Recent Work on Computing Lorentzian Spin Foams

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Outline of Talk

- Review computational challenges for Lorentzian spin foam models
- Summary of existing method for the tetrahedral network (6J)
- Recoupling Theory for SL(2,C)
- The analogue of the Christensen-Egan algorithm for the Lorentzian 10J
- Outlook for numerical implementation

Spin Foam Models of Quantum Gravity

- Assign partition function to 2-complexes in spacetime

$$Z = \sum_{\text{colorings}} \prod_{\text{faces}} A_F(f) \prod_{\text{edges}} A_E(e) \prod_{\text{vertices}} A_V(v)$$

- Would like to study numerically to investigate phase structure, semiclassical limit
- Definition involves a sum over labellings of 2-complexes, and possibly over different complexes as well.
- But evaluating summand is computationally hard for just one labeling, because the vertex amplitude A_V is hard to compute

Why are they computationally hard?

- For Riemannian models (Spin(4) gauge group) efficient algorithm known that re-expresses 10J as a sum over 6J symbols (Racah coefficients)
- For Lorentzian models (SL(2,C) gauge group) no such efficient algorithm was known; 6J symbols themselves are hard.
- Why? They are defined by integrals that are high-dimensional with oscillatory integrands.

$$6\text{J} = \int_{H^4} \prod_{i=1}^{4} dx_i \; K_{\rho_1}(x_1, x_2) \cdots K_{\rho_6}(x_3, x_4)$$

$$10\text{J} = \int_{H^5} \prod_{i=1}^{5} dx_i \; K_{\rho_1}(x_1, x_2) \cdots K_{\rho_{10}}(x_4, x_5)$$

$$K_\rho(x, y) = \frac{\sin(\rho r)}{\rho \sinh r} \qquad r = d_{\text{hyp}}(x, y)$$

A Better Algorithm for Lorentzian 6J

- Using group-theoretic techniques, can re-express the Lorentzian 6J as a sum of products of Clebsch-Gordan coefficients for SL(2,C). Analogous to similar formula for SU(2) Racah coefficients:

$$6\mathrm{J} \propto \sum_J (2J+1) C^{\,0\rho_1\ 0\rho_5\ 0\rho_4}_{\,00\ \ J0\ \ J0} C^{\,0\rho_6\ 0\rho_4\ 0\rho_3}_{\,00\ \ J0\ \ J0} C^{\,0\rho_2\ 0\rho_5\ 0\rho_3}_{\,00\ \ J0\ \ J0}$$

- These coefficients can be calculated recursively; thus, very efficiently.
- Much more efficient than direct integration, but convergence can still require many terms.
- Can further speed convergence by using asymptotic form of Clebsch-Gordan coefficients (this is the hard part of both the derivation and coding).

Tet(1,1,1,1,1,1)

Vegas Monte-Carlo Integration

Calls	Value	Time (sec)
10^3	$0.041267 \pm 21.4\%$	0.0070
10^4	$0.126242 \pm 4.03\%$	0.0430
10^5	$0.122350 \pm 1.53\%$	0.4309
10^6	$0.118190 \pm 0.490\%$	4.933
10^7	$0.117902 \pm 0.192\%$	69.99
10^8	$0.118459 \pm 0.0532\%$	436.6

Summation Algorithm

Terms	Value	Time (sec)
10^2	0.118087292	≈ 0.00002
10^3	0.118283570	≈ 0.0002
10^4	0.118306260	0.002
10^5	0.118299794	0.0200
10^6	0.118300212	0.198
10^7	0.118300200	1.98
10^8	0.118300196	19.9

Accelerated Summation Algorithm

Terms	Value	Time (sec)
10^2	0.1183001969	≈ 0.0002
10^3	0.1183001969	≈ 0.002
10^4	0.1183001969	0.0170

Toward the 10J

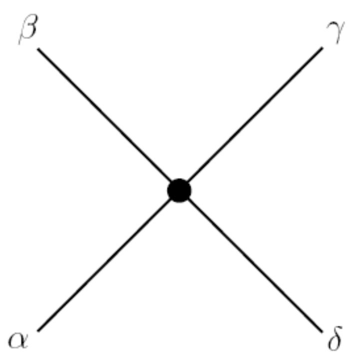
- The reason for calculating the 6J is to use it in calculating the 10J, hoping that this method is more efficient or more accurate than the direct integration.
- To do this, we need to use recoupling theory for SL(2,C) in the same way that the Riemannian algorithms rely on recoupling theory for SU(2).
- This can be done, and leads to diagrammatic techniques similar to those used for SU(2) spin networks
- Such techniques can be proven using known identities for SL(2,C) matrix elements and Clebsch-Gordan coefficients.

Example: Expanding the 4-Valent Vertex

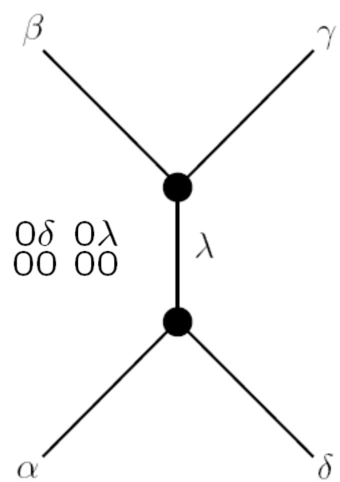
- All manipulations are based on re-expressing the kernels for Lorentzian spin networks in terms of matrix elements on SL(2,C).
- Use the identity:

$$D^{0\alpha}_{J_\alpha M_\alpha 00}(g)\, D^{0\beta}_{J_\beta M_\beta 00}(g) = \int \lambda^2 d\lambda\, \overline{C}{}^{0\alpha\ 0\beta\ 0\lambda}_{00\ 00\ 00} \sum_{JM} C^{\ 0\alpha\ \ \ 0\beta\ \ \ 0\lambda}_{J_\alpha M_\alpha\ J_\beta M_\beta\ JM}\, D^{0\lambda}_{JM00}(g)$$

twice to prove the diagrammatic relation:

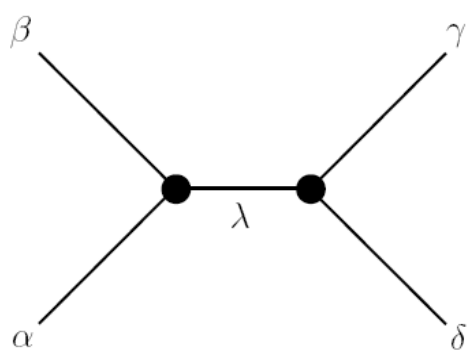


$$= \int \lambda^2 d\lambda\, \overline{C}{}^{0\beta\ 0\gamma\ 0\lambda}_{00\ 00\ 00}\, C^{0\alpha\ 0\delta\ 0\lambda}_{00\ 00\ 00}$$

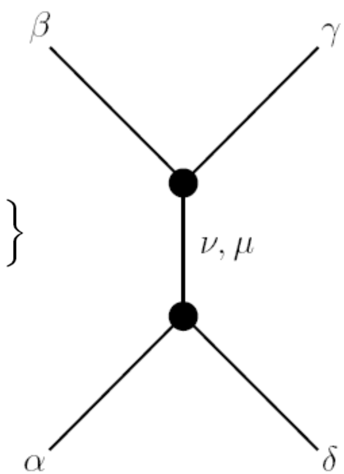


SL(2,C) Recoupling

- After suitably renormalizing the 3-valent vertex, can prove recoupling for SL(2,C) spin networks

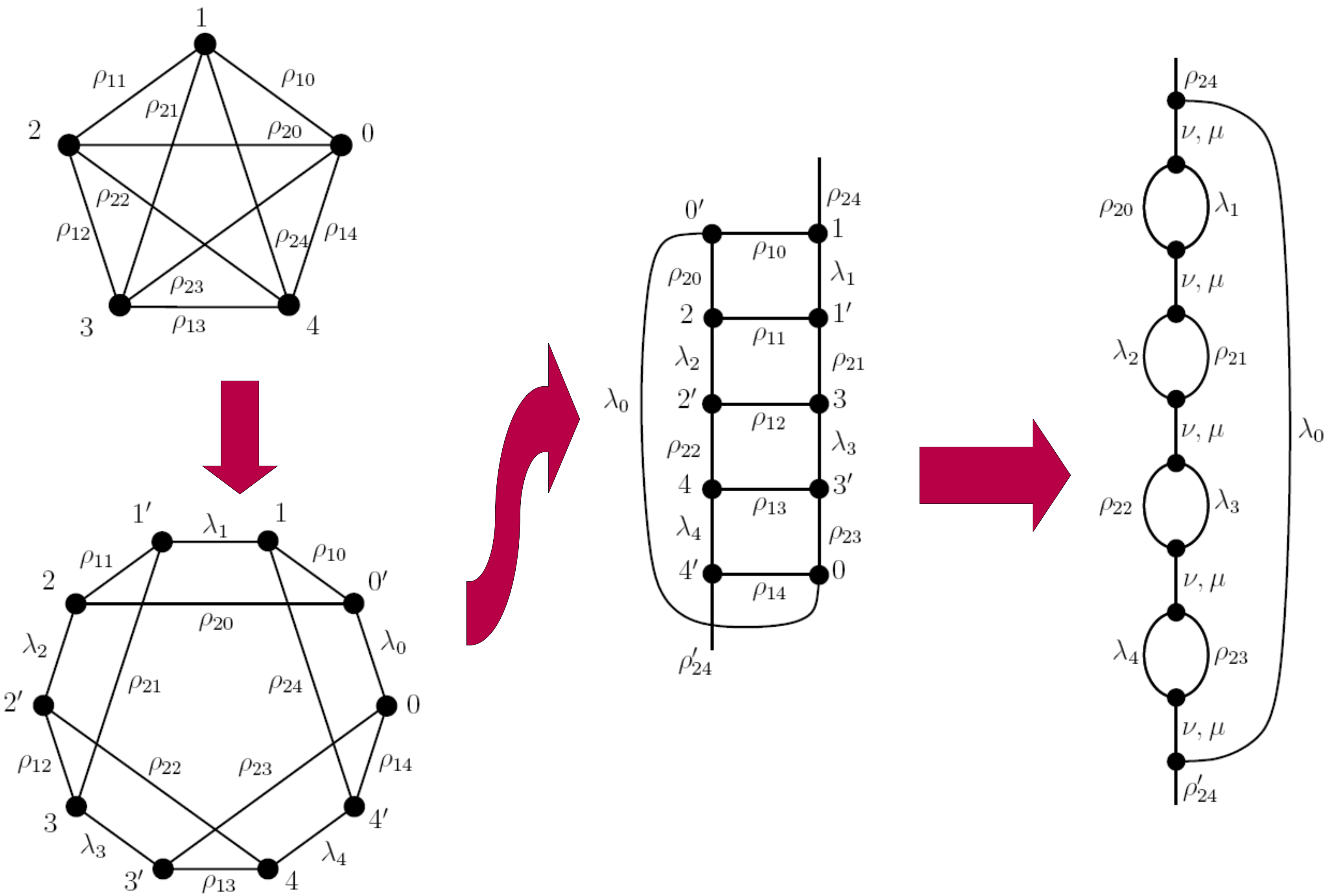


$$= \sum_{\nu} \int (\mu^2 + 4\nu^2) d\mu \left\{ \begin{matrix} \alpha & \beta & \nu,\mu \\ \gamma & \delta & \lambda \end{matrix} \right\}$$



- Note the appearance of a non-simple representation in the recoupling formula: this is unavoidable and an exactly analogous situation occurs in the Riemannian case, when we consider recoupling for Spin(4) spin networks.

Evaluating the 10J



A Formula for 10J's in terms of 6J's

- Combining all of these steps we get a formula analgous to the Christensen-Egan algorithm for the Riemannian 10J:

$$10J = \int \prod_{i=0}^{4} \left(\lambda_i^2 \, d\lambda_i\right) \sum_{\nu} \int (\mu^2 + 4\nu^2) \, d\mu \ \text{(Prod of CG's)}$$

$$\begin{Bmatrix} \rho_{20} & \lambda_0 & \nu,\mu \\ \rho_{24} & \lambda_1 & \rho_{10} \end{Bmatrix} \begin{Bmatrix} \rho_{21} & \lambda_1 & \nu,\mu \\ \rho_{20} & \lambda_2 & \rho_{11} \end{Bmatrix} \begin{Bmatrix} \rho_{22} & \lambda_2 & \nu,\mu \\ \rho_{21} & \lambda_3 & \rho_{12} \end{Bmatrix} \begin{Bmatrix} \rho_{23} & \lambda_3 & \nu,\mu \\ \rho_{22} & \lambda_4 & \rho_{13} \end{Bmatrix} \begin{Bmatrix} \rho_{24} & \lambda_4 & \nu,\mu \\ \rho_{23} & \lambda_0 & \rho_{14} \end{Bmatrix}$$

- Expresses 10J as a six-dimensional integral and one-dimensional sum over 6J symbols.
- Thus, dimension of integral is reduced (from 9 to 6) and experimentation seems to indicate the integrand is in general less oscillatory. When triangle inequalities are violated it decays exponentially.

Numerical Implementation

- This has been implemented, but at present is not as fast as the existing direct integration
  - Need to improve asymptotics in 6J ➡ faster 6J
  - Use importance sampling in integrals to take advantage of exponential decay.
  - Other methods of evaluating 6J?
- Hope to test these improvements in next couple of months.
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