Relating LQC to LQG:

Sectors, embeddings and questions

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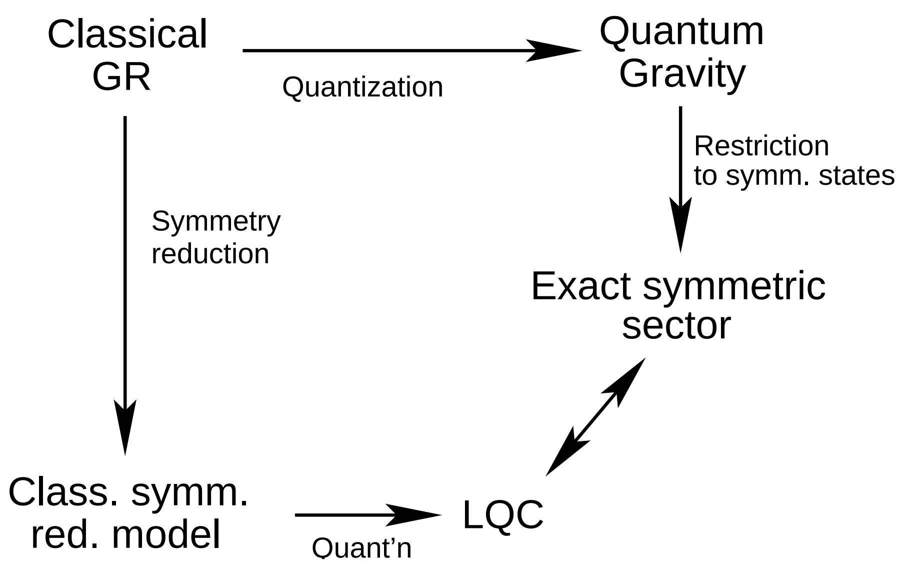
Outline

- Introduction & motivation
- Ways of imposing symmetry
- Recall lessons from KG theory
- ‘c’ symmetry in relating LQG to LQC
- ‘b’ symmetry in relating LQG to LQC
- Extension to gauge and diffeomorphism-invariant level
- Summary and questions

Introduction: basic motivating questions

1. What is the meaning of symmetry in QFT? That is, what is the "exactly symmetric sector" we are trying to approximate?
2. What is the meaning of specific *states* in LQC? Can we identify them with (symmetric) states in LQG? (Embedding)



3. Does LQC accurately model the (exactly) homogeneous and isotropic sector of LQG, physically selected in an appropriate way?

Note: We are not talking about inhomogeneities. That is a different question which can be asked meaningfully only in as much as the above questions are answered.

Elementary motivation for different approach to symmetry

Quantum analogues of classical conditions on states:

The analogue of $\mathcal{O} = \lambda$ is $\hat{\mathcal{O}}\Psi = \lambda\Psi$.

So we ask, given a symmetry group G with action on classical phase space:

> What is the quantum analogue of classical G-symmetry, in the above sense?

Suppose $\{C_i\}$ are classical constraints sufficient to isolate the classical symmetric sector. We propose to define the quantum analogue to be roughly

$$\{\Psi \mid \hat{C}_i \Psi = 0\}.$$

That is the idea.

This is different from invariance under the (induced) action of G on quantum states.

Additional condition: this symmetric sector should be chosen in such a way that it is preserved under dynamics (approximately when theory is interacting).

Ways of imposing symmetry via constraints, and corresponding embeddings

'c'-symmetry: symmetry imposed only on config. vars	$\longrightarrow$	$\iota_c :$	$\delta^S_{\mathring{\varphi}_S} \mapsto \delta_{\mathring{\varphi}_S}$
'b'-symmetry: symmetry imposed using 'annihilation ops.', so that symm. is imposed on both config. and momenta.	$\longrightarrow$	$\iota_b :$	$\psi^{coh,S}_{\xi_S} \mapsto \Psi^{coh}_{\xi_S}$

Lessons from KG theory:

'b'-symmetry is selected by the dynamics.

One has exact commutation of symmetry reduction and quantization at even the *dynamical* level with 'b'-symmetry (for KG case), and fluctuations from symmetry are minimized in a precise sense.

- For non-linear theories, it is expected only approximate commutation at the dynamical level will possible.
- With 'c'-symmetry, commutation at the dynamical level is not even approximate.
- 'Invariance' symmetry doesn't even give commutation at the *kinematical* level (in the KG case), does not have same control on fluctuations from symmetry, and is vacuous in quantum gravity anyway.

Notations for LQG and LQC

Choose action $\mathcal{E}$ of Euclidean group on $SU(2)$ principal fiber bundle over Σ

$\Rightarrow$ induces action on Σ

$\Rightarrow$ $\mathring{q}_{ab}$ unique upto scaling; fix one.

$$\begin{aligned}
\mathcal{A} &:= SU(2) \text{ connections over } \Sigma \\
\Gamma &:= T^*\mathcal{A} = \{(A_a^i, \tilde{E}_i^a)\} \\
\mathcal{A}_{inv} &:= SU(2) \text{ connections over } \Sigma \text{ invariant under } \mathcal{E} \\
&\quad = \{c\mathring{\omega}\}_{c\in\mathbb{R}} \subset \mathcal{A} \\
\Gamma_S &:= T^*(\mathcal{A}_{inv}) \cong T^*\mathbb{R} = \{(c,p)\}
\end{aligned}$$

where $\mathring{\omega}$ may be chosen s.t. $\mathring{q}_{ab} = \mathring{\omega}_a^i \mathring{\omega}_{bi}$, and $\exists$ coord. system s.t. $\mathring{\omega}_a^i = \delta_a^i$.

$$\begin{aligned}
\mathrm{Cyl} &:= \text{functions on } \mathcal{A} \text{ dep. on finite \# of hol.s} \\
\underline{\mathrm{Cyl}} &:= \text{functions on } \mathcal{A} \text{ dep. on finite \# of hol.s along straight edges} \\
\mathrm{Cyl}_S &:= \text{functions on } (\mathbb{R} \cong \mathcal{A}_{inv}) \text{ dep. on finite \# of exponentials } e^{\frac{i\mu c}{2}} \\
&= \text{space of almost periodic functions}
\end{aligned}$$

‘c’-symmetry in LQG

Sector

$$\mathcal{V}_c := \text{distributions in Cyl}^* \text{ that vanish outside of } \mathcal{A}_{inv}.$$

All $\Psi \in \mathcal{V}_c$ satisfy

$$(g \cdot \hat{A})(e)\Psi = \hat{A}(e)\Psi \quad \forall g \in \mathcal{E}, e.$$

Embedding

$r : \mathbb{R} \longrightarrow \mathcal{A},\ c \mapsto c\mathring{\omega}$

Define $\iota_c : \mathrm{Cyl}^*_S \longrightarrow (\underline{\mathrm{Cyl}})^*$ by

$$\iota_c[\delta_{c'}] := \delta_{r(c')} \quad \forall c' \in \mathbb{R}.$$

Explicitly, for general $\big(\psi\big| \in \mathrm{Cyl}^*_S$,

$$\iota_c\Big[\big(\psi\big|\Big]\big|\Phi\big\rangle := \big(\psi\big|r^*\Phi\big\rangle$$

$\mathrm{Im}\,\iota_c \approx \mathcal{V}_c$

Intertwining: $F(\hat{A}) \circ \iota_c = \iota_c \circ F\left(r\left(\hat{c}\right)\right), \qquad \forall F \in \underline{\mathrm{Cyl}}$

‘b’-symmetry in LQG

Structures needed

$(A^{\mathbb{C}})^i_a,\ (\hat{A}^{\mathbb{C}})^i_a$	—	$SL(2,\mathbb{C})$ connection, complex coordinate on $\Gamma = T^*\mathcal{A} \cong \mathcal{A}^{\mathbb{C}}$; annihilation operator.
$a,\ \hat{a}$	—	complex coord. on $\Gamma_S = \mathbb{R}^2 \cong \mathbb{C}$; annihilation operator.

Corresponding coherent states $\{\Psi^{coh}_\xi\}$, $\{\psi^{coh}_{\xi_S}\}$.

Sector

$$\mathcal{V}_b :\approx \text{span}\{\Psi^{coh}_\xi\}_{\xi\in\Gamma_S}$$

close in weak-star topology?

All $\Psi \in \mathcal{V}_b$ satisfy

$$(g\cdot \hat{A}^{\mathbb{C}})(e)\Psi = \hat{A}^{\mathbb{C}}(e)\Psi \quad \forall g \in \mathcal{E}, e.$$

‘b’-symmetry in LQG

Embedding

Define $\iota_b : \mathrm{Cyl}^*_S \longrightarrow \underline{\mathrm{Cyl}}^*$ by

$$\iota_b : \psi^{coh}_{\xi_S} \mapsto \Psi^{coh}_{\xi_S}.$$

Explicitly, for general $\big(\psi\big| \in \mathrm{Cyl}^*_S$,

$$\iota_b\Big[\big(\psi\big|\Big]\big|\Phi\big\rangle := \big(\psi\big|\pi\Phi\big\rangle$$

where π is similar to r^*, but defined using holomorphic representations.

$\mathrm{Im}\,\iota_b \approx \mathcal{V}_b$

Intertwining:

Define $\beta : \mathbb{C} \longrightarrow \mathcal{A}^{\mathbb{C}}$ by

$$\beta(a[\xi_S]) = A^{\mathbb{C}}[\xi_S]$$

so that β is the representation of the inclusion map $\Gamma_S \hookrightarrow \Gamma$ in the coords a and $A^{\mathbb{C}}$.
Then

$$F(\hat{\mathcal{A}}^{\mathbb{C}}) \circ \iota_b = \iota_b \circ F(\beta(\hat{a})), \qquad F \text{ hol. and cyl. along str. edges}$$

Extension to gauge and diffeomorphism invariant levels

Gauss: Can define $P_G : \mathrm{Cyl}^* \to (\mathrm{Cyl}^*)_{Gauss-inv.}$:

$$
\begin{aligned}
P_G\Big[\big(\Psi\big|\Big]\big|\Phi\big\rangle &:= \left(\int_{g\in\mathcal{G}} \mathcal{D}g U^*_{g^{-1}}\Big[\big(\Psi\big|\Big]\right)\big|\Phi\big\rangle \\
&= \int_{g\in\mathcal{G}} \mathcal{D}g\big(\Psi\big|U_{g^{-1}}\big|\Phi\big\rangle = \int_{g\in\mathcal{G}} \mathcal{D}g\big(\Psi\big|U_g\big|\Phi\big\rangle \\
&= \big(\Psi\big|P_G\big|\Phi\big\rangle
\end{aligned}
$$

For $P_G : \underline{\mathrm{Cyl}}^* \to (\underline{\mathrm{Cyl}}^*)_{Gauss-inv.}$, derivation and final formula similar.

$$
\mathcal{V}_c^G := P_G\big[\mathcal{V}_b\big] \qquad \mathcal{V}_b^G := P_G\big[\mathcal{V}_b\big] \tag{1}
$$

$$
\iota_c^G := P_G \circ \iota_b \qquad \iota_b^G := P_G \circ \iota_b \tag{2}
$$

Diffeo:

1. Definition of $P_{Diff} : \mathrm{Cyl}^* \to (\mathrm{Cyl}^*)_{Diff-inv.}$ more difficult.
2. In trying to derive $P_{Diff} : \underline{\mathrm{Cyl}}^* \to (\underline{\mathrm{Cyl}}^*)_{Diff-inv.}$, cannot even write down 1st formal line b/c diffeo group does not act on $\underline{\mathrm{Cyl}}^*$!

Summary

- goal clarified: we ultimately want a "sector" approximately preserved by the dynamics that is appropriately symmetric.
- 'c' and 'b' symmetric sectors of LQG defined, each with corresponding embedding of LQC (of limited, but precise well-definedness).
- Intertwining of certain operators: Consistency of both ι_c and ι_b with existing LQC quantization.
- Extension to $SU(2)$-gauge invariant level: easy
- Extension to Diffeo invariant level: hard. If an embedding of LQC is desired, furthermore, there is a problem with straight edges.

Questions:

1. Type of closure needed in defining 'b'-symmetric sector?
2. Try to define diff-invariant ('c' and) 'b' symmetric sectors.
3. Approximately dynamical 'b'-symmetric sector? Approx. dyn. coherent states in LQG would achieve this. General theory?

If these can be sufficiently answered, one will at least have a proposal for the *exact, physical symmetric sector of LQG.*

3. Can the constructions of the 'c' and 'b' embeddings presented here be extended to this exact physical symmetric sector?

Holomorphic representations and π

$\mathfrak{Z}^i_a := \overline{(A^{\mathbb{C}})^i_a}$ — $SL(2,\mathbb{C})$ connection, complex coordinate on Γ.

$z := \overline{a}$ — complex coordinate on Γ_S.

$$(U\Phi)(\mathfrak{Z}[\xi]) := \left(\Psi^{coh}_{\xi}\middle|\Phi\right\rangle$$
$$(U_S\phi)(z[\xi_S]) := \left(\psi^{coh}_{\xi_S}\middle|\phi\right\rangle$$

Define $s : \mathbb{C} \to \mathcal{A}^{\mathbb{C}}$ by

$$s(z[\xi_S]) = \mathfrak{Z}[\xi_S].$$

so that s is the representation of the inclusion map $\Gamma_S \hookrightarrow \Gamma$ in the complex coordinates z and $\mathfrak{Z}$. Then

$$\boxed{\pi := U_S^{-1} \circ s^* \circ U}$$

$$U \quad : \quad \underline{\mathrm{Cyl}} \to \underline{\mathrm{Cyl}}^{hol}(\mathcal{A}^{\mathbb{C}})$$

$$s^* \quad : \quad \underline{\mathrm{Cyl}}^{hol}(\mathcal{A}^{\mathbb{C}}) \to \mathrm{Cyl}^{hol}_S(\mathbb{C})$$

If s is holomorphic and complex coords are chosen appropriately!

$$U_S^{-1} \quad : \quad \mathrm{Cyl}^{hol}_S(\mathbb{C}) \to \mathrm{Cyl}_S$$

Thus

$$\boxed{\pi : \underline{\mathrm{Cyl}} \to \mathrm{Cyl}_S}$$