GENERALIZED PATH DEPENDENT REPRESENTATIONS FOR GAUGE THEORIES

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PLAN OF THE TALK

- **INTRODUCTION: THE CONCEPT OF PATH DEPENDENCE**
- **DEFINITION OF THE PATH OPERATOR**
- **GEOMETRICAL INTERPRETATION: THE GENERATOR OF CURVES**
- **APPLICATION: DERIVATION OF COVARIANT TAYLOR SERIES**
- **CONCLUSIONS**

THE NOTION OF PATH DEPENDENCE

- **Dirac's work on the nonintegrability of the phase of wave functionals in quantum mechanics** P. A. M. Dirac, Proc. Roy. Soc. Lond. A 133, 60 (1931).

- **Mandelstam first introduce an end point derivative in gauge theory** S. Mandelstam, Annals Phys. 19, 1 (1962), Annals Phys. 19, 25 (1962).

- **Integral formulation of Wu and Yang** C. N. Yang, Phys. Rev. Lett. 33, 445 (1974); T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975).

- **The loop representation in loop quantum gravity** R. Gambini and A. Trias, Phys. Rev. D 22, 1380 (1980), Phys. Rev. D 23, 553 (1981); X. Fustero, R. Gambini and A. Trias, Phys. Rev. D 31 (1985) 3144.

MOTIVATION: TOWARD A UNIFIED VIEWPOINT

We concentrate in gauge theories where several and different definitions of path dependent operators have been made. They depend essentially on

- **The space where path dependent functionals take values is either the space of open or closed curves.**

- **The nature of the variation is due to a point or many points, which have been usually called end point derivatives and area derivatives respectively.**

- **The place where the variation is appended, is on the curve or in other place on the manifold.**

DEFINITION OF THE PATH OPERATOR

- **We define the path derivative of the functional $\Psi(\alpha)$ for a given path α by**

$$\mathcal{D}\Psi(\alpha) = \Delta\Psi(\alpha) - \Psi(\alpha) \tag{1}$$

where the action of $\Delta : \Psi(\alpha) \longrightarrow \Psi'(\alpha')$, is to displace infinitesimally and continuously the initial curve α to a deformed curve α' with some transforming action on Ψ.

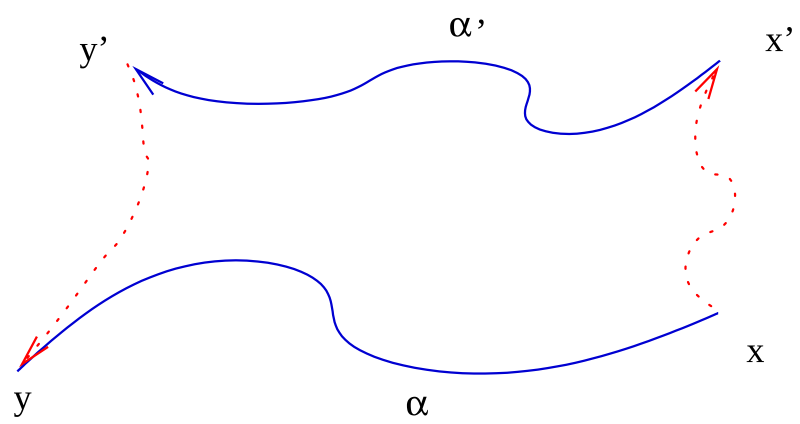

Figure 1: Deformation of the curve $\alpha \longrightarrow \alpha'$.

• We assume a transformation of a matrix functional $\Psi_{AB}(\alpha)$ under the action of the deformation by

$$\Delta\Psi_{AB}(\alpha) = U_A^{A'}(\delta y^{-1})\,\Psi_{A'B'}(\,\alpha')\,U_B^{B'}(\delta x) \tag{2}$$

where the elements $U_B^{B'}(\delta x)$ and $U_A^{A'}(\delta y^{-1})$ are functions of the paths δx and δy^{-1}.

• We take U elements to be parallel propagators

• For the curve deformation that just moves one point along a straight line, we define the point deformations

$$\mathcal{D}_{\delta y}\Psi_{y,x} = \delta\Psi_{y,x} + \delta y^{\mu} A_{\mu}(y)\,\Psi_{y,x}, \tag{3}$$

$$\mathcal{D}_{\delta x}\Psi_{y,x} = \delta\Psi_{y,x} - \Psi_{y,x}\,\delta x^{\mu} A_{\mu}(x) \tag{4}$$

And for the curve deformation with x and y fixed but that encloses some area, the loop deformation

$$\mathcal{D}_{L}\Psi_{y,x} = \delta\Psi_{y,x} \tag{5}$$

ACTION OF THE GROUP OF LOOPS

The construction can be understood in terms of the action of the group of loops L on arbitrary paths γ. Let us consider the same path α as before and focus on the loop $l = \delta x \, o \, \alpha' \, o \, \delta y^{-1} \, o \, \alpha^{-1}$ with composition $l \, o \, \alpha = \delta x \, o \, \alpha' \, o \, \delta y^{-1}$. The variation of a functional $\Delta\Psi(\alpha)$ will be represented by an operator $U(l)$ with $l \in L$ as,

$$\Delta\Psi(\alpha) = \Psi(l \, o \, \alpha) = U(l)\Psi(\alpha), \tag{6}$$

and therefore we have

$$\Psi(\alpha') = U(\delta y) \Big[U(l) \, \Psi(\alpha) \Big] U(\delta x^{-1}) \, . \tag{7}$$

COVARIANT DIFFERENTIATION OF GAUGE OBJECTS

Here we compute the action of the path derivative on phase factors. Let us consider the ordered phase factor of the same path α as before,

$$U_{y,x}(\alpha) = \mathcal{P}_\sigma \left(\exp \int_0^1 -A_\mu(\sigma) \frac{d\alpha^\mu(\sigma)}{d\sigma}\, d\sigma \right), \tag{8}$$

We partition the paths α and α' in N segments.

$$U(\alpha') = \prod_{i=0}^{N} U(\alpha'_{i+1,i}) = U'_{N+1,N} \cdots U'_{2,1}\, U'_{1,0}, \tag{9}$$

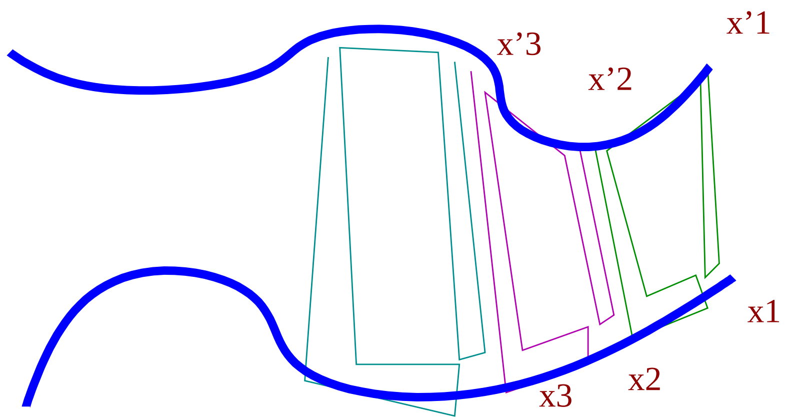

Figure 2: Intermediate paths

We have

$$\mathcal{D}U(\alpha) = \prod_{i=0}^{N} \Big(U(\alpha_{i+1,i})\, H(x_i) \Big) - U(\alpha), \tag{10}$$

Using the non abelian Stokes theorem to lowest order

$$H(x_i) = 1 - \int_0^1 \mathcal{F}_{\mu\nu}(x_i)\, N^{\mu}(\sigma_i, t) \frac{\partial x_i^{\nu}}{\partial \sigma} d\sigma_i dt, \tag{11}$$

where $\mathcal{F}_{\mu\nu}(x_i) = U(\delta x_i) F_{\mu\nu}(x_i) U(\delta x_i^{-1})$ is the parallel transported curvature. Replacing, we have

$$\mathcal{D}(N)\, U(\alpha) = -\int_0^1 \sum_{i=0}^{N} U_{i+1,i}\, \mathcal{F}_{\mu\nu}(x_i)\, N^{\mu}(\sigma_i, t)\, \frac{\partial x_i^{\nu}}{\partial \sigma}\, d\sigma_i\, dt \tag{12}$$

The continuum limit of the above equation gives

$$\mathcal{D}(N)\, U(\alpha) = -\int_0^1 dt \int_0^1 d\sigma\, U_{y,x(\sigma,t)}\, \mathcal{F}_{\mu\nu}(x(\sigma,t))\, U_{x(\sigma,t),x}\, N^{\mu}(\sigma,t)\, \frac{\partial x^{\nu}(\sigma,t)}{\partial \sigma}$$

THE GENERATOR OF CURVES

We introduce a family of deformed curves $\alpha_t(\sigma)$

$$\Psi(\alpha_{n+1}) = \Psi(\alpha_n) + \Psi(\alpha_n)A_{x_n} - A_{y_n}\Psi(\alpha_n) + \mathcal{D}(N_n)\Psi(\alpha_n), \qquad (13)$$

Iterating the above equation

$$\Psi(\alpha') = U(\alpha(1))\left[\mathcal{P}_t\left(\exp\int_0^1 dt\,\mathcal{D}_t\right)\Psi(\alpha)\right]U(\alpha^{-1}(0)), \qquad (14)$$

therefore we identify $U(l) = \mathcal{P}_t\left(\exp\int_0^1 dt\,\mathcal{D}_t\right)$

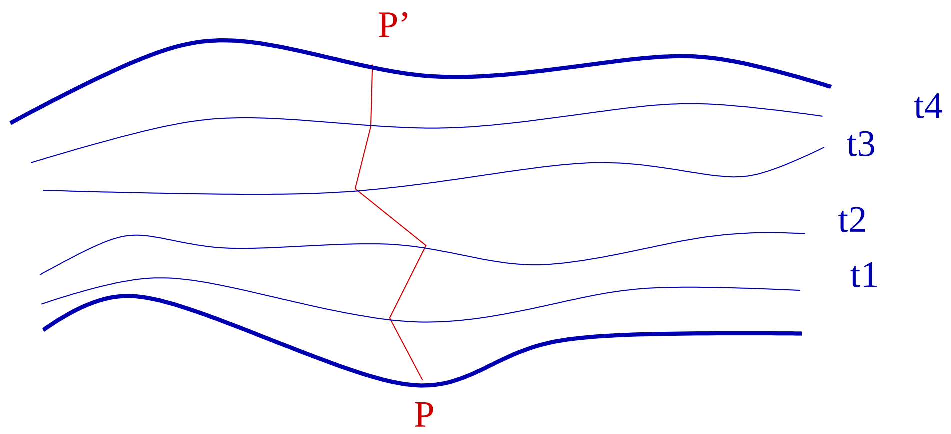

Figure 3: Flow of the point P under the diffeomorphism

COVARIANT TAYLOR SERIES

- **Covariant Taylor expansions were developed as part of a method of calculation to found the effective action in quantum field theories.**

A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. 119, 1 (1985); S. M. Kuzenko and I. N. McArthur, JHEP 0305, 015 (2003).

STANDARD DERIVATION

- Using the transport equation $\frac{D\dot{x}^\nu}{dt} = \dot{x}^\mu D_\mu \dot{x}^\nu(t) = 0$ one can show that for the scalar function $f(x(t))$ one has for all n,

$$\frac{d^n f(x(t))}{dt^n} = [D_{\nu_n} \dots D_{\nu_1} f(x)]_{x=x(t)} \; \dot{x}^{\nu_1} \dots \dot{x}^{\nu_n}. \tag{15}$$

Then, considering the expansion and defining,

$$\sigma^\mu(x_1, x_2) = (t_2 - t_1) \left[\frac{dx^\mu(t)}{dt}\right]_{t=t_1}, \tag{16}$$

we arrive to the expression,

$$f(x_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\nu_1}(x_1, x_2) \dots \sigma^{\nu_n}(x_1, x_2) D_{\nu_n} \dots D_{\nu_1} f(x_1). \tag{17}$$

We consider the field composed with the two parallel propagators as $U(x',x)\,\varphi(x)\,U(x,x')$. Since the composition behaves as a scalar with respect to the point x

$$\begin{aligned} U(x'',x')\,\varphi(x')\,U(x',x'') &= \sum_{n=0}^{\infty} \frac{1}{n!}\,\sigma^{\nu_1}(x,x')\ldots\sigma^{\nu_n}(x,x')\,D^{x}_{\nu_n}\ldots D^{x}_{\nu_1} \\ &\times\; U(x'',x)\,\varphi(x)\,U(x,x''). \end{aligned} \tag{18}$$

Using the identity

$$\sigma^{\nu_1}(x,x')\ldots\sigma^{\nu_n}(x,x')\,D^{x}_{\nu_n}\ldots D^{x}_{\nu_1}\,U(x',x) = 0, \tag{19}$$

We obtain the covariant Taylor series for the field $\varphi(x)$, taking $x'' = x'$ and multiplying by $U^{-1}(x,x')$ and $U^{-1}(x',x)$,

$$U(x,x')\,\varphi(x')\,U(x',x) = \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^{\nu_1}(x,x')\ldots\sigma^{\nu_n}(x,x')\,D_{\nu_n}\ldots D_{\nu_1}\,\varphi(x). \tag{20}$$

GEOMETRICAL DERIVATION

Let us define the path dependent field $\Psi(\gamma) = U(\gamma_2)\,\phi(x)\,U(\gamma_1)$

- The first deformation is $\Psi(\gamma') = U(D)\Psi(\gamma)$
- The second deformation is $\Psi(\gamma'') = U(\gamma_2^{-1})\,[\,U(D')\Psi(\gamma')\,]\,U(\gamma_1^{-1})$

Both deformations can be viewed as one point deformation $D_{\delta x}$

$$\Psi(\gamma'') = U(\gamma_2^{-1})[\,U(D_{\delta x})\Psi(\gamma)\,]\,U(\gamma_1^{-1}). \tag{21}$$

We obtain covariant Taylor expansions with $\delta x = x' - x$

$$U(x, x')\,\phi(x')\,U(x', x) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta x^{\nu_1} \ldots \delta x^{\nu_n} \, D_{\nu_n} \ldots D_{\nu_1} \phi(x), \tag{22}$$

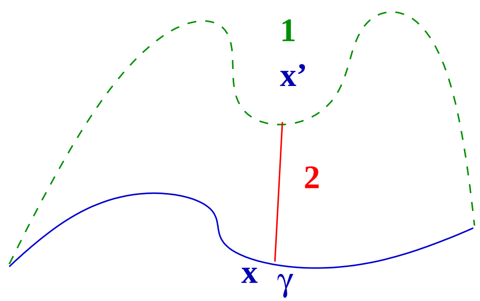

Figure 4: Two deformations of the curve γ.

DISCUSION Y CONCLUSION

- **We have defined a path dependent operator in gauge theory, which is covariant by construction, and acts by continuous deformations on the space of smooth curves $\Gamma(M)$**

- **We have established a relation between the path derivative introduced here and the area and end point derivative.**

- **We have calculated the finite variation of a functional when its argument is changed by successive infinitesimal deformations.**

- **We have derived covariant Taylor expansions for non Abelian fields by considering the deformation of open curves.**