Quantum Evolution in an Expanding Hilbert Space

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Isabeau Prémont-Schwarz

Perimeter Institute and Univ. of Waterloo

In collaboration with: David Kribs and Fotini Markopoulou-Kalamara

OUTLINE

Introduction and Motivation

- **Motivation:** Existence of situations and models where Hilbert Space Dimension varies with spacial volume.
  - Naive model of one qubit per elementary volume in an expanding universe.
  - Quantum Black Holes with one qubit per elementary area during expansion or evaporation.
  - QFT where there is one Harmonic oscillator per spacetime point in an expanding universe.
- Problem: Find a framework for Quantum evolution when the Hilbert space's dimension increases, subject to the condition that there is no information loss (in the sense that information can always be recovered).

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- Either some extra information is created,
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Information Creation

- Create information such that the existing information is still recoverable: This is basically a Quantum correctible code.
- It can be shown (David Krips) that, in all generality, a quantum correctible code consists of:
  - Tensoring the existing state with some randomly initialised state.
  - Then applying a unitary transformation to the whole thing.

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Information Creation – Example

Simplest possible example: Add a randomly initialised qubit at every evolution step.

In other words ...

- One of the simplest possible example that can be thought of is simply adding one “bit" of information.
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Information Dilution – Basic Notation

- Let $\mathscr{H}_n$ be the Hilbert space at step n in the expansion process,
- $\{O_n^i\}_{i \in A_n}$ a complete set of operators on $\mathscr{H}_n$,
- $\mathscr{E}_n : \mathscr{H}_n \mapsto \mathscr{H}_{n+1}$ the evolution maps for the states,
- $\mathfrak{f}_n$, algebra homomorphisms, the evolution maps for the operators.

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Information Dilution – Conditions

Conditions for “Quasi-Unitarity"

1. There exists unitary operators U_n such that
$$U_{n-1}\mathscr{H}_n = \mathfrak{C}_n \oplus \mathscr{H}_{n-1}.$$
2. $U_{n-1}\mathscr{E}_n = \mathbb{1}_{\mathscr{H}_{n-1}}$
3. $A_n \subset A_{n+1}$ and $\forall i \in A_n, \mathfrak{f}_n O^i_n = O^i_{n+1}$ such that $\forall i \in A_n, U_n O^i_{n+1} U^\dagger_n|_{\mathscr{H}_{n-1}} = O^i_{n-1}$

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Example – what makes it work

Due to the standard decomposition of three qubits into fundamental representations, $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$, one can isometrically map a single qubit to a totally symmetric subspace of the $\frac{1}{2} \oplus \frac{1}{2}$ subspace of the three qubit space in the following manner:

The $\mathscr{E}$ Map

$$|+\rangle \overset{\mathscr{E}}{\mapsto} \frac{1}{\sqrt{3}}(|-\rangle|+\rangle|+\rangle + j|+\rangle|-\rangle|+\rangle + j^2|+\rangle|+\rangle|-\rangle)$$

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Example – illustrated

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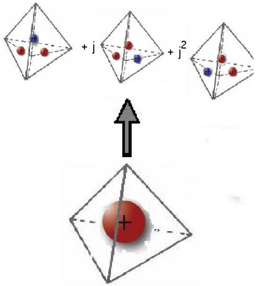

Example – prior definitions

and the three operators $\{\mathbf{a}, \mathbf{a}^\dagger, \mathrm{H}\}$, the annihilation, creation and the unique element of the Cartan subalgebra of the fundamental representation of $\mathfrak{su}(2)$, are evolved using the standard coproduct:

The Standard Coproduct

$$\begin{aligned} \mathbf{a} &\overset{\mathfrak{f}}{\longmapsto} \mathbf{a}\otimes\mathbb{1}_2\otimes\mathbb{1}_2+\mathbb{1}_2\otimes\mathbf{a}\otimes\mathbb{1}_2+\mathbb{1}_2\otimes\mathbb{1}_2\otimes\mathbf{a} \\ \mathbf{a}^\dagger &\overset{\mathfrak{f}}{\longmapsto} \mathbf{a}^\dagger\otimes\mathbb{1}_2\otimes\mathbb{1}_2+\mathbb{1}_2\otimes\mathbf{a}^\dagger\otimes\mathbb{1}_2+\mathbb{1}_2\otimes\mathbb{1}_2\otimes\mathbf{a}^\dagger \\ H &\overset{\mathfrak{f}}{\longmapsto} H\otimes\mathbb{1}_2\otimes\mathbb{1}_2+\mathbb{1}_2\otimes H\otimes\mathbb{1}_2+\mathbb{1}_2\otimes\mathbb{1}_2\otimes H \end{aligned}$$

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\end{aligned}$$

Example – TWO models

With this, we have two different simple options for the evolution map:

1) Exponential Expansion

- $\mathscr{H}_n = (\mathbb{C}^2)^{\otimes 3^{(n-1)}}$
- $\mathscr{E}_n = \mathscr{E}^{\otimes 3^{(n-1)}}$
- $\mathfrak{f}_n = \sum_{k=1}^{3^{(n-1)}} \underbrace{\underbrace{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathfrak{f}}_{k} \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{3^{(n-1)}}$
- $U_n = \mathbb{1}$

Example – TWO models

With this, we have two different simple options for the evolution map:

2) Linear Expansion

- $\mathscr{H}_n = (\mathbb{C}^2)^{\otimes(2n-1)}$
- $\mathscr{E}_n = \sum_{k=1}^{(2n-1)} \underbrace{\underbrace{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathscr{E}}_{k} \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{(2n-1)}$
- $\mathfrak{f}_n = \sum_{k=1}^{3^{(n-1)}} \underbrace{\underbrace{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathfrak{f}}_{k} \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{(2n-1)}$
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Conclusion and Outlook

- In general, one could imagine that the dynamics could combine a mixture of both processes.
- In summary:
  - There are situations where the Hilbert Space changes.
  - Increase Hilbert Space by adding information.
  - Increase Hilbert Space by diluting information.
  - ... or both.
- Outlook:One important question that should be asked at this point is whether there is, or whether there should be any link between the evolution (especially the "Quasi-Unitary" evolution) and the energy or a Hamiltonian.

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