Exploring the diffeomorphism invariant Hilbert space

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(See also gr-qc/0609032)

motivation

Spatial diff invariant states in LQG: In the dual of $\mathcal{H}_{\text{kin}}$. Labeled among other things by diff equivalence classes of graphs.

Important because

- ✘ home of the scalar constraint (→ Thiemann)
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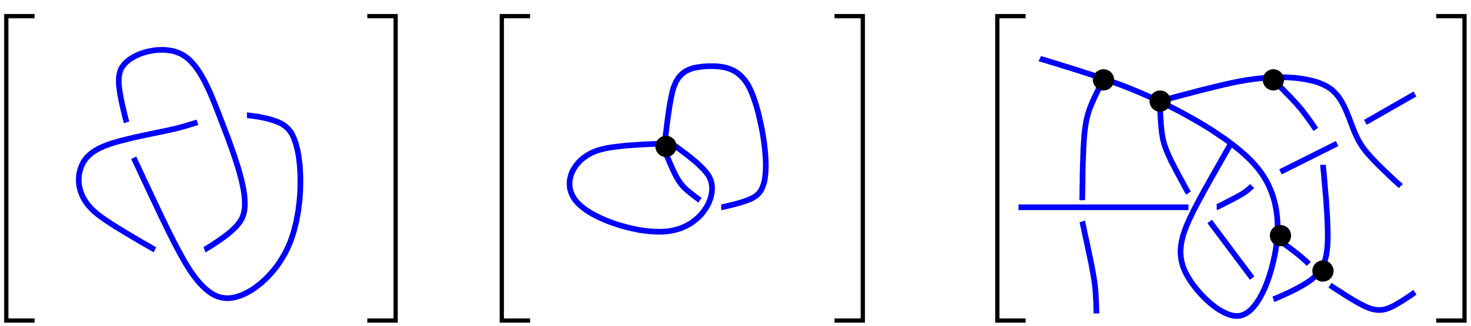

Any of these a homogenous isotropic universe?

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- ✘ Total volume
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Idea: Start with toy model – Quantum scalar field in LQG-like rep.
(⟶ Thiemann, Starodubtsev, Ashtekar + Lewandowski + Sahlmann)

In particular: Space of spatially diffeo invariant states $\mathcal{H}_{\text{diff}}$, and operators thereon.

what we'll do

Starting point: Scalar field/U(1) sigma model quantized a la LQG:

$$T_{x,\lambda} = \exp(i\lambda\phi(x)), \qquad \pi(f) = \int \pi(y) f(y), \qquad \lambda \in \mathcal{I}(\equiv \mathbb{R}, \mathbb{Z} \text{ resp.})$$

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First exercise: Characterize $\mathcal{H}_{\text{diff}}$ explicitly for this model.
Second exercise: Quantize the diffeomorphism invariant quantities

$$L_\alpha = \int \pi(x) \exp[i\alpha\phi(x)]$$

$$\{L_\alpha\, ,\, L_{\alpha'}\} = i(\alpha - \alpha')L_{\alpha+\alpha'}, \qquad \overline{L_\alpha} = L_{-\alpha}$$

(these generate "target space diffeos").

$\mathcal{H}_{\textbf{\textit{kin}}}$

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Representation of basic variables:

$$T_{x,\lambda} |\underline{\lambda}\rangle = |\underline{\lambda} + \lambda \delta_x\rangle, \qquad \pi(f)|\underline{\lambda}\rangle = \sum_{x \in \Sigma} \lambda_x f(x) |\underline{\lambda}\rangle$$

Spatial diffeos unitarily implemented.

Definition of $\mathcal{H}_{\text{diff}}$: Space of linear forms on Cyl with scalar product. Obtained via group averaging map Γ. Morally:

$$(\Gamma\Psi)(\Phi) = (\text{Vol}(\text{Diff}))^{-1} \int_{\text{Diff}} D\varphi \, \langle \varphi * \Psi \, | \, \Phi \rangle.$$

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More precise formulation gives ($\rightarrow$ ALMMT):

$$(\Gamma\Psi_\gamma)(\Phi) = \sum_{\varphi_1 \in \text{Diff} \,/\, \text{Diff}_\gamma} F(\,|\text{GS}_\gamma|\,) \sum_{\varphi_2 \in \text{GS}_\gamma} \langle \varphi_1 * \varphi_2 * \Psi_\gamma \,|\, \Phi \rangle.$$

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Scalar product given by $(\Gamma\Psi \,|\, \Gamma\Psi') := (\Gamma\Psi)(\Psi')$.

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Not satisfied for example for $\Sigma = S^1$. Will say more, later.

...

Note: From assumption follows: Quantities

$$N^{\underline{\lambda}}_{\alpha} = \sum_x \delta(\lambda_x, \alpha) \qquad (= \text{number of "charges" } \alpha \text{ in } |\underline{\lambda}\rangle)$$

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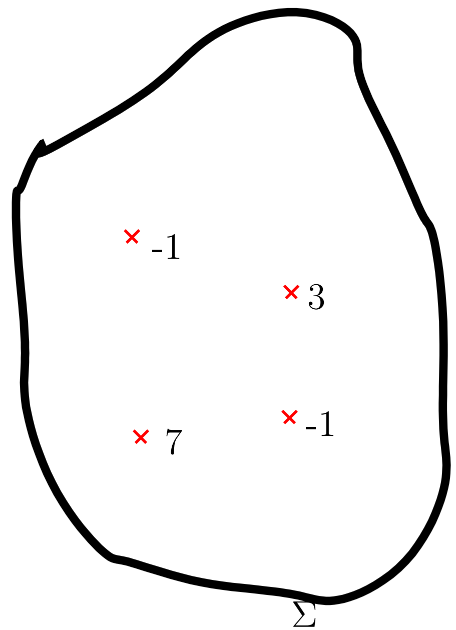

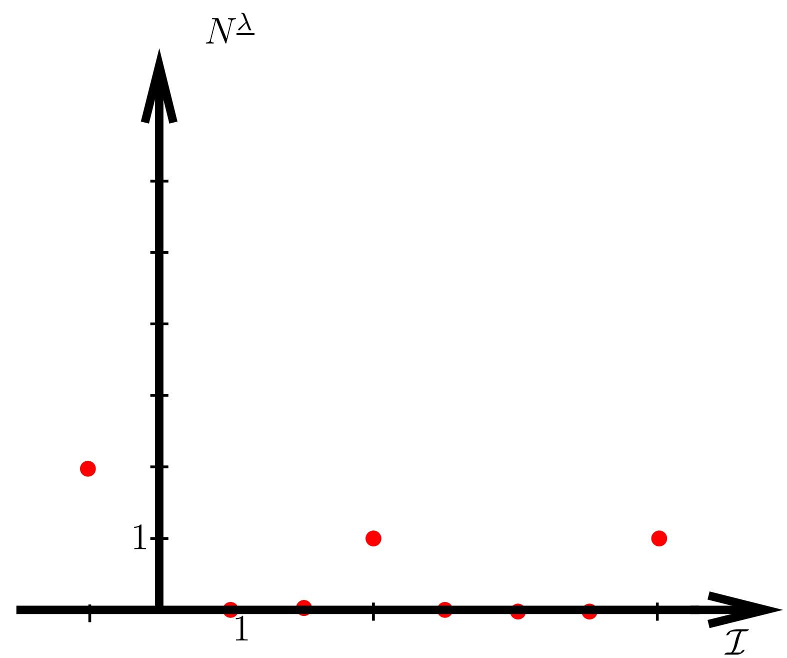

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- Scalar product on $\mathcal{H}_{\text{diff}}$ is $(\cdot \mid \cdot)$ from above.

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Note: Not the usual Fourier coefficients of the field. Graph-changing.

aside: $\Sigma = S^1$

In this case $\mathcal{H}_{\text{diff}}$ more complicated ("knotting"):

- ✘ $|\lambda_1, \lambda_2 \ldots, \lambda_N) \neq |\lambda_2, \lambda_1 \ldots, \lambda_N)$ in general.

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$$[a_\alpha, a_{\alpha'}] = 0, \qquad [a^\dagger_\alpha, a^\dagger_{\alpha'}] = 0, \qquad [a_\alpha, a^\dagger_\alpha] = 1$$

but $[a_\alpha, a^\dagger_{\alpha'}] = -R_{\alpha\to\alpha'}$ for $\alpha \neq \alpha'$.

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However:

Proposition: The $\widehat{L}_\alpha$ have **no** densely defined adjoints.

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Lemma:

$$\widetilde{S}^\dagger_\alpha = \sum_\lambda \lambda a^\dagger_{\lambda+\alpha} a_\lambda = \widetilde{S}_{-\alpha} - \alpha \sum_\lambda a^\dagger_{\lambda+\alpha} a_\lambda .$$

So they don't yet satisfy the adjointness relations. But that's expected.

. . .

Use symmetric ordering:

$$\widehat{L}_\alpha := \frac{1}{2}(\widetilde{S}_\alpha + \widetilde{S}^\dagger_{-\alpha}) = \sum_\lambda \left(\lambda - \frac{\alpha}{2}\right) a^\dagger_{\lambda-\alpha} a_\lambda$$

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Lemma:

$$[\widehat{L}_\alpha, \widehat{L}_{\alpha'}] = (\alpha - \alpha')\widehat{L}_{\alpha+\alpha'} + \frac{1}{4}\alpha\alpha' \left(a^\dagger_{-\alpha} a_{\alpha'} - a^\dagger_{-\alpha'} a_\alpha\right)$$

Algebra gets **extended**.

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with $$G(n) = \frac{F((n+1)!)}{F((n+2)!)} - \frac{F(n!)}{F((n+1)!)} + 1$$

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Proposition: In the case of $\Sigma = S^1$ additional correction

$$\frac{1}{4}\alpha\alpha' \left(a_{\alpha'} a^\dagger_{-\alpha} - a_\alpha a^\dagger_{-\alpha'} \right)$$

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$\longrightarrow$ non-anomalous representation on $\mathcal{H}_{\text{diff}}$ for scalar field + gravity if scalars constrained to sit on gravity vertices.

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To do:

- Do something analogous for gauge theory.
- Connection to vertex operators for bosonic string?