On Modules and Complexes without Self-extensions

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Abstract

Let Λ be an artin algebra over a commutative artinian ring, k. If M is a finitely generated left Λ-module, we denote by $\Omega(M)$ the kernel of $\eta_M : P_M \to M$ a minimal projective cover. We prove that if M and N are finitely generated left Λ-modules and $\mathrm{Ext}^1_\Lambda(M,M) = 0$, $\mathrm{Ext}^1_\Lambda(N,N) = 0$, then $M \cong N$ if and only if $M/\mathrm{rad}M \cong N/\mathrm{rad}N$ and $\Omega(M) \cong \Omega(N)$.

Now if k is an algebraically closed field and $(d_i)_{i\in\mathbb{Z}}$ is a sequence of non negative integers almost all of them zero, then we prove that the family of objects $X \in \mathcal{D}^b(\Lambda)$, the bounded derived category of Λ, with $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X, X[1]) = 0$ and $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$, has only a finite number of isomorphism classes (see [9]).

1 Introduction

Let Λ be an artin algebra over a commutative artinian ring k. We denote by $\mathrm{Mod}\,\Lambda$ the category of left Λ-modules, by $\mathrm{mod}\,\Lambda$, $\mathrm{proj}\,\Lambda$ we denote the full subcategories of $\mathrm{Mod}\,\Lambda$ whose objects are respectively, the finitely generated Λ-modules and the finitely generated projective Λ-modules. By $\mathcal{D}^b(\Lambda)$ we denote the bounded derived category of Λ.

For $M \in \mathrm{mod}\,\Lambda$, consider $P^0_M \overset{\eta_M}{\to} M$ a minimal projective cover and $\Omega(M) = \ker(\eta_M)$. Here we prove the following: if M, N are in $\mathrm{mod}\,\Lambda$ and $\mathrm{Ext}^1_\Lambda(M,M) \cong \mathrm{Ext}^1_\Lambda(N,N) = 0$, then $M \cong N$ if and only if $M/\mathrm{rad}M \cong N/\mathrm{rad}N$ and $\Omega(M) \cong \Omega(N)$.

For $M \in \mathrm{mod}\,\Lambda$ with finite projective dimension consider a minimal projective resolution:

$$0 \to P_M^{-m(M)} \to P_M^{-m(M)+1} \to .... \to P^0_M \overset{\eta_M}{\to} M \to 0.$$

Suppose that M, N are in $\mathrm{mod}\,\Lambda$, $\mathrm{Ext}^1_\Lambda(M,M) \cong \mathrm{Ext}^1_\Lambda(N,N) = 0$ and for all $j > 0$, $\mathrm{Ext}^j_\Lambda(M, \Omega^{j-1}(M)) \cong \mathrm{Ext}^j_\Lambda(N, \Omega^{j-1}(N)) = 0$, then we prove using the result above that $M \cong N$ if and only if $m(M) = m(N)$ and for all j, $P_M^{-j} \cong P_N^{-j}$.

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Now let Λ be a finite-dimensional algebra over an algebraically closed field k and $\mathbf{d} = (d_i)_{i \in \mathbb{Z}}$ be a collection of non-negative integers with almost all $d_i = 0$. Then the family $\mathcal{U}(\mathbf{d})$ of objects $X \in \mathcal{D}^b(\Lambda)$ such that $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X, X[1]) = 0$ has only a finite number of isomorphism classes in $\mathcal{D}^b(\Lambda)$. This result is closely related with Corollary 9 of [9].

For the proof of the above mentioned results we see that some problems involving upper bounded complexes of finitely generated projective Λ-modules with bounded homology can be reduced to problems involving complexes of fixed size of finitely generated projective Λ-modules (see Proposition 4.7). Then in this last case we can apply properties of lift categories introduced by W.W Crawley-Boevey in [6].

2 Exact structures and exact subcategories

Here we recall the notion of exact category. For $(\mathcal{A}, \mathcal{E})$, an exact category, $\mathcal{B}, \mathcal{C}$ full subcategories of $\mathcal{A}$ closed under extensions we consider the category of morphisms $f : W \to Z$ with W an object of $\mathcal{B}$ and W an object of $\mathcal{C}$. Then we introduce an exact structure on this category of morphisms.

Definition 2.1 Let $\mathcal{A}$ be an additive category. A pair of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

is called exact if i is kernel of d and d is cokernel of i.

Let $\mathcal{E}$ be a class of exact composable sequences (i, d) in $\mathcal{A}$ closed under isomorphisms; we call $(i, d) \in \mathcal{E}$ a conflation, i an inflation and d a deflation.

$\mathcal{E}$ is an exact structure if it satisfies the following axioms:

$K1)$ 1_0 is a deflation.

$K2)$ Composition of deflations is a deflation.

$K3)$ For every $h \in \mathcal{A}(Z, Z_0)$ and all deflation $d_0 \in \mathcal{A}(Y_0, Z_0)$ there exists a pullback

$$\begin{array}{ccc} Y & \xrightarrow{d} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y_0 & \xrightarrow{d_0} & Z_0 \end{array}$$

where d is a deflation

$K3^{op})$ For every $f \in \mathcal{A}(X, X_0)$ and all inflation $i \in \mathcal{A}(Y, Z)$ there exists a pushout

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_0 & \xrightarrow{i_0} & Y_0 \end{array}$$

where i_0 is an inflation

$K4)$ Retractions in $\mathcal{A}$ have kernels.

In this situation we say that $(\mathcal{A},\mathcal{E})$ is an exact category. For simplicity, if $\mathcal{E}$ is an exact structure we use extension instead of conflation.

Remark 2.2 It is known ([7]) that the above axioms imply their duals.

Moreover, in an exact category the next claims are true ([7]); if dd' is a deflation then d is a deflation, if $i'i$ is an inflation then i is an inflation.

Also, $K3$ induces a diagram of extensions

$$\begin{array}{ccccc} X & \overset{i}{\to} & Y & \overset{d}{\to} & Z \\ \| & & {}^{g}\downarrow & & \downarrow^{h} \\ X_0 & \overset{i_0}{\to} & Y_0 & \overset{d_0}{\to} & Z_0 \end{array}$$

where the right square is a pullback and a pushout, and $K3^{op}$ a diagram of extensions

$$\begin{array}{ccccc} X & \overset{i}{\to} & Y & \overset{d}{\to} & Z \\ {}^{f}\downarrow & & {}^{g}\downarrow & & \| \\ X_0 & \overset{i_0}{\to} & Y_0 & \overset{d_0}{\to} & Z_0 \end{array}$$

where the left square is a pullback and a pushout.

The following is a well known result.

Proposition 2.3 *Let $(\mathcal{A},\mathcal{E})$ be an exact category, and $\mathcal{B}$ a full subcategory closed under direct summands and extensions. Then $(\mathcal{B},\mathcal{E}_{\mathcal{B}})$ is an exact structure, where $\mathcal{E}_{\mathcal{B}}$ is the restriction of the class $\mathcal{E}$ to $\mathcal{B}$.*

Definition 2.4 Let $\mathcal{A}$ be a category, and $\mathcal{B}$ and $\mathcal{C}$ subcategories of $\mathcal{A}$. We define the category $Morph\,(\mathcal{B},\mathcal{C})$ as follows: the objects are the morphisms $f: X\to Y$ in $\mathcal{A}$ such that $X\in\mathcal{B}$ and $Y\in\mathcal{C}$, and a morphism from $f: X\to Y$ to $f': X'\to Y'$ is a pair (u,v) of morphisms $u: X\to X'$ in $\mathcal{B}$ and $v: Y\to Y'$ in $\mathcal{C}$ such that $f'u=vf$.

Proposition 2.5 *Let $(\mathcal{A},\mathcal{E})$ be an exact category, and $\mathcal{B}$ and $\mathcal{C}$ full subcategories of $\mathcal{A}$ closed under direct summands and extensions. Then $\left(Morph\,(\mathcal{B},\mathcal{C}),\mathcal{E}_{\mathcal{C}}^{\mathcal{B}}\right)$ is an exact category, where $\mathcal{E}_{\mathcal{C}}^{\mathcal{B}}$ is the class of the pairs $((u_0,v_0),(u_1,v_1))$ such that $(u_0,u_1)\in\mathcal{E}_{\mathcal{B}}$ and $(v_0,v_1)\in\mathcal{E}_{\mathcal{C}}$.*

Proof. Let $f_1: X_1\to Y_1$, $f_2: X_2\to Y_2$ and $f_3: X_3\to Y_3$ be objects in $Morph\,(\mathcal{B},\mathcal{C})$ and $\eta: f_1 \overset{(u_1,v_1)}{\longrightarrow} f_2 \overset{(u_2,v_2)}{\longrightarrow} f_3$ an element of $\mathcal{E}_{\mathcal{C}}^{\mathcal{B}}$.

Clearly $\mathcal{E}_{\mathcal{C}}^{\mathcal{B}}$ is closed under isomorphisms.

Now we check that η is an exact pair.

Suppose we have an object $f: X\to Y$ and a morphism $(u,v): f\to f_2$ in $Morph\,(\mathcal{B},\mathcal{C})$ such that $(u_2,v_2)(u,v)=0$. Then there exist unique morphisms $s: X\to X_1$ and $t: Y\to Y_1$ in $\mathcal{A}$ such that $u_1 s=u$ and $v_1 t=v$. We have $v_1(f_1 s-tf)=f_2u_1s-vf=0$, then $f_1 s=tf$ following that (s,t) is a morphism.

The proof of (u_2,v_2) being cokernel is dual.

$K1$, $K2$ and $K4$ are immediate from the proposition 2.3.

In order to prove $K3$, let f be as before, take $(u,v) : f \to f_3$ a morphism in $Morph(\mathcal{B},\mathcal{C})$, and consider the pullback diagrams

$$\begin{array}{ccccc} X_1 & \stackrel{i}{\to} & X_0 & \stackrel{d}{\to} & X \\ \| & & \downarrow^{s} & & \downarrow^{u} \\ X_1 & \stackrel{u_1}{\to} & X_2 & \stackrel{u_2}{\to} & X_3 \end{array}$$

$$\begin{array}{ccccc} Y_1 & \stackrel{i'}{\to} & Y_0 & \stackrel{d'}{\to} & Y \\ \| & & \downarrow^{t} & & \downarrow^{v} \\ Y_1 & \stackrel{v_1}{\to} & Y_2 & \stackrel{v_2}{\to} & Y_3 \end{array}$$

for the property of the pullback, there exists a unique morphism $f_0 : X_0 \to Y_0$ such that $d'f_0 = fd$ and $tf_0 = f_2 s$.

Suppose there is an object $g : W \to Z$ and morphisms $(\lambda_1, \lambda_2) : g \to f_2$ and $(\mu_1, \mu_2) : g \to f_3$ in $Morph(\mathcal{B},\mathcal{C})$ such that $(u_2, v_2)(\lambda_1, \lambda_2) = (\mu_1, \mu_2)(u, v)$. Then, by the pullback property, there exist unique morphisms $\alpha : W \to X_0$ and $\beta : Z \to Y_0$ in $\mathcal{A}$ such that $s\alpha = \lambda_1$, $d\alpha = \mu_1$, $t\beta = \lambda_2$ and $d'\beta = \mu_2$.

Now observe that $d'(f_0\alpha - \beta g) = fd\alpha - \mu_2 g = 0$ and $t(f_0\alpha - \beta g) = f_2 s\alpha - \lambda_2 g = f_2\lambda_1 - \lambda_2 g = 0$. By the pullback property (α, β) is a morphism in $Morph(\mathcal{B},\mathcal{C})$.

A dual argument proves $K3^{op}$.

□

3 Lift categories

Here we recall the properties of lift categories needed in our work. Now suppose $\mathcal{A}$ is a Krull-Schmidt category and X, Y are objects in $\mathcal{A}$. We consider the category of morphisms in $\mathcal{A}$, $f : W \to Z$, with W a finite direct sum of direct summands of X and Z a finite direct sum of direct summands of Y. We see that this category of morphisms can be seen as a lift category.

Definition 3.1 A lift pair (R, ξ) is given by a ring R and an exact sequence of $R-$bimodules

$$\xi : \quad 0 \quad \to \quad M \stackrel{i}{\to} \quad E \stackrel{\pi}{\to} \quad R \quad \to \quad 0$$

Definition 3.2 Given a lift pair (R, ξ) we define the lift category $\xi(R)$ as follows: the objects are pairs (P, e) where P is a projective $R-$module and $e : P \to E \otimes_R P$ is an $R-$morphism such that the composition

$$P \quad \stackrel{e}{\to} \quad E \otimes_R P \stackrel{\pi \otimes 1}{\to} \quad R \otimes_R P \stackrel{\cong}{\to} \quad P$$

is 1_P. A morphism $f : (P, e) \to (P', e')$ is an $R-$morphism $f : P \to P'$ such that the following diagram is commutative

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ {}^{e}\downarrow & & \downarrow^{e'} \\ E\otimes_R P & \xrightarrow{1\otimes f} & E\otimes_R P' \end{array}$$

An object (P,e) in $\xi(R)$ is called *finite* if and only if P is a finitely generated $R-$module.

Definition 3.3 Let $\xi(R)$ be a lift category and $F_0 : \xi(R) \to R-$Proj the forgetful functor. We define $\mathcal{H}$ as the class of sequences $Y \xrightarrow{i} Z \xrightarrow{d} X$ in $\xi(R)$ such that the sequence $0 \to F_0(Y) \to F_0(Z) \to F_0(X) \to 0$ is exact. It is known ([5]) that $\mathcal{H}$ is an exact structure, and we will always associate this structure to any lift category.

Definition 3.4 Let $\mathcal{A}$ be an additive category. For X an object in the category $\mathcal{A}$ and $\Gamma_X = \mathrm{End}_{\mathcal{A}}(X)^{op}$, we denote by $G_X : \mathcal{A} \to \mathrm{Mod}\,\Gamma_X$ the evaluation functor $\mathrm{Hom}_{\mathcal{A}}(X,?)$.

Proposition 3.5 (II.2.1 [1]) *Let $\mathcal{A}$ be an additive Krull-Schmidt category with splitting idempotents. Let X be in $\mathcal{A}$, then:*

1. $G_X : \mathrm{Hom}_{\mathcal{A}}(W,Z) \to \mathrm{Hom}_{\Gamma_X}(G_X(W), G_X(Z))$ *is an isomorphism for W in $addX$ and Z in $\mathcal{A}$.*
2. *If W is in $addX$ then $G_X(W)$ is in $\mathcal{P}(\Gamma_X)$.*
3. *$G_X|_{addX} : addX \to \mathcal{P}(\Gamma_X)$ is an equivalence of categories.*

Remark 3.6 Let $\mathcal{A}$ be an additive Krull-Schmidt category with splitting idempotents and $X,Y \in \mathcal{A}$.

Assume $X = \oplus_i^n X_i$ and $Y = \oplus_t^m Y_t$, where each summand is indecomposable and the summands are pairwise non-isomorphic. It is clear that $G_X(Y)$ is a $\Gamma_X - \Gamma_Y-$bimodule.

Let e_i be the idempotent of Γ_X determined by X_i and $W \cong \oplus_i^n c_i X_i$, then $\amalg_i^n c_i \Gamma_X e_i \cong G_X(W)$ as Γ_X-modules.

Now let Z be in $addY$, there is a Γ_X-isomorphism $\phi_Z : G_X(Y) \otimes_{\Gamma_Y} G_Y(Z) \to G_X(Z)$ given by $u \otimes v \mapsto vu$.

Moreover, if $g : Z \to Z'$ is a morphism in $addY$ we have a commutative diagram of Γ_X-modules:

$$\begin{array}{ccc} G_X(Y)\otimes_{\Gamma_Y} G_Y(Z) & \xrightarrow{\phi_Z} & G_X(Z) \\ \downarrow^{1\otimes G_Y(g)} & & \downarrow^{G_X(g)} \\ G_X(Y)\otimes_{\Gamma_Y} G_Y(Z') & \xrightarrow{\phi_{Z'}} & G_X(Z') \end{array}$$

This remark ends with the next convention: if $\mathcal{A}$ is an additive Krull-Schmidt category with splitting idempotents, it always has the exact structure of the trivial extensions; if it is not indicated in other way, we think in $\mathcal{A}$ as an exact category with trivial extensions.

Proposition 3.7 *Let $\mathcal{A}$ be an additive Krull-Schmidt category with splitting idempotents, X and Y in $\mathcal{A}$ where $X = \oplus_i^n X_i$, $Y = \oplus_t^m Y_t$, and $(X_1, ..., X_n)$ and $(Y_1, ..., Y_m)$ are pairwise non-isomorphic indecomposable objects in $\mathcal{A}$. Then there is an equivalence of categories $\Theta : Morph\,(addX, addY) \rightarrow \xi_Y^X\left(R_Y^X\right)$, where the lift category is determined by the splitting lift pair $\left(R_Y^X, \xi_Y^X\right)$:*

$$0 \rightarrow \begin{pmatrix} 0 & Hom_{\mathcal{A}}(X,Y) \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \Gamma_X & Hom_{\mathcal{A}}(X,Y) \\ 0 & \Gamma_Y \end{pmatrix} \rightarrow \begin{pmatrix} \Gamma_X & 0 \\ 0 & \Gamma_Y \end{pmatrix} \rightarrow 0$$

Moreover the functor Θ is an exact functor, i.e., it sends $\mathcal{E}_{addY}^{addX}$-extensions to $\mathcal{H}$-extensions in $\xi_Y^X\left(R_Y^X\right)$.

Proof. $\xi_Y^X\left(R_Y^X\right)$ is equivalent to the category of Γ_X-morphisms $t : P_X \rightarrow \mathrm{Hom}_{\mathcal{A}}(X,Y) \otimes_{\Gamma_Y} P_Y$, where P_X and P_Y are Γ_X-projective and Γ_Y-projective modules respectively.

For $\alpha \in \mathrm{Hom}_{\mathcal{A}}(W,Z)$ in $Morph\,(addX, addY)$ put $\Theta(\alpha) = \phi_Z^{-1} G_X(\alpha)$. Let $\alpha' \in \mathrm{Hom}_{\mathcal{A}}(W',Z')$ and $(f,g) : \alpha \rightarrow \alpha'$ be in $Morph\,(addX, addY)$ we define $\Theta(f,g) = \left(G_X(f), 1_{\mathrm{Hom}_{\mathcal{A}}(X,Y)} \otimes G_Y(g)\right)$.

The functor Θ is dense by proposition 3.5.3. By proposition 3.5.1 and remark 3.6 it follows that Θ is a full and faithful functor. The exactness is immediate. □

4 Complexes and projective resolutions

In this section we see some relations between the homotopy category of upper bounded complexes over $\mathrm{Proj}\,\Lambda$ with bounded homology and the complexes of fixed size over $\mathrm{Proj}\,\Lambda$ (see proposition 4.7).

Notation 4.1 Let $\mathcal{A}$ be an additive category.

1. Denote by $C(\mathcal{A})$ the category of complexes over $\mathcal{A}$, a complex $X \in C(\mathcal{A})$ is a sequence $\left(X^i, d_X^i\right)_{i\in\mathbb{Z}}$ with $X^i \in \mathcal{A}$ and $d_X^i : X^i \rightarrow X^{i+1}$ morphisms in $\mathcal{A}$ such that $d_X^{i+1} d_X^i = 0$. If $X = \left(X^i, d_X^i\right)_{i\in\mathbb{Z}}$ and $Y = \left(Y^i, d_Y^i\right)_{i\in\mathbb{Z}}$ are two complexes, a morphism $f : X \rightarrow Y$ is a sequence of morphisms in $\mathcal{A}$, $\left(f^i : X^i \rightarrow Y^i\right)_{i\in\mathbb{Z}}$ such that $f^{i+1} d_X^i = d_Y^i f^i$.

2. If $X \in C(\mathcal{A})$ and $s \in \mathbb{Z}$, the translation functors are defined by $(X[s])^i = X^{i+s}$ and $\left(d_{X[s]}\right)^i = (-1)^s (d_X)^{i+s}$.

3. Recall that $f, g \in \mathrm{Hom}_{C(\mathcal{A})}(X,Y)$ are homotopic if there are morphisms $h^i : X^i \rightarrow Y^{i-1}$ such that $f^i - g^i = h^{i+1}(d_X)^i + (d_Y)^{i-1} h_i$ for all $i \in \mathbb{Z}$. For X and Y complexes in $C(\mathcal{A})$, we denote by $\mathrm{Hom}_K(X,Y)$ the homomorphisms in the homotopy category.

Definition 4.2

1. We denote by $C^{\leq m}(\mathcal{A})$ the full subcategory of complexes $X \in C(\mathcal{A})$ such that $X^i = 0$ for $i > m$, and by $C^{\geq m}(\mathcal{A})$ the full subcategory of complexes $X \in C(\mathcal{A})$ such that $X^i = 0$ for $i < m$. For $C^{[m,n]}(\mathcal{A})$ we mean the intersection $C^{\leq n}(\mathcal{A}) \cap C^{\geq m}(\mathcal{A})$.

2. Let us denote by $t_m : C(\mathcal{A}) \to C^{\leq m}(\mathcal{A})$ the "erase at right" functor, given in objects as $t_m(X) = ((t_m(X))^i, d^i_{t_m(X)})$:

$$(t_m(X))^i = \left\{ \begin{array}{cc} X^i & \text{if } m \geq i \\ 0 & \text{otherwise} \end{array} \right\}, \quad \left(d_{t(m)(X)}\right)^i = \left\{ \begin{array}{cc} d^i_X & \text{if } i < m \\ 0 & \text{otherwise} \end{array} \right\}$$

If $f : X \to Y$ is a morphism of complexes then $t_m(f) = ((t_m(f)^i)$ where

$$t_m(f)^i = \left\{ \begin{array}{cc} f^i & \text{if } m \geq i \\ 0 & \text{otherwise} \end{array} \right\}$$

Dually we define the functor $l_m : C(\mathcal{A}) \to C^{\geq m}(\mathcal{A})$ "erase at left".

Now we denote by $s_m : C(\mathcal{A}) \to C^{\geq m}(\mathcal{A})$ the functor "erase and pull", given in objects as follows:

$$s_m(X)^i = \left\{ \begin{array}{cc} X^{i+1} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{array} \right\}, \quad d^i_{s_m(X)} = \left\{ \begin{array}{cc} d^{i+1}_X & \text{if } i \geq m \\ 0 & \text{otherwise} \end{array} \right\}$$

If $f : X \to Y$ is a morphism of complexes then:

$$(s_m(f))^i = \left\{ \begin{array}{cc} f^{i+1} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{array} \right\}$$

3. We define the m-bending functor

$$\urcorner_m : C(\mathcal{A}) \to Morph\left(C^{\leq m}(\mathcal{A}), C^{\geq m}(\mathcal{A})\right)$$

as follows: $\urcorner_m(X) = \left(u^i\right) : t_m(X) \to s_m(X)$ where $u^m = d^m_X$ and $u^i = 0$ for $i \neq m$, and for a morphism $f : X \to Y$ we have $\urcorner_m(f) = (t_m(f), s_m(f))$.

Remark 4.3 In $C(\mathcal{A})$ there is a natural exact structure $\mathcal{E}$ given by composable pairs $f : X \to Y$, $g : Y \to Z$ such that $0 \to X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \to 0$ are split exact for all $i \in \mathbb{Z}$. The exact category $(C(\mathcal{A}), \mathcal{E})$ has enough projectives and enough injectives, moreover the projectives coincide with the injectives. The stable category $\underline{C}(\mathcal{A})$, which is the category with the same objects as $C(\mathcal{A})$ and morphisms those in $C(\mathcal{A})$ modulo the morphisms which are factorized through projectives, coincides with $K(\mathcal{A})$, the homotopy category of $C(\mathcal{A})$.

Remark 4.4 Observe that by proposition 2.3, there are canonical induced exact structures on $C^{\leq n}(\mathcal{A})$, $C^{\geq m}(\mathcal{A})$ and on $C^{\leq n}(\mathcal{A}) \cap C^{\geq m}(\mathcal{A})$.

The definitions introduced in this section allow us to describe in a short way the next result.

Lemma 4.5 *$\urcorner_m : C(\mathcal{A}) \to Morph\left(C^{\leq m}(\mathcal{A}), C^{\geq m}(\mathcal{A})\right)$ is an exact isomorphism of categories for any $m \in \mathbb{Z}$.*

Since $(C(\operatorname{Proj}\Lambda), \mathcal{E})$ is an exact category for X, Y in $C(\operatorname{Proj}\Lambda)$ and n a positive integer we have the extension group $\operatorname{Ext}^n_{C(\operatorname{Proj}\Lambda)}(X,Y)$, see 12.3 of [8]. For $n = 1$, $\operatorname{Ext}^1_{C(\operatorname{Proj}\Lambda)}(X,Y)$ coincides, as in abelian categories, with the set of equivalence classes of sequences in $\mathcal{E}$, $Y \to E \to X$. For X, Y in $\mathcal{U}$, a full subcategory of $C(\operatorname{Proj}\Lambda)$ closed under extensions, we have the extension groups $\operatorname{Ext}^j_{\mathcal{U}}(X,Y)$ corresponding to the induced exact structure on $\mathcal{U}$. Through the paper we say that W an object of $\mathcal{U}$, is projective (respectively injective) if for all $X \in \mathcal{U}$, $\operatorname{Ext}^1_{\mathcal{U}}(W,X) = 0$ (respectively $\operatorname{Ext}^1_{\mathcal{U}}(X,W) = 0$).

For Y in the category $C(\operatorname{Proj}\Lambda)$ there is an exact sequence in $\mathcal{E}$, $Y \to W \to Y[1]$ with W injective. Then for all $X \in C(\operatorname{Proj}\Lambda)$ we have $\operatorname{Ext}^1_{C(\operatorname{Proj}\Lambda)}(X,Y) \cong \operatorname{Hom}_{C(\operatorname{Proj}\Lambda)}(X, Y[1])/I(X, Y[1]) \cong \operatorname{Hom}_K(X, Y[1])$, where $I(X, Y[1])$ is the subspace of morphisms which are factorized through injectives. If u is any integer and P a projective Λ-module we define the complex $J_u(P)$ in $C(\operatorname{Proj}\Lambda)$ as follows: $J_u(P)^i = 0$ for $i \neq u$, $i \neq u+1$, $J_u(P)^u = J_u(P)^{u+1} = P$, $d^u_{J_u(P)} = id_P$. The objects $J_u(P)$ are projectives and injectives in $C(\operatorname{Proj}\Lambda)$.

For integers m, n with $n \geq m+1$ and P a projective Λ-module we define the following complexes in $C^{[m,n]}(\operatorname{Proj}\Lambda)$:

$S(P)$ given by $S(P)^i = 0$ for $i \neq m$, $S(P)^m = P$; $T(P)$ defined by $T(P)^i = 0$ for $i \neq n$, $T(P)^n = P$. The projectives in $C^{[m,n]}(\operatorname{Proj}\Lambda)$ are the objects $J_u(P)$, $u \in [m, n-1]$ and $T(P)$, the injectives in $C^{[m,n]}(\operatorname{Proj}\Lambda)$ are the complexes $J_u(P)$, $u \in [m, n-1]$ and $S(P)$ (see corollary 3.3 of [4]). For $Y \in C^{[m,n]}(\operatorname{Proj}\Lambda)$ we have the $\mathcal{E}$-sequence:

$$Y \to \oplus_{u=n-1}^{m-1} J_u(Y^{u+1}) \to Y[1],$$

Taking l_m of the above sequence we obtain the $\mathcal{E}$-sequence in $C^{[m,n]}(\operatorname{Proj}\Lambda)$:

$$Y \to S(Y^m) \oplus_{u=m}^{n-1} J_u(Y^{u+1}) \to l_m(Y[1]).$$

Observe that if W is an injective in $C^{[m,n]}(\operatorname{Proj}\Lambda)$, any morphism $h : Y \to W$ is the sum of morphisms factorized through $S(Y^m)$ or through some $J_u(Y^{u+1})$, for $u \in [m, n-1]$. For $X, Y \in C^{[m,n]}(\operatorname{Proj}\Lambda)$, we denote by $I(X,Y)$ the subspace of morphisms which are factorized through injectives. The space $I(X,Y)$ is generated as k-module by the morphisms which are factorized through objects of the form $S(P)$ or $J_u(P)$ for $u \in [m, n-1]$.

We denote by $\overline{C^{[m,n]}}(\operatorname{Proj}\Lambda)$ the category with the same objects as those of $C^{[m,n]}(\operatorname{Proj}\Lambda)$ and morphisms the morphisms in $C^{[m,n]}(\operatorname{Proj}\Lambda)$ modulo those which are factorized through injectives. The homomorphisms from X to Y in this category are denoted by $\overline{\operatorname{Hom}}_{C^{[m,n]}(\operatorname{Proj}\Lambda)}(X,Y)$.

Lemma 4.6 *Let $Y \in C^{[m,n]}(\mathrm{Proj}\,\Lambda)$ with $H^{m+1}(Y) = 0$. Take $Y \xrightarrow{u} W \xrightarrow{v} Y[1]$ be an $\mathcal{E}$-sequence in $C^{[m,n]}(\mathrm{Proj}\,\Lambda)$ with W injective. Then we have:*

$$\mathrm{Ext}^1_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(X, Y) \cong \overline{\mathrm{Hom}}_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(X, Y[1]).$$

Proof. We have the exact sequence of k-modules: $\mathrm{Hom}_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(X, W) \xrightarrow{\mathrm{Hom}(1,v)} \mathrm{Hom}_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(X, Y[1]) \to \mathrm{Ext}^1_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(X, Y) \to 0$.

For proving our claim we prove that $\mathrm{ImHom}(1, v) = I(X, Y[1])$. For this we only need to prove that a morphism $h : X \to Y[1]$ which factorizes through $S(P)$ is in the image of $\mathrm{Hom}(1, v)$. But any morphism $h_1 : S(P) \to Y[1]$ is factorized by v if $\mathrm{Ext}^1_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(S(P), Y) = 0$. Now $\mathrm{Ext}^1_{C^{[m,n]}(\mathrm{Proj}\,\Lambda)}(S(P), Y) \cong \mathrm{Ext}^1_{C(\mathrm{Proj}\,\Lambda)}(S(P), Y) \cong \mathrm{Hom}_K(S(P), Y[1])$. Take $h : S(P) \to Y[1]$ a morphism of complexes then $h^m : P \to Y^{m+1}$ is such that $d_Y^{m+1} h^m = 0$. Since $H^{m+1}(Y) = 0$, there is a $g : P \to Y^m$ with $d_Y^m g = h^m$. This implies that h is homotopic to zero. Consequently $\mathrm{Hom}_K(S(P), Y[1]) = 0$, proving our claim.
□

Proposition 4.7 *Let W and Z be complexes in $C^{\leq 0}(\mathrm{Proj}\,\Lambda)$ with $H^i(W) = 0$ and $H^i(Z) = 0$ for $i \leq -t$ for some positive integer number t. Then, for $j > 0$ and $m \geq 0$*

$$\mathrm{Ext}^j_{C(\mathrm{Proj}\,\Lambda)}(W, Z) \cong \mathrm{Ext}^1_{C(\mathrm{Proj}\,\Lambda)}\left(l_{-(j+t+m)} W, (l_{-(1+t+m)} Z)[j-1]\right)$$

as $\mathrm{End}_{C(\mathrm{Proj}\,\Lambda)}(Z) - \mathrm{End}_{C(\mathrm{Proj}\,\Lambda)}(W)$ −bimodules.

Proof. We denote by $\mathcal{L}^{[-s,0]}$ the full subcategory of $K^{\leq 0}(\mathrm{Proj}\,\Lambda)$, the homotopy category of $C^{\leq 0}(\mathrm{Proj}\,\Lambda)$, whose objects are those X such that $H^i(X) = 0$ for $i \leq -s$. We recall (see for instance Corollary 5.7 of [4]) that l_{-s} induces an equivalence:

$$l_{-s} : \mathcal{L}^{[-s,0]} \to \overline{C^{[-s,0]}}(\mathrm{Proj}\,\Lambda).$$

For $s \geq j$, $l_{-t-s}W, l_{-t-s}(Z[j])$ are in $\mathcal{L}^{[-t-s,0]}$, then $\mathrm{Ext}^j_{C(\mathrm{Proj}\,\Lambda)}(W, Z) \cong \mathrm{Hom}_K(W, Z[j]) \cong \overline{\mathrm{Hom}}_{C^{[-t-s,0]}(\mathrm{Proj}\,\Lambda)}(l_{-t-s}W, l_{-t-s}(Z[j]))$. Observe we have $l_{-s-t}(Z[j]) = (l_{-s-t+j}Z)[j] = l_{-t-s}[(l_{-t-s+j-1}Z)[j]]$. Thus:

$$\mathrm{Ext}^j_{C(\mathrm{Proj}\,\Lambda)}(W, Z) \cong \overline{\mathrm{Hom}}_{C^{[-t-s,0]}(\mathrm{Proj}\,\Lambda)}(l_{-t-s}W, l_{-t-s}[(l_{-t-s+j-1}Z[j]])$$

Now $(l_{-t-s+j-1}Z)[j] = (l_{-t-s+j-1}Z)[j-1][1]$. We have that the complex $(l_{-t-s+j-1}Z)[j-1] \in C^{[-t-s,0]}(\mathrm{Proj}\,\Lambda)$. Moreover $((l_{-t-s+j-1}Z)[j-1])^{-t-s} = Z^{-t-s+j-1}$, and $((l_{-t-s+j-1}Z)[j-1])^{-t-s+1} = Z^{-t-s+j}$. Since $s \geq j$, $-t-s+j = -t-(s-j) \leq -t$. Therefore $H^{-t-s+1}((l_{-t-s+j-1}Z)[j-1])) = 0$. Then by our previous lemma we have:

$$\begin{aligned}
&\mathrm{Ext}^j_{C(\mathrm{Proj}\,\Lambda)}(W, Z) \cong \\
&\overline{\mathrm{Hom}}_{C^{[-t-s,0]}(\mathrm{Proj}\,\Lambda)}(l_{-t-s}W, l_{-t-s}[(l_{-t-s+j-1}Z[j-1][1]]) \\
&\cong \mathrm{Ext}^1_{C^{[-t-s,0]}(\mathrm{Proj}\,\Lambda)}(l_{-s-t}W, (l_{-t-s+j-1}Z)[j-1]) \\
&\cong \mathrm{Ext}^1_{C(\mathrm{Proj}\,\Lambda)}(l_{-s-t}W, (l_{-t-s+j-1}Z)[j-1]).
\end{aligned}$$

Taking $s = j + m$ we obtain our result.
□

For $M \in \operatorname{mod}\Lambda$ we choose a minimal projective resolution:

$$... \to P_M^{-m} \overset{d_M^{-m}}{\to} P_M^{-m+1} \overset{d_M^{-m+1}}{\to} ...P_M^{-1} \overset{d_M^{-1}}{\to} P_M^0 \overset{\eta_M}{\to} M \to 0$$

We denote by P_M the complex in $C^{\leq 0}(\operatorname{proj}\Lambda)$:

$$... \to P_M^{-m} \overset{d_M^{-m}}{\to} P_M^{-m+1} \overset{d_M^{-m+1}}{\to} ...P_M^{-1} \overset{d_M^{-1}}{\to} P_M^0 \to 0 \to 0...$$

Corollary 4.8 Let L and N be $\Lambda-$modules and P_L, P_N as above. Then, for $j \geq 0$ and $m \geq 0$

$$\operatorname{Ext}_\Lambda^j(M,N) \cong \operatorname{Ext}_{C(\operatorname{Proj}\Lambda)}\left(l_{-(1+j+m)}(P_M), l_{-(2+m)}(P_N)[j-1]\right).$$

as $\operatorname{End}_\Lambda(M) - \operatorname{End}_\Lambda(N)-$bimodules.

Proof. We know that

$$\operatorname{Ext}^j_\Lambda(M,N) \cong \operatorname{Hom}_K(P_M, P_N[j]) \cong \operatorname{Ext}^j_{C(\operatorname{Proj}\Lambda)}(P_M, P_N).$$

Now in proposition 4.7 put $W = P_M, Z = P_N$, then $t = 1$ and we obtain our result.
□

We will need the following results.

Lemma 4.9 *Suppose* $Y \in C^{[-m+1,0]}(\Lambda-\text{proj})$ *is such that* $\operatorname{Im}d_Y^i \subset \operatorname{rad}Y^{i+1}$ *for all* $i \in \mathbb{Z}$ *and* $\dim_k H^j(Y) \leq c$ *for all* j *and for some* $u \in [-m+2, ..., 0]$, $\dim_k Y^u \leq d_u$, *then* $\dim_k Y^{u-1} \leq (d_u + c)L$, *with* $L = \dim_k \Lambda$.

Proof. We have $\dim_k Y^{u-1}/\operatorname{Ker}d_Y^{u-1} = \dim_k \operatorname{Im}d_Y^{u-1} \leq d_u$, moreover we know that $\dim_k \operatorname{Ker}d_Y^{u-1}/\operatorname{Im}d_Y^{u-2} \leq c$. Therefore $\dim_k Y^{u-1}/\operatorname{Im}d_Y^{u-2} \leq c + d_u$.

Here $\operatorname{Im}d_Y^{u-2} \subset \operatorname{rad}Y^{u-1}$, thus $\dim_k Y^{u-1}/\operatorname{rad}Y^{u-1} \leq \dim_k Y^{u-1}/\operatorname{Im}d_Y^{u-2}$. Consequently, $\dim_k Y^{u-1} \leq (c + d_u)L$.
□

Lemma 4.10 *Let* $Y \in C^{[-m+1,0]}(\operatorname{proj}\Lambda)$, *with* $\operatorname{Im}d_Y^i \subset \operatorname{rad}Y^{i+1}$ *for all* $i \in \mathbb{Z}$, *such that for all* $j \in \mathbb{Z}$, *we have the inequality* $\dim_k H^j(Y) \leq c$ *for some fixed* c. Then

$$\dim_k Y \leq c(mL + (m-1)L^2 + (m-2)L^3 + ... + 2L^{m-1} + L^m).$$

Proof. Here $Y^1 = 0$, then by our previous lemma, $\dim_k Y^0 \leq cL$. Then again by lemma 4.9 we have, $\dim_k Y^{-1} \leq c(L + L^2)$, $\dim_k Y^{-2} \leq c(L + L^2 + L^3)$, ..., $\dim_k Y^{-m+1} \leq c(L + L^2 + ... + L^m)$. From here we obtain our result.
□

Theorem 4.11 (See corollary 9 in [9]) *Let* Λ *be a finite-dimensional algebra over an algebraically closed field* k *and* $\mathbf{d} = (d_i)_{i\in\mathbb{Z}}$ *be a collection of non-negative* *integers with almost all $d_i = 0$. Then the family $\mathcal{U}(\mathbf{d})$ of objects $X \in \mathcal{D}^b(\Lambda)$ such that $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X, X[1]) = 0$ has only a finite number of isomorphism classes in $\mathcal{D}^b(\Lambda)$.*

Proof. We may assume $d_i = 0$ for all $i \leq t$ and $i > 0$. Consider now the family $\mathcal{V}(\mathbf{d})$ of those $P \in C^{\leq 0}(\mathrm{proj}\,\Lambda)$ such that $\dim_k H^i(P) = d_i$ for all $i \in \mathbb{Z}$, $\mathrm{Hom}_K(P, P[1]) = 0$ and for all $i \in \mathbb{Z}$, $\mathrm{Im} d_P^i \subset \mathrm{rad} P^{i+1}$. For each $X \in \mathcal{U}(\mathbf{d})$ we may choose a quasi-isomorphism $P_X \to X$ with $P_X \in C^{\leq 0}(\mathrm{proj}\,\Lambda)$.

We have $\mathrm{Hom}_K(P_X, P_X[1]) \cong \mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X, X[1])$. Clearly the assignament $X \mapsto P_X$ stablishes a bijection between the isomorphism classes of $\mathcal{U}(\mathbf{d})$ and those of $\mathcal{V}(\mathbf{d})$. If $P \in \mathcal{V}(\mathbf{d})$, then $P \in \mathcal{L}^{[-t-1,0]}$.

By proposition 4.7 for $P \in \mathcal{V}(\mathbf{d})$ we have: $0 = \mathrm{Hom}_K(P, P[1]) \cong$
$\mathrm{Ext}^1_{C(\mathrm{Proj}\,\Lambda)}(P, P) \cong \mathrm{Ext}^1_{C^{[-t-1,0]}(\mathrm{Proj}\,\Lambda)}(l_{-t-1}P, l_{-t-1}P)$.

Using Lemma 4.10 one can prove that there is a number $n(\mathbf{d})$, such that if $P \in \mathcal{V}(\mathbf{d})$ then $\sum_{i \in \mathbb{Z}} \dim_k (l_{-t-1}P)^i \leq n(\mathbf{d})$. Therefore the functor l_{-t-1} induces a bijection between the isomorphism classes of $\mathcal{V}(\mathbf{d})$ and the isomorphism classes of a subfamily of the family $\mathcal{F}(n(\mathbf{d}))$ consisting of the complexes $Z \in C^{[-t-1,0]}(\mathrm{proj}\,\Lambda)$ which have not injectives in this category as direct summands, $\mathrm{Ext}^1_{C^{[-t-1,0]}(\mathrm{proj}\,\Lambda)}(Z, Z) = 0$, and $\sum_{i \in \mathbb{Z}} \dim_k (Z)^i \leq n(\mathbf{d})$.

The category $C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda)$ is an abelian category with enough projectives, the projectives in this category are the complexes $T(P), J_u(P), u \in [-t-1, -1]$ introduced before. Then taking $H = \oplus_{u=-t-1}^{-1} J_u(\Lambda) \oplus T(\Lambda)$ and $\Gamma = \mathrm{End}_{C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda)}(H)$, the functor

$$F = \mathrm{Hom}_{C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda)}(H, -) : C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda) \to \mathrm{Mod}\,\Gamma$$

is an equivalence of abelian categories.

Now there is a number $m(\mathbf{d})$ such that for all $Z \in \mathcal{F}(n(\mathbf{d}))$, $\dim_k F(Z) \leq m(\mathbf{d})$.

Since the category $C^{[-t-1,0]}(\mathrm{proj}\,\Lambda)$ is a full subcategory of the category $C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda)$, closed under extensions, then for $Z \in \mathcal{F}(n(\mathbf{d}))$,
$0 = \mathrm{Ext}^1_{C^{[-t-1,0]}(\mathrm{proj}\,\Lambda)}(Z, Z) = \mathrm{Ext}^1_{C^{[-t-1,0]}(\mathrm{Mod}\,\Lambda)}(Z, Z) \cong \mathrm{Ext}^1_{\Gamma}(F(Z), F(Z))$.
Therefore F gives a bijection between the isomorphism classes of $\mathcal{F}(n(\mathbf{d}))$ and the isomorphism classes of a family of Γ-modules M with $\dim_k M \leq m(\mathbf{d})$ and $\mathrm{Ext}_\Gamma(M, M) = 0$. But by a result of D. Voigt ([10]), this last family has only a finite number of isomorphism classes. This implies that our family $\mathcal{U}(\mathbf{d})$ has only a finite number of isomorphism classes.

□

5 An application to modules

Let Λ be an artin algebra over a commutative artinian ring k. In this section we study under which conditions two finitely generated Λ-modules M and N with $\mathrm{Ext}_\Lambda(M, M) = 0$ and $\mathrm{Ext}_\Lambda(N, N) = 0$ are isomorphic. As before for $M \in \mathrm{mod}\,\Lambda$

we choose a minimal projective resolution:

$$... \to P_M^{-m} \overset{d_M^{-m}}{\to} P_M^{-m+1} \overset{d_M^{-m+1}}{\to} ...P_M^{-1} \overset{d_M^{-1}}{\to} P_M^0 \overset{\eta_M}{\to} M \to 0.$$

We denote by P_M the complex in $C^{\leq 0}(\operatorname{proj}\Lambda)$:

$$... \to P_M^{-m} \overset{d_M^{-m}}{\to} P_M^{-m+1} \overset{d_M^{-m+1}}{\to} ...P_M^{-1} \overset{d_M^{-1}}{\to} P_M^0 \to 0 \to 0...$$

For $M \in \operatorname{mod}\Lambda$ we put $\Omega(M) = \operatorname{Ker}(\eta_M)$.

Theorem 5.1 *Let M and N be in $\operatorname{mod}\Lambda$ such that $\operatorname{Ext}^1_\Lambda(M,M) = 0$ and $\operatorname{Ext}^1_\Lambda(N,N) = 0$. Then $M \cong N$ if and only if $M/\operatorname{rad}M \cong N/\operatorname{rad}N$ and $\Omega(M) \cong \Omega(N)$.*

Proof. Neccesity is obvious, let us to prove sufficiency.

Let $P'_M = l_{-2}(P_M)$ and $P'_N = l_{-2}(P_N)$.

By lemma 4.5 we have an exact isomorphism of categories

$$\urcorner_{-1} : C(\operatorname{Proj}\Lambda) \to Morph\left(C^{\leq -1}(\operatorname{Proj}\Lambda), C^{\geq -1}(\operatorname{Proj}\Lambda)\right)$$

wich induces an exact isomorphism of categories

$$\urcorner : C^{[-2,0]}(\operatorname{Proj}\Lambda) \to Morph\left(C^{[-2,-1]}(\operatorname{Proj}\Lambda), C^{[-1,-1]}(\operatorname{Proj}\Lambda)\right).$$

We denote by $\mathcal{M}_2$ the category $Morph\left(C^{[-2,-1]}(\operatorname{Proj}\Lambda), C^{[-1,-1]}(\operatorname{Proj}\Lambda)\right)$.

By corollary 4.8 we have $\operatorname{Ext}^1_{C(\operatorname{Proj}\Lambda)}(P'_M, P'_M) \cong \operatorname{Ext}^1_\Lambda(M,M) = 0$. Then

$$(1) \quad \operatorname{Ext}_{\mathcal{M}_2}(\urcorner(P'_M), \urcorner(P'_M)) = 0.$$

In a similar way

$$(2) \quad \operatorname{Ext}_{\mathcal{M}_2}(\urcorner(P'_N), \urcorner(P'_N)) = 0.$$

The categories of bounded complexes over the finitely generated projective Λ-modules are Krull-Schmidt categories, so using the functors "erase at right" and "pull and erase" we get $t_{-1}(P'_M) \cong \oplus_{i=1}^u a_i X_i$ and $s_{-1}(P'_M) \cong \oplus_{j=1}^v b_j Y_j$, $t_{-1}(P'_N) \cong \oplus_{i=1}^u c_i X_i$ $s_{-1}(P'_N) \cong \oplus_{j=1}^v h_j Y_j$, where all decompositions are sums of pairwise non-isomorphic indecomposable objects. Now $t_{-1}(P'_M)$ and $t_{-1}(P'_N)$ correspond to minimal projective resolutions of $\Omega(M)$ and $\Omega(N)$ respectively, since $\Omega(M) \cong \Omega(N)$ then $a_i = c_i$ for all $i \in \{1, ..., v\}$. On the other hand the only non zero module in the complexes $s_{-1}(P'_M)$ and $s_{-1}(P'_M)$ are P_M^0 and P_N^0 respectively, since $M/\operatorname{rad}M \cong N/\operatorname{rad}N$, $P_M^0 \cong P_N^0$. Therefore $s_{-1}(P'_M) \cong s_{-1}(P'_N)$ and this implies that $b_j = h_j$ for $j \in \{1, ..., u\}$. Let $X = \oplus_{i=1}^u X_i$ and $Y = \oplus_{j=1}^v Y_j$.

By proposition 3.7 we have an exact equivalence of categories

$$\Theta : Morph(addX, addY) \to \xi_Y^X\left(R_Y^X\right)$$

Then by (1) and (2) we have:

$$\mathrm{Ext}_{\xi_Y^X\left(R_Y^X\right)}\left(\Theta\urcorner\left(P'_M\right),\Theta\urcorner\left(P'_M\right)\right)=0,\quad \mathrm{Ext}_{\xi_Y^X\left(R_Y^X\right)}\left(\Theta\urcorner\left(P'_N\right),\Theta\urcorner\left(P'_N\right)\right)=0.$$

It follows by 5.1 of [5] that $\Theta\urcorner\left(P'_M\right)\cong\Theta\urcorner\left(P'_N\right)$. Then $P'_M\cong P'_N$ and consequently $M\cong N$.
□

Theorem 5.2 *Assume* M, N *in* $\mathrm{mod}\,\Lambda$ *have finite projective dimension and* $\mathrm{Ext}^1_\Lambda(M,M)\cong\mathrm{Ext}^1_\Lambda(N,N)=0$ *and, for all* $j>1$ $\mathrm{Ext}^j_\Lambda(M,\Omega^{j-1}(M))\cong\mathrm{Ext}^j_\Lambda(N,\Omega^{j-1}(N))=0$, *then* $M\cong N$ *if and only for all* j, $P_M^{-j}\cong P_N^{-j}$.

Proof. Neccesity is obvious, let us to prove sufficiency by induction on $m=max\left(p(M),p(N)\right)$, where $p(M)$ and $p(N)$ are the projective dimension of M and N respectively. If $m=0$ our claim is trivial. Suppose our claim proved for $m-1$, we will prove it for m. But $\mathrm{Ext}^1_\Lambda(\Omega(M),\Omega(M))\cong\mathrm{Ext}^2_\Lambda(M,\Omega(M))=0$ and for all $j>1$, $\mathrm{Ext}^j_\Lambda(\Omega(M),\Omega^{j-1}(\Omega(M)))\cong\mathrm{Ext}^{j+1}_\Lambda(M,\Omega^j(M))=0$. Thus $\Omega(M)$ satisfies the hypothesis of our theorem, similarly $\Omega(N)$ also satisfies the hypothesis of our theorem, since $p(\Omega(M))=p(\Omega(N))=m-1$ and $P^1_M/\mathrm{rad}P^1_M\cong\Omega(M)/\mathrm{rad}\Omega(M)\cong P^1_N/\mathrm{rad}P^1_N\cong\Omega(N)/\mathrm{rad}\Omega(N)$, by the induction hypothesis, $\Omega(M)\cong\Omega(N)$. Here $P^0_M\cong P^0_N$, then by theorem 5.1, we obtain $M\cong N$.
□

Example 5.3 We present an example of two non-isomorphic modules with both having finite minimal projective resolution and the same projectives. Let Λ be the $k-$algebra of the quiver

$$\circ_1 \quad \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \quad \circ_2 \quad \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \quad \circ_3$$

with the relation $\{\beta_1\alpha_1,\beta_2\alpha_2\}$. The representations

$$k \quad \underset{0}{\overset{1}{\rightrightarrows}} \quad k \quad \underset{1}{\overset{0}{\rightrightarrows}} \quad k \qquad\qquad k \quad \underset{1}{\overset{0}{\rightrightarrows}} \quad k \quad \underset{0}{\overset{1}{\rightrightarrows}} \quad k$$

have trivial selfextensions group but both have a minimal projective resolution with data $(...,0,...,0,\Lambda e_3,\Lambda e_2,\Lambda e_1)$.

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