Quantization of string-like sources coupled to BF theory : transition amplitudes and topological invariance

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Outline

- Introduction
- Geometrical interpretation
- Quantization : physical inner product
- Topological invariance

• 2 + 1 gravity : Particle worldline $\gamma \equiv$ local conical singularity in spacetime curvature

$$F[A] = p\,\delta_\gamma \qquad (1)$$

• How to extend the same idea to higher dimensions ?

$\Rightarrow$ Replace particles by $(d-3)$-branes

(Baez, Wise, Crans - 06; Baez, Perez - 06)

• Framework : d-dimensional BF theory - compact, semi-simple structure group G

-M : d-dimensional spacetime manifold

-(A, B) : $\mathfrak{g}$-valued fields on M (connection one-form and $(d-2)$-form respectively)

$$S_{BF} = \int_M \mathrm{tr}\,(B \wedge F[A]) \qquad (2)$$

• Idea :

- Fix $W \subset M \equiv (d-2)$-dimensional worldsheet embedded into M
- Put a $\mathfrak{g}$-valued $(d-3)$-form q and a G-valued function λ on W, fix a constant unit element $v \in \mathfrak{g}$ and define the 'momentum density' $p = \tau Ad_\lambda(v) \in C^\infty(W, \mathfrak{g})$
- Add the following term to the free action

(de Sousa Gerbert - 90; Baez, Perez - 06)

$$S = \int_M \mathrm{tr}\,(B \wedge F[A]) - \int_W \mathrm{tr}\,((B + d_A q)\,p)\,, \qquad (3)$$

$\rightarrow$ Equations of motion

$$\begin{aligned} F[A] &= p\,\delta_W \quad , \quad & d_A B &= [p,q]\,\delta_W \\ d_A p &= 0 \quad , \quad & d_A q &= -B \end{aligned}$$

• Symmetries :

-standard YM gauge symmetries

$$\forall g \in C^\infty(M,G), \qquad \begin{array}{rcl} B & \mapsto & B = gBg^{-1} \\ A & \mapsto & A = gAg^{-1} + gdg^{-1} \\ \lambda & \mapsto & g\lambda \\ q & \mapsto & gqg^{-1} \end{array} \tag{4}$$

-'topological' transformations

$$\forall \eta \in \Omega^{d-3}(M,\mathfrak{g}), \qquad \begin{array}{rcl} B & \mapsto & B + d_A\eta \\ A & \mapsto & A \\ \lambda & \mapsto & \lambda \\ q & \mapsto & q - \eta \end{array} \tag{5}$$

Geometrical interpretation

In this section, $d = 4$ and $G = \mathrm{SO}(\eta)$ $(\eta = (\pm, +, +, +))$ with $V_\eta = \mathbb{R}\{e_I\}_I$ the vector representation space of G

• Two questions :

-What is the physical meaning of the algebraic variables λ and q ?

-Does the theory relate to solutions to GR ?

$\longrightarrow$ Physical interpretation : Matter on flat backgrounds

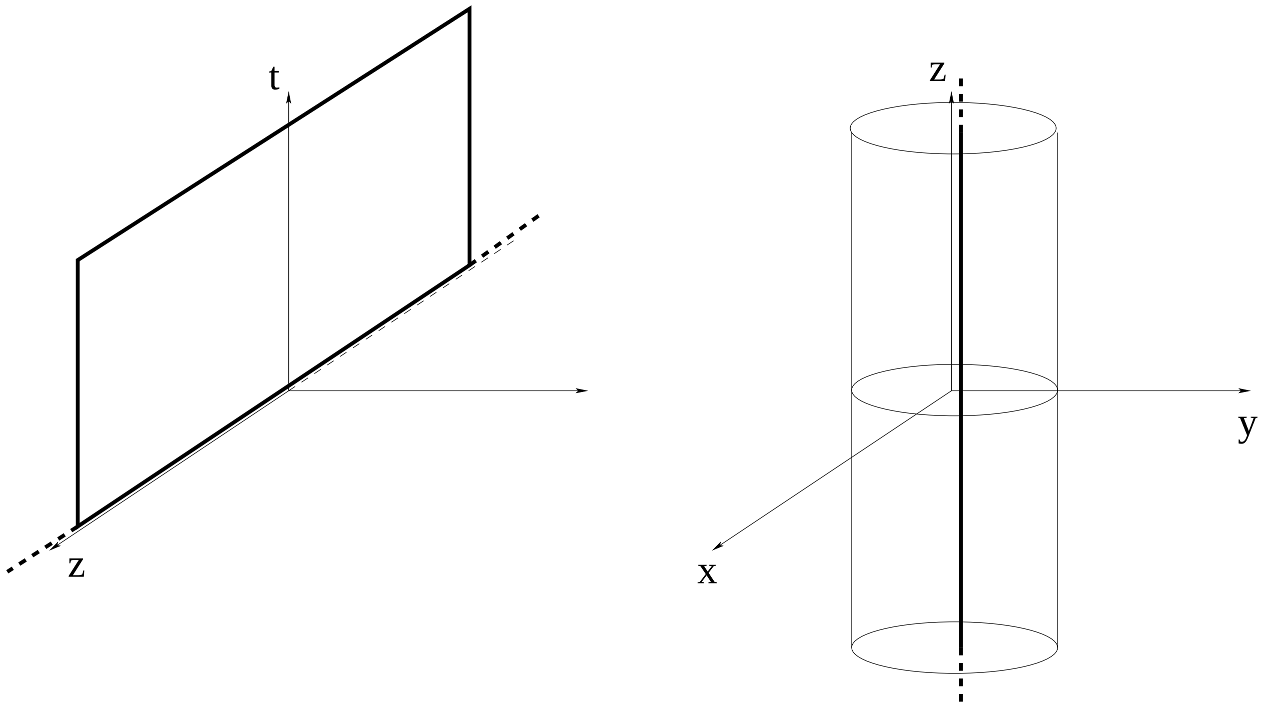
(cf Alejandro's talk)

→ Geometrical interpretation : Cosmic string solutions

-Cosmic string ≡ infinitely long and thin, straight string

-Let $\mathscr{S}$ be a spinless cosmic string with mass per unit length τ

-In local cylindrical coordinates centered on $\mathscr{S}$ in which the string lies along the z axis,



spacetime is described by the dual co-frame $e^I = e^I_\mu dx^\mu$

$$\begin{aligned} e^0 &= dt \\ e^1 &= \cos\varphi dr - 4\tau r \sin\varphi d\varphi \\ e^2 &= \sin\varphi dr + 4\tau r \cos\varphi d\varphi \\ e^3 &= dz, \end{aligned} \tag{6}$$

and the connection

$$A = A^{IJ}_\mu \sigma_{IJ} dx^\mu = 4\tau\, \sigma_{12}\, d\varphi \tag{7}$$

-The associated spacetime curvature $F = F^{12}\sigma_{12}$ is singular at the location of the string worldsheet :

$$F^{12} = 8\pi\tau\delta^2(r) dx \wedge dy \tag{8}$$

-This cosmic string solution can be generated by the Palatini action (+ interaction)

$$S = \int_M \mathrm{tr}\,((*e \wedge e) \wedge F[A]) - \tau \int_W \mathrm{tr}\,((*e \wedge e)\sigma_{12}) \tag{9}$$

or equivalently by the Plebanski-like action

$$S = \int_M \mathrm{tr}\,(B \wedge F[A]) - \tau \int_W \mathrm{tr}\,(Bv) + \frac{1}{2}\int_M \mathrm{tr}\,(B \wedge \Phi(B)) \tag{10}$$

where we have set $v = \sigma_{12}$

⇒ This is the action of string-like sources coupled to BF theory (in a particular gauge) augmented by the Plebanski term

Quantization : physical inner product

- Canonical analysis : $M = \Sigma \times \mathbb{R}$ - $\mathscr{S} = \Sigma \cap W$

$\rightarrow$ Gauss law : $G_i = \epsilon^{abc} D_a B_{bci} + \int_{\mathscr{S}} \dot{x}^a [q_a, p]_i \delta_{\mathscr{S}} = 0$

$\rightarrow$ Curvature constraint : $C_i^a = \epsilon^{abc} F_{bci} - \int_{\mathscr{S}} \dot{x}^a p_i \delta_{\mathscr{S}} = 0$

- Canonical quantization of the kinematics:

$\rightarrow \mathcal{H}_{kin} = \mathbb{C}\{\Psi_\alpha\}_\alpha$

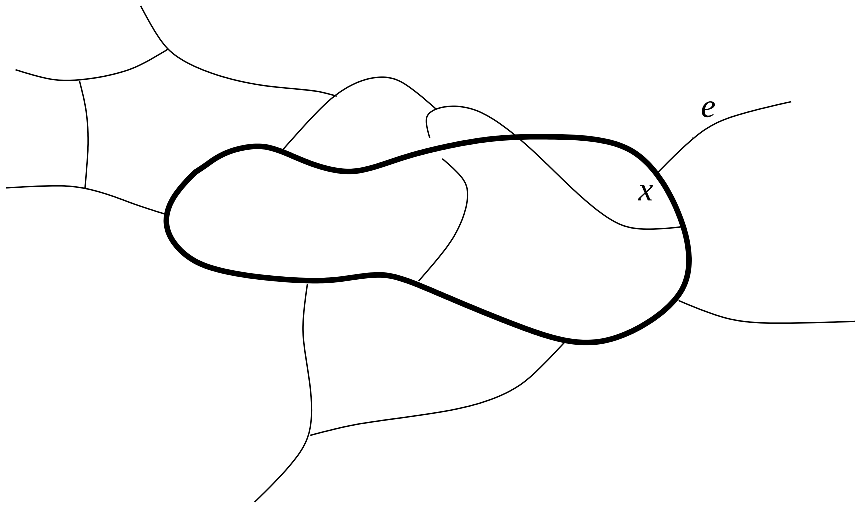
$\Psi_\alpha \equiv$ string spin networks (SSN) (Thiemann - 97; Baez, Perez - 06)

- SSN : Open graph Γ - finite set of points X on $\mathscr{S}$

-Edges and endpoints $\rightarrow$ unitary, irreducible representations ρ of G

-Vertices (including endpoints) $\rightarrow$ intertwining operators ι

$$\Psi_{\Gamma,X}[A,\lambda] = \left[\bigotimes_{e\in\Gamma} \rho_e[g_e] \bigotimes_{x\in X} \rho_x[\lambda_x]\right] \cdot \bigotimes_{v\in\Gamma} \iota_v, \tag{11}$$



- Kinematical inner product $< \Psi_{\Gamma,X}, \Psi'_{\Gamma,X} >$: AL-measure

$\rightarrow$ Haar integrals assigned to the edges $(\prod_e \int_G dg_e)$ and endpoints $(\prod_x \int_G d\lambda_x)$ of (Γ, X)

• Physical inner product : Formal definition

→ Introduce the rigging map (Rovelli,Reisenberger - 97)

$\eta_{\text{phys}} : \text{Cyl} \to \text{Cyl}^*;\ \Psi \mapsto \delta(\hat{C})\Psi$

$\Rightarrow \eta_{\text{phys}}(\text{Cyl}) = \text{Cyl}^*_{\text{phys}} \subset \text{Cyl}^*$

with $\text{Cyl}^*_{\text{phys}}$ $\equiv$ vector space of solutions to the curvature constraint

→ Physical inner product :

$$< \eta_{\text{phys}}(\Psi_1), \eta_{\text{phys}}(\Psi_2) >_{\text{phys}} \;=\; [\eta_{\text{phys}}(\Psi_2)](\Psi_1) \tag{12}$$

$$= \;< \Psi_1,\, \delta(\hat{C})\, \Psi_2 > \tag{13}$$

• Regularization of the physical inner product

→ make sense of $\delta(\hat{C})$

$$\delta(\hat{C}) = \prod_{x\in\Sigma} \delta(\hat{C}(x)) = \int_{\mathcal{N}} \mathcal{D}N \, \exp\left(i \int_{\Sigma} \text{tr}(N \wedge \hat{C}) \right) \tag{14}$$

→ Interesting duality :

$$\int_\Sigma \mathrm{tr}(N \wedge C) = \int_\Sigma \mathrm{tr}(N \wedge F) + \int_{\mathscr{S}} \mathrm{tr}(Np) \quad (15)$$

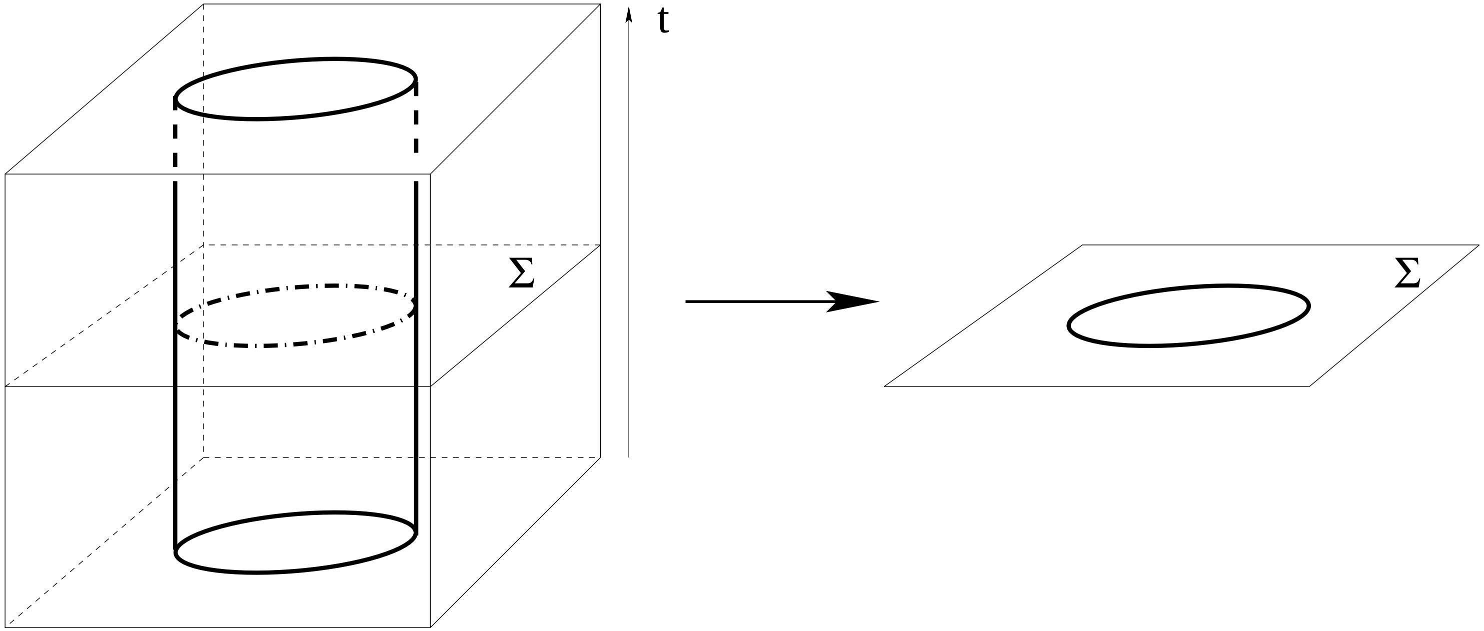
$$= S^{3d}_{\mathrm{BF+particle}}, \quad (16)$$

⇒ Action of 3d GR coupled to a point particle (Freidel, Louapre - 04)
Introducing the linear form $P : \mathrm{Cyl} \to \mathbb{C};\ \Psi \mapsto <\Omega,\ \delta(\hat{C})\Psi>$, we have furthermore ($d > 3$)

$$P(\Omega)^d_{\mathrm{BF}+(d-3)-\mathrm{branes}} = \mathcal{Z}^{d-1}_{\mathrm{BF}+(d-4)-\mathrm{branes}} \quad (17)$$

where $\mathcal{Z}^{d-1} \equiv$ path integral of $(d-1)$-dimensional BF theory coupled to branes

Quantization : physical inner product



⇒ Transition amplitudes of 4d BF theory coupled to strings dual to evaluations of Feynman loops coupled to 3d quantum gravity

(Barrett - 04, 05; Freidel, Livine - 05)

• Regularization of $P[\Psi_\Gamma]$ - (Very!) heuristic idea

-Pick a cellular decomposition $\mathcal{T}_\epsilon^* = (v^*, e^*, f^*)$ of Σ adapted to Γ, i.e., s.t. $\Gamma \subset \partial f^* \subset \mathcal{T}_\epsilon^*$

-Impose $F = 0$ ($g_{f^*} = \mathbb{1}$) around all two-cells f^* except for the two-cells circling the string where $F = p$ ($g_{f^*} = \exp p$)

-The physical inner product yields $P[\Psi] = \lim_{\epsilon \to 0} P[\mathcal{T}_\epsilon^*; \Psi]$, with

$$P[\mathcal{T}_\epsilon^*; \Psi] =< \Omega \, , \left[\prod_{f^* \notin \mathscr{S}} \hat{\delta}(g_{f^*}) \prod_{f^* \in \mathscr{S}} \hat{\delta}(g_{f^*} \exp p) \right] \Psi > \qquad (18)$$

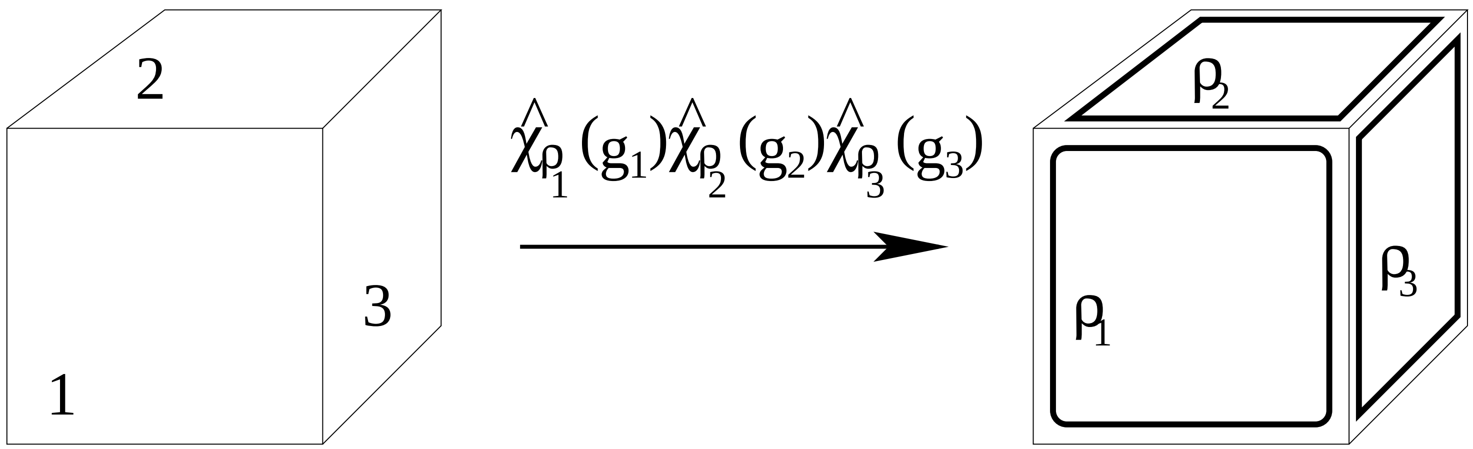
-Remark : There are in fact many subtleties (reducibility of the constraints ($D_a C_i^a = 0$), presence of open SN edges...)

$\longrightarrow$ The full regulator R_ϵ is not simply the cellular complex $\mathcal{T}_\epsilon^*$; it contains other structures (a maximal tree etc...)

-Each delta function can then be given an operational meaning

$\rightarrow$ Peter-Weyl decomposition $\delta(g) = \sum_\rho \dim(\rho)\chi_\rho(g)$

$\rightarrow$ $\hat{\chi}_\rho(g)$ self-adjoint Wilson loop operator on $\mathcal{H}_{kin}$



• How to remove the regulator R_ϵ ?

$\longrightarrow$ We show that

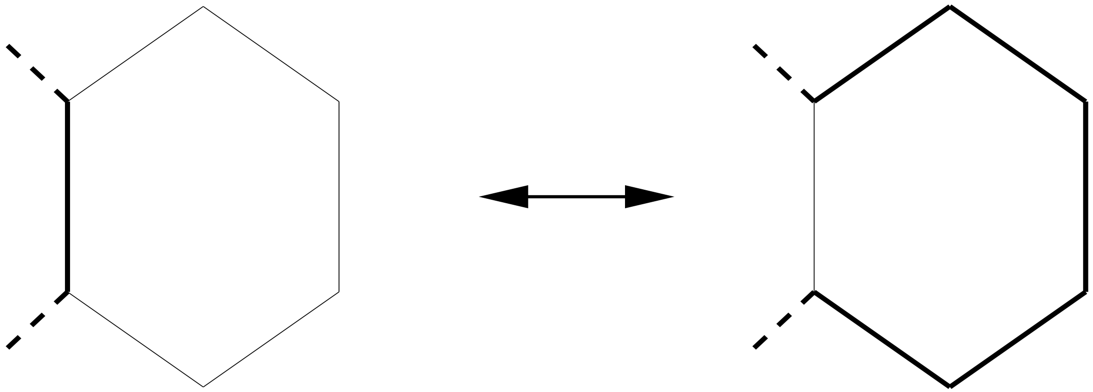
$$\forall \Psi \in \mathcal{H}_{kin}, \quad P[R_\epsilon ; \Psi] = P[R'_\epsilon ; \Psi] \tag{19}$$

where,

$R_\epsilon \mapsto R'_{\epsilon'}$ finite combination of discrete moves acting on each component of the full regulator R_ϵ (3d Pachner moves, elementary tree moves ...)

• Furthermore, we can show the invariance of the physical inner product under elementary SN edge moves

(map PL-paths into ambient isotopic PL-paths)



⇒ Putting all this together, we can show that

$$\forall \Psi_\Gamma \in \mathcal{H}_{kin}, \quad P[R_\epsilon;\Psi_\Gamma] = P[[\Sigma];\Psi_{[\Gamma]}] \tag{20}$$

where $[\Sigma]$ and $[\Gamma]$ denote the equivalence classes of topological manifolds and one-complexes up to homeomorphisms and ambient isotopy respectively

• The road towards a clear physical picture of the topological models discussed here is open

• The physical inner product between any two states can be explicitly computed

• The transition amplitudes are topological invariants of the canonical manifold together with the embedded string spin networks

• To do :
-Write the covariant model
-Re-express the amplitudes as Feynman diagrams of a QFT (String field theory ?)
-Add the Plebanski term (contact with conventional ST ?)
- ...