Braided Quantum Field Theories and Their Symmetries

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1. Introduction

In recent years, there has been interesting conceptual progress in noncommutative field theories.

Hopf algebra symmetry on noncommutative spacetime

Examples

1. Moyal plane: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$

- Invariant under the twisted Poincaré transformation.

Chaichian, et al (2004), etc

- Various proposals to implement the twisted Poincaré invariance in quantum field theories.

Balachandran, et al (2006)
Tureanu (2006)
Bu, et al (2006)
Abe (2006)
Fiore, Wess (2007)
Joung, Mourad (2007), etc.

☑ Quantization in $\theta^{0i} \neq 0$ case?

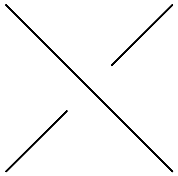
⇨ Desirable to start from a solid framework of quantum field theory.

2. Lie type noncommutativity $[x^i, x^j] = i\kappa\epsilon^{ijk}x_k \ (i,j,k = 1,2,3)$

- Usual momentum conservation is violated. Imai, Sasakura (2000)

  No invariance under $x^i \to x^i + a^i$.

- This noncommutative field theory with a nontrivial braiding was derived as an effective field theory of 3D quantum gravity coupled scalar particles. Freidel, Livine (2005)



- With this braiding, there exists a kind of conserved energy-momentum in the amplitudes, and the energy-momentum generators have Hopf algebra structures.

⇒ This quantum field theory should have a Hopf algebra translational symmetry.

Our aim

Systematically understand these Hopf algebra symmetries in noncommutative quantum field theories in the framework of braided quantum field theories proposed by Oeckl.

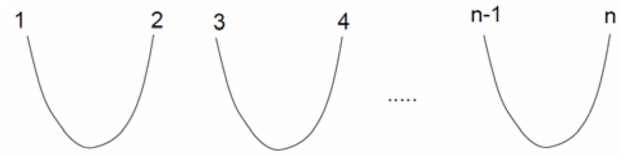
Contents

2. Review of braided quantum field theory

Oeckl (1999)

In usual free quantum field theory,

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_0 = \frac{\int \phi(x_1)\cdots\phi(x_n)e^{-S_0}}{\int e^{-S_0}}$$

$=$ 

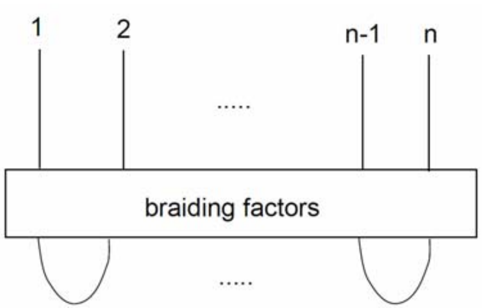
+ permutations.

In braided quantum field theory, path integral measure is defined such that

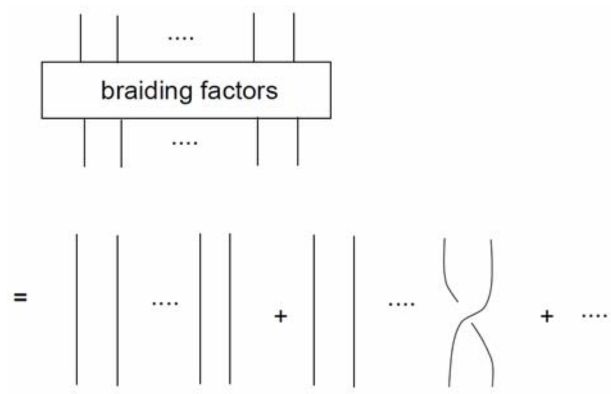
$$\int \frac{\delta}{\delta\phi(x_a)}\left(\phi(x_1)\cdots\phi(x_k)e^{-S_0}\right) = 0.$$

This gives nontrivial braiding Wick theorem,

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_0 =$$



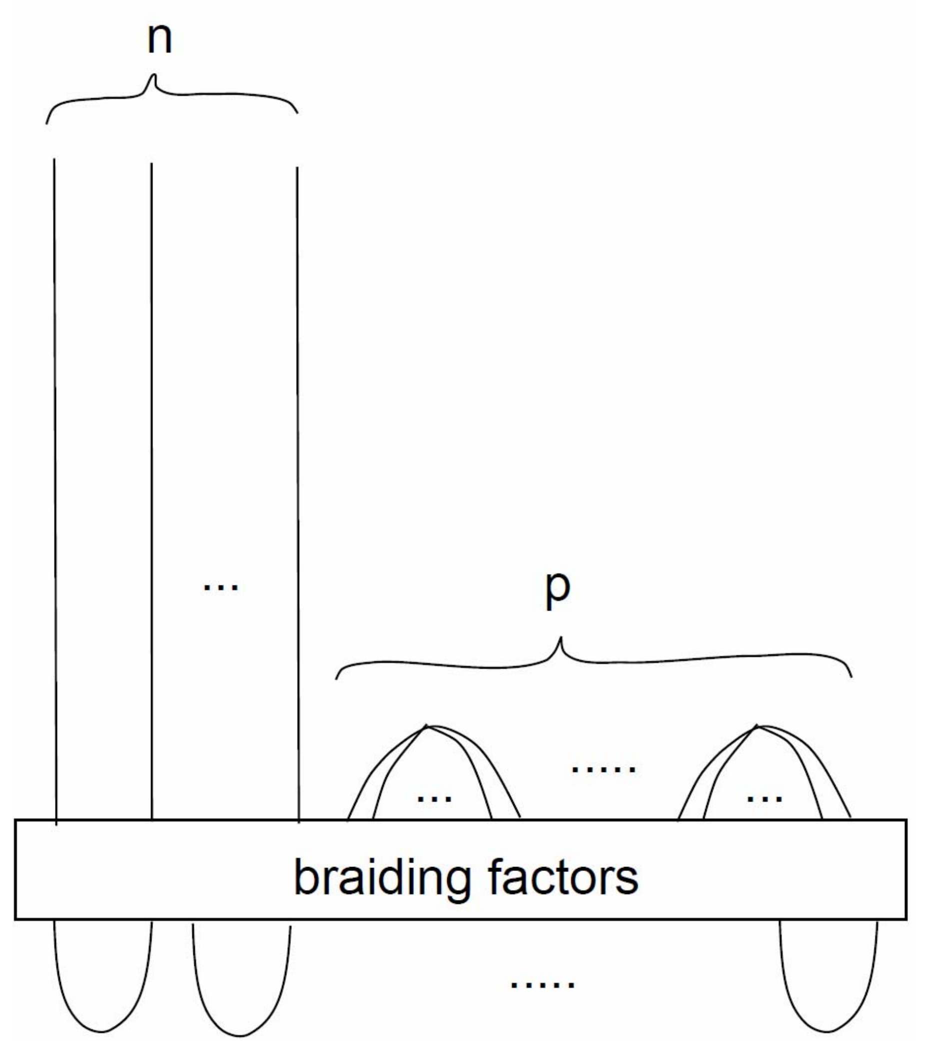
, where



If the action includes interaction terms $S = S_0 + \lambda S_{int},$

the diagram of n-point function at p-th order of perturbation is given by

$$\langle \phi(x_1) \cdots \phi(x_n) (S_{int})^p \rangle_0 =$$



3. Symmetries in braided quantum field theory

$\mathcal{A}ction$ of a general Hopf algebra on vector spaces

$action$

$\alpha_V : \mathcal{A} \otimes V \to V,$ ($\mathcal{A}$: arbtrary Hopf algebra)

We shortly write $a \triangleright V,$ $a \in \mathcal{A}.$

(We take V as a space of a field $\phi(x)$)

Axioms

- $a \triangleright \mathbf{1} = \epsilon(a)\mathbf{1}$ $\mathbf{1} \in V$ ϵ : counit
- $a \triangleright (V \otimes W) = \Delta a \triangleright (V \otimes W)$

$$= \sum_i a^i_{(1)} \triangleright V \otimes a^i_{(2)} \triangleright W$$

Δ : coproduct

Ex) $\Delta(a) = a \otimes \mathbf{1} + \mathbf{1} \otimes a$ for $a \in$ Lie alg.

$\Delta(a) \triangleright (V \otimes W) = (a \triangleright V) \otimes W + V \otimes (a \triangleright W)$ ⇐ Leibnitz rule

- We assume coassociativity of coproduct.

What is the implication of a symmetry in quantum field theory?

In usual quantum field theories, symmetries give nonperturbative relations among correlation functions such that,

$$\sum_{i=1}^{n} \langle \phi(x_1) \cdots \delta_a \phi(x_i) \cdots \phi(x_n) \rangle = 0$$

⟸ Ward-Takahashi (WT) identity

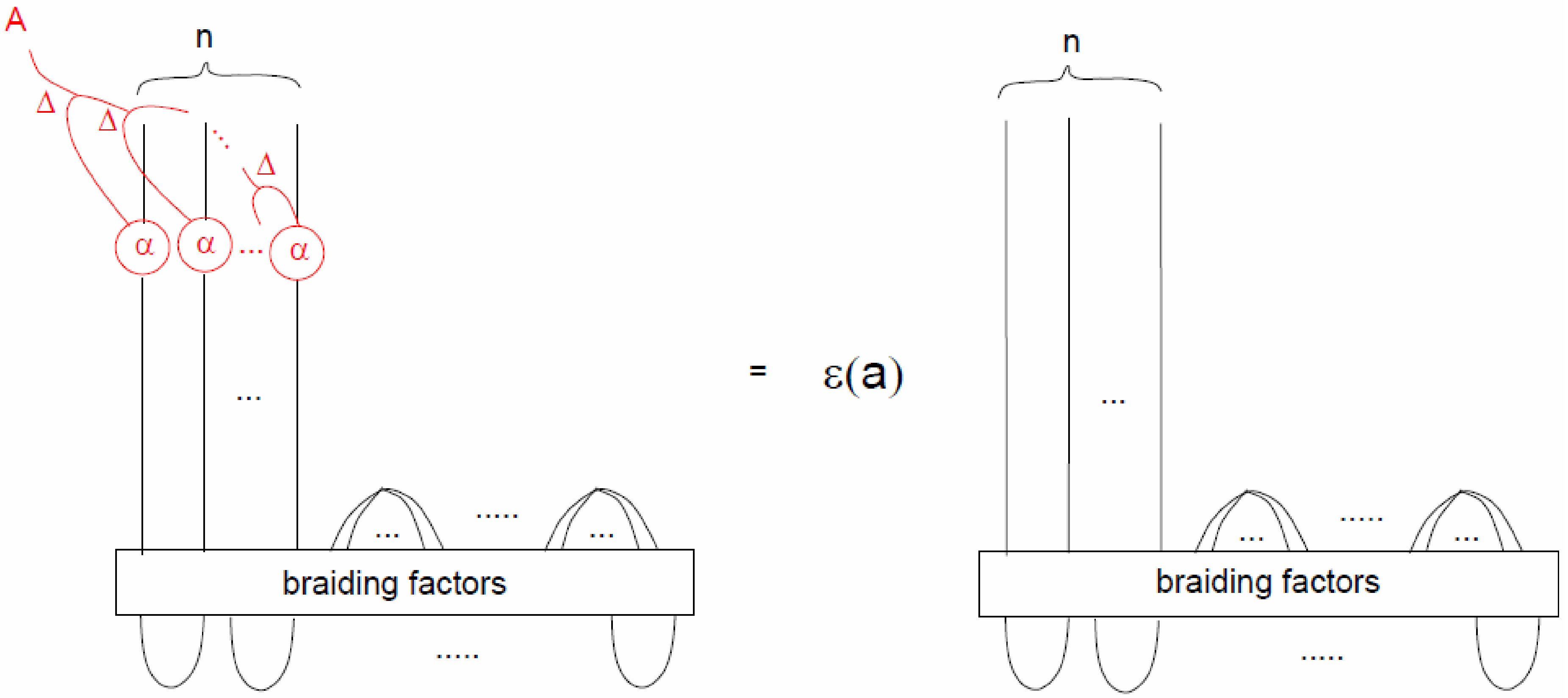
$\delta_a \phi(x)$: a variation of a field under an usual transformation.

If the coproduct of a symmetry algebra is not the usual one,
the WT identity becomes

$$\begin{aligned}
&c_a^{(bi)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \phi(x_n) \rangle \\
&+ c_a^{(bi)(cj)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \delta_c \phi(x_j) \cdots \phi(x_n) \rangle \\
&+ c_a^{(bi)(cj)(dk)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \delta_c \phi(x_j) \cdots \delta_d \phi(x_k) \cdots \phi(x_n) \rangle \\
&+ \cdots = 0,
\end{aligned}$$

$c_a^{\cdots}$: some coefficients.

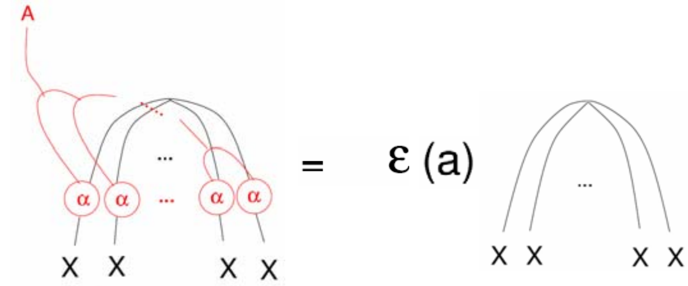
Diagrammatically,



Conditions to satisfy WT identity

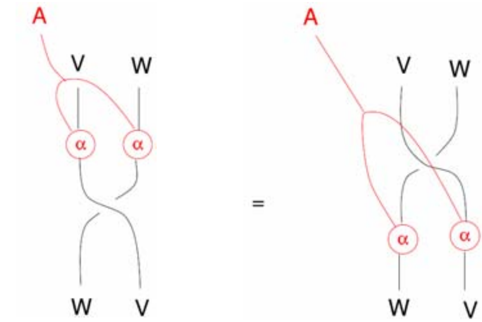
- (Condition1) S_{int} must satisfy

$$a \triangleright S_{int} = \epsilon(a) S_{int}.$$



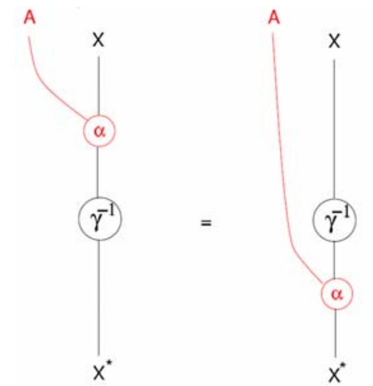
- (Condition2) The braiding ψ must satisfy

$$\psi(a \triangleright (V \otimes W)) = a \triangleright \psi(V \otimes W).$$



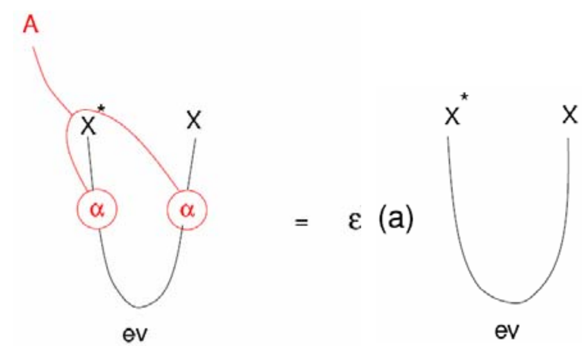
- (Condition3) γ^{-1} and a are commutative

$$a \triangleright (\gamma^{-1}(V)) = \gamma^{-1}(a \triangleright V).$$

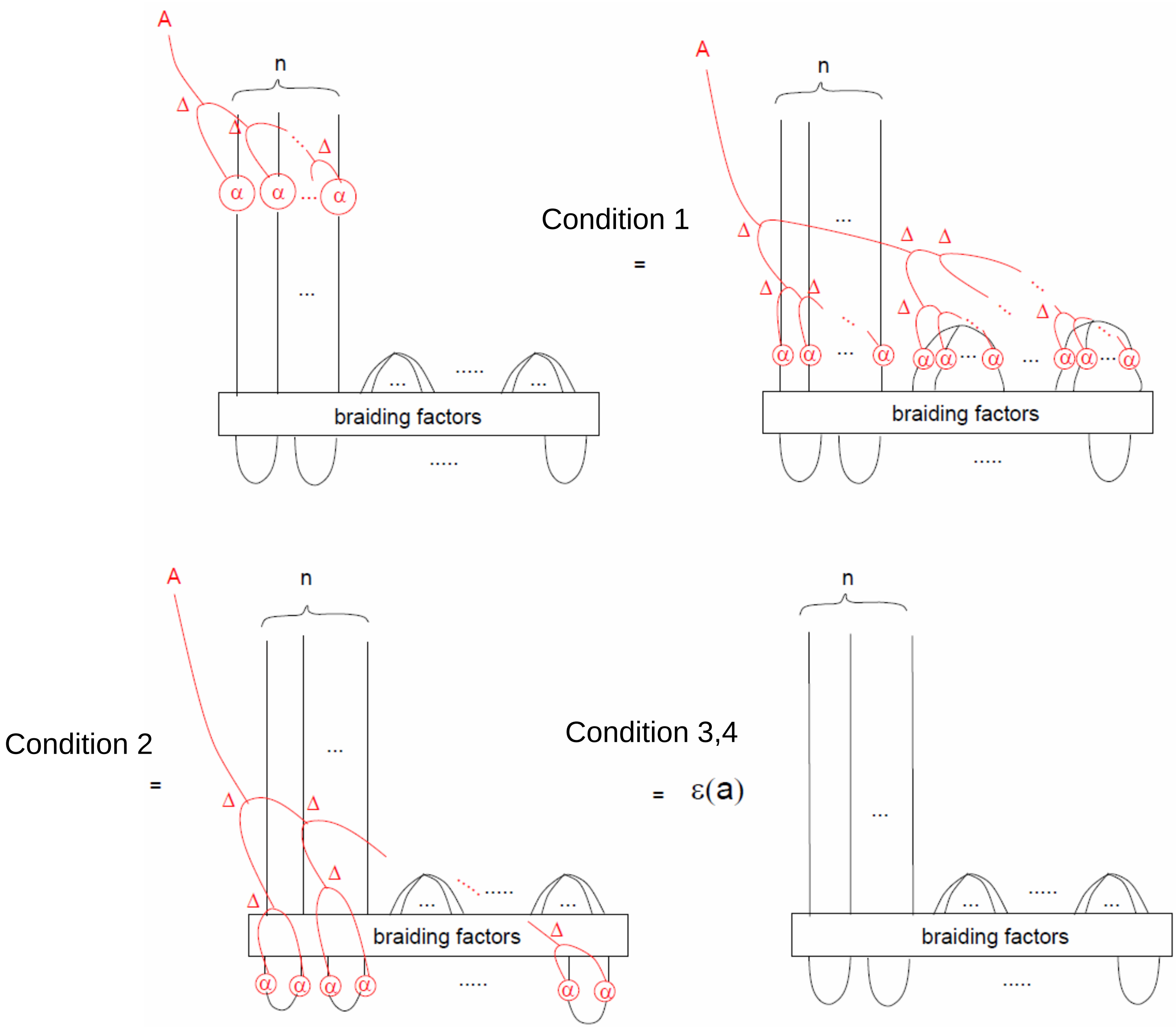


- (Condition4) Under an action a,
  the evaluation map follows

$$\mathrm{ev}(a \triangleright (X^* \otimes X)) = \epsilon(a)\mathrm{ev}(X^* \otimes X).$$



Diagrammatic proof of WT identity



4. Examples

Ex1) Symmetries of the effective noncommutative field theory of 3D quantum gravity coupled with scalar particles

$\phi(x)$: scalar field

Fourier transformation

$$\phi(x) = \int dg \tilde{\phi}(g) e^{iP^i(g)x_i},$$

where $g = P^0 - i\kappa P^i \sigma_i \in \mathrm{SO}(3)$

Star product

$$e^{iP^i(g_1)x_i} \star e^{iP^i(g_2)x_i} = e^{iP^i(g_1 g_2)x_i}, \qquad g_1 g_2 = P^0(g_1 g_2) - i\kappa P^i(g_1 g_2)\sigma_i.$$

Action

$$S = \frac{1}{8\pi\kappa^3}\int d^3x \left[\frac{1}{2}(\partial_i\phi \star \partial_i\phi)(x) - \frac{1}{2}M^2(\phi\star\phi)(x) + \frac{\lambda}{3!}(\phi\star\phi\star\phi)(x)\right]$$

Its momentum representation is given by

$$S = \frac{1}{2}\int dg\big(P^2(g) - M^2\big)\tilde{\phi}(g)\tilde{\phi}(g^{-1}) + \frac{\lambda}{3!}\int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3)\tilde{\phi}(g_1)\tilde{\phi}(g_2)\tilde{\phi}(g_3)$$

Braiding

$$\psi(\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2)$$

Translational operator

$$P^i \triangleright \tilde{\phi}(g) = P^i(g)\tilde{\phi}(g)$$

$$P^0 \triangleright \tilde{\phi}(g) = P^0(g)\tilde{\phi}(g)$$

$$\begin{cases} P^i \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = P^i(g_1g_2)\tilde{\phi}(g_1)\tilde{\phi}(g_2) \\ \qquad = (P_1^0P_2^i + P_2^0P_1^i + \kappa\epsilon^{ijk}P_1^jP_2^k)\tilde{\phi}(g_1)\tilde{\phi}(g_2), \\ P^0 \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = (P_1^0P_2^0 - \kappa^2 P_1^iP_{2i})\tilde{\phi}(g_1)\tilde{\phi}(g_2). \end{cases}$$

This determines the coproduct of P^i and P^0

$$\Delta(P^i) = P^0 \otimes P^i + P^i \otimes P^0 + \kappa\epsilon^{ijk}P^j \otimes P^k,$$

$$\Delta(P^0) = P^0 \otimes P^0 - \kappa^2 P^i \otimes P_i.$$

Freidel, Livine (2005)

From the Hopf algebra axiom, the counit of P^i, P^0 is given by

$$\epsilon(P^i) = \epsilon(P^0) = 0.$$

Check condition 2

LHS: $\psi(P^i \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2))) = P^i(g_1 g_2)(\tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1} g_1 g_2))$

RHS: $P^i \triangleright \psi(\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = P^i(g_2 g_2^{-1} g_1 g_2)(\tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1} g_1 g_2))$

Thus condition 2 is satisfied.

The other conditions are also satisfied.

⇨ The effective braided noncommutaitive field theory of 3D quantum gravity coupled with scalar particles has the (Hopf algebraic) translational symmetry.

Physical meaning of the WT identity

A WT identity is given by

$$P_i \triangleright \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle = P_i(g_1 \cdots g_n) \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle = 0.$$

This gives a selection rule for the correlation function.

$$\langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle \neq 0 \quad \Longrightarrow \quad \begin{aligned} & P_i(g_1 \cdots g_n) \\ =& P_i(g_1) + \cdots + P_i(g_n) + \mathcal{O}(\kappa) \\ =& 0 \end{aligned}$$

(modified) momentum conservation law

$$P_i(g_1 \cdots g_n) \neq 0 \quad \Longrightarrow \quad \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle = 0$$

Ex2) Twisted Poincaré symmetry of noncommutative field theory on Moyal plane

Action

$$S=\int d^Dx\left[\frac{1}{2}(\partial_\mu\phi*\partial^\mu\phi)(x)-\frac{1}{2}m^2(\phi*\phi)(x)+\frac{\lambda}{3!}(\phi*\phi*\phi)(x)\right],$$

Star product

$$\phi(x)*\phi(x)=e^{\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi(x)\phi(y)\Big|_{x=y}.$$

In the momentum representation,

$$S=\int d^Dp\left[\frac{1}{2}(p^2-m^2)\tilde{\phi}(p)\tilde{\phi}(-p)\right.$$
$$\left.+\frac{\lambda}{3!}\int d^Dp_1d^Dp_2d^Dp_3e^{-\frac{i}{2}p_{1\mu}\theta^{\mu\nu}p_{2\nu}}\delta(p_1+p_2+p_3)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\right].$$

Twisted Poincaré symmetry.

$$\Delta(P^\mu)=P^\mu\otimes 1+1\otimes P^\mu,$$
$$\epsilon(P^\mu)=0,$$
$$\Delta(M^{\mu\nu})=M^{\mu\nu}\otimes 1+1\otimes M^{\mu\nu}$$
$$-\frac{1}{2}\theta^{\alpha\beta}[(\delta^\mu_\alpha P^\nu-\delta^\nu_\alpha P^\mu)\otimes P_\beta+P_\alpha\otimes(\delta^\mu_\beta P^\nu-\delta^\nu_\beta P^\mu)],$$
$$\epsilon(M^{\mu\nu})=0.$$

If the braiding is trivial, the condition 2 is not satisfied.

$$\psi(M^{\mu\nu} \triangleright (\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2))) \neq M^{\mu\nu} \triangleright \psi(\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)).$$

In order to keep the invariance, the braiding must be taken as

$$\boxed{\psi(\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)) = e^{i\theta^{\alpha\beta} p_{2\alpha} \otimes p_{1\beta}} (\tilde{\phi}(p_2) \otimes \tilde{\phi}(p_1)).}$$

, which is in agreement with Oeckl (2000), Balachandran, et.al (2006)

5. Summary

- Symmetries in noncommutative field theories have been discussed by considering a generalized WT identity.

- We have obtained the algebraic conditions for a quantum field theory to satisfy the WT identity.

- In the former example, we can understand the braiding between fields from the viewpoint of the translational symmetry of the noncommutative field theory on a Lie-algebraic noncommutative spacetime.

- In the latter example, we reproduced that the twisted Poincaré symmetry on Moyal plane is a symmetry of the quantum field theory only after the inclusion of the nontrivial braiding factor, which is in agreement with the previous proposal.

Oeckl (2000), Balacahndran, et.al (2006)

Possible future applications

- Applications to other noncommutative field theories?

  Noncommutative field theory on κ -Poincaré spacetime?

  … etc.

- Generalization to local transformations?