

DERIVED ENDO-DISCRETE ARTIN ALGEBRAS

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ABSTRACT. Let Λ be an artin algebra. We prove that for each sequence of non-negative integers $(h_i)_{i \in \mathbb{Z}}$ there are only a finite number of isomorphism classes of indecomposables $X \in \mathcal{D}^b(\Lambda)$, the bounded derived category of Λ , with $\text{length}_{E(X)} H^i(X) = h_i$ for all $i \in \mathbb{Z}$ and $E(X)$ the endomorphism ring of X in $\mathcal{D}^b(\Lambda)$ if and only if $\mathcal{D}^b(\text{Mod } \Lambda)$ the bounded derived category of the category $\text{Mod } \Lambda$ of all left Λ -modules, has not generic objects in the sense of [3].

1. INTRODUCTION

Let Λ be an artin algebra over a commutative artinian ring k and $\mathcal{D}^b(\Lambda)$ be its bounded derived category. We consider $\text{Mod } \Lambda$ the category of left Λ -modules. We denote by $\text{mod } \Lambda$, $\text{Proj } \Lambda$ and $\text{proj } \Lambda$, the full subcategories of $\text{Mod } \Lambda$ consisting of the finitely generated, the projectives and the finitely generated projectives Λ -modules, respectively. By $\mathcal{D}^b(\text{Mod } \Lambda)$ we denote the bounded derived category of $\text{Mod } \Lambda$, we recall that $\mathcal{D}^b(\Lambda)$ is the bounded derived category of the category $\text{mod } \Lambda$. If $X = (X^i, d_X^i)_{i \in \mathbb{Z}}$ is an object in $\mathcal{D}^b(\Lambda)$ an invariant of it is given by its homology dimension $\mathbf{hdim} = (h_i)_{i \in \mathbb{Z}}$ with $h_i = \dim_k H^i(X)$.

A sequence of non negative integers $\mathbf{h} = (h_i)_{i \in \mathbb{Z}}$ is called a homology dimension if for all but finitely many $i \in \mathbb{Z}$, $h_i = 0$. We recall that according with [4], $\mathcal{D}^b(\Lambda)$ is called discrete and Λ derived discrete if there are only finitely many isoclasses of indecomposables $X \in \mathcal{D}^b(\Lambda)$ with fixed homology dimension.

We recall that $X \in \mathcal{D}^b(\text{Mod } \Lambda)$ is called endofinite if for all $i \in \mathbb{Z}$, $H^i(X)$ has finite length as left $E(X) = \text{End}_{\mathcal{D}^b(\text{Mod } \Lambda)}(X)$ -module. In case X is endofinite its homology endolength is defined as

$$\mathbf{hendol}(X) = (\text{length}_{E(X)} H^i(X))_{i \in \mathbb{Z}}.$$

Observe that all objects in $\mathcal{D}^b(\Lambda)$ are endofinite. The category $\mathcal{D}^b(\Lambda)$ is called endofinite discrete and Λ derived endo-discrete if for each homology dimension \mathbf{h} there only a finite number of isomorphism classes of indecomposable objects X in $\mathcal{D}^b(\Lambda)$ with $\mathbf{hendol}(X) = \mathbf{h}$.

We recall from [3] that $G \in \mathcal{D}^b(\text{Mod } \Lambda)$ is called generic if G is not in $\mathcal{D}^b(\Lambda)$, G is endofinite and indecomposable.

In this paper we prove the following.

Theorem 1.1. *Let Λ be an artin algebra over k then:*

- i) Λ is not derived endo-discrete if and only if $\mathcal{D}^b(\text{Mod } \Lambda)$ has a generic object;*
- ii) If k has infinite cardinality, then Λ is not derived discrete if and only if the category $\mathcal{D}^b(\text{Mod } \Lambda)$ has a generic object.*

In [4] it has been proved that if k is an algebraically closed field, then Λ is derived discrete if and only if $\mathcal{D}^b(\Lambda)_{prf}$, the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are the perfect complexes is discrete. In this paper we prove that the same result holds for artin algebras (see i) of Proposition 2.4).

For the proof of Theorem 1.1, we consider in section 2, $\mathbf{C}_m(\text{proj } \Lambda)$ which is the category of complexes $X = (X^i, d_X^i)$ of finitely generated projective Λ -modules with $X^i = 0$ for i outside the interval $[1, \dots, m]$. We denote by $\mathbf{C}_m^1(\text{proj } \Lambda)$ the full subcategory of $\mathbf{C}_m(\text{proj } \Lambda)$ whose objects are the complexes $X = (X^i, d_X^i)$ such that $\text{Im} d_X^{i-1} \subset \text{rad} X^i$ for all $i \in \mathbb{Z}$.

In general if \mathcal{C} is a k -category a morphism $f : M \rightarrow N$ in \mathcal{C} is called radical if for any split monomorphism $\sigma : X \rightarrow M$ and any split epimorphism $\pi : M \rightarrow Y$, $\pi f \sigma : X \rightarrow Y$ is not isomorphism. If P and Q are projective Λ -modules, $f : P \rightarrow Q$ is a radical morphism if and only if $\text{Im} f \subset \text{rad} Q$.

2. COMPLEXES OF FIXED SIZE

Let Y be a complex in $\mathbf{C}_m(\text{Proj } \Lambda)$, we denote by $E_C(Y)$ the endomorphism ring of Y in the category of complexes and by $E_K(Y)$ the endomorphism ring in the homotopy category, the ring $E_K(Y)$ is a quotient ring of $E_C(Y)$.

We need the following two results.

Lemma 2.1. *Suppose $Y = (Y^i, d_Y^i)_{i \in \mathbb{Z}} \in \mathbf{C}_m^1(\text{proj } \Lambda)$ is such that*

$$\text{length}_{E_K(Y)} H^j(Y) \leq c$$

for all j and for some $u \in [2, \dots, m]$, $\text{length}_{E_C(Y)} Y^u \leq d_u$, then

$$\text{length}_{E_C(Y)} Y^{u-1} \leq (d_u + c)L,$$

with $L = \text{length}_k \Lambda$.

Proof. We have $\text{length}_{E_C(Y)} Y^{u-1} / \text{Ker} d_Y^{u-1} = \text{length}_{E_C(Y)} \text{Im} d_Y^{u-1} \leq d_u$, moreover we know that $\text{length}_{E_C(Y)} \text{Ker} d_Y^{u-1} / \text{Im} d_Y^{u-2} \leq c$.

Therefore $\text{length}_{E_C(Y)} Y^{u-1} / \text{Im} d_Y^{u-2} \leq c + d_u$. Here $\text{Im} d_Y^{u-2} \subset \text{rad} Y^{u-1}$, thus $\text{length}_{E_C(Y)} Y^{u-1} / \text{rad} Y^{u-1} \leq \text{length}_{E_C(Y)} Y^{u-1} / \text{Im} d_Y^{u-2}$.

Consequently, $\text{length}_{E_C(Y)} Y^{u-1} \leq (c + d_u)L$. \square

Lemma 2.2. *Let $Y = (Y^i, d_Y^i)_{i \in \mathbb{Z}} \in \mathbf{C}_m^1(\text{proj } \Lambda)$ such that for some fixed c and all $j \in [1, m]$, we have $\text{length}_{E_C(Y)} H^j(Y) \leq c$. Then*

$$\text{length}_{E_C(Y)} Y \leq c(mL + (m-1)L^2 + (m-2)L^3 + \dots + 2L^{m-1} + L^m).$$

Proof. Here $Y^{m+1} = 0$, then by our previous lemma, $\text{length}_{E_C(Y)} Y^m \leq cL$. Then again by lemma 2.1 we have: $\text{length}_{E_C(Y)} Y^{m-1} \leq c(L + L^2)$, $\text{length}_{E_C(Y)} Y^{m-2} \leq c(L + L^2 + L^3)$, ..., $\text{length}_{E_C(Y)} Y^1 \leq c(L + L^2 + \dots + L^m)$. From here we obtain our result. \square

The object $Y \in \mathbf{C}_m(\text{Proj } \Lambda)$ is called endofinite if for all $i \in \mathbb{Z}$, Y^i has finite length as $E_C(Y)$ -module, in this case we put $\text{endol}(Y) = \sum_{i=1}^m \text{length}_{E_C(Y)} Y^i$. In case $Y \in \mathbf{C}_m(\text{proj } \Lambda)$ we say that Y is a finite object. Now an object $X \in \mathbf{C}_m(\text{Proj } \Lambda)$ is called generic if it is endofinite, indecomposable and it is not finite.

Definition 2.3. *The category $\mathbf{C}_m(\text{proj } \Lambda)$ is called endo-discrete if for all natural number d there are only finite number of isomorphism classes of indecomposable objects X with $\text{endol}(X) \leq d$.*

For P a projective Λ -module we consider the objects $J_u(P)$, for $u \in [1, m-1]$, $T(P)$ and $S(P)$ in $\mathbf{C}_m(\text{Proj } \Lambda)$ defined as follows: $J_u(P)^i = 0$ for $i \neq u, u+1$, and $P = J_u(P)^u = J_u(P)^{u+1}$, $d_{J_u(P)}^u = \text{id}_P$; $S(P)^i = 0$ for $i \neq 1$ and $S(P)^1 = P$; $T(P)^i = 0$ for $i \neq m$ and $T(P)^m = P$. If $h : P \rightarrow Q$ is a morphism between projective Λ -modules $S(h) : S(P) \rightarrow S(Q)$ is the morphism with $S(h)^1 = h$.

Consider in $\mathbf{C}_m(\text{Proj } \Lambda)$ the class \mathcal{E} of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that for all $i \in [1, m]$ the sequences $0 \rightarrow X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \rightarrow 0$ are exact. Then the pair $(\mathbf{C}_m(\text{Proj } \Lambda), \mathcal{E})$ is an exact category.

The indecomposable \mathcal{E} -projectives (respectively \mathcal{E} -injectives) are the complexes $T(P)$ and $J_u(P)$, $u \in [1, m-1]$, (respectively $J_u(P)$, $S(P)$) with P indecomposable projective Λ -module.

Consider $\mathbf{K}^{\leq m, b}(\text{Proj } \Lambda)$ the homotopy category of those complexes X over $\text{Proj } \Lambda$ such that $H^i(X) = 0$ for almost all i and $X^j = 0$ for $j > m$.

Take $F : \mathbf{K}^{\leq m, b}(\text{Proj } \Lambda) \rightarrow \mathbf{C}_m(\text{Proj } \Lambda)$ the functor given in objects by $F(X)^i = 0$ for $i < m$ and $F(X)^j = X^j$ for $j \geq 1$. If $f : X \rightarrow Y$ is a morphism in $\mathbf{K}^{\leq m, b}(\text{Proj } \Lambda)$, $F(f) : F(X) \rightarrow F(Y)$ is defined by $F(f)^s = f^s$ for $s \in [1, m]$.

We know from Corollary 5.7 of [1] that F induces an equivalence

$$\overline{F} : \mathcal{L}_m \rightarrow \overline{\mathbf{C}_m}(\text{Proj } \Lambda)$$

where \mathcal{L}_m is the full subcategory of $\mathbf{K}^{\leq m, b}(\text{Proj } \Lambda)$ whose objects are those X with $H^i(X) = 0$ for $i \leq 1$. The category $\overline{\mathbf{C}_m}(\text{Proj } \Lambda)$ is the category with the same objects as $\mathbf{C}_m(\text{Proj } \Lambda)$ and morphisms the morphisms in $\mathbf{C}_m(\text{Proj } \Lambda)$ modulo those which are factorized through \mathcal{E} -injectives.

Proposition 2.4. *Let Λ be an artin algebra then:*

- i) $\mathcal{D}^b(\Lambda)$ is endo-discrete (respectively discrete) if and only if for all m , $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete (respectively discrete);
- ii) $\mathcal{D}^b(\text{Mod } \Lambda)$ has a generic complex if and only if for some m , $\mathbf{C}_m(\text{Proj } \Lambda)$ has a generic object.

Proof. Assume $\mathcal{D}^b(\Lambda)$ is endo-discrete. Take now a family of pairwise non-isomorphic indecomposable objects $\{Y_s\}_{s \in I}$ in some $\mathbf{C}_m(\text{proj } \Lambda)$ with $\text{endol}(Y_s) \leq d$ for a fixed d and all $s \in I$. Clearly $\text{length}_{E(Y_s)} H^i(Y_s) \leq d$ and the objects Y_s are pairwise non isomorphic in $\mathcal{D}^b(\Lambda)$. Since $\mathcal{D}^b(\Lambda)$ is endo-discrete the set I is finite, this proves that $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete for all m .

Suppose now that for all m , $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete. Take $\{Z_s\}_{s \in I}$ a family of pairwise non-isomorphic indecomposable complexes in $\mathcal{D}^b(\Lambda)$ with fixed homology endolength $\mathbf{h} = (h_i)_{i \in \mathbb{Z}}$. We take $c = \max\{h_i\}_{i \in \mathbb{Z}}$. After a shifting we may assume that for some m , $h_i = 0$ for i outside the interval $[2, m]$. For each $s \in I$ take a quasi-isomorphism $P_s \rightarrow Z_s$ with P_s an indecomposable object in $\mathbf{K}^{\leq m, b}(\text{proj } \Lambda)$. Since $H^i(P_s) \cong H^i(Z_s)$ we have that $P_s \in \mathcal{L}_m$. Now $\overline{F} : \mathcal{L}_m \rightarrow \overline{\mathbf{C}_m}(\text{Proj } \Lambda)$ is an equivalence, this implies that the objects $Y_s = \overline{F}(Z_s)$ are pairwise non-isomorphic in $\overline{\mathbf{C}_m}(\text{Proj } \Lambda)$ and therefore they are pairwise non-isomorphic in $\mathbf{C}_m(\text{Proj } \Lambda)$. Using Lemma 2.2 we have for all $s \in I$, $\text{endol}(Y_s) \leq c(mL + (m-1)L^2 + \dots + 2L^{m-1} + L^m)$ with $L = \text{length}_k \Lambda$. Since $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete the set I is finite. This

proves that $\mathcal{D}^b(\Lambda)$ is endo-discrete. The corresponding statement for discrete is proved in a similar way.

Take now Y a generic complex in $\mathcal{D}^b(\text{Mod } \Lambda)$. As before we may assume that if $\text{hendol} Y = \mathbf{h} = (h_i)_{i \in \mathbb{Z}}$, then there is a m such that $h_i = 0$ for i outside the interval $[2, m]$. Take $P_Y \rightarrow Y$ a quasi-isomorphism with $P_Y \in \mathbf{K}^{\leq \mathbf{m}, \mathbf{b}}(\text{Proj } \Lambda)$. We have $H^i(P_Y) \cong H^i(Y)$ for all $i \in \mathbb{Z}$, thus $P_Y \in \mathcal{L}_m$. Using Lemma 2.2 we may prove that $\overline{F}(P_Y)$ is endofinite. Since \overline{F} is an equivalence P_Y is also indecomposable, therefore $\overline{F}(P_Y)$ is a generic object in $\mathbf{C}_m(\text{Proj } \Lambda)$. \square

3. A CATEGORY OF MORPHISMS

For $m \geq 1$, we consider the following category \mathcal{M}_m of morphisms in $\mathbf{C}_m^1(\text{Proj } \Lambda)$. The objects of \mathcal{M}_m are radical morphisms $f : S(P) \rightarrow X$ in $\mathbf{C}_m^1(\text{Proj } \Lambda)$ with P a projective Λ -module and X an object in $\mathbf{C}_m^1(\text{Proj } \Lambda)$. The morphisms from $f : S(P) \rightarrow X$ to $f' : S(P') \rightarrow X'$ are given by pairs of morphisms $u = (u_1, u_2)$, $u_1 : P \rightarrow P'$, $u_2 : X \rightarrow X'$ such that $u_2 f = f' S(u_1)$. If $u = (u_1, u_2)$ is a morphism from $f : S(P) \rightarrow X$ to $f' : S(P') \rightarrow X'$ and $v = (v_1, v_2)$ is a morphism from $f' : S(P') \rightarrow X'$ to $f'' : S(P'') \rightarrow X''$, then $vu = (v_1 u_1, v_2 u_2)$. The identity morphism in the object $f : S(P) \rightarrow X$ is given by the pair (id_P, id_X) .

An object $f : S(P) \rightarrow X$ is called endofinite if P and all X^i have finite length as $E(f) = \text{End}_{\mathcal{M}_m}(f)$ -modules. In this case $\text{endol}(f) = \text{length}_{E(f)} P + \sum_i \text{length}_{E(f)} X^i$. The object $f : S(P) \rightarrow X$ is called finite if P is a finitely generated Λ -module and X is an object in $\mathbf{C}_n^1(\text{proj } \Lambda)$. We put $\text{length}(f) = \text{length}_k P + \sum_i \text{length}_k X^i$.

Proposition 3.1. *There is a functor $G : \mathcal{M}_m \rightarrow \mathbf{C}_{m+1}^1(\text{Proj } \Lambda)$ with the following properties:*

- i) G is an equivalence of categories;
- ii) $f \in \mathcal{M}_m$ is endofinite (respectively finite) if and only if $G(f)$ is endofinite (respectively finite) and

$$\text{endol}(f) = \text{endol}(G(f)) \quad \text{length}(f) = \text{length} G(f).$$

Proof. Take $f : S(P) \rightarrow X$ an object in \mathcal{M}_m . We have the morphism $f^1 : P \rightarrow X^1$, f is a radical morphism, thus $\text{Im} f^1 \subset \text{rad} X^1$, moreover f is a morphism of complexes, we have $d_X^1 f^1 = f^2 d_P^1 = 0$. Therefore we have the complex $G(f)$ in $\mathbf{C}_{m+1}^1(\text{Proj } \Lambda)$ given by $G(f)^i = 0$ for i outside the interval $[1, \dots, m+1]$, $G(f)^1 = P$, $G(f)^{i+1} = X^i$ for $i = 1, \dots, m$, $d_{G(f)}^1 = f^1$, $d_{G(f)}^{i+1} = d_X^i$ for $i = 1, \dots, m$.

Now if $u = (u_1, u_2)$ is a morphism from $f : S(P) \rightarrow X$ to $f' : S(P') \rightarrow X'$, we define $G(u)$ in the following way: $G(u)^i = 0$ for i outside the interval $[1, \dots, m+1]$, $G(u)^1 = u_1 : G(f)^1 = P \rightarrow G(f')^1 = P'$, $G(u)^{i+1} = u_2 : G(f)^{i+1} = X^i \rightarrow G(f')^{i+1} = (X')^i$ for $i = 1, \dots, m$.

We have $d_{G(f)}^1 G(u)^1 = (f')^1 u_1 = (u_2)^1 f' = G(u)^2 d_{G(f)}^1$. For $i = 1, \dots, m$ we have $d_{G(f')}^{i+1} G(u)^{i+1} = d_{G(f')}^{i+1} u_2^i = u_2^{i+1} d_X^i = G(u)^{i+2} d_{G(f)}^{i+1}$. From here we conclude that $G(u) : G(f) \rightarrow G(f')$ is a morphism of complexes. We have $G(id_f) = id_{G(f)}$. Now if v is a morphism from $f' : S(P') \rightarrow X'$ to $f'' : S(P'') \rightarrow X''$, $G(v)G(u) = G(vu)$. Clearly G is a full, faithful dense functor and ii) holds. \square

Suppose now that $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete. Let Z_1, \dots, Z_n be a complete representative set of the isomorphism classes of indecomposables X with $\text{endol}(X) \leq d$

or $X \cong S(P)$ for some projective Λ -module P , take $Z = Z_1 \oplus \dots \oplus Z_n$ and Y the sum of those Z_i which are isomorphic to some $S(P)$.

Consider now $R = \text{End}_{\mathbf{C}_{n-1}^1(\text{proj } \Lambda)}(Z)^{op}$ and f the projection of Z on Y followed of the corresponding inclusion of Y in Z , f is an idempotent and $fRf \cong \text{End}_{\mathbf{C}_{n-1}(\text{proj } \Lambda)}(Y)^{op}$.

Take $A = \begin{pmatrix} fRf & 0 \\ 0 & R \end{pmatrix}$ and consider the following exact sequence of A - A -bimodules:

$$\xi: 0 \rightarrow \begin{pmatrix} 0 & f\text{rad}R \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} fRf & f\text{rad}R \\ 0 & R \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} fRf & 0 \\ 0 & R \end{pmatrix} \rightarrow 0.$$

In the following we put F equal to the middle term of the above sequence. We recall that the lift category $\xi(A)$ is defined as follows:

1. The objects in $\xi(A)$ are pairs (P, e) consisting of a projective A -module P and a morphism of A -modules $e : P \rightarrow F \otimes_A P$, such that $(\pi \otimes 1_P)e = 1_P$ for $\pi \otimes 1_P : F \otimes_A P \rightarrow P$.
2. The morphisms from (P, e) to (P', e') are those morphisms of A -modules $\theta : P \rightarrow P'$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\theta} & P' \\ e \downarrow & & \downarrow e' \\ F \otimes_R P & \xrightarrow{1 \otimes \theta} & F \otimes_R P'. \end{array}$$

An object $X = (P, e) \in \xi(A)$ is called endofinite if P has finite length as $E(X) = \text{End}_{\xi(A)}(X)$ -module, in case X is endofinite we put $\text{endol}(X) = \text{length}_{E(X)} P$. An object $Y = (Q, f) \in \xi(A)$ is called finite if Q has finite length as k -module in such case we put $\text{length} X = \text{length}_k Q$. An object $X \in \xi(A)$ is called generic if X is not finite and it is indecomposable and endofinite.

Definition 3.2. *The category $\xi(A)$ is called endo-discrete (respectively discrete) if for all natural number d , there are only finite number of isomorphisms classes of finite indecomposable objects in $\xi(A)$ having endolength (respectively length) less or equal to d .*

From Theorem 9.5 of [2] we obtain the following.

Theorem 3.3. *The category $\xi(A)$ is not endo-discrete if and only if it contains a generic object. In case k has infinitely many elements, $\xi(A)$ is not discrete if and only if it contains a generic object.*

In the proof of next proposition we need the following lemma.

Lemma 3.4. *Let X be a finitely generated left Λ -module and Y a $\Lambda - B$ -bimodule with B a k -algebra. Then if Y has finite length as right B -module, the right B -module $\text{Hom}_\Lambda(X, Y)$ has finite length. Moreover*

$$\text{length}_B \text{Hom}_\Lambda(X, Y) \leq t(X) \text{length}_B Y,$$

where $t(X)$ is the minimal number of generators of the Λ -module X .

Proof. We have an epimorphism $\Lambda^t \rightarrow X$, for $t = t(X)$. From here we obtain a monomorphism right B -modules:

$$\text{Hom}_\Lambda(X, Y) \rightarrow \text{Hom}_\Lambda(\Lambda^t, Y) \cong \oplus_{i=1}^t Y$$

from this we obtain our result. \square

Assume $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete. We denote by $\mathcal{M}_{m,d}$ the full subcategory of \mathcal{M}_m whose objects are of the form $f : S(P) \rightarrow X$, with X a direct sum of direct summands of Z .

Proposition 3.5. *Suppose Λ is a basic artin algebra, then there is a functor $U : \mathcal{M}_{m,d} \rightarrow \xi(A)$ with the following properties:*

- i) U is an equivalence;
- ii) for $Y \in \mathcal{M}_{m,d}$, and $t(Z)$ the maximal of the numbers $t(Z^i)$, $i \in [1, m]$ for $Z = (Z^i, d_Z^i)$ and $t(Z^i)$ the minimal number of generators of Z^i as Λ -module we have

$$\begin{aligned} \text{endol}(Y) &\leq \text{endol}(U(Y)) \leq t(Z)\text{endol}(Y) \\ \text{length}(Y) &\leq \text{length}(U(Y)) \leq t(Z)\text{length}(Y); \end{aligned}$$

- iii) $Y \in \mathcal{M}_{m,d}$ is a generic object if and only if $U(Y)$ is a generic object in $\xi(A)$.

Proof. A projective A -module is given by a pair (P_1, P_2) with P_1 a projective fRf -module and P_2 a projective R -module. If (P, e) is an object of $\xi(A)$, then e is given by a radical morphism of fRf -modules $\phi : P_1 \rightarrow fP_2$. Therefore $\xi(A)$ is equivalent to the category \mathcal{W} whose objects are radical morphisms of fRf -modules $\phi : P_1 \rightarrow fP_2$ with P_1 a projective fRf -module and P_2 a projective R -module. The morphisms from $\phi : P_1 \rightarrow fP_2$ to $\psi : Q_1 \rightarrow fQ_2$ are given by pairs (u_1, u_2) with $u_1 : P_1 \rightarrow Q_1$ a morphism of fRf -modules and $u_2 : P_2 \rightarrow Q_2$ a morphism of R -modules such that $\psi u_1 = f u_2 \phi$.

Now if P_1 and P_2 are as before we have the following natural isomorphisms:

$$\text{Hom}_{fRf}(P_1, fP_2) \cong \text{Hom}_{fRf}(P_1, \text{Hom}_R(Rf, P_2)) \cong \text{Hom}_R(Rf \otimes_{fRf} P_1, P_2).$$

Therefore the category \mathcal{W} is equivalent to the category \mathcal{U} whose objects are radical morphisms of R -modules $u : Q_1 \rightarrow Q_2$, where Q_1 is a direct sum of direct summands of Rf and Q_2 is a projective R -module. Now \mathcal{C}_Z the full subcategory of $\mathbf{C}_m(\text{Proj } \Lambda)$ whose objects are direct summands of arbitrary sums of direct summands of Z is equivalent to the category $\text{Proj } R$. In a similar way, \mathcal{C}_Y , the full subcategory of \mathcal{C}_Z whose objects are arbitrary sums of direct summands of Y is equivalent to the full subcategory of $\text{Proj } R$, whose objects are arbitrary sums of direct summands of Rf . Consequently the category \mathcal{U} is equivalent to the category $\mathcal{M}_{m,d}$ whose objects are radical morphisms in $\mathbf{C}_m^1(\text{Proj } \Lambda)$, $h : S(P) \rightarrow X$, with $S(P)$ a sum of direct summands of Y and X a sum of direct summands of Z . Then we have an equivalence $U : \mathcal{M}_{m,d} \rightarrow \xi(A)$ such that

$$U(h : S(P) \rightarrow X) = (\text{Hom}_{\mathbf{C}_m(\text{proj } \Lambda)}(Y, S(P)) \oplus \text{Hom}_{\mathbf{C}_m(\text{proj } \Lambda)}(Z, X), e_h).$$

Then if the object $h : S(P) \rightarrow X$ is endofinite and $E(h) = \text{End}_{\mathcal{M}_{m,d}}(h)$, both $S(P)$ and X are $\Lambda - E(h)^{op}$ -bimodules. Thus by Lemma 3.4 both $\text{Hom}_{\mathbf{C}_m(\text{proj } \Lambda)}(Y, S(P))$ and $\text{Hom}_{\mathbf{C}_m(\text{proj } \Lambda)}(Z, X)$ have finite length as $E(h)$ -modules, therefore both have finite length as $E(U(h))$ -modules. Consequently $U(h : S(P) \rightarrow X)$ is endofinite, then if h is generic its image under U is also generic.

Moreover using Lemma 3.4 we have $\text{endol}(U(h)) \leq t(Z)\text{endol}(h)$ and if h is a finite object $\text{length}(U(h)) \leq t(Z)\text{length}(h)$.

Take $1 = \sum_j e_j$ a decomposition into orthogonal primitive idempotents. Consider Λe_i , then for $u \in [1, m-1]$, $J_u(\Lambda e_i) \cong W$ for some $W \in \{Z_1, \dots, Z_n\}$. Denote by f_W the projection of Z on W followed by the inclusion of W in Z . We have :

$f_W \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(Z, X) \cong \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(J_u(\Lambda e_i), X) \cong \text{Hom}_{\Lambda}(\Lambda e_i, X^u) \cong e_i X^u$. Then $\text{length}_{E(h)} e_i X^u = \text{length}_{E(U(h))} f_W \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(Z, X)$, taking now $T(\Lambda e_i)$ instead of $J_u(\Lambda e_i)$, we obtain $\text{length}_{E(h)} e_i X^m = \text{length}_{E(U(h))} f_W \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(Z, X)$, for $W \in \{Z_1, \dots, Z_n\}$ isomorphic to $T(\Lambda e_i)$.

Since Λ is basic we have:

$$\sum_{u=1}^m \text{length}_{E(h)} X^u \leq \text{length}_{E(U(h))} \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(Z, X).$$

In a similar way we obtain:

$$\text{length}_{E(h)} S(P) \leq \text{length}_{E(U(h))} \text{Hom}_{\mathbf{C}_m(\text{Proj } \Lambda)}(Y, S(P)).$$

From here we have $\text{endol}(h) \leq \text{endol}(U(h))$. Similarly we also have $\text{length}(h) \leq \text{length}(U(h))$. Therefore if $U(h)$ is endofinite, h is endofinite and this implies that if $U(h)$ is generic h is generic. \square

Theorem 3.6. *For $m \geq 1$, the category $\mathbf{C}_m(\text{proj } \Lambda)$ is not endo-discrete if and only if $\mathbf{C}_m(\text{Proj } \Lambda)$ has a generic object. In case k has infinitely many elements, $\mathbf{C}_m(\text{proj } \Lambda)$ is not discrete if and only if $\mathbf{C}_m(\text{Proj } \Lambda)$ has a generic object.*

Proof. For $m = 1$, $\mathbf{C}_1(\text{proj } \Lambda) \cong \text{proj } \Lambda$, therefore this category is endo-discrete and $\mathbf{C}_1(\text{Proj } \Lambda) \cong \text{Proj } \Lambda$ does not contain indecomposable objects of infinite length over k , then our result holds in this case.

Assume our result proved for m , we shall prove it for $m + 1$. Suppose that the category $\mathbf{C}_{m+1}(\text{proj } \Lambda)$ is not endo-discrete, then for some d there are infinitely many isomorphism classes of indecomposables in $\mathbf{C}_{m+1}(\text{proj } \Lambda)$ with endolength equal to d . If $\mathbf{C}_m(\text{proj } \Lambda)$ is not endo-discrete then by the induction hypothesis there is a generic object $G \in \mathbf{C}_m(\text{proj } \Lambda)$, thus there is a generic object in $\mathbf{C}_{m+1}(\text{proj } \Lambda)$. Therefore we may assume $\mathbf{C}_m(\text{proj } \Lambda)$ is endo-discrete. Consider the equivalence $G : \mathcal{M}_n \rightarrow \mathbf{C}_{m+1}(\text{proj } \Lambda)$. We have an infinite family of pairwise non-isomorphic indecomposable objects $\{Y_s\}_{s \in I}$ in $\mathbf{C}_{m+1}(\text{proj } \Lambda)$ with $\text{endol}(Y_s) \leq d$ for all $s \in I$. For each $s \in I$ there is an object $h_s : S(P_s) \rightarrow X_s$ in \mathcal{M}_m with $G(h_s) \cong Y_s$. We have $\text{endol}(X_s) \leq \text{length}_{E(h_s)} X_s \leq d$. Thus each X_s is a finite direct sum of indecomposable finite objects with endolength smaller or equal to d . Therefore each $h_s \in \mathcal{M}_{m,d}$ and by ii) of Proposition 3.5 $\text{endol}(U(h_s)) \leq t(Z)d$. Therefore $\xi(A)$ is not endo-discrete, by Theorem 3.3 the category $\xi(A)$ contains a generic object G . Then $G \cong U(g)$, and by iii) of Theorem 3.3, g is a generic object in $\mathcal{M}_{m,d}$. But then $G(g)$ is a generic object in $\mathbf{C}_{m+1}(\text{Proj } \Lambda)$.

Conversely, suppose Y is a generic object in $\mathbf{C}_{m+1}(\text{Proj } \Lambda)$ with $\text{endol}(Y) \leq d$. Then $Y \cong G(h)$ with $h : S(P) \rightarrow X$ in \mathcal{M}_m . Here $\text{endol}(X) \leq \text{length}_{E(h)}(X) \leq d$. Then X is endofinite and therefore is the direct sum of indecomposable objects of endolength less or equal to d . By the induction hypothesis $\mathbf{C}_m(\text{Proj } \Lambda)$ does not contain generic objects, therefore X is the direct sum of direct summands of Z , then $h \in \mathcal{M}_{m,d}$. By iii) of Proposition 3.5, $U(h)$ is a generic object in $\xi(A)$. Again by Theorem 3.3, $\xi(A)$ is not endo-discrete. Thus there is an infinite family $\{Y_s\}_{s \in I}$ of pairwise non-isomorphic finite indecomposable objects in $\xi(A)$ with $\text{endol}(Y_s) \leq u$ for all $s \in I$ and some u . For each Y_s there is a $h_s \in \mathcal{M}_{m,d}$ finite object such that $U(h_s) \cong Y_s$. By ii) of Proposition 3.5, $\text{endol}(h_s) \leq \text{endol}(Y_s) \leq u$. Now $G(h_s)$ is a

finite object in $\mathbf{C}_{\mathbf{m}+1}(\text{Proj } \Lambda)$ and $\text{endol}(h_s) = \text{endol}(G(h_s))$. Then $\mathbf{C}_{\mathbf{m}+1}(\text{proj } \Lambda)$ is not endo-discrete. The second part of our theorem is proved in a similar way. \square

Proof of Theorem 1.1. Follows from Proposition 2.4 and Theorem 3.6.

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