Canonical Quantization of Non-commutative Holonomies in 2+1 LQG

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Part I
The case of vanishing cosmological constant

- Purpose: quantization of 2+1 gravity, without matter, in euclidean signature, **with non-zero cosmological constant**.

- Method: Dirac program of quantization

- Previous work: Karim Noui, Alejandro Perez — "Three-Dimensional Loop Quantum Gravity: Physical Scalar Product and Spin-foam Models", 2004, gr-qc/0402110

- Noui & Perez solved the problem with **zero** cosmological constant. Following their approach, we shall see ourselves naturally led to the study of a certain **quantum group**.

- The Dirac quantization program means:
  - Find an "auxiliary" Hilbert space $\mathcal{H}_{aux}$ on which the phase space variables of the theory act as operators and promote $\{\cdot, \cdot\}$ to $-\frac{\mathrm{i}}{\hbar}[\cdot, \cdot]$
  - Promote the constraints of the theory to self-adjoint operators in $\mathcal{H}_{aux}$
  - Characterize the space of the solutions of the constraints and define on it an inner product in order to get a notion of physical probability. This will be the "physical" Hilbert space $\mathcal{H}_{phys}$
  - Find a complete set of gauge invariant observables (i.e., operators commuting with the constraints)

- For Euclidean General Relativity (expressed in connection variables), without cosmological constant, the action is

$$S\left(A,\, e\right) = \int_M \mathrm{Tr}\left[e \wedge F\left(A\right)\right]$$

  and the constraints are:

  - the Gauss constraint:

  $$G_i = D_a E^a_i$$

  - the vector constraints:

  $$V_a = E^b_i F^i_{ab}$$

  - the scalar constraint:

  $$S = \epsilon^{ij}_k E^a_i E^b_j F^k_{ab}$$

- Solving the quantized scalar constraint is difficult.
- Fortunately, in 2+1 dimensions, the vector and scalar constraints are equivalent to the curvature constraint:

$$F^i_{ab} = 0$$

- $\mathcal{H}_{aux}$ is the completion of the space of cylindrical functions.
- Solving the Gauss constraint leads to the “auxiliary” space $\mathcal{H}_{kin} \subset \mathcal{H}_{aux}$ of spin-network states.
- Solving the curvature constraint leads to solutions in the dual of $\mathcal{H}_{kin}$.

- The solutions of the curvature constraint are of the form Ps, where $s \in \mathcal{H}_{kin}$ and P is defined formally as

$$P = \prod_{x \in \Sigma} \delta\left(\hat{F}\,(A)\right) = \int_{su(2)} D\,[N]\ \mathrm{e}^{\mathrm{i} \int_{\Sigma} \mathrm{Tr}[N \cdot F(A)]}$$

- The physical scalar product will thus be

$$\langle s,\, s' \rangle_{phys} = \langle Ps,\, Ps' \rangle = \langle Ps,\, s' \rangle$$

and the rightmost term is defined by a regularization (of which it proves to be independent).

- In the process of constructing this regularization, one uses the fact that

$$U\left[A\right] = 1 + \epsilon^2\, F\left(A\right) + \mathcal{O}\left(\epsilon^2\right)$$

where the curvature is computed in a point and the holonomy is considered along a curve of diameter smaller than ϵ around that point.

- One obtains that

$$\langle s,\, s'\rangle_{phys} = \lim_{\epsilon \longrightarrow 0} \left\langle \prod_{p \in triangulation} \sum_{j_p} (2\, j_p + 1)\, \chi_{j_p}\left(U_p\right)\; s,\; s' \right\rangle$$

where the product is over all the plaquettes in the regularization and the sum over all spins.

Part II

The case of non-vanishing cosmological constant

- When a non-zero cosmological action is added, the action becomes

$$S\left(A,\, e\right) = \int_M \mathrm{Tr}\left[e \wedge F\left(A\right)\right] + \frac{\Lambda}{6}\mathrm{Tr}\left[e \wedge e \wedge e\right]$$

- The curvature constraints now become:

$$F^i_{ab}\left(A_\lambda\right) = 0$$

where

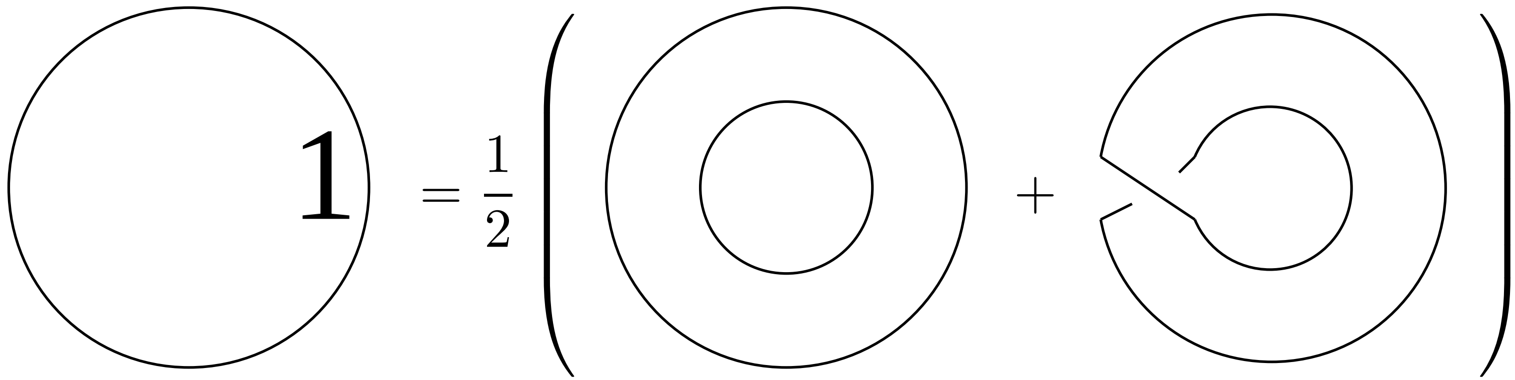
$$\left(A_\lambda\right)^i_a = A^i_a + \frac{1}{2}\lambda\, \epsilon_{ab} E^b_i$$

and $\lambda = \sqrt{\Lambda}$.

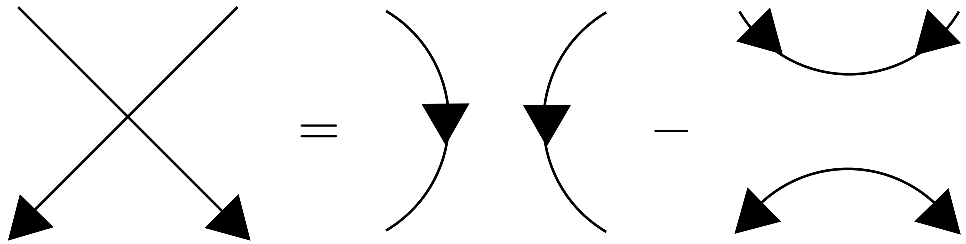
- In analogy to the case $\Lambda = 0$, we can use the formula

$$U\left[A_\lambda\right] = 1 + \epsilon^2\, F\left(A_\lambda\right) + \mathcal{O}\left(\epsilon^2\right)$$

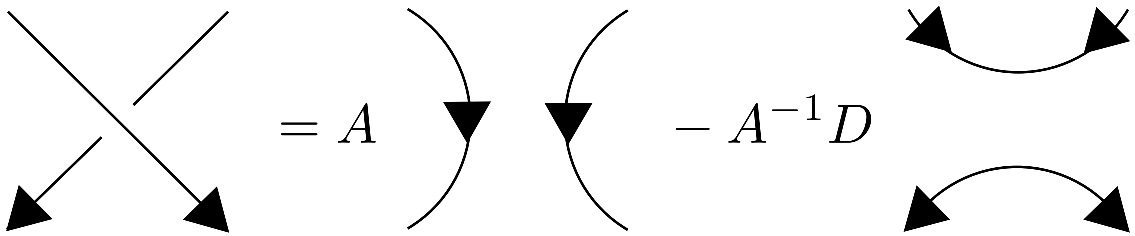
- What are the modifications induced by $\Lambda \neq 0$ to the theory?
- Consider a loop of spin 1. Diagramatically we can write:



- In the case $\Lambda = 0$, one deals with the $SU(2)$ representation theory, which is encoded in the **binor identity**:



- For $\Lambda \neq 0$, we get a **quantum** binor identity:



where $A = e^{2i\sqrt{\Lambda}}$ and $D = \frac{A^2 + A^{-2}}{2}$ (the quantum dimension).

- We consider the path-ordered expression of the holonomy:

$$h_\eta\left(A_\lambda\right) = 1 + \sum_{1\le n} (-1)^n \int_0^1 \mathrm{d}t_1 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n\, A_\lambda\left(t_1\right) \ldots A_\lambda\left(t_n\right)$$

- We **quantize** it by replacing A_a^i and E_i^b with their corresponding operators.

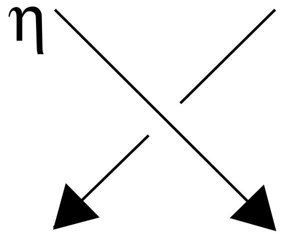
- It is easy to check that

$$h_\eta\left(A_\lambda\right)|0\rangle = h_\eta\left(A\right)|0\rangle = h_\eta\left(A\right)$$

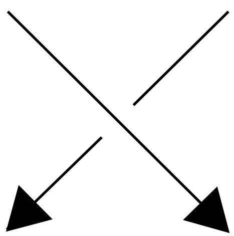
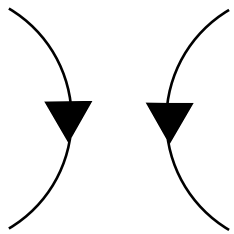
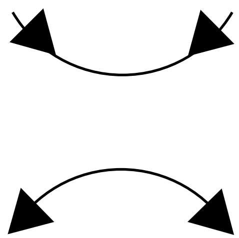
- Let now η act on some pre-existing γ. Formally, we want to study

$$h_\eta(A_\lambda) \triangleright h_\gamma(A_\lambda) = h_\eta(A_\lambda) \triangleright h_\gamma(A)$$

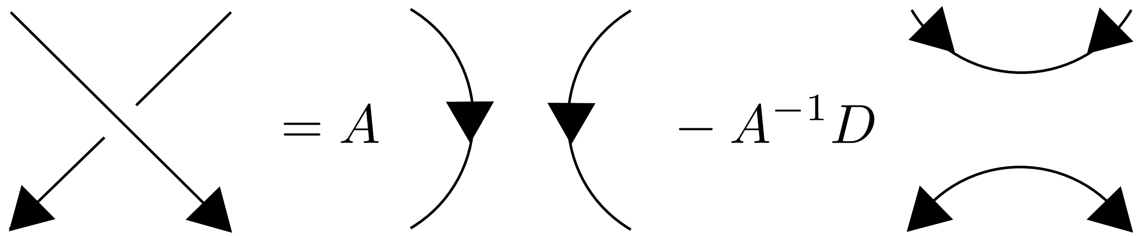
or, graphically,

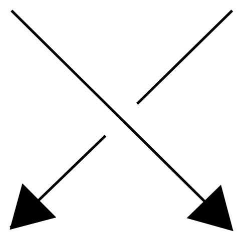


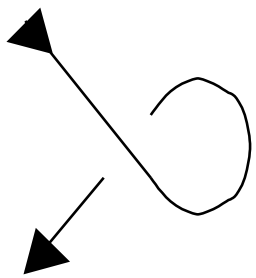
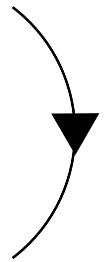
- The result is of the form

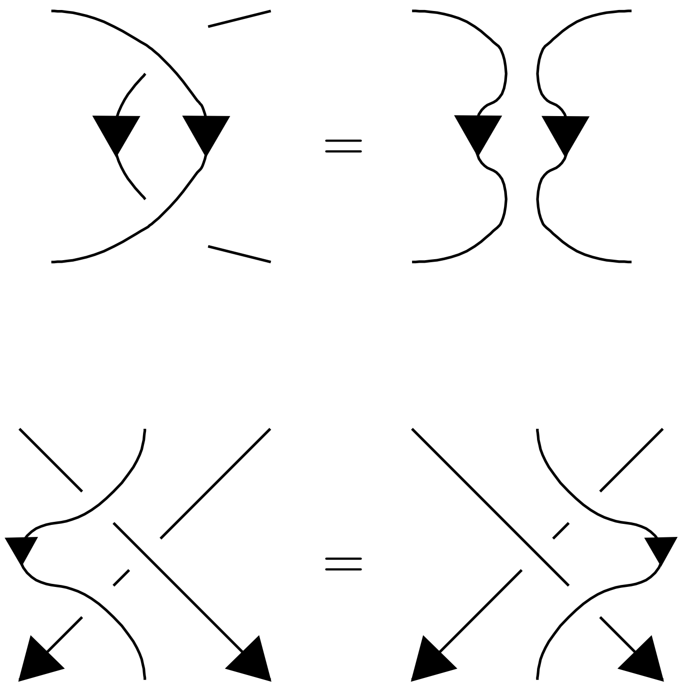
 $= A(\lambda)$  $+ B(\lambda)$ 

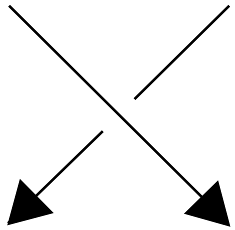
- By a reordering of certain products of matrices (analogue to the normal ordering in QFT) one gets



- The objects  satisfy the three Reidemeister moves:

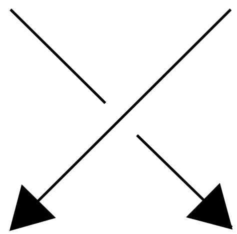
 $= \left(2A - A^{-1}D\right)$ 



- Let us denote by $R : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ the object  .

- In matrix form, it is:

$$R = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A - A^{-1}D & A^{-1}D & 0 \\ 0 & A^{-1}D & A - A^{-1}D & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

- By the second Reidemeister move, we get that  $=$ R^{-1}.

- With these notations, the third Reidemeister move can be written algebraically as

$$(R \otimes I)\,(I \otimes R)\left(R^{-1} \otimes I\right) = \left(I \otimes R^{-1}\right)(R \otimes I)\,(I \otimes R)\,,$$

  which is the braid equation.

- We can apply now the Faddeev-Reshetikhin-Takhtadjian construction and obtain a bi-algebra. An antipodal map can also be constructed, thus obtaining a Hopf algebra.