Hamilton

An alternative to the loop algebra of q-Quantum Gravity?

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Loops '07 Morelia
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Ashtekar Formulation of Canonical Gravity

General relativity with canonical variables (A, E) on Σ

$$\{A_a^i(x), E^{bj}(y)\} = i\, G_* \delta_a^b \delta^{ij} \delta^3(y, x).$$

G_* Newton's constant plus factors of π and 2

Constraints:

$$\begin{aligned} \mathcal{G}^i &:= D_a E^{ai} \approx 0 \\ \mathcal{V}_a &:= E^{bi} F_{ab}^i \approx 0 \\ \mathcal{H} &:= \epsilon^{ijk} E^{bj} E^{ck} (F_{bc}^i + \frac{\Lambda}{6} \epsilon_{abc} E^{ci}) \approx 0 \end{aligned}$$

- 3-diffeos generated by linear combo of gauss and vector
- The magnetic-electric duality

$$B^{ai} = -\frac{\Lambda}{3} E^{ai}$$

solves the Hamiltonian constraint.

- To recover GR we must implement "reality conditions"

Quantum Theory:

Quantum Theory: Heuristics

Quantum Theory: Heuristics

Connection representation: E is promoted to

$$\widehat{E}^{ai} = \hbar G_* \frac{\delta}{\delta A_a^i} = \ell_p^2 \frac{\delta}{\delta A_a^i},$$

defining the 'Planck length' as $\ell_p^2 = \hbar G_*$.

With $\{\,,\,\} = \cdot \to [\,,\,] = i\hbar\cdot$ the commutator becomes

$$\left[\widehat{A}_a^i(x), \widehat{E}^{bj}(y)\right] = \ell_p^2 \delta_a^b \delta^{ij} \delta^3(x,y).$$

Note factors of i.

Quantum Theory: Heuristics

Seek wavefunctions $\psi(A)$ such that

$$\begin{aligned}
\widehat{\mathcal{G}}^i\psi(A) &:= \ell_p^2 D_a \frac{\delta}{\delta A_a^i}\psi(A) = 0 \\
\widehat{\mathcal{V}}^a\psi(A) &:= F_{ab}^i \frac{\delta}{\delta A_b^i}\psi(A) = 0 \\
\widehat{\mathcal{H}}\psi(A) &:= \ell_p^4 \epsilon^{ijk} \frac{\delta}{\delta A_b^j}\frac{\delta}{\delta A_c^k}\left[F_{bc}^i + \frac{\ell_p^2\Lambda}{6}\epsilon_{abc}\frac{\delta}{\delta A_c^i}\right]\psi(A) = 0
\end{aligned}$$

An advantageous choice of operator ordering - doesn't muck up constraint algebra and ...

Can we find a solution?

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An advantageous choice of operator ordering - doesn't muck up constraint algebra and ...

Can we find a solution? Sure, Kodama did.

Quantum Theory: Kodama State

With the Chern-Simons form

$$S_{CS} = \int_\Sigma (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\, d^3x$$

Let

$$\psi(A) = \mathcal{N} \exp\left(-\frac{3}{\Lambda \ell_p^2} S_{CS} \right)$$

$\mathcal{N}$ possibly topology-dependent norm (Soo gr-qc/0109046). The handy fact

$$\frac{\delta}{\delta A_a^i} S_{CS} = \epsilon^{abc} F_{bc}^i = B^{ai}$$

ensures that the Kodama state satisfies the electric-magnetic duality and thus the Hamiltonian constraint.
Also (small) gauge and diffeomorphism invariant.

Quantum Theory: Loop Transform

What about the loop transform?

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$$\Psi(s) = \int d\mu(A) T_s(A)\, \psi(A) = \mathcal{N} \int d\mu(A) T_s(A) \exp\left[\frac{3}{\Lambda \ell_p^2} S_{CS}\right]?$$

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Witten showed that the path integral

$$\Psi(L) = \int d\mu(A) T_L(A) \text{ exp} \left[\frac{ik}{4\pi} S_{CS} \right]$$

is, for knots and links L, equivalent to an invariant, the Kauffman bracket!

Key point: The invariant is sensitive to twists of the spin net edges. PI only defined for framed links - "tubes with stripes" or "ribbons"

Quantum Gravity?

Beautiful Picture:

- State(s) of Quantum Gravity!
- Includes the cosmological constant!
- Knot classes are "quantum numbers" for states!

$$\Psi(s) = K(s)$$

- Has DeSitter as a semiclassical limit!
- Cosmological constant - particle statistics connection?

\- composite particle statistics determined by framing in theory of fractional QHE

Key new feature: depends on framed spin networks.

Quantum Gravity?

Obviously this is too good to actually hold.

• Kodama state is in Lorenztian framework. While Witten's result is in YM theory, with real-valued connections

$$K(L) = \int d\mu(A) W(L; A) \exp\left[\frac{ik}{4\pi} S_{CS}(A)\right],$$

does not (obviously) hold for a complex connection. Like defining the inverse Laplace transform due care is required in the choice on contour.

• Is the state normalizable? Not in linearized Lorentzian case [Freidel-Smolin CQG 21 (2004) 3831]

• Violates CPT (relevance? NPT vs. QFT)

• Using the variational calculus methods, the "invariant" for graphs acquires tangent space sensitivity (SM hep-th/9810071)

Notes on Euclidean results

• Due to invariance under large gauge transformations, k is an integer

• Equating YM and Kodama coefficients

$$\frac{ik}{4\pi} = \frac{3i}{\Lambda \ell_p^2} \implies k = \frac{12\pi}{\Lambda \ell_p^2}$$

So $\frac{12\pi}{\Lambda \ell_p^2}$ is an integer. Note: Small Λ means large k.

• The deformation parameter - measure of twist - is $q = \exp\left(\frac{\pi i}{k+2}\right)$, a root of unity.

• Kauffman bracket is a polynomial in q, may be expressed in terms of quantum integers

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$$

and as the evaluation of q spin nets using graphical recoupling theory.

q-Quantum Gravity

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Yes. Define the action of loop variables, usual algebra for loops on spin net (α with color n) works [SM-Smolin NPB 473 (1996) 267] (for single intersection)

$$\left[\widehat{T}_q[\alpha], \widehat{T}_q^a[\beta](s)\right] = i\ell_p^2\, \Delta^a[\alpha, \beta](s)\widehat{T}_q[\alpha \cup_s \beta]$$

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Removing everything but the combinatorics the operators of q-Quantum Gravity on state $\mid n_\alpha, n_\beta\rangle$ for loops α, β

$$S_\alpha \mid n,\, 0\rangle := \mid n+1,\, 0\rangle + \mid n-1,\, 0\rangle$$
$$T_\beta \mid n,\, 0\rangle := n \mid n,\, 1\rangle$$

The algebra above is

$$\left[S_\alpha, T_\beta\right] = S_{\alpha\cup\beta}.$$

It is a choice -

q-Quantum Gravity

Are there other choices for the action of T, e.g. $[n]$, such that the algebra is consistent with combinatorics of q-spin nets at a root of unity (or more general deformation parameter)?

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q-deformed algebra or "qummutator"

$$[a, b]_\lambda := ab - \lambda(q)ba.$$

i.e. for some $t_n \neq n$ and $\lambda(q)$ is

$$S_\alpha \mid n,\, 0\rangle := \mid n+1,\, 0\rangle + \mid n-1,\, 0\rangle$$
$$T_\beta \mid n,\, 0\rangle := t_n \mid n,\, 1\rangle$$
$$\Big[S_\alpha, T_\beta\Big]_\lambda = S_{\alpha\cup\beta}$$

consistent? Essentially, no.

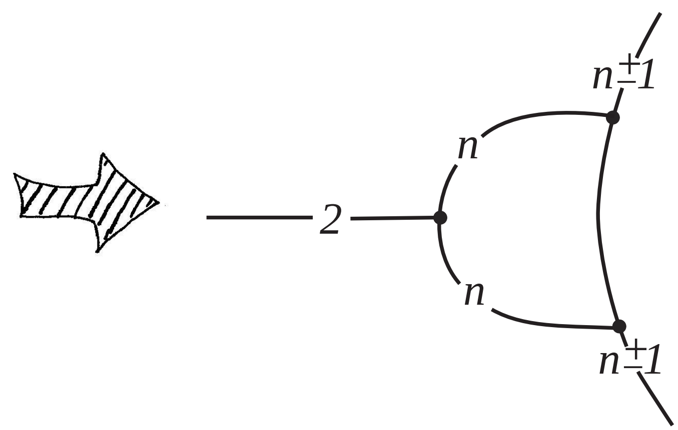
A deformation of q-Quantum Gravity?

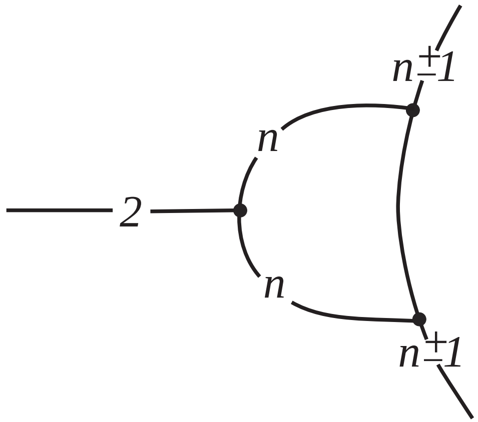
Acting with algebra on state $|\, n, 0\rangle$.

$$\left[S_\alpha, T_\beta\right]_\lambda |\, n,\, 0\rangle = S_{\alpha\cup\beta} \,|\, n,\, 0\rangle.$$

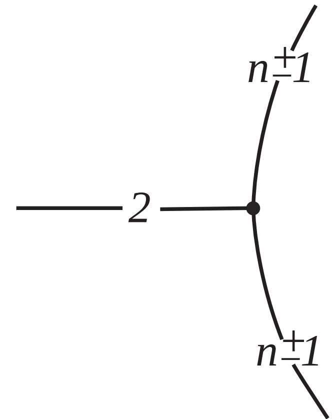
First term is simply $t_n(|\, n+1\rangle + |\, n-1\rangle)$. For the second and third terms graphical methods are useful

$S_\alpha \,|\, n,\, 1\rangle$

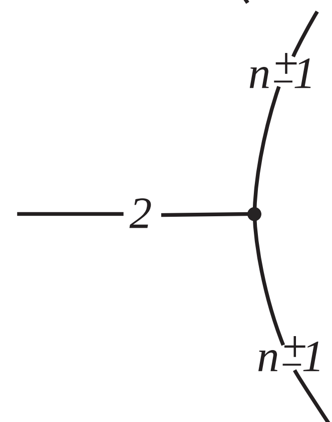




$$= \frac{\text{Tet}\begin{bmatrix} n\pm 1 & n\pm 1 & 1\\ n & n & 2\end{bmatrix}}{\theta(n\pm 1, n\pm 1, 2)}$$



$$= \begin{cases} \dfrac{[n+2][n-1]}{[n]} & \text{for } n-1 \\ 1 & \text{for } n+1 \end{cases}$$



$$\Longrightarrow S_\alpha \mid n,\ 1\rangle = \begin{cases} \dfrac{[n+2][n-1]}{[n]} \mid n+1,\ 1\rangle \\ 1 \mid n-1,\ 1\rangle \end{cases}$$

A deformation of q-Quantum Gravity?

$$\begin{aligned} t_{n+1} \mid n+1,\, 1\rangle + t_{n-1} \mid n-1,\, 1\rangle - \lambda t_n \frac{[n+2][n-1]}{[n][n+1]} \mid n-1,\, 1\rangle \\ -\lambda t_n \mid n+1,\, 1\rangle \\ = \mid n+1,\, 1\rangle - \frac{[n-1]}{[n+1]} \mid n-1,\, 1\rangle \end{aligned}$$

A deformation of q-Quantum Gravity?

$$
\begin{aligned}
t_{n+1} \mid n+1,\, 1\rangle + t_{n-1} \mid n-1,\, 1\rangle - \lambda t_n \frac{[n+2][n-1]}{[n][n+1]} \mid n-1,\, 1\rangle \\
-\lambda t_n \mid n+1,\, 1\rangle \\
= \mid n+1,\, 1\rangle - \frac{[n-1]}{[n+1]} \mid n-1,\, 1\rangle
\end{aligned}
$$

Hence

$$
\begin{aligned}
& t_{n+1} - \lambda t_n = 1, && \text{for } 1 \leq n \leq k-1 \\
& t_{n-1} - \lambda \frac{[n+2][n-1]}{[n][n+1]} t_n = -\frac{[n-1]}{[n+1]}, && \text{for } 2 \leq n \leq k
\end{aligned}
$$

The first equation immediately gives

$$
t_n = 1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1} = \frac{1 - \lambda^{-n}}{1 - \lambda}
$$

A deformation of q-Quantum Gravity?

Then for a consistent algebra $\lambda(q)$ satisfies

$$1+\lambda+\lambda^2+...+\lambda^{n-2}-(\lambda+\lambda^2+...+\lambda^n)\frac{[n+2][n-1]}{[n][n+1]}+\frac{[n-1]}{[n+1]}=0.$$

Question becomes: Does λ exists for any k and all $2 \leq n \leq k$? ($k < 2$ trivial)

Case by case analysis shows no solutions except:

For large k arbitrarily accurate approximate solutions $\lambda = 1 \implies t_n = n$ exist for $n << k$. In the limit $k \to \infty$ the relation is exact.

Note: During inflation $k \sim 10^5$ which implies roots for $n = 2$ and $n = 100$ differ by one part in 10^{14}.

A deformation of q-Quantum Gravity?

Summary

• On the question of the possible deformation: There does not exist deformations of the loop algebra that are consistent with the combinatorics of q spin nets at a root of unity. Approximate solutions exist.

• If the Lorentzian loop transform is defined simply by analytic continuation then k is complex. The above conclusions appear unaltered.

• Implies that quantum deformed area $\sim \sqrt{[n][n+2]}$ is not consistent with loop algebra and q spin net combinatorics.

• On the bigger picture: To be compelling, work must be completed on the definition of the loop transform (see Paternoga, Graham PRD 62 084005 for definition of triad transform)

References

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- Major, Smolin "Quantum Deformation of Quantum Gravity" Nuc. Phys. B 473 (1996) 267 **gr-qc/9512020**
- Borissov, Major, Smolin "The Geometry of Spin Networks" Class. Quant. Grav. 13 (1996) 3181 **gr-qc/9512043**
- Markopoulou, Smolin "Quantum Geometry with intrinsic local causality" Phys Rev D 58 (1998) 084032 **gr-qc/9712067**
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I am an old man now. Let me give you a little advice: Do not shy away from using deformation parameters that are roots of unity. Otherwise you miss the fun in life.

– J. Frölich