2-GROUPS AND TOPOLOGICAL ACTION

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NEW STUFF

THEOREM

Let Λ be a closed and oriented combinatorial d-manifold, $d \in \{3,4\}$, and $(G, H, \triangleright, t)$ be a Lie (finite) 2-group. The partition function

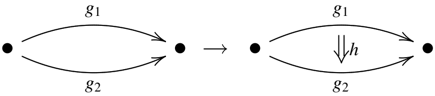
$$Z = |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\left(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\right)\left(\prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell}\right)$$
$$\times \left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(t(h_{jk\ell})g_{jk}g_{k\ell}g_{j\ell}^{-1}\big)\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(h_{j\ell m}h_{jk\ell}(g_{jk}\triangleright h_{k\ell m}^{-1})h_{jkm}^{-1}\big)\right).$$

is invariant under Pachner moves and therefore well defined on equivalence classes of combinatorial manifolds.

The g's decorate the edges, the h's decorate the faces. Λ_i set of i-simplices.

WHY 2-GROUPS?

- Generalization of lattice gauge theory:



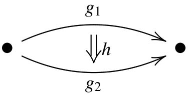
- Parallel transport of strings.
- Construct more sensitive topological invariant, *i.e.* information on π_2.
- Make sense of the topological symmetry $B \to B + d_A y$ (when B is a 2-form).

2-GROUPS IN A PEDESTRIAN WAY

[Baez, hep-th/0206130]

A *strict Lie (resp. finite) 2-group* is given by (G, H, t, α) where

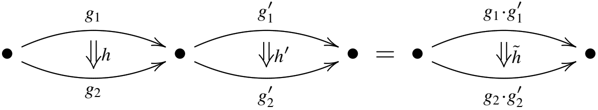
- G and H Lie (resp. finite) groups.
- $t : H \to G$ is a Lie (resp. finite) group morphism.
- $\alpha : G \times H \to H$ is an action, *i.e.* $\alpha(g)(h) \equiv g \triangleright h$.
- $t(g \triangleright h) = g\, t(h) g^{-1}, \quad t(h) \triangleright h' = hh'h^{-1}$.



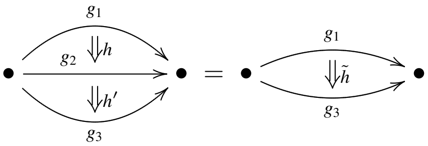
We really have in mind that h transport g_1 into g_2. t is called the target map

$$t(h)g_1 = g_2.$$

2-GROUPS MULTIPLICATIONS

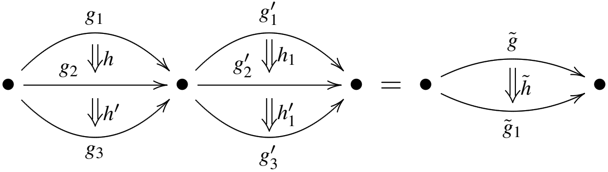


$$(h_1, g_1) * (h', g_1') = (h \cdot (g_1 \triangleright h'), g_1 g_1').$$



$$(h_1, g_1) \circ (h', g_1') = (h' \cdot h_1, g_1)$$

There are identity and inverse for each one.


g_1
h
g_2
h'
g_3
g_1'
h_1
g_2'
h_1'
g_3'
$=$
$\tilde{g}$
$\tilde{h}$
$\tilde{g}_1$

Examples

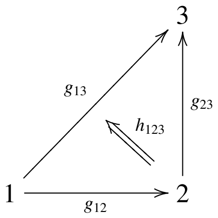
- *Poincaré 2-group:*
  $(H = \mathbb{R}^{n+d}, G = SO(n,d), t \equiv 1, \alpha \equiv$ action by rotation$)$.
- *Adjoint 2-group:*
  $(H = \mathcal{G}, G, t \equiv 1, \alpha \equiv$ adjoint action$))$.
- *Automorphism 2-group:*
  $(H, G = Aut(H), t : H \to Inn(H), \alpha \equiv$ action of the automorphisms$))$.
  $Inn(H)$ is set of automorphisms of H of the type $a_h(h') = hh'h^{-1}$

HOLONOMY AND 2-HOLONOMY

This structure allows to circumvent the Eckman-Hilton no go theorem [stating that any group decoration of surfaces need to be abelian].

Holonomy is given by

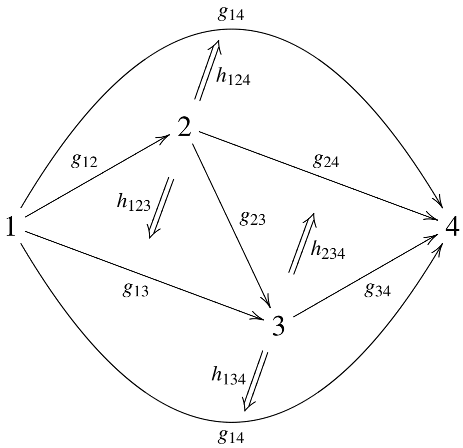
$$t(h_{123})g_{12}g_{23} = g_{13} \quad \Leftrightarrow \quad t(h_{123})g_{12}g_{23}g_{13}^{-1} = 1$$



Use the composition of morphisms to define the 2-holonomy.

2-holonomy from tetraedron:

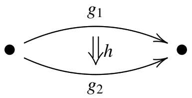
$$\tilde{h} = h_{134} h_{123} (g_{12} \triangleright h_{234}^{-1}) h_{124}^{-1}$$



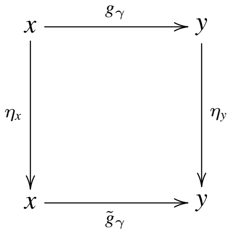
2-GAUGE TRANSFORMATIONS

[F. Girelli and H. Pfeiffer, hep-th/0309173]

Consider a 2-lattice gauge where the fundamental lattice is given by



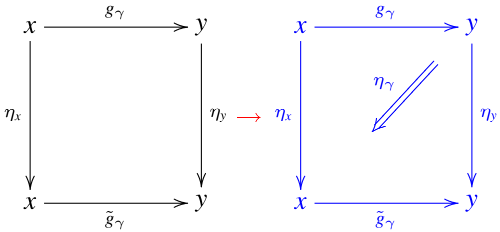
Usual gauge transformations: $\tilde{g}_\gamma = \eta_x^{-1} g_\gamma \eta_y$,



2-GAUGE TRANSFORMATIONS

Usual gauge transformations → 2-gauge transformation

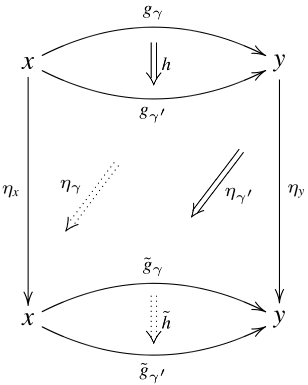
$$\tilde{g}_\gamma = \eta_x^{-1} g_\gamma \eta_y \quad \longrightarrow \quad \tilde{g}_\gamma = \eta_x^{-1} t(\eta_\gamma) g_\gamma \eta_y$$



2-GAUGE TRANSFORMATIONS

We have also a transformation for the faces:

$$\tilde{h} = \eta_x^{-1} \rhd \left(\eta_{\gamma'} \cdot h \cdot \eta_\gamma^{-1}\right)$$



LIE 2-ALGEBRAS

In the same way one can define a Lie 2-algebra. A *strict Lie (resp. finite) 2-group* is given by $(\mathcal{H}, \mathcal{G}, \tau, d\alpha)$ where

- $\mathcal{G}$ and $\mathcal{H}$ Lie Lie algebras.
- $\tau : \mathcal{H} \to \mathcal{H}$ is a Lie algebra morphism.
- $d\alpha : \mathcal{G} \times \mathcal{H} \to \mathcal{H}$ is an action, *i.e.* $d\alpha(X)(Y) \equiv X \triangleright Y, \forall X \in \mathcal{G}, Y \in \mathcal{H}$.
- $\tau(X \triangleright Y) = [X, \tau(Y)], \quad \tau(Y) \triangleright Y' = [Y, Y']$

$$\tau(Y) = X_2 - X_1, \;\; X_i \in \mathcal{G}, \;\; Y \in \mathcal{H}.$$

A Lie 2-group can be seen as the exponential of a Lie 2-algebra. [Baez, Crans, math/0307263]

DIFFERENTIAL PICTURE

We consider a 2-principal bundle [Baez, Schreiber, hep-th/0412325].
Connection $(A_\mu, B_{\mu\nu})$:

$$\begin{aligned} g_\mu(0) &\sim e^{iaA_\mu}, \\ h_{\mu\nu}(0) &\sim e^{ia^2 B_{\mu\nu}} \end{aligned}$$

Curvature $(\mathcal{F}_{\mu\nu}, G_{\alpha\mu\nu})$:

$$\begin{aligned} t(h_{12\ell})g_{23}g_{k\ell}g_{13}^{-1} &\rightarrow \mathcal{F} = dA + \tfrac{1}{2}[A,A] + \tau(B), \\ h = h_{134}h_{123}(g_{12} \triangleright h_{234}^{-1})h_{124}^{-1} &\rightarrow G = dB + A \triangleright B. \end{aligned}$$

But careful! We have by construction $\mathcal{F}_{\mu\nu} = 0$, since $t(h_{jk\ell})g_{jk}g_{k\ell}g_{j\ell}^{-1} = 1$.

2-GAUGE TRANSFORMATIONS

$$
\begin{aligned}
A &\mapsto A + \delta A, && \text{where} && \delta A = d_A(X) + \tau(Y) \\
B &\mapsto B + \delta B, && \text{where} && \delta B = d_A(Y) + X \triangleright B, \\
\mathcal{F} &\mapsto \mathcal{F} + \delta \mathcal{F}, && \text{where} && \delta \mathcal{F} = [\mathcal{F}, X], \\
G &\mapsto G + \delta G, && \text{where} && \delta G = \alpha \triangleright G.
\end{aligned}
$$

X is a scalar with value in $\mathcal{G}$. Y is a 1-form with value in $\mathcal{H}$. [To have well defined transformations, it is essential that $\mathcal{F} = 0$].

Note the topological symmetry!

EXAMPLE OF AN ACTION

Consider the adjoint 2-group $(H = \mathcal{G}, G, t = 1, \alpha_{adjoint})$.
Introduce Σ and C, resp. $(d-2)$-forms with value in $\mathcal{G}$ and $(d-3)$-form with value in $\mathcal{H}$.

$$\begin{array}{ccc} \Sigma \mapsto \Sigma + \delta\Sigma & \text{with} & \delta\Sigma = [\Sigma, X] \\ C \mapsto C + \delta C & \text{with} & \delta C = X \triangleright C \end{array}$$

$$\mathcal{S} = \int_M \mathrm{tr}_{\mathcal{G}}(\Sigma \wedge F_A) + \mathrm{tr}_{\mathcal{H}}(C \wedge G),$$

Equations of motion:

$$\begin{array}{rcl} d_A(\Sigma) + [B, C] & = & 0 \\ d_A(C) & = & 0 \\ F_A & = & 0 \\ G & = & 0 \end{array}$$

In the 3d case, this action can be interpreted as topological matter coupled to gravity ($\Sigma\Phi EA$ model). [R. B. Mann and E. M. Popescu, gr-qc/0607076]

PARTITION FUNCTION

Just as in the standard BF case, we can discretize the partition function

$$\mathcal{Z} = \int [\mathcal{D}C][\mathcal{D}B][\mathcal{D}A][\mathcal{D}\Sigma] e^{i\int_M \{Tr_{\mathcal{G}}\{\Sigma\wedge F\} + Tr_{\mathcal{H}}\{C\wedge G\}\}}$$
$$\downarrow$$
$$\mathcal{Z} = \int [\mathcal{D}A][\mathcal{D}B]\, \delta(F)\, \delta(G).$$

$$\begin{aligned} \delta(F) &\rightarrow \delta_G\left(g_{ij}g_{jk}(g_{ik})^{-1}\right), \\ \delta(G) &\rightarrow \delta_H\left(h_{j\ell m}h_{jk\ell}(g_{jk} \triangleright h^{-1}_{k\ell m})h^{-1}_{jkm}\right) \end{aligned}$$

$$\begin{aligned} \int [\mathcal{D}A] &\mapsto \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \\ \int [\mathcal{D}\Sigma] &\mapsto \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell}. \end{aligned}$$

$\Lambda_i \equiv$ set of i-simplicies.

TOPOLOGICAL INVARIANCE

THEOREM

Let Λ be a closed and oriented combinatorial d-manifold, $d \in \{3,4\}$, and $(G,H,\triangleright,t)$ be a Lie (finite) 2-group. The partition function

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$$\times \left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\left(t(h_{jk\ell})g_{jk}g_{k\ell}g_{j\ell}^{-1}\right)\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\left(h_{j\ell m}h_{jk\ell}(g_{jk}\triangleright h_{k\ell m}^{-1})h_{jkm}^{-1}\right)\right).$$

is invariant under Pachner moves and therefore well defined on equivalence classes of combinatorial manifolds.

OUTLOOK

- Using Lie 2-groups, we have constructed a topological invariant Some finite versions existed already *c.f.* work by Yetter, Porter, Mackaay.
- We need a 2-Peter-Weil theorem to construct the state sum in terms of 2-representations (mathematicians are working on it).
- Diagrammatic to show the topological invariance? let's ask Robert!
- What is the notion of 2-Quantum group to regularize the partition function?
- We should find application of this general scheme to models of interests, such as the ones dealing with strings presented by Alejandro and Winston...
- Is there a relation with Area metric introduced by F. Schuller and co?