f(a_i)| + |f(a_i) - f_k(a_i)| + |f_k(a_i) - f_k(x)| < 3\epsilon.$$

Since $x \in K$ was arbitrary this implies $f \in f_k + U_{K,3\epsilon} \subseteq f_k + U$. We conclude that $\overline{F}$ is covered by the set $\{f_1, \dots, f_m\} + U$. Since U was an arbitrary neighborhood of 0 this means that $\overline{F}$ is totally bounded. □

Proposition 3.29. *Let S be a locally compact space. Let $F \subseteq \mathrm{C}(S,\mathbb{K})$ be totally bounded in the topology of compact convergence. Then, F is equicontinuous.*

Proof. **Exercise.** □

3.5 The Hahn-Banach Theorem

Theorem 3.30 (Hahn-Banach)**.** *Let V be a vector space over $\mathbb{K}$, p be a seminorm on V, $A \subseteq V$ a vector subspace. Let $f : A \to \mathbb{K}$ be a linear map such that $|f(x)| \le p(x)$ for all $x \in A$. Then, there exists a linear map $\tilde{f} : V \to \mathbb{K}$, extending f (i.e., $\tilde{f}(x) = f(x)$ for all $x \in A$) and such that $|\tilde{f}(x)| \le p(x)$ for all $x \in V$.*

Proof. We first consider the case $\mathbb{K} = \mathbb{R}$. Suppose that A is a proper subspace of V. Let $v \in V \setminus A$ and define B to be the subspace of V spanned by A and v. In a first step we show that there exists a linear map $\tilde{f} : B \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in A$ and $|f(y)| \le p(y)$ for all $y \in B$. Since any vector $y \in B$ can be uniquely written as $y = x + \lambda v$ for some $x \in A$ and some $\lambda \in \mathbb{R}$, we have $\tilde{f}(y) = f(x) + \lambda \tilde{f}(v)$, i.e, $\tilde{f}$ is completely determined by its value on v. For all $x, x' \in A$ we have

$$f(x) + f(x') = f(x + x') \le p(x + x') \le p(x - v) + p(x' + v)$$

and thus,

$$f(x) - p(x - v) \le p(x' + v) - f(x').$$

In particular, defining a to be the supremum for $x \in A$ on the left and b to be the infimum for $y \in A$ on the right we get

$$a = \sup_{x \in A} \{f(x) - p(x - v)\} \le \inf_{x' \in A} \{p(x' + v) - f(x')\} = b.$$

Now choose $c \in [a, b]$ arbitrary. We claim that by setting $\tilde{f}(v) := c$, $\tilde{f}$ is bounded by p as required. For $x \in A$ and $\lambda > 0$ we get

$$\tilde{f}(x + \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) + c \right) \le \lambda\, p\left(\lambda^{-1} x + v\right) = p\left(x + \lambda v\right)$$
$$\tilde{f}(x - \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) - c \right) \le \lambda\, p\left(\lambda^{-1} x - v\right) = p\left(x - \lambda v\right).$$

Thus, we get $\tilde{f}(x) \le p(x)$ for all $x \in B$. Replacing x by $-x$ and using that $p(-x) = p(x)$ we obtain also $-\tilde{f}(x) \le p(x)$ and thus $|\tilde{f}(x)| \le p(x)$ as required.

We proceed to the second step of the proof, showing that the desired linear form $\tilde{f}$ exists on V. We will make use of Zorn's Lemma. Consider the set of pairs $(W, \tilde{f})$ of vector subspaces $A \subseteq W \subseteq V$ with linear forms $\tilde{f} : W \to \mathbb{R}$ that extend f and are bounded by p. These pairs are partially ordered by extension, i.e., $(W, \tilde{f}) \le (W', \tilde{f}')$ iff $W \subseteq W'$ and $\tilde{f}'|_W = \tilde{f}$. Moreover, for any totally ordered subset of pairs $\{(W_i, \tilde{f}_i)\}_{i \in I}$ there is an upper bound given by $(W_I, \tilde{f}_I)$ where $W_I := \bigcup_{i \in I} W_i$ and $\tilde{f}_I(x) := \tilde{f}_i(x)$ for $x \in W_i$. Thus, by Zorn's Lemma there exists a maximal pair $(W, \tilde{f})$. Since the first part of the proof has shown that for any proper vector subspace of V we can construct an extension, i.e., a pair that is strictly greater with respect to the ordering, we must have $W = V$. This concludes the proof in the case $\mathbb{K} = \mathbb{R}$.

We turn to the case $\mathbb{K} = \mathbb{C}$. Let $f_r(x) := \Re f(x)$ for all $x \in A$ be the real part of the linear form $f : A \to \mathbb{C}$. Since the complex vector spaces A and V are also real vector spaces and p reduces to a real seminorm, we can apply the real version of the proof to f_r to get a real linear map $\tilde{f}_r : V \to \mathbb{R}$ extending f_r and being bounded by p. We claim that $\tilde{f} : V \to \mathbb{C}$ given by

$$\tilde{f}(x) := \tilde{f}_r(x) - \mathrm{i} \tilde{f}_r(\mathrm{i}x) \quad \forall x \in V$$

is then a solution to the complex problem. We first verify that $\tilde{f}$ is complex linear. Let $x \in V$ and $\lambda \in \mathbb{C}$. Then, $\lambda = a + \mathrm{i}b$ with $a, b \in \mathbb{R}$ and

$$\begin{aligned}\tilde{f}(\lambda x) &= a\tilde{f}(x) + b\tilde{f}(\mathrm{i}x)\\ &= a\tilde{f}_r(x) - a\mathrm{i}\tilde{f}_r(\mathrm{i}x) + b\tilde{f}_r(\mathrm{i}x) + b\mathrm{i}\tilde{f}_r(x)\\ &= (a + \mathrm{i}b)\left(\tilde{f}_r(x) - \mathrm{i}\tilde{f}_r(\mathrm{i}x)\right)\\ &= \lambda\tilde{f}(x).\end{aligned}$$

We proceed to verify that $\tilde{f}(x) = f(x)$ for all $x \in A$. For all $x \in A$,

$$\tilde{f}(x) = \Re f(x) - \mathrm{i}\Re f(\mathrm{i}x) = \Re f(x) - \mathrm{i}\Re(\mathrm{i}f(x)) = \Re f(x) + \mathrm{i}\Im(f(x)) = f(x).$$

It remains to show that $\tilde{f}$ is bounded by p. Let $x \in V$. Choose $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\tilde{f}(x) \in \mathbb{R}$. Then,

$$\left|\tilde{f}(x)\right| = \left|\lambda\tilde{f}(x)\right| = \left|\tilde{f}(\lambda x)\right| = \left|\tilde{f}_r(\lambda x)\right| \leq p(\lambda x) = p(x).$$

This completes the proof. □

Corollary 3.31. *Let V be a seminormed vector space, $c > 0$, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a linear form satisfying $|f(x)| \leq c\|x\|$ for all $x \in A$. Then, there exists a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A and satisfies $|\tilde{f}(x)| \leq c\|x\|$ for all $x \in V$.*

Proof. Immediate. □

Theorem 3.32. *Let V be a locally convex tvs, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a continuous linear form. Then, there exists a continuous linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A.*

Proof. Since f is continuous on A, the set $U := \{x \in A : |f(x)| \leq 1\}$ is a neighborhood of 0 in A. Since A carries the subset topology, there exists a neighborhood $\tilde{U}$ of 0 in V such that $\tilde{U} \cap A \subseteq U$. By local convexity, there exists a convex and balanced subneighborhood $W \subseteq \tilde{U}$ of 0 in V. The associated Minkowski functional $\|\cdot\|_W$ is a seminorm on V according to Proposition 2.35 and we have $|f(x)| \leq \|x\|_W$ for all $x \in A$. Thus, we may apply the Hahn-Banach Theorem 3.30 to obtain a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on the subspace A and is bounded by $\|\cdot\|_W$. Since $\|\cdot\|_W$ is continuous this implies that $\tilde{f}$ is continuous. □

Corollary 3.33. *Let V be a locally convex Hausdorff tvs. Then, $\mathrm{CL}(V, \mathbb{K})$ separates points in V. That is, for any pair $x, y \in V$ such that $x \neq y$, there exists $f \in \mathrm{CL}(V, \mathbb{K})$ such that $f(x) \neq f(y)$.*

Proof. **Exercise.** □

Proposition 3.34. *Let X be a locally convex Hausdorff tvs. Then, any finite dimensional subspace of X admits a closed complement.*

Proof. We proceed by induction in dimension. Let $A \subseteq X$ be a subspace of dimension 1 and $v \in A \setminus \{0\}$. Define the linear map $\lambda : A \to \mathbb{K}$ by $\lambda(v) = 1$. Then, the Hahn-Banach Theorem in the form of Theorem 3.32 ensures that λ extends to a continuous map $\tilde{\lambda} : X \to \mathbb{K}$. Then, clearly $\ker \tilde{\lambda}$ is a closed complement of A in X. Now suppose we have shown that for any subspace of dimension n a closed complement exists in X. Let N be a subspace of X of dimension $n+1$. Choose an n-dimensional subspace $M \subset N$. This has a closed complement C by assumption. Moreover, C is a locally convex Hausdorff tvs in its own right. Let $A = N \cap C$. Then, A is a one-dimensional subspace of C and we can apply the initial part of the proof to conclude that it has a closed complement D in C. But D is closed also in X since C is closed in X and it is a complement of N. □

3.6 More examples of function spaces

Definition 3.35. Let T be a locally compact space. A continuous function $f : T \to \mathbb{K}$ is said to *vanish at infinity* iff for any $\epsilon > 0$ the subset $\{x \in T : |f(x)| \geq \epsilon\}$ is compact in T. The set of such functions is denoted by $\mathrm{C}_0(T, \mathbb{K})$.

Exercise 21. Let T be a locally compact space. Show that $\mathrm{C}_0(T, \mathbb{K})$ is complete in the topology of uniform convergence, but not in general complete in the topology of compact convergence.

Definition 3.36. Let U be a non-empty open subset of $\mathbb{R}^n$. For a multi-index $l \in \mathbb{N}_0^n$ we denote the corresponding partial derivative of a function $f : \mathbb{R}^n \to \mathbb{K}$ by

$$D^l f := \frac{\partial^{l_1} \dots \partial^{l_n}}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} f.$$

Let $k \in \mathbb{N}_0$. If all partial derivatives with $|l| := l_1 + \cdots + l_n \leq k$ for a function f exist and are continuous, we say that f is k times continuously differentiable. We denote the vector space of k times *continuously differentiable* functions on U with values in $\mathbb{K}$ by $\mathrm{C}^k(U, \mathbb{K})$. We say a function $f : U \to \mathbb{K}$ is *infinitely differentiable* or *smooth* if it is k times continuously differentiable for any $k \in \mathbb{N}_0$. The corresponding vector space is denoted by $\mathrm{C}^\infty(U, \mathbb{K})$.

Definition 3.37. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0$. We denote by $\mathrm{C}^k(\overline{U}, \mathbb{K})$ the set of continuous functions $f : \overline{U} \to \mathbb{K}$ that are k times continuously differentiable on U, and such that any partial derivative $D^l f$ with $|l| \leq k$ extends continuously to $\overline{U}$. Similarly, we denote by $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ the set of continuous functions $f : \overline{U} \to \mathbb{K}$, smooth in U and such that any partial derivative extends continuously to $\overline{U}$.

Example 3.38. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Let $l \in \mathbb{N}_0^n$ and define the seminorm $p_l : \mathrm{C}^k(\overline{U}, \mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_l(f) := \sup_{x \in \overline{U}} \left| \left(D^l f \right)(x) \right|$$

for $k \in \mathbb{N}_0$ with $k \geq |l|$ or for $k = \infty$. For any $k \in \mathbb{N}_0$ the set of seminorms $\{p_l : l \in \mathbb{N}_0^n, |l| \leq k\}$ makes $\mathrm{C}^k(\overline{U}, \mathbb{K})$ into a normable vector space. Similarly, the set of seminorms $\{p_l : l \in \mathbb{N}_0^n\}$ makes $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ into a locally convex mvs.

Exercise 22. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Show that $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ with the topology defined above is complete, but not normable.

Proposition 3.39. *Let T be a σ-compact space. Then, $\mathrm{C}(T, \mathbb{K})$ with the topology of compact convergence is metrizable.*

Proof. **Exercise.** □

Example 3.40. Let U be a non-empty open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. Let W be an open and bounded subset of $\mathbb{R}^n$ such that $\overline{W} \subseteq U$ and let $l \in \mathbb{N}_0^n$ such that $|l| \leq k$. Define the seminorm $p_{\overline{W},l} : \mathrm{C}^k(U, \mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_{\overline{W},l}(f) := \sup_{x \in \overline{W}} \left| \left(D^l f \right)(x) \right|.$$

The set of these seminorms makes $\mathrm{C}^k(U, \mathbb{K})$ into a locally convex tvs.

Exercise 23. Let $U \subseteq \mathbb{R}^n$ be non-empty and open and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that $\mathrm{C}^k(U, \mathbb{K})$ is complete and metrizable, but not normable.

Exercise 24. Let $0 \leq k < m \leq \infty$. (a) Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded. Show that the inclusion map $\mathrm{C}^m(\overline{U}, \mathbb{K}) \to \mathrm{C}^k(\overline{U}, \mathbb{K})$ is injective and continuous, but does not in general have closed image. (b) Let $U \subseteq \mathbb{R}^n$ be non-empty and open. Show that the inclusion map $\mathrm{C}^m(U, \mathbb{K}) \to \mathrm{C}^k(U, \mathbb{K})$ is injective and continuous, but is in general neither bounded nor has closed image.

Exercise 25. Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded, let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that the inclusion map $\mathrm{C}^k(\overline{U}, \mathbb{K}) \to \mathrm{C}^k(U, \mathbb{K})$ is injective and continuous. Show also that its image is in general not closed.

Exercise 26. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{K})$ consider the operator $D(f) := f'$. (a) Show that $D : \mathrm{C}^{k+1}([0,1], \mathbb{K}) \to \mathrm{C}^k([0,1], \mathbb{K})$ is continuous. (b) Show that $D : \mathrm{C}^{k+1}(\mathbb{R}, \mathbb{K}) \to \mathrm{C}^k(\mathbb{R}, \mathbb{K})$ is continuous.

Exercise 27. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}(\mathbb{R}, \mathbb{K})$ consider the operator

$$(I(f))(y) := \int_0^y f(x)\,\mathrm{d}x.$$

(a) Show that $I : \mathrm{C}^k([0,1], \mathbb{K}) \to \mathrm{C}^{k+1}([0,1], \mathbb{K})$ is continuous. (b) Show that $I : \mathrm{C}^k(\mathbb{R}, \mathbb{K}) \to \mathrm{C}^{k+1}(\mathbb{R}, \mathbb{K})$ is continuous.

Definition 3.41. Let D be a non-empty, open and connected subset of $\mathbb{C}$. We denote by $\mathcal{O}(D)$ the vector space of holomorphic functions on D. If D is also bounded we denote by $\mathcal{O}(\overline{D})$ the vector space of complex continuous functions on $\overline{D}$ that are holomorphic in D.

Exercise 28. (a) Show that $\mathcal{O}(\overline{D})$ is complete with the topology of uniform convergence. (b) Show that $\mathcal{O}(D)$ is complete with the topology of compact convergence.

Theorem 3.42 (Montel)**.** *Let $D \subseteq \mathbb{C}$ be non-empty, open and connected and $F \subseteq \mathcal{O}(D)$. Then, the following are equivalent:*

1. *F is relatively compact.*
2. *F is totally bounded.*
3. *F is bounded.*

Proof. 1.⇒2. $\overline{F}$ is compact and hence totally bounded by Proposition 1.82. Since F is a subset of $\overline{F}$ it must also be totally bounded. 2.⇒3. This follows from Proposition 2.14. 3.⇒1. Since D is locally compact, it is easy to see that boundedness is equivalent to the following property: For each point $z \in D$ there exists a neighborhood $U \subseteq D$ and a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in U$ and all $f \in F$. It can then be shown that F is equicontinuous [Notes on Complex Analysis, Theorem 5.28]. The Arzela-Ascoli Theorem 3.28 then ensures that F is relatively compact. □

Definition 3.43. Let X be a measurable space, μ a measure on X and $p > 0$. Define

$$\mathcal{L}^p(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f|^p \text{ integrable}\}.$$

Also define

$$\mathcal{L}^\infty(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f| \text{ bounded almost everywhere}\}.$$

We recall the following facts from real analysis.

Example 3.44. The set $\mathcal{L}^p(X, \mu, \mathbb{K})$ for $p \in (0, \infty]$ is a vector space.

1. $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_\infty := \inf\{\|g\|_{\sup} : g = f \text{ a.e. and } g : X \to \mathbb{K} \text{ bounded measurable}\}$$

defines a seminorm on $\mathcal{L}^\infty(X, \mu, \mathbb{K})$, making it into a complete seminormed space.

2. If $1 \leq p < \infty$, then $\|\cdot\|_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_p := \left(\int_X |f|^p \right)^{1/p}$$

defines a seminorm on $\mathcal{L}^p(X, \mu, \mathbb{K})$, making it into a complete seminormed space.

3. If $p \leq 1$, then $s_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$s_p(f) := \int_X |f|^p$$

defines a pseudo-seminorm on $\mathcal{L}^p(X, \mu, \mathbb{K})$, making it into a complete pseudometrizable space.

Example 3.45. For any $p \in (0, \infty]$, the closure $N := \overline{\{0\}}$ of zero in $\mathcal{L}^p(X, \mu, \mathbb{K})$ is the set of measurable functions that vanish almost everywhere. The quotient space $\mathrm{L}^p(X, \mu, \mathbb{K}) := \mathcal{L}^p(X, \mu, \mathbb{K})/N$ is a complete mvs. It carries a norm (i.e., is a Banach space) for $p \geq 1$ and a pseudo-norm otherwise. In the case $p = 2$ the norm comes from an inner product making the space into a Hilbert space.

3.7 The Banach-Steinhaus Theorem

Definition 3.46. Let S be a topological space. A subset $C \subseteq S$ is called *nowhere dense* iff its closure $\overline{C}$ does not contain any non-empty open set. A subset $C \subseteq S$ is called *meager* iff it is the countable union of nowhere dense subsets.

Proposition 3.47. *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X, Y)$. Then A is equicontinuous iff for any neighborhood W of 0 in Y there exists a neighborhood V of 0 in X such that*

$$f(V) \subseteq W \quad \forall f \in A.$$

Proof. Immediate. □

Theorem 3.48 (Banach-Steinhaus)**.** *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X, Y)$. For $x \in X$ define $A(x) := \{f(x) : f \in A\} \subseteq Y$. Define $B \subseteq X$ as*

$$B := \{x \in X : A(x) \text{ is bounded}\}.$$

If B is not meager in X, then $B = X$ and A is equicontinuous.

Proof. We suppose that B is not meager. Let U be an arbitrary neighborhood of 0 in Y. Choose a closed and balanced subneighborhood W of 0. Set

$$E := \bigcap_{f \in A} f^{-1}(W)$$

and note that E is closed and balanced, being an intersection of closed and balanced sets. If $x \in B$, then $A(x)$ is bounded, there exists $n \in \mathbb{N}$ such that $A(x) \subseteq nW$ and hence $x \in nE$. Therefore,

$$B \subseteq \bigcup_{n=1}^{\infty} nE.$$

If all sets nE were meager, their countable union would be meager and also the subset B. Since by assumption B is not meager, there must be at least one $n \in \mathbb{N}$ such that nE is not meager. But since the topology of X is scale invariant, this implies that E itself is not meager. Thus, the interior $\mathring{E} = \mathring{\overline{E}}$ is not empty. Also, $\mathring{E}$ is balanced since E is balanced and thus must contain 0. In particular, $\mathring{E}$, being open, is therefore a neighborhood of 0 and so is E itself. Thus,

$$f(E) \subseteq W \subseteq U \quad \forall f \in A.$$

This means that A is equicontinuous at 0 and hence equicontinuous by linearity (Proposition 3.47). Let now $x \in X$ arbitrary. Since x is bounded, there exists $\lambda > 0$ such that $x \in \lambda E$. But then, $f(x) \in f(\lambda E) \subseteq \lambda U$ for all $f \in A$. That is, $A(x) \subseteq \lambda U$, i.e., $A(x)$ is bounded and $x \in B$. Since x was arbitrary, $B = X$. □

Proposition 3.49. *Let S be a complete metric space and $C \subseteq S$ a meager subset. Then, C does not contain any non-empty open set. In particular, $C \neq S$.*

Proof. Since C is meager, there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of nowhere dense subsets of S such that $C = \bigcup_{n\in\mathbb{N}} C_n$. Define $U_n := S \setminus \overline{C_n}$ for all $n \in \mathbb{N}$. Then, each U_n is open and dense in S. Thus, by Baire's Theorem 1.86 the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in S. Thus, its complement $\bigcup_{n\in\mathbb{N}} \overline{C_n}$ cannot contain any non-empty open set. The same is true for the subset $C \subseteq \bigcup_{n\in\mathbb{N}} \overline{C_n}$. □

Corollary 3.50. *Let X be a complete Hausdorff mvs, Y be a tvs and $A \subseteq \mathrm{CL}(X,Y)$. Suppose that $A(x) := \{f(x) : f \in A\} \subseteq Y$ is bounded for all $x \in X$. Then, A is equicontinuous.*

Proof. **Exercise.** □

Corollary 3.51. *Let X be a Banach space, Y a normed vector space and $A \subseteq \mathrm{CL}(X,Y)$. Suppose that*

$$\sup_{f\in A} \|f(x)\| < \infty \quad \forall x \in X.$$

Then, there exists $M > 0$ such that

$$\|f(x)\| < M\|x\| \quad \forall x \in X, \forall f \in A.$$

Proof. **Exercise.** □

3.8 The Open Mapping Theorem

Theorem 3.52 (Open Mapping Theorem)**.** *Let X be a complete Hausdorff mvs, Y a Hausdorff tvs, $f \in \mathrm{CL}(X,Y)$ and $f(X)$ not meager in Y. Then, Y is a complete Hausdorff mvs and f is open and surjective.*

Proof. Suppose U is a neighborhood of 0 in X. Let $V \subseteq U$ be a balanced subneighborhood of 0. Since every point of X is bounded we have

$$X = \bigcup_{n\in\mathbb{N}} nV \quad \text{and hence} \quad f(X) = \bigcup_{n\in\mathbb{N}} nf(V).$$

But $f(X)$ is not meager, so $nf(V)$ is not meager for at least one $n \in \mathbb{N}$. But then scale invariance of the topology of Y implies that $f(V)$ itself is not meager. Thus, $\overset{\circ}{\overline{f(V)}}$ is not empty, is open and balanced (since V is balanced) and thus forms a neighborhood of 0 in Y. Consequently, $\overline{f(V)}$ is also a neighborhood of 0 in Y and so is $\overline{f(U)}$.

Consider now a compatible pseudonorm on X. Let U be a neighborhood of 0 in X. There exists then $r > 0$ such that $B_r(0) \subseteq U$. Let $y_1 \in \overline{f(B_{r/2}(0))}$. We proceed to construct sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ by induction. Supposed we are given $y_n \in \overline{f(B_{r/2^n}(0))}$. By the first part of the proof $\overline{f(B_{r/2^{n+1}}(0))}$ is a neighborhood of 0 in Y. Thus,

$$f(B_{r/2^n}(0)) \cap \left(y_n + \overline{f(B_{r/2^{n+1}}(0))}\right) \neq \emptyset.$$

In particular, we can choose $x_n \in B_{r/2^n}(0)$ such that

$$f(x_n) \in y_n + \overline{f(B_{r/2^{n+1}}(0))}.$$

Now set $y_{n+1} := y_n - f(x_n)$. Then, $y_{n+1} \in \overline{f(B_{r/2^{n+1}}(0))}$ as the latter is balanced.

Since in the pseudonorm $\|x_n\| < r/2^n$ for all $n \in \mathbb{N}$, the partial sums $\{\sum_{n=1}^m x_n\}_{m\in\mathbb{N}}$ form a Cauchy sequence. (Use the triangle inequality). Since X is complete, they converge to some $x \in X$ with $\|x\| < r$, i.e., $x \in B_r(0)$. On the other hand

$$f\left(\sum_{n=1}^m x_n\right) = \sum_{n=1}^m f(x_n) = \sum_{n=1}^m (y_n - y_{n+1}) = y_1 - y_{m+1}.$$

Since f is continuous the limit $m \to \infty$ exists and yields

$$f(x) = y_1 - y \quad \text{where} \quad y := \lim_{m\to\infty} y_m.$$

Note that our notation for the limit y implies uniqueness which indeed follows from the fact that Y is Hausdorff.

We proceed to show that $y = 0$. Suppose the contrary. Again using that Y is Hausdorff there exists a closed neighborhood C of 0 in Y that does not contain y. Its preimage

$f^{-1}(C)$ is a neighborhood of 0 in X by continuity and must contain a ball $B_{r/2^n}(0)$ for some $n \in \mathbb{N}$. But then $f(B_{r/2^n}(0)) \subseteq C$ and $\overline{f(B_{r/2^n}(0))} \subseteq C$ since C is closed. But $y_k \in \overline{f(B_{r/2^n}(0))} \subseteq C$ for all $k \geq n$. So no y_k for $k \geq n$ is contained in the open neighborhood $Y \setminus C$ of y, contradicting convergence of the sequence to y. We have thus established $f(x) = y_1$. But since $x \in B_r(0)$ and $y_1 \in \overline{f(B_{r/2}(0))}$ was arbitrary we may conclude that $\overline{f(B_{r/2}(0))} \subseteq f(B_r(0)) \subseteq f(U)$. By the first part of the proof $\overline{f(B_{r/2}(0))}$ is a neighborhood of 0 in Y. So we may conclude that $f(U)$ is also a neighborhood of 0 in Y. This establishes that f is open at 0 and hence open everywhere by linearity.

Since f is open the image $f(X)$ must be open in Y. On the other hand $f(X)$ is a vector subspace of Y. But the only open vector subspace of a tvs is the space itself. Hence, $f(X) = Y$, i.e., f is surjective.

Let now $C := \ker f$. Since f is surjective, Y is naturally isomorphic to the quotient space X/C as a vector space. Since f is continuous and open Y is also homeomorphic to X/C by Proposition 2.19.3 and hence isomorphic as a tvs. But then Propositions 2.29 and 3.18 imply that Y is metrizable and complete. □

Corollary 3.53. *Let X, Y be complete Hausdorff mvs and $f \in \mathrm{CL}(X,Y)$ surjective. Then, f is open.*

Proof. **Exercise.** □