Reduction of a Quantum Theory
Cosmological Degrees of Freedom in Loop Quantum Gravity

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Overview

What are the cosmological degrees of freedom of LQG?

1. Classical insight:

"Symmetry reduction in class. systems is furnished by a pull-back under a Poisson map (\( \eta^* \))"

\( \Rightarrow \) Construct "quantum Poisson map"

2. Quantum Reduction:

"Use a mechanism similar to Morita equivalence and Rieffel induction"

3. Application to Loop Quantum Gravity \( \Rightarrow \) Extract Cosmology
Observables: Restricted Sensitivity

- Poisson system: Observables in focus (Symmetry assumption $\Rightarrow$ Sensitivity of measurements restricted)
  - Observable algebra: $\mathcal{A}_{\text{class}} = C_c^\infty(\Gamma)$ (smooth functions of compact support on phase space)
  - Algebraic structure by point wise operations $+, \times, .*, ||.||$
  - Poisson-structure: $\{., .\} : \mathcal{A}_{\text{class}} \times \mathcal{A}_{\text{class}} \rightarrow \mathcal{A}_{\text{class}}$

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- Restricted Sensitivity (measurements separate only points in $\Gamma_o$): Measurements insensitive to functions vanishing on $\Gamma_o$
  $\Rightarrow$ Ideal of functions $\mathcal{I} = \{ f \in \mathcal{A}_{\text{class}} : f|_{\Gamma_o} = 0 \}$
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  $\Rightarrow \eta^*\mathcal{A}_{\text{class}} = \mathcal{A}_{\text{class}}/\mathcal{I}$, i.e. equivalence classes of observables, that agree on $\Gamma_o$.
  $\{ \eta^* f, \eta^* g \} = \eta^* \{ f, g \} \ \forall f, g \in \mathcal{A}_{\text{class}}$ (Poisson map)
  $\Rightarrow$ Find ”Quantum Poisson Maps”!

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Reduction of a Quantum Theory
Problem: Quantum Phase space is non commutative

Goal: ”Extend Phase Space Reduction to Quantum Systems”
Problem: ”Γ_{quant} is non commutative” : $e^{iax} e^{ibp} e^{-iax} e^{-ibp} = e^{iab}$
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⇒ Operator ordering is important (QFTs need particular ordering to be well defined!)
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• Example:

\[ H_{class} = p^2 + V(q) + c = f(q)pf^{-2}(q)pf(q) + V(q) + c =: H'_{class} \]

using a definite solution of \( f'' + (V + c)f = 0: \)

\[ \hat{H} = \hat{p}^2 + V(\hat{q}) \] whereas \( \hat{H}' = \hat{p}^2 \)
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⇒ Quantization and Reduction do generally not commute!
Objective

- We do **not** want to ”reduce and then quantize”, but:
”reduce the quantum system”, i.e. we seek: $E$ for $(A, \pi, H)$, s.t.

\[
\begin{align*}
\text{(1) Matching VEVs:} & \quad \forall T \in D: \exists S \leftrightarrow T (D\text{ dense in } A) \omega(S) = \omega_o(T) \\
\text{(2) "Correct" reduced observable algebra} & \quad E_{A} \rightarrow A_{o} \hbar \rightarrow 0 \downarrow \eta^{*} \downarrow \hbar \rightarrow 0 \quad C(\Gamma) \rightarrow C(\Gamma_{o})
\end{align*}
\]

Definition: Quantum Poisson map $E$ together with $S \leftrightarrow T$. 
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Definition: **Quantum Poisson map** $\mathcal{E}$ together with $S \leftrightarrow T$. 
Embedding a Space

$S^1 \rightarrow T^2$

$T^2$
Embedding a Vector Bundle
Projecting to the Embedded Base Space
Non commutative Embeddings

- Non commutative topol. space = $C^*$-algebra $\mathcal{A}$
- Non commutative vector bundle = Hilbert-$C^*$-modules $\mathcal{E}$ over $\mathcal{A}$
  $\mathcal{E}$ is an $\mathcal{A}$-module with a sesquilinear structure
  $\langle . , . \rangle_\mathcal{A}$, with dense range in $\mathcal{A}$, satisfying certain properties.
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- Rieffel induction: $\omega_\mathcal{B}(e_1 \langle e_2, . \rangle_{\mathcal{A}}) := \omega_\mathcal{A}(\langle e_1, e_2 \rangle_{\mathcal{A}})$.
- $C(\mathbb{X})$ often admits pre-Hilb.-$C^*$-module over $\mathcal{A}_{\text{quant.}}$ (\mathbb{X}=\text{config. space})
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  $\Rightarrow$ Embedding $m : X_0 \to X$ allows: $P : C(X) \to C(X_0) : f \mapsto m^*f$ and an "inverse" $i : C(X_0) \to C(X)$ s.t. $P \circ i = id_{C(X_0)}$ and $i \circ P = id_{\text{Img}_i}$.
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  $\Rightarrow$ ”quantum embedding” : $(P, i)$ induces reduced $\mathcal{A}_o = \{ T_{g_1,g_2} : f_o \mapsto P(i(P(g_1)) \langle i(P(g_2)), i(f_o) \rangle_\mathcal{A}) : g_1, g_2 \in C(X) \}$ and representation $\omega_\mathcal{B}(T_{g_1,g_2}) := \omega_\mathcal{A}(\langle i(P(g_1)), i(P(g_2)) \rangle_\mathcal{A})$. 
Non straight Holonomies of isotropic connections

- ODE: $\dot{h}(t) = i\epsilon \epsilon_i(t) \tau^i h(t)$; fix $t$ to arc length

$$\Rightarrow \ddot{h}_{11} = \frac{\dddot{\epsilon}_1 + i\dddot{\epsilon}_2}{\dddot{\epsilon}_1 + i\dddot{\epsilon}_2} \dot{h}_{11} + (i\epsilon \epsilon_3 - \epsilon_3 \frac{\dddot{\epsilon}_1 + i\dddot{\epsilon}_2}{\dddot{\epsilon}_1 + i\dddot{\epsilon}_2} - c^2) h_{11}, \ h_{12} \text{ similarly}$$
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(Brunnemann/Fleischhack independently T.K.): spiral holonomies:

\[
e(t) = \frac{1}{\sqrt{R^2 + b^2}} (R \sin(t), R \cos(t), bt)
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\Rightarrow \text{spiral holonomy matrix elements: } \exp\left( \frac{it}{2} \sqrt{(c - A)^2 + B^2} \right).
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- Notice that spiral holonomies are \textbf{not} almost periodic, but:

\textit{Matrix elements are asymptotically} \( |c| \gg A, B \text{ almost periodic.} \)

\textbf{Conjecture 1:} All holonomies are asymptotically almost periodic.
(already proved, whenever Liouville-Green ansatz converges.)
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**Conjecture 1:** All holonomies are asymptotically almost periodic.
(already proved, whenever Liouville-Green ansatz converges.)

**ODE is approximated by an approximation of paths through piecewise spirals.**

**Conjecture 2:** All holonomy matrix elements are in the uniform closure of spiral holonomy matrix elements.
Construction needs second countability of $\mathcal{A}$ ⇒ partially gauge-fix extended diffeos. ⇒ ”scaffold” ⇒ ”allowed graphs” and ”allowed surfaces”, enough for all diffeo.classes of $\text{Cyl}(A) \circ \mathcal{W}(E)$. 

• piecewise spirals with rational radii, heights
• ”umbrella shaped” surfaces for Weyl-operators with unit area
• vertices at points with rational coordinates
⇒ Only very small diffeos. are needed in gauge fixing
⇒ Holonomy ODE is approximated

$\mathcal{P} : \text{Cyl} \gamma \mapsto \text{Cyl} \phi_{\text{scaff}}(\gamma)$ | $\mathcal{A}_{\text{iso}}$ and $i$ by ”partially inverting” $\mathcal{P} E = \text{Cyl}$ with $\langle \text{Cyl}_1, \text{Cyl}_2 \rangle_{\mathcal{A}} : \text{Cyl} \mapsto \text{Cyl}_1 \langle \pi(\text{Cyl}_2) \Omega, \pi(\text{Cyl}_2) \Omega \rangle$ full

• kinematical constraints solved by restriction ($\text{Dom}(\mathcal{P}), \text{Img}(i)$) to constraint surface in full LQG
• New combinatorial problems in finding kernel of scalar constraint!
Cosmology from LQG

Construction needs second countability of $\mathcal{A} \Rightarrow$ partially gauge-fix extended diffeos. $\Rightarrow$ ”scaffold”
$\Rightarrow$ ”allowed graphs” and ”allowed surfaces”, enough for all diffeo.classes of $\text{Cyl}(A) \circ \mathcal{W}(E)$.
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\begin{align*}
P : \text{Cyl}_\gamma & \mapsto \text{Cyl}_{\phi_{\text{scaff.}}(\gamma)}|_{A_{iso}} \text{ and } i \text{ by } ”\text{partially inverting” } P \\
\mathcal{E} = \text{Cyl} \text{ with } \langle \text{Cyl}_1, \text{Cyl}_2 \rangle_{\mathfrak{A}} : \text{Cyl} & \mapsto \text{Cyl}_1\langle \pi(\text{Cyl}_2)\Omega, \pi(\text{Cyl})\Omega \rangle_{\text{full}}
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$P : Cyl_{\gamma} \mapsto Cyl_{\phi_{scaff.}(\gamma)}|_{A_{iso}}$ and $i$ by ”partially inverting” $P$

$\mathcal{E} = Cyl$ with $\langle Cyl_1, Cyl_2 \rangle_{\mathcal{A}} : Cyl \mapsto Cyl_1\langle \pi(Cyl_2)\Omega, \pi(Cyl)\Omega \rangle_{full}$

- kinematical constraints solved by restriction $(Dom(P), Img(i))$ to constraint surface in full LQG
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Conclusions

- Symmetry reduction of Poisson system through Poisson-mapping $\eta^*$
- certain important Poisson reductions $\eta^*$ can be cast as $(P, i)$
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- Certain important Poisson reductions $\eta^*$ can be cast as $(P, i)$
- Used $(P, i)$ to reduce Quantum System, s.t.
  - Reduced quantum algebra "relevant part of the system"
  - Induced representation (matching as matrix elements)
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Cosmology from LQG:

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- Dynamics induced from LQG (Conj. 2)
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  - Systematic construction of Cosmology from LQG (no $\mu_0$!)
  - Dynamics induced from LQG (Conj. 2)
- Future work:
  - Explore scalar constraint surface $\Rightarrow$ Cosmology
  - Use holomorphic cosmological states (proposed by J. Engle)