Entanglement Entropy in Loop Quantum Gravity

William Donnelly

University of Waterloo

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Black Hole Entropy

Observers outside a black hole horizon see thermal radiation.

The entropy of a black hole of horizon area A is

$$S_{\mathrm{BH}} = \frac{A}{4}\frac{c^3}{\hbar G}$$

Entropy occurs also for *cosmological horizons* and *acceleration horizons*.

Problem: find a statistical description of this entropy

$$S_{\mathrm{BH}} = -\mathrm{Tr}(\rho \log \rho)$$

Entanglement Entropy

Consider a QFT on $\mathcal{M} = \mathbb{R} \times \Sigma$ and a state $|\psi\rangle \in \mathcal{H}_\Sigma$.

- For each $\Omega \subseteq \Sigma$, $\mathcal{H}_\Sigma = \mathcal{H}_\Omega \otimes \mathcal{H}_{\Omega^c}$
- Get a density matrix $\rho_\Omega = \mathrm{Tr}_{\mathcal{H}_{\Omega^c}} |\psi\rangle\langle\psi|$

Definition

The *entanglement entropy* of Ω is the von Neumann entropy of ρ_Ω

$$S_E(\Omega) \equiv S(\rho_\Omega) = -\mathrm{Tr} \rho_\Omega \log \rho_\Omega$$

Proposal: Black hole entropy is entanglement entropy [1]

$$S_{\mathrm{BH}} = S_E$$

[1] Bombelli, Koul, Lee, Sorkin. Phys. Rev. D 1986.

Entanglement Entropy in Loop Quantum Gravity

Plan: Compute S_E in loop quantum gravity,

- $\mathcal{H}_\Omega = \mathrm{Cyl}(\Omega)$, cylindrical functions of an $su(2)$ connection

$$\mathrm{Cyl}(\Omega) \equiv \{\Psi : \Psi(A) = f\left(U(A,\gamma_1), \ldots, U(A,\gamma_L)\right)\}$$

  Functions depending on finitely many holonomies.

- $|\psi\rangle$ a spin network state

The Schmidt Decomposition

Every state $|\psi\rangle \in \mathcal{H}_\Omega \otimes \mathcal{H}_{\Omega^c}$ has a *Schmidt decomposition*:

$$|\psi\rangle = \sum_{i \in \mathcal{I}} \sqrt{\lambda_i} \left|\psi_i^\Omega\right\rangle \otimes \left|\psi_i^{\Omega^c}\right\rangle$$

Where

- $\{|\psi_i^\Omega\rangle\}$ is an orthonormal set in $\mathcal{H}_\Omega$.
- $\left\{\left|\psi_i^{\Omega^c}\right\rangle\right\}$ is an orthonormal set in $\mathcal{H}_{\Omega^c}$.
- $\lambda_i > 0$ and $\sum_{i \in \mathcal{I}} \lambda_i = 1$.

The numbers $\{\lambda_i\}$ are called the *Schmidt coefficients*.
The number of elements in $\mathcal{I}$ is the *Schmidt rank*.

The Schmidt Decomposition

Suppose we know the Schmidt decomposition

$$|\psi\rangle = \sum_{i\in\mathcal{I}} \sqrt{\lambda_i} \left|\psi_i^{\Omega}\right\rangle \otimes \left|\psi_i^{\Omega^C}\right\rangle$$

Then we can compute the reduced density matrices in diagonal form

$$\rho_\Omega = \sum_{i\in\mathcal{I}} \lambda_i \left|\psi_i^{\Omega}\middle\rangle\middle\langle\psi_i^{\Omega}\right| \qquad \rho_{\Omega^C} = \sum_{i\in\mathcal{I}} \lambda_i \left|\psi_i^{\Omega^C}\middle\rangle\middle\langle\psi_i^{\Omega^C}\right|$$

Note: both reduced density matrices have the same nonzero spectrum.

The entanglement entropy is symmetric:

$$S_E(\Omega) = S_E(\Omega^C) = -\sum_{i\in\mathcal{I}} \lambda_i \log \lambda_i$$

Link states

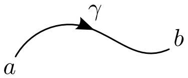
The *link state* $|\gamma, j, a, b\rangle$ is a matrix element of the holonomy of the connection A

$$\langle A|\gamma, j, a, b\rangle \equiv R^j(U(A,\gamma))^a_b$$

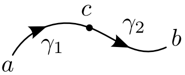
Split $\gamma = \gamma_1 \circ \gamma_2$, giving $\mathcal{H}_\gamma = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$

Insert a normalized identity intertwiner:

$$|\gamma, j, a, b\rangle = \frac{1}{\sqrt{2j+1}} \sum_{c=1}^{2j+1} |\gamma_1, j, a, c\rangle \otimes |\gamma_2, j, c, b\rangle$$



$$= \frac{1}{\sqrt{2j+1}} \sum_{c=1}^{2j+1}$$

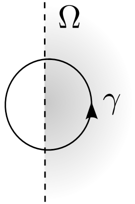


This is a Schmidt decomposition of $|\gamma, j, a, b\rangle$

Entanglement of Wilson loops

Let $|\gamma, j\rangle$ be a Wilson loop state for γ

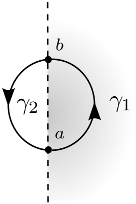
$$|\gamma, j\rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{a=1}^{2j+1} |\gamma, j, a, a\rangle$$



Add intertwiners at the boundary

$$|\gamma, j\rangle = \frac{1}{2j+1} \sum_{a,b=1}^{2j+1} \underbrace{|\gamma_1, j, a, b\rangle}_{\in \mathcal{H}_\Omega} \otimes \underbrace{|\gamma_2, j, b, a\rangle}_{\in \mathcal{H}_{\Omega^C}}$$

$$= \frac{1}{2j+1} \sum_{a,b=1}^{2j+1}$$



This is the Schmidt decomposition of $|\gamma, j\rangle$

$$S_E(\Omega) = 2 \log(2j+1)$$

Entanglement of Wilson loops

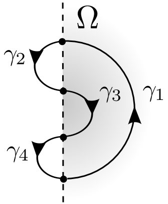
Suppose γ intersects $\partial\Omega$ at n points

$$|\gamma, j\rangle = \frac{1}{\sqrt{2j+1}^n} \sum_{a_1,\dots,a_n} |\gamma_1, j, a_1, a_2\rangle \otimes \cdots \otimes |\gamma_n, j, a_n, a_1\rangle$$

$$= \frac{1}{\sqrt{2j+1}^n} \sum_{a_1,\dots,a_n} \underbrace{\left(|\gamma_1, j, a_1, a_2\rangle \otimes \cdots\right)}_{\in \mathcal{H}_\Omega} \otimes \underbrace{\left(|\gamma_2, j, a_2, a_3\rangle \otimes \cdots\right)}_{\in \mathcal{H}_{\Omega^C}}$$

The Schmidt rank is $(2j+1)^n$, so

$$S_E(\Omega) = n \log(2j+1)$$



Entanglement entropy counts intersections of γ with $\partial\Omega$

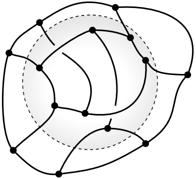
Entanglement of spin networks

Let $S = (\Gamma, j_l, i_n)$ be a spin network,

$$|S\rangle = \left(\bigotimes_n i_n\right) \circ \left(\bigotimes_l |\gamma_l, j_l, a_l, b_l\rangle\right)$$

Let $\mathcal{P}$ be the set of "punctures", insert identity intertwiner at each $p \in \mathcal{P}$:

$$|S\rangle = \frac{1}{\sqrt{N}} \sum_{a_p=1}^{2j_p+1} |S_\Omega, a_p\rangle \otimes |S_{\Omega^c}, a_p\rangle$$



The Schmidt rank is $N = \prod(2j_p + 1)$ and

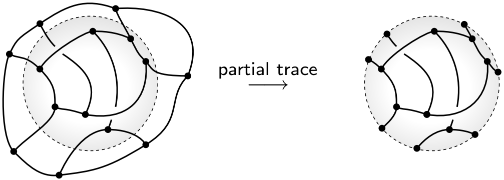
$$|S_\Omega, a_p\rangle \equiv \left(\bigotimes_{n\in\Omega} i_n\right) \circ \left(\bigotimes_{l\in\Omega} |\gamma_l, j_l, a_l, b_l\rangle\right)$$

Entanglement of spin networks

The entanglement entropy is

$$S_E(\Omega) = \sum_{p \in \mathcal{P}} \log(2j_p + 1)$$

The density matrix ρ_Ω is a gauge-invariant "mixed spin network state".



A pure spin network cannot have endpoints; a mixed spin network can.

Relation to Isolated Horizons

Suppose we treat the horizon as an inner boundary.

Construct a boundary space $\mathcal{H}_{\partial\Omega}$ such that

- For each $|S\rangle \in \mathcal{H}_\Omega \otimes \mathcal{H}_{\Omega^C}$ there exists $|S'\rangle \in \mathcal{H}_\Omega \otimes \mathcal{H}_{\partial\Omega}$
- $|S\rangle$ and $|S'\rangle$ agree on Ω:

$$\mathrm{Tr}_{\mathcal{H}_{\Omega^C}} |S\rangle\langle S| = \mathrm{Tr}_{\mathcal{H}_{\partial\Omega}} |S'\rangle\langle S'|$$

Then the state of the boundary is

$$\rho_{\partial\Omega} = \mathrm{Tr}_{\mathcal{H}_\Omega} |S'\rangle\langle S'|$$

This state is maximally mixed on a subspace of $\mathcal{H}_{\partial\Omega}$ with dimension

$$\mathrm{rank}(\rho_{\partial\Omega}) = \prod_{p\in\mathcal{P}} (2j_p + 1)$$

Relation to Isolated Horizons

The isolated horizon approach[2] has exactly such a Hilbert space

$$\mathcal{H}_{\mathrm{IH}} = \bigoplus_{\mathcal{P}} \underbrace{\mathcal{H}^{\mathcal{P}}_{\Omega}}_{\substack{\text{Open spin networks}\\ \text{ending at } \mathcal{P}}} \otimes \underbrace{\mathcal{H}^{\mathcal{P}}_{\partial\Omega}}_{\substack{\text{U(1) Chern-Simons}\\ \text{states on } \partial\Omega - \mathcal{P}}}$$

Trace over $\mathcal{H}^{\mathcal{P}}_{\Omega}$ gives $\rho_{\partial\Omega}$ maximally mixed on $\mathcal{H}^{\mathcal{P}}_{\partial\Omega}$

$$\dim(\mathcal{H}^{\mathcal{P}}_{\partial\Omega}) \sim \prod_{p\in\mathcal{P}} (2j_p + 1)$$

We get the same result without having to quantize an isolated horizon.

[2] Ashtekar, Baez, Corichi, Krasnov. Phys. Rev. Lett. 1998

Corrections to S_{BH}

For an arbitrary diffeomorphism-invariant Lagrangian, the classical black hole entropy is [3]

$$S = 2\pi \oint_{\partial\Omega} Q$$

Where Q is a Noether charge depending on the Lagrangian.

Open Question: Is there a quantity Q such that when quantized

$$\left(\widehat{2\pi \oint_{\partial\Omega} Q}\right) |S\rangle = \sum_{p\in\mathcal{P}} \log(2j_p + 1) |S\rangle$$

Knowing Q tells us corrections to the Lagrangian.

[3] Wald. Phys. Rev. D 1993.

Conclusion

Entanglement provides a quantum source for black hole entropy.

- It can be computed as a sum over punctures.
- It agrees asymptotically with results from isolated horizons.
- It applies to arbitrary horizons.

Open question:

- Does $S_E(\Omega)$ correspond to a geometric quantity?
- Can we use this to predict corrections to the gravitational action?