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The cosmological constant in 3d gravity: towards a unified approach

June 28, 2007

Loops 07, Morelia, Mexico

ref: C. Meusburger, Geometrical (2+1)-gravity and the Chern-Simons formulation: grafting, Dehn twists, Wilson loop observables and the cosmological constant gr-qc/0607121, to appear in Commun. Math. Phys.

C. Meusburger, B.J. Schroers, in preparation

Contents:

① $\boxed{\text{3d gravity}}$

- Euclidean signature $\eta_E = \mathrm{diag}(1,1,1)$
- Lorentzian signature $\eta_L = \mathrm{diag}(1,-1,-1)$
- $\Lambda > 0$, $\Lambda = 0$, $\Lambda < 0$

geometrical viewpoint

- Einstein-Hilbert action

$$S_{EH} = \int_M e^a \wedge \left(d\omega_a + \tfrac{1}{2}\varepsilon_{abc}\,\omega^b \wedge \omega^c\right) + \frac{\Lambda}{6}\,\varepsilon_{abc}\, e^a \wedge e^b \wedge e^c$$

- solution of Einstein equations as quotients of model spacetimes $\mathbb{X}_{\Lambda,S}$

		$\boxed{\mathbb{X}_{\Lambda,S}}$	$\boxed{\mathrm{Isom}(\mathbb{X}_{\Lambda,S})}$
Euclidean	$\Lambda < 0$	$\mathbb{H}^3_\Lambda$	$SL(2,\mathbb{C})$
	$\Lambda = 0$	$\mathbb{E}^3$	$SU(2) \ltimes \mathbb{R}^3$
	$\Lambda > 0$	S^3_Λ	$SU(2) \times SU(2)$
Lorentzian	$\Lambda < 0$	dS^3_Λ	$SL(2,\mathbb{C})$
	$\Lambda = 0$	$\mathbb{M}^3$	$SL(2,\mathbb{R}) \ltimes \mathbb{R}^3$
	$\Lambda > 0$	AdS^3_Λ	$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

Chern-Simons viewpoint

- Chern-Simons action $S_{CS} = \int_M \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \rangle$ $\quad A = e^a P_a + \omega^a J_a$
- gauge group $H_\Lambda = \mathrm{Isom}(\mathbb{X}_\Lambda)$
- Lie algebra $\mathfrak{h}_\Lambda = \mathrm{Lie}\, H_\Lambda$: generators J_a, P_a, $a = 0,1,2$
  - Lie bracket $[J_a, J_b] = \varepsilon_{abc} J^c$ $\quad [J_a, P_b] = \varepsilon_{abc} P^c$ $\quad [P_a, P_b] = \Lambda\, \varepsilon_{abc} J^c$
  - Ad-invariant symmetric BLFs $\quad \langle J_a, J_b \rangle = 0$ $\quad \langle J_a, P_b \rangle = \eta_{ab}$ $\quad \langle P_a, P_b \rangle = 0$

    $s(J_a, J_b) = \eta_{ab}$ $\quad s(J_a, P_b) = 0$ $\quad s(P_a, P_b) = \Lambda\, \eta_{ab}$

phase space

on manifolds $\mathbb{R} \times S$: Hamiltonian formulation

phase space = { flat H_Λ-connections on S } / gauge transformati

- finite dimensional
- parametrised by holonomies along generators of $\pi_1(S)$
- physical observables: Wilson loops

Poisson structure:

- given by classical r-matrix for h_Λ
- particles $\hat{=}$ dual Poisson-Lie structure
  handles $\hat{=}$ Heisenberg double Poisson structure

$\Rightarrow$ depending on signature and sign of Λ:

1. different Lie groups H_Λ (gauge / isometry groups)
2. different Poisson-Lie structures (phase space)
3. different quantum groups (quantum symmetries)

$\Rightarrow$ • unified description for $\Lambda>0$, $\Lambda=0$, $\Lambda<0$ in which structural similarities apparent?

• Λ as a deformation parameter?

→ relate Lie groups H_Λ

→ relate Poisson-Lie structures

→ relate quantum groups

② Lie groups in 3d gravity – a unified description

(2A) unified description of the Lie algebras in 3d gravity

commutative ring $R_\Lambda = (\mathbb{R}^2, +, \cdot)$

- elements $a + \theta b$, $a, b \in \mathbb{R}$
- addition $(a+\theta b) + (c + \theta d) = (a+c) + \theta(b+d)$
- multiplication

$$(a+\theta b)\cdot(c+\theta d) = (ac + \Lambda bd) + \theta(ad+bc)$$

→ generalises construction of $\mathbb{C}$
→ formal parameter θ, $\theta^2 = \Lambda$

unified description of the gravity Lie algebras $\mathfrak{h}_\Lambda$

- 3d Lorentz/rotation algebra

$$[J_a, J_b] = \epsilon_{abc} J^c \qquad \kappa(J_a, J_b) = \eta_{ab}$$

- extend Lie bracket and Killing form bilinearly to R_Λ
- with identification $P_a = \theta J_a$

→ recover Lie bracket of $\mathfrak{h}_\Lambda$
→ recover the Ad-inv symmetric BLFs on $\mathfrak{h}_\Lambda$

$$\kappa = S + \theta \cdot \langle\, ,\rangle$$

⇒ gravity Lie algebras $\mathfrak{h}_\Lambda$:

$$\boxed{\begin{array}{l} \mathfrak{h}_\Lambda = su(2, R_\Lambda) \text{ for Euclidean signature} \\ \mathfrak{h}_\Lambda = sl(2, R_\Lambda) \text{ for Lorentzian signature} \end{array}}$$

2B

The unified description of the gravity Lie groups H_Λ

→ idea: use identification of $SL(2,\mathbb{R})$, $SU(2)$ with (pseudo) quaternions

(pseudo) quaternions $\mathbb{H}$

associative algebra over $\mathbb{R}$ with generators e_a, $a=0,1,2$ and relations

$$e_a \cdot e_b = \eta_{ab} + \varepsilon_{abc} e^c$$

Clifford algebra relations + Lie bracket

$$\mathbb{H} = \{ q_3 \cdot 1 + q^a e_a \mid q_3, q^a \in \mathbb{R} \}$$

group of unit (pseudo) quaternions

$$\mathbb{H}_1 = \{ q_3 \cdot 1 + q^a e_a \mid q_3, q^a \in \mathbb{R},\ q_3^2 + \eta_{ab} q^a q^b = 1 \}$$

Euclidean: $\mathbb{H}_1^E \cong SU(2)$ Lorentzian $\mathbb{H}_1^L \cong SL(2,\mathbb{R})$

unified description of the gravity Lie groups H_Λ

quaternions over ring R_Λ

$$\mathbb{H}(R_\Lambda) = \{ q_3 \cdot 1 + q^a e_a \mid q_3, q^a \in R_\Lambda \}$$

⇒ $\boxed{\text{Theorem}}$

The unit quaternions over the ring R_Λ form a group isomorphic to the gravity Lie groups H_Λ

$$H_\Lambda \cong \mathbb{H}_1(R_\Lambda) = \{ q_3 \cdot 1 + q^a e_a \mid q_3, q^a \in R_\Lambda,\ q_3^2 + \eta_{ab} q^a q^b = 1 \}$$

③ Application: Wilson loop observables

Wilson loop observables: conjugation invariant functions of holonomies along closed curves

identification $H_\Lambda \cong \mathbb{H}_1(R_\Lambda)$:

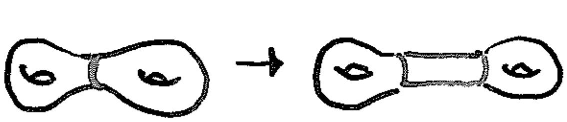
$\Rightarrow$ Wilson loop observables for gravity groups H_Λ from Wilson loop observables of rotation group/Lorentz group

$\Rightarrow$ canonical pair of Wilson loop observables as real and θ-component of R_Λ-valued observable

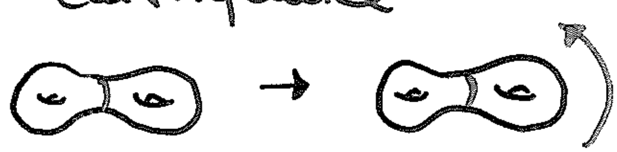
Theorem (Lorentzian $\Lambda>0, \Lambda=0, \Lambda<0$, Euclidean $\Lambda<0$)

The two canonical Wilson loop observables associated to closed, simple geodesic generate via the Poisson bracket the two fundamental geometry changing transformations: grafting and earthquake

grafting



earthquake



- real component $\hat{=}$ grafting $\rightarrow$ "momentum of geodesic"
  θ - component $\hat{=}$ earthquake $\rightarrow$ "angular momentum of geodesic"
- grafting = earthquake with formal parameter θ

(4) The unified description of the phase space via Poisson-Lie structures

Lorentzian $\Lambda>0, \Lambda=0, \Lambda<$
Euclidean $\Lambda<$

Poisson-structure given by classical r-matrix and associated Poisson-structures from theory of Poisson-Lie groups: particle ~ dual Poisson-Lie structure
handle ~ Heisenberg double

→ unified description of phase space from unified description of Poisson-Lie groups

classical r-matrices: $r = S_a \otimes J^a \quad S_a = P_a + \varepsilon_{abc} n^b J^c \quad n^2 = -\Lambda$
(classical doubles) $[J_a, J_b] = \varepsilon_{abc} J^c \quad [J_a, S_b] = \varepsilon_{abc} S^c - (n^b J_a - \eta_{a}$

"natural" coordinates:

classical double → factorisation of H_Λ

$$H_\Lambda \ni g = P(g)\cdot Q(g) \quad P \in \mathbb{H}_1,\ Q \in \mathbb{R} \ltimes \mathbb{R}^2$$

factorisation + identification $H_\Lambda = \mathbb{H}_1(R_\Lambda)$ ⇒ coordinates

$$P = \sqrt{1 - p^2/4} + p^a J_a \quad Q = \sqrt{1 + (qn)^2/4} + q^a S_a$$

dual Poisson-Lie structure and Heisenberg double

dual Poisson-Lie structure

$$\{q^a, q^b\} = \sqrt{1 + (qn)^2/4}\ \varepsilon^{abc} q_c$$
$$\{p^a, p^b\} = \sqrt{1 - p^2/4}\ (n^a p^b - n^b p^a)$$
$$\{q^a, p^b\} = \sqrt{1 + (qn)^2/4}\ \varepsilon^{abc} p_c - \sqrt{1 - p^2/4}\ (n^b q^a - \eta^{ab}(qn))$$

- "natural" coordinates common to H_Λ ($L, \Lambda \gtreqless 0$, $E\Lambda<0$) in terms of which Poisson structure of particularly simple form
- unified description of phase space in which Λ is a parameter (via n, $n^2 = \Lambda$)

5. Outlook and Conclusions

- unified descriptions of Lie groups in 3d gravity as unit (pseudo) quaternions over commutative ring

$$\left.\begin{array}{l} SL(2,\mathbb{C}) \\ SU(2)\ltimes\mathbb{R}^3 \\ SU(2)\times SU(2)\end{array}\right\} = \mathbb{H}_1^E(R_\Lambda) \qquad \left.\begin{array}{l} SL(2,\mathbb{C}) \\ SL(2,\mathbb{R})\ltimes\mathbb{R}^3 \\ SL(2,\mathbb{R})\times SL(2,\mathbb{R})\end{array}\right\} = \mathbb{H}_1^L(R_\Lambda)$$

- unified description of Lie bialgebra and Poisson-Lie structures in 3d gravity (Lorentzian $\Lambda>0$, $\Lambda=0$, $\Lambda<0$, Euclidean $\Lambda<0$)
  - classical r-matrices
  - factorisation
  - dual Poisson-Lie structure and Heisenberg double

$\Rightarrow$ Physical applications:
  - unified description of phase space and Poisson structure
  - "natural" coordinates
  - canonical Wilson loop observables with clear physical interpretation (Hamiltonians for geometry change via earthquakes and grafting)

open questions

- applications to quantisation?
- unified description of quantum groups in 3d gravity?
- "cosmological deformation" of quantum groups?
- limit $\Lambda\nearrow 0$ $\Lambda\searrow 0$?
- representation theory?