Reduction of a Quantum Theory

Cosmological Degrees of Freedom in Loop Quantum Gravity

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June 26,
at "Loops 07" in Morelia, Mexico

Overview

What are the cosmological degrees of freedom of LQG?

1. Classical insight:
   "Symmetry reduction in class. systems is furnished by a pull-back under a Poisson map(η^*)"
   $\Rightarrow$ Construct "quantum Poisson map"
2. Quantum Reduction:
   "Use a mechanism similar to Morita equivalence and Rieffel induction"
3. Application to Loop Quantum Gravity $\Rightarrow$ Extract Cosmology

Observables: Restricted Sensitivity

Poisson system: Observables in focus (Symmetry assumption $\Rightarrow$ Sensitivity of measurements restricted)

- Observable algebra: $\mathfrak{A}_{class} = C_c^\infty(\Gamma)$ (smooth functions of compact support on phase space)
- Algebraic structure by point wise operations $+, \times, .^*, ||.||$
- Poisson-structure: $\{., .\} : \mathfrak{A}_{class} \times \mathfrak{A}_{class} \to \mathfrak{A}_{class}$

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$\Rightarrow \eta^* \mathfrak{A}_{class} = \mathfrak{A}_{class}/\mathfrak{I}$, i.e. equivalence classes of observables, that agree on Γ_o.
$\{\eta^* f, \eta^* g\} = \eta^* \{f, g\} \; \forall f, g \in \mathfrak{A}_{class}$ (Poisson map)
$\Rightarrow$ Find "Quantum Poisson Maps"!

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$\Rightarrow$ Quantization and Reduction do generally not commute!

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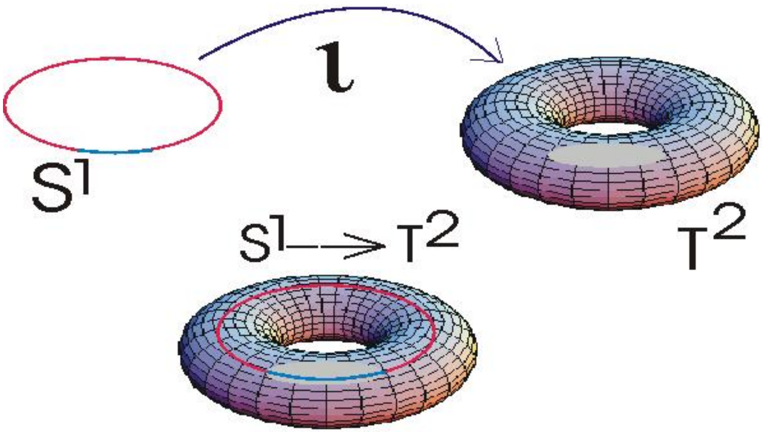
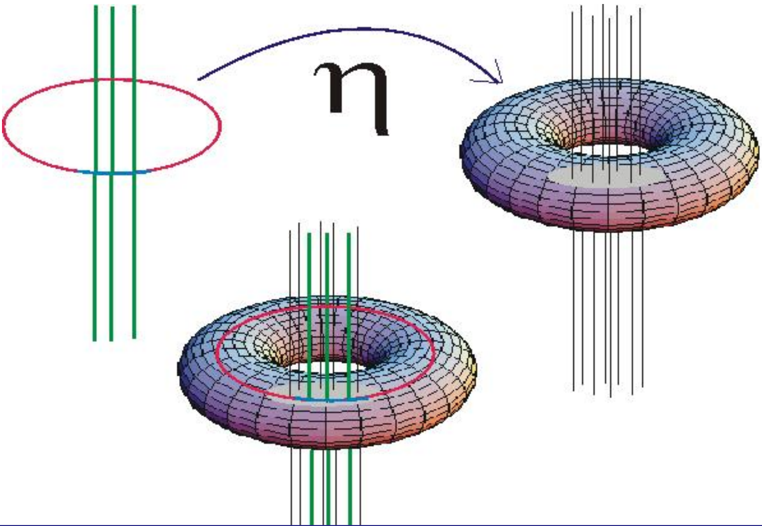
Definition: **Quantum Poisson map** $\mathcal{E}$ together with $S \leftrightarrow T$.

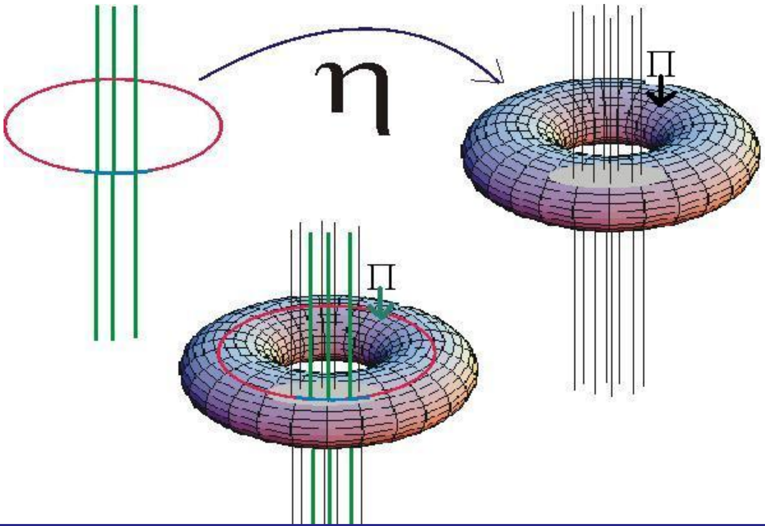
Embedding a Space



Embedding a Vector Bundle



Projecting to the Embedded Base Space



Non commutative Embeddings

- Non commutative topol. space = C^*-algebra $\mathfrak{A}$
- Non commutative vector bundle = Hilbert-C^*-modules $\mathcal{E}$ over $\mathfrak{A}$

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$\Rightarrow$ Embedding $m : \mathbb{X}_o \to \mathbb{X}$ allows: $P : C(\mathbb{X}) \to C(\mathbb{X}_o) : f \mapsto m^* f$ and an "inverse" $i : C(\mathbb{X}_o) \to C(\mathbb{X})$ s.t. $P \circ i = id_{C(X_o)}$ and $i \circ P = id_{Img_i}$

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$\Rightarrow$ "quantum embedding": (P, i) induces reduced $\mathfrak{A}_o = \overline{\{T_{g_1,g_2} : f_o \mapsto P(i(P(g_1)) \langle i(P(g_2)), i(f_o) \rangle_{\mathfrak{A}}) : g_1, g_2 \in C(\mathbb{X})\}}$ and representation $\omega_{\mathfrak{B}}(T_{g_1,g_2}) := \omega_{\mathfrak{A}}(\langle i(P(g_1)), i(P(g_2)) \rangle_{\mathfrak{A}})$.

Non straight Holonomies of isotropic connections

- ODE: $\dot{h}(t) = ic\dot{e}_i(t)\tau^i h(t)$; fix t to arc length
$\Rightarrow \ddot{h}_{11} = \frac{\overline{\ddot{e}_1 + i\ddot{e}_2}}{\dot{e}_1 + i\dot{e}_2}\dot{h}_{11} + (ic\dot{e}_3 - e_3\frac{\overline{\ddot{e}_1 + i\ddot{e}_2}}{\dot{e}_1 + i\dot{e}_2} - c^2)h_{11}$, h_{12} similarly

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- Notice that spiral holonomies are **not** almost periodic, but:

Matrix elements are asymptotically $|c| >> A, B$ *almost periodic.*

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ODE is approximated by an approximation of paths through piecewise spirals.

Conjecture 2: All holonomy matrix elements are in the uniform closure of spiral holonomy matrix elements.

Cosmology from LQG

Construction needs second countability of $\mathfrak{A} \Rightarrow$ partially gauge-fix extended diffeos. $\Rightarrow$ "scaffold"
$\Rightarrow$ "allowed graphs" and "allowed surfaces", enough for all diffeo.classes of $Cyl(A) \circ \mathcal{W}(E)$.

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- kinematical constraints solved by restriction $(Dom(P), Img(i))$ to constraint surface in full LQG
- New combinatorial problems in finding kernel of scalar constraint!

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- Future work:
  - Explore scalar constraint surface $\Rightarrow$ Cosmology
  - Use holomorphic cosmological states (proposed by J. Engle)