A New Perspective on Covariant Canonical Gravity

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Why not gauge fix?

- Discreteness at the Planck scale is closely related to compactness of gauge group
  - What happens when full gauge group is retained?
- Frees us from timelike evolution:
  - Any manifold locally can be turned into $R \times S^3$
- Spin foam models don't gauge fix
  - Could avoid trouble gluing SU(2) boundary spin networks to SO(3,1) spin foam amplitudes
- Immirzi ambiguity
  - Immirzi term might not be necessary
- Kodama state
  - Indications that the Kodama state is best understood without gauge fixing

Outline

•Will show that canonical analysis can be done without gauge fixing to time gauge

•Exploit a canonical approach that avoids Legendre transform, thereby avoiding primary constraints on momenta

•True dynamical variables are unconstrained Spin(3,1) spin connection and tetrad, both pulled back to 3-space

•Poisson algebra of the constraints closes, and is a deformation of the de Sitter Lie algebra

•In contrast to other approaches, components of spin connection commute under Poisson bracket

Conventions

- Will use a Clifford algebra formalism

$$\omega = \omega^{IJ}\,\tfrac{1}{4}\gamma_{[I}\gamma_{J]} \qquad e = e^{I}\,\tfrac{1}{2}\gamma_{I} \qquad \star = -i\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$$

- Wedge products and explicit traces will be dropped

$$\frac{1}{4k}\int_M \varepsilon_{IJKL}\, e^I \wedge e^J \wedge R^{KL} = \frac{1}{k}\int \star\, e\, e\, R$$

- Clifford elements form basis of de Sitter Lie algebra

$$\left[\tfrac{1}{2}\gamma^{[I}\gamma^{J]}, \tfrac{1}{2}\gamma^{[K}\gamma^{L]}\right] = \tfrac{1}{2}\left(\eta^{JK}\gamma^{[I}\gamma^{L]} - \eta^{IK}\gamma^{[J}\gamma^{L]} - \eta^{JL}\gamma^{[I}\gamma^{K]} + \eta^{IL}\gamma^{[J}\gamma^{K]}\right)$$

$$\left[\tfrac{1}{2}\gamma^{[I}\gamma^{J]}, \tfrac{i}{2r_0}\gamma^{K}\right] = \tfrac{i}{2r_0}\left(\eta^{KJ}\gamma^{I} - \eta^{KI}\gamma^{J}\right)$$

$$\left[\tfrac{i}{2r_0}\gamma^{I}, \tfrac{i}{2r_0}\gamma^{J}\right] = -\tfrac{1}{r_0^2}\,\tfrac{1}{2}\gamma^{[I}\gamma^{J]} \qquad r_0 = \sqrt{\tfrac{3}{\lambda}}$$

Toy Model

- Begin with a modified Holst action

$$S = \frac{1}{k}\int_M \star e \wedge e \wedge R - \frac{1}{2\beta} T \wedge T \; - \frac{\lambda}{6} \star e \wedge e \wedge e \wedge e$$

$$\frac{1}{\beta} e \wedge e \wedge R - \frac{1}{2\beta} d(e \wedge T)$$

- Dynamical variables are connection and frame

Position	Momentum	Primary Constraint
ω	$\Pi_\omega = \frac{1}{k}\Sigma$	$\Sigma = \star e \wedge e$
e	$\Pi_e = -\frac{1}{k\beta} T$	$T = De$

Canonical Constraints

- Symplectic structure defines naïve Poisson bracket:

$$\{A,B\} = k\int_\Sigma \frac{\delta A}{\delta\omega}\wedge\frac{\delta B}{\delta\Sigma} - \beta\frac{\delta A}{\delta e}\wedge\frac{\delta B}{\delta T} \; - \; (A\leftrightarrow B)$$

- Naïve Constraints (prior to primary constraints) are

$$C_D = \tfrac{1}{k}\int_\Sigma \mathcal{L}_{\bar{N}}\omega\wedge\Sigma - \tfrac{1}{\beta}\mathcal{L}_{\bar{N}}e\wedge T \qquad \bar{t} = \bar{\eta} + \bar{N}$$

$$C_G = -\tfrac{1}{k}\int_\Sigma D\alpha\wedge\Sigma + \tfrac{1}{\beta}[\alpha, e]\wedge T \qquad \alpha\in so(3,1)$$

$$C_H = \tfrac{1}{k}\int_\Sigma [\eta, e]\wedge(\star R - \tfrac{\lambda}{3}\Sigma) + \tfrac{1}{\beta}D\eta\wedge T \qquad \eta\equiv e(\bar{\eta})$$

Constraint Algebra

•Need to compute constraint algebra.

$$\{C_D(\bar{N}_1), C_D(\bar{N}_2)\} = C_D([\bar{N}_1, \bar{N}_2])$$
$$\{C_D(\bar{N}), C_G(\lambda)\} = C_G(\mathcal{L}_{\bar{N}}\lambda)$$
$$\{C_D(\bar{N}), C_H(\eta)\} = C_H(\mathcal{L}_{\bar{N}}\eta)$$

$$\{C_G(\lambda_1), C_G(\lambda_2)\} = C_G([\lambda_1, \lambda_2])$$
$$\{C_G(\lambda), C_H(\eta)\} = C_H([\lambda, \eta])$$
$$\{C_H(\eta_1), C_H(\eta_2)\} = -\tfrac{\lambda}{3} C_G([\eta_1, \eta_2])$$

•Naïve constraints close, and algebra is isomorphic to de Sitter Lie algebra with diffeomorphisms!

$$\mathcal{A}_C \simeq Lie(dS_4 \rtimes Diff_3)$$

Lessons Learned

- The true dynamical variables are the spin connection and tetrad pulled back to the 3-space
  - The momentum variables add no new degrees of freedom

- The Hamiltonian constraint is vectorial
  - Its generators are closely related to pseudo-translations

- Hamiltonian degrees of freedom (DOF) counted as follows:

$$\begin{aligned} DOF_{Total} &= (DOF(e) + DOF(\omega))/2 - DOF(C_G + C_D + C_H) \\ &= (3 \times 4 + 3 \times 6)/2 - (6 + 3 + 4) \\ &= 2 \end{aligned}$$

- The true constraint algebra likely to be a deformation of the de Sitter Lie algebra with diffeomorphisms

The non-Canonical Poisson Bracket

- It is possible define non-canonical Poisson bracket without performing Legendre transform

- First define symplectic structure:

$$\boldsymbol{\delta} S = (\boldsymbol{\delta} S)_{boundary} + (\boldsymbol{\delta} S)_{bulk} \longrightarrow \begin{array}{l} \boldsymbol{J} \equiv (\boldsymbol{\delta} S)_{boundary} \\ \boldsymbol{\Omega} = -\boldsymbol{\delta} \boldsymbol{J} \end{array}$$

- Associate a canonical vector field to every functional, f :

$$\boldsymbol{\Omega}(\bar{\boldsymbol{X}}_f, \ ) = \boldsymbol{\delta} f$$ (partially defines $\bar{\boldsymbol{X}}_f$)

- Poisson bracket is defined in a coordinate free way by

$$\{f, g\} \equiv \boldsymbol{\Omega}(\bar{\boldsymbol{X}}_g, \bar{\boldsymbol{X}}_f)$$

- Hamilton's equations are

$$\boldsymbol{\Omega}(\bar{\boldsymbol{t}}, \ ) = \boldsymbol{\delta} H$$

The Symplectic Form

- Return to the Einstein-Cartan action:

$$S = \frac{1}{k}\int_M \star e e R - \frac{\lambda}{6}\star e e e e$$

- The symplectic form for this action is given by

$$(\boldsymbol{\delta} S)_{boundary} = \frac{1}{k}\int_\Sigma \star e e \,\boldsymbol{\delta}\omega \qquad \boldsymbol{\Omega} = \int_\Sigma \star \boldsymbol{\delta}\omega \wedge (\boldsymbol{\delta} e\, e + e\,\boldsymbol{\delta} e)$$

- The components of the canonical vector field are

$$\bar{\boldsymbol{X}}_f = \int_\Sigma \delta_f e \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta} e} + \delta_f \omega \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\omega}$$

- Symplectic form only partially determines components

$$e \star \delta_f \omega + \star \delta_f \omega\, e = \frac{\delta f}{\delta e} \qquad \star\delta_f (e\, e) = -\frac{\delta f}{\delta \omega}$$

The Constraints

- Hamiltonian is a sum of constraints
  - Constraints are equations of motion pulled back to boundary

$$C_D(\bar{N}) = \frac{1}{k}\int_\Sigma \mathcal{L}_{\bar{N}}\omega \star e\, e \qquad \bar{t} = \bar{\eta} + \bar{N}$$

$$C_G(\lambda) = \frac{1}{k}\int_\Sigma -D\lambda \star e\, e \qquad \lambda = -\omega(\bar{\eta})$$

$$C_H(\eta) = \frac{1}{k}\int_\Sigma -\star[\eta, e]\left(R - \tfrac{\lambda}{3} e\, e\right) \qquad \eta = \eta_I \tfrac{1}{2}\gamma^I = e(\bar{\eta})$$

- Hamilton's equations give remaining components of Einstein equations

$$\boldsymbol{\Omega}(\bar{\boldsymbol{t}},\ ) = \boldsymbol{\delta}(C_D + C_G + C_H) \longrightarrow \begin{array}{l} i_{\bar{\eta}} D(\star e\, e) = 0 \\ i_{\bar{\eta}}\left[e, \star R - \frac{\lambda}{3}\star e\, e\right] = 0 \end{array}$$

The True Constraint Algebra

- The true constraint algebra is given by

$$\{C_X, C_Y\} = \mathbf{\Omega}(\bar{\boldsymbol{X}}_{C_Y}, \bar{\boldsymbol{X}}_{C_X})$$

- Most of the constraint algebra can be evaluated straightforwardly with no surprises

$$\{C_D(\bar{N}_1), C_D(\bar{N}_2)\} = C_D([\bar{N}_1, \bar{N}_2])$$

$$\{C_D(\bar{N}), C_G(\lambda)\} = C_G(\mathcal{L}_{\bar{N}}\lambda)$$

$$\{C_D(\bar{N}), C_H(\eta)\} = C_H(\mathcal{L}_{\bar{N}}\eta)$$

$$\{C_G(\lambda_1), C_G(\lambda_2)\} = C_G([\lambda_1, \lambda_2])$$

$$\{C_G(\lambda), C_H(\eta)\} = C_H([\lambda, \eta])$$

$$\{C_H(\eta_1), C_H(\eta_2)\} = \ ???$$

(This requires more work)

Evaluating the Final Commutator

- We will use the Ricci decomposition of the curvature tensor (pulled back to 3-space):

$$R^{IJ} = \tfrac{1}{2}(\varepsilon^I \wedge R^J - \varepsilon^J \wedge R^I) - \tfrac{1}{6}\varepsilon^I \wedge \varepsilon^J R + C^{IJ}$$

$$R = \tfrac{1}{2}\left(e\,\overset{\circ}{R} + \overset{\circ}{R}\,e\right) - \tfrac{1}{6} e\,e\,\overset{\bullet}{R} + C$$

$${}^4e^I(\bar{\varepsilon}_J) = \delta^I_J$$

$$\overset{\circ}{R} \equiv \tfrac{1}{2}\gamma_J R^{IJ}(\bar{\varepsilon}_I,\ )$$

$$\overset{\bullet}{R} \equiv R^{IJ}(\bar{\varepsilon}_I, \bar{\varepsilon}_J)$$

- Using above expression, the commutator can be evaluated

$$\{C_H(\eta_1), C_H(\eta_2)\} = \frac{1}{k}\int_\Sigma \star[\eta_1,\eta_2]\,[T,\overset{\circ}{R}] - \frac{1}{6}\overset{\bullet}{R}\star[\eta_1,\eta_2]\,[T,e]$$
$$+2\star\left(\eta_1 C(\bar{\eta}_2) - \eta_2 C(\bar{\eta}_1)\right)T$$

Properties of the Commutator

Commutator vanishes weakly!

- All terms depend explicitly on torsion
- Torsion vanishes on constraint manifold

$$\{C_H(\eta_1), C_H(\eta_2)\} \approx 0$$

- Constraint algebra is deformation of de Sitter algebra with diffeomorphisms
    - Partially solve equations of motion:

$$\overset{\circ}{R} = \lambda e \qquad \overset{\bullet}{R} = 4\lambda$$

$$\{C_H(\eta_1), C_H(\eta_2)\} \approx -\tfrac{\lambda}{3} C_G([\eta_1, \eta_2]) - C_G(C(\bar{\eta}_1, \bar{\eta}_2))$$

Conclusions

- The constraint algebra can be computed without gauge fixing

- The constraint algebra closes
  - No primary constraints
  - No second-class constraints

- The algebra is a deformation of the de Sitter Lie algebra together with diffeomorphisms
  - The Hamiltonian constraint is closely related to the generator of de Sitter pseudo-translations

- In contrast to other approaches, the components of the spin connection commute

Solving Quantum Constraints

- In connection-tetrad representation define the operators

$$\hat{\omega} = \omega \qquad \hat{\Sigma} = -ik\frac{\delta}{\delta\omega}$$

$$\hat{e} = e \qquad \hat{T} = ik\beta\frac{\delta}{\delta e}$$

$$\Psi = \Psi[\omega, e]$$

- Can solve all of the naïve quantum constraints by a version of the Kodama state

$$\Psi[\omega, e] = \exp\left[\frac{3i}{2k\lambda}\int_\Sigma \star Y[\omega] + \frac{1}{\beta}Y[\omega] - \frac{\lambda}{3\beta}e\wedge De\right]$$

$$\hat{C}_{\{H,G,D\}}\Psi = 0$$