On Modules and Complexes without Self-extensions
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Abstract
Let $\Lambda$ be an artin algebra over a commutative artinian ring, $k$. If $M$ is a finitely generated left $\Lambda$-module, we denote by $\Omega(M)$ the kernel of $\eta_M : P_M \to M$ a minimal projective cover. We prove that if $M$ and $N$ are finitely generated left $\Lambda$-modules and $\text{Ext}_1^\Lambda(M, M) = 0$, then $M \cong N$ if and only if $M/\text{rad}M \cong N/\text{rad}N$ and $\Omega(M) \cong \Omega(N)$.

Now if $k$ is an algebraically closed field and $(d_i)_{i \in \mathbb{Z}}$ is a sequence of non negative integers almost all of them zero, then we prove that the family of objects $X \in \mathcal{D}^b(\Lambda)$, the bounded derived category of $\Lambda$, with $\text{Hom}_{\mathcal{D}^b(\Lambda)}(X, X[1]) = 0$ and $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$, has only a finite number of isomorphism classes (see [9]).

1 Introduction
Let $\Lambda$ be an artin algebra over a commutative artinian ring $k$. We denote by $\text{Mod}\Lambda$ the category of left $\Lambda$-modules, by $\text{mod}\Lambda$, $\text{proj}\Lambda$ we denote the full subcategories of $\text{Mod}\Lambda$ whose objects are respectively, the finitely generated $\Lambda$-modules and the finitely generated projective $\Lambda$-modules. By $\mathcal{D}^b(\Lambda)$ we denote the bounded derived category of $\Lambda$.

For $M \in \text{mod}\Lambda$, consider $P_M^{\eta_M} M$ a minimal projective cover and $\Omega(M) = \text{ker}(\eta_M)$. Here we prove the following: if $M$, $N$ are in $\text{mod}\Lambda$ and $\text{Ext}_1^\Lambda(M, M) \cong \text{Ext}_1^\Lambda(N, N) = 0$, then $M \cong N$ if and only if $M/\text{rad}M \cong N/\text{rad}N$ and $\Omega(M) \cong \Omega(N)$.

For $M \in \text{mod}\Lambda$ with finite projective dimension consider a minimal projective resolution:

$$0 \to P_M^{-m(M)} \to P_M^{-m(M)+1} \to \ldots \to P_M^0 \xrightarrow{\eta_M} M \to 0.$$

Suppose that $M, N$ are in $\text{mod}\Lambda$, $\text{Ext}_1^\Lambda(M, M) \cong \text{Ext}_1^\Lambda(N, N) = 0$ and for all $j > 0$, $\text{Ext}_1^\Lambda(M, \Omega_j^{-1}(M)) \cong \text{Ext}_1^\Lambda(N, \Omega_j^{-1}(N)) = 0$, then we prove using the result above that $M \cong N$ if and only if $m(M) = m(N)$ and for all $j$, $P_M^{-j} \cong P_N^{-j}$.

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Now let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field $k$ and $d = (d_i)_{i \in \mathbb{Z}}$ be a collection of non-negative integers with almost all $d_i = 0$. Then the family $\mathcal{U}(d)$ of objects $X \in D^b(\Lambda)$ such that $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$ and $\text{Hom}_{D^b(\Lambda)}(X, X[1]) = 0$ has only a finite number of isomorphism classes in $D^b(\Lambda)$. This result is closely related with Corollary 9 of [9].

For the proof of the above mentioned results we see that some problems involving upper bounded complexes of finitely generated projective $\Lambda$-modules with bounded homology can be reduced to problems involving complexes of fixed size of finitely generated projective $\Lambda$-modules (see Proposition 4.7). Then in this last case we can apply properties of lift categories introduced by W.W Crawley-Boevey in [6].

2 Exact structures and exact subcategories

Here we recall the notion of exact category. For $(\mathcal{A}, \mathcal{E})$, an exact category, $\mathcal{B}, \mathcal{C}$ full subcategories of $\mathcal{A}$ closed under extensions we consider the category of morphisms $f : W \to Z$ with $W$ an object of $\mathcal{B}$ and $W$ an object of $\mathcal{C}$. Then we introduce an exact structure on this category of morphisms.

**Definition 2.1** Let $\mathcal{A}$ be an additive category. A pair of composable morphisms

\[
X \xrightarrow{i} Y \xrightarrow{d} Z
\]

is called exact if $i$ is kernel of $d$ and $d$ is cokernel of $i$.

Let $\mathcal{E}$ be a class of exact composable sequences $(i, d)$ in $\mathcal{A}$ closed under isomorphisms; we call $(i, d) \in \mathcal{E}$ a conflation, $i$ an inflation and $d$ a deflation.

$\mathcal{E}$ is an exact structure if it satisfies the following axioms:

$K1)$ $1_0$ is a deflation.

$K2)$ Composition of deflations is a deflation.

$K3)$ For every $h \in \mathcal{A}(Z, Z_0)$ and all deflation $d_0 \in \mathcal{A}(Y_0, Z_0)$ there exists a pullback

\[
\begin{array}{ccc}
Y & \xrightarrow{d} & Z \\
\downarrow g & & \downarrow h \\
Y_0 & \xrightarrow{d_0} & Z_0
\end{array}
\]

where $d$ is a deflation.

$K3^{op}$) For every $f \in \mathcal{A}(X, X_0)$ and all inflation $i \in \mathcal{A}(Y, Z)$ there exists a pushout

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow g \\
X_0 & \xrightarrow{i_0} & Y_0
\end{array}
\]

where $i_0$ is an inflation.
K4) Retractions in $\mathcal{A}$ have kernels.

In this situation we say that $(\mathcal{A}, \mathcal{E})$ is an exact category. For simplicity, if $\mathcal{E}$ is an exact structure we use extension instead of conflation.

**Remark 2.2** It is known ([7]) that the above axioms imply their duals.

Moreover, in an exact category the next claims are true ([7]): if $dd'$ is a deflation then $d$ is a deflation, if $i'j$ is an inflation then $i$ is an inflation.

Also, $K3$ induces a diagram of extensions

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{g} & & \downarrow{h} \\
X_0 & \xrightarrow{i_0} & Y_0 \\
\end{array}
\]


where the right square is a pullback and a pushout, and $K3^{op}$ a diagram of extensions

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{f} & & \downarrow{d} \\
X_0 & \xrightarrow{i_0} & Y_0 \\
\end{array}
\]

where the left square is a pullback and a pushout.

The following is a well known result.

**Proposition 2.3** Let $(\mathcal{A}, \mathcal{E})$ be an exact category, and $\mathcal{B}$ a full subcategory closed under direct summands and extensions. Then $(\mathcal{B}, \mathcal{E}_B)$ is an exact structure, where $\mathcal{E}_B$ is the restriction of the class $\mathcal{E}$ to $\mathcal{B}$.

**Definition 2.4** Let $\mathcal{A}$ be a category, and $\mathcal{B}$ and $\mathcal{C}$ subcategories of $\mathcal{A}$. We define the category $\text{Morph}(\mathcal{B}, \mathcal{C})$ as follows: the objects are the morphisms $f : X \to Y$ in $\mathcal{A}$ such that $X \in \mathcal{B}$ and $Y \in \mathcal{C}$, and a morphism from $f : X \to Y$ to $f' : X' \to Y'$ is a pair $(u, v)$ of morphisms $u : X \to X'$ in $\mathcal{B}$ and $v : Y \to Y'$ in $\mathcal{C}$ such that $f' u = v f$.

**Proposition 2.5** Let $(\mathcal{A}, \mathcal{E})$ be an exact category, and $\mathcal{B}$ and $\mathcal{C}$ full subcategories of $\mathcal{A}$ closed under direct summands and extensions. Then $\text{Morph}(\mathcal{B}, \mathcal{C})$ is an exact category, where $\mathcal{E}_C^B$ is the class of the pairs $((u_0, v_0), (u_1, v_1))$ such that $(u_0, u_1) \in \mathcal{E}_B$ and $(v_0, v_1) \in \mathcal{E}_C$.

**Proof.** Let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ and $f_3 : X_3 \to Y_3$ be objects in $\text{Morph}(\mathcal{B}, \mathcal{C})$ and $\eta : f_1 \xrightarrow{(u_1, v_1)} f_2 \xrightarrow{(u_2, v_2)} f_3$ an element of $\mathcal{E}_C^B$.

Clearly $\mathcal{E}_C^B$ is closed under isomorphisms.

Now we check that $\eta$ is an exact pair.

Suppose we have an object $f : X \to Y$ and a morphism $(u, v) : f \to f_2$ in $\text{Morph}(\mathcal{B}, \mathcal{C})$ such that $(u_2, v_2)(u, v) = 0$. Then there exist unique morphisms $s : X \to X_1$ and $t : Y \to Y_1$ in $\mathcal{A}$ such that $u_1 s = u$ and $v_1 t = v$. We have $v_1 (f_1 s - t f) = f_2 u_1 s - v f = 0$, then $f_1 s = t f$ following that $(s, t)$ is a morphism.

The proof of $(u_2, v_2)$ being cokernel is dual.
$K1$, $K2$ and $K4$ are immediate from the proposition 2.3.

In order to prove $K3$, let $f$ be as before, take $(u, v) : f \to f_3$ a morphism in $\text{Morph}(B, C)$, and consider the pullback diagrams

$$
\begin{array}{c}
X_1 \rightarrow X_0 \xrightarrow{d} X \\
\parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\
X_1 \xrightarrow{u} X_2 \xrightarrow{u_2} X_3
\end{array}
$$

$$
\begin{array}{c}
Y_1 \rightarrow Y_0 \xrightarrow{d'} Y \\
\parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \\
Y_1 \xrightarrow{v} Y_2 \xrightarrow{v_2} Y_3
\end{array}
$$

for the property of the pullback, there exists a unique morphism $f_0 : X_0 \to Y_0$ such that $d'f_0 = fd$ and $tf_0 = f_2s$.

Suppose there is an object $g : W \to Z$ and morphisms $(\lambda_1, \lambda_2) : g \to f_2$ and $(\mu_1, \mu_2) : g \to f_3$ in $\text{Morph}(B, C)$ such that $(u_2, v_2)(\lambda_1, \lambda_2) = (\mu_1, \mu_2)(u, v)$. Then, by the pullback property, there exist unique morphisms $\alpha : W \to X_0$ and $\beta : Z \to Y_0$ in $\mathcal{A}$ such that $sa = \lambda_1$, $da = \mu_1$, $t\beta = \lambda_2$ and $d'\beta = \mu_2$.

Now observe that $d'(f_0\alpha - \beta g) = f_2d\alpha - \mu_2g = 0$ and $t(f_0\alpha - \beta g) = f_2s\alpha - \lambda_2g = f_2\lambda_1 - \lambda_2g = 0$. By the pullback property $(\alpha, \beta)$ is a morphism in $\text{Morph}(B, C)$.

A dual argument proves $K3^{op}$.

\hspace{1cm} $\square$

3 Lift categories

Here we recall the properties of lift categories needed in our work. Now suppose $\mathcal{A}$ is a Krull-Schmidt category and $X, Y$ are objects in $\mathcal{A}$. We consider the category of morphisms in $\mathcal{A}$, $f : W \to Z$, with $W$ a finite direct sum of direct summands of $X$ and $Z$ a finite direct sum of direct summands of $Y$. We see that this category of morphisms can be seen as a lift category.

Definition 3.1 A lift pair $(R, \xi)$ is given by a ring $R$ and an exact sequence of $R$–bimodules

$$
\xi : \quad 0 \quad \rightarrow \quad M \rightarrow \quad E \xrightarrow{\pi} \quad R \rightarrow \quad 0
$$

Definition 3.2 Given a lift pair $(R, \xi)$ we define the lift category $\xi(R)$ as follows: the objects are pairs $(P, e)$ where $P$ is a projective $R$–module and $e : P \to E \otimes_R P$ is an $R$–morphism such that the composition

$$
P \xrightarrow{e} \quad E \otimes_R P \xrightarrow{\pi \otimes 1} \quad R \otimes_R P \xrightarrow{\pi} \quad P
$$

is $1_P$. A morphism $f : (P, e) \to (P', e')$ is an $R$–morphism $f : P \to P'$ such that the following diagram is commutative
An object \((P, e)\) in \(\xi(R)\) is called finite if and only if \(P\) is a finitely generated \(R\)-module.

**Definition 3.3** Let \(\xi(R)\) be a lift category and \(F_0 : \xi(R) \to R\text{-}\text{Proj}\) the forgetful functor. We define \(\mathcal{H}\) as the class of sequences \(Y \xrightarrow{i} Z \xrightarrow{d} X\) in \(\xi(R)\) such that the sequence \(0 \to F_0(Y) \to F_0(Z) \to F_0(X) \to 0\) is exact. It is known ([5]) that \(\mathcal{H}\) is an exact structure, and we will always associate this structure to any lift category.

**Definition 3.4** Let \(\mathcal{A}\) be an additive category. For \(X\) an object in the category \(\mathcal{A}\) and \(\Gamma_X = \text{End}_\mathcal{A}(X)^{op}\), we denote by \(G_X : \mathcal{A} \to \text{Mod} \Gamma_X\) the evaluation functor \(\text{Hom}_\mathcal{A}(X, ?)\).

**Proposition 3.5** (II.2.1 [1]) Let \(\mathcal{A}\) be an additive Krull-Schmidt category with splitting idempotents. Let \(X\) be in \(\mathcal{A}\), then:

1. \(G_X : \text{Hom}_\mathcal{A}(W, Z) \to \text{Hom}_\Gamma_X(G_X(W), G_X(Z))\) is an isomorphism for \(W\) in \(\text{add}\mathcal{A}\) and \(Z\) in \(\mathcal{A}\).
2. If \(W\) is in \(\text{add}\mathcal{A}\) then \(G_X(W)\) is in \(\mathcal{P}(\Gamma_X)\).
3. \(G_X|_{\text{add}\mathcal{A}} : \text{add}\mathcal{A} \to \mathcal{P}(\Gamma_X)\) is an equivalence of categories.

**Remark 3.6** Let \(\mathcal{A}\) be an additive Krull-Schmidt category with splitting idempotents and \(X, Y \in \mathcal{A}\).

Assume \(X = \bigoplus^n X_i\) and \(Y = \bigoplus^m Y_i\), where each summand is indecomposable and the summands are pairwise non-isomorphic. It is clear that \(G_X(Y)\) is a \(\Gamma_X - \Gamma_Y\)-bimodule.

Let \(e_i\) be the idempotent of \(\Gamma_X\) determined by \(X_i\) and \(W \cong \bigoplus^c_i c_i X_i\), then \(\Pi^c_i \Gamma_X e_i \cong G_X(W)\) as \(\Gamma_X\)-modules.

Now let \(Z\) be in \(\text{add}\mathcal{Y}\), there is a \(\Gamma_X\)-isomorphism \(\phi_Z : G_X(Y) \otimes_{\Gamma_Y} G_Y(Z) \to G_X(Z)\) given by \(u \otimes v \mapsto vu\).

Moreover, if \(g : Z \to Z'\) is a morphism in \(\text{add}\mathcal{Y}\) we have a commutative diagram of \(\Gamma_X\)-modules:

\[
\begin{array}{ccc}
G_X(Y) \otimes_{\Gamma_Y} G_Y(Z) & \xrightarrow{\phi_Z} & G_X(Z) \\
\downarrow^{1 \otimes G_Y(g)} & & \downarrow^{G_X(g)} \\
G_X(Y) \otimes_{\Gamma_Y} G_Y(Z') & \xrightarrow{\phi_{Z'}} & G_X(Z')
\end{array}
\]

This remark ends with the next convention: if \(\mathcal{A}\) is an additive Krull-Schmidt category with splitting idempotents, it always has the exact structure of the trivial extensions; if it is not indicated in other way, we think in \(\mathcal{A}\) as an exact category with trivial extensions.
In this section we see some relations between the homotopy category of upper 4 Complexes and projective resolutions bounded complexes over Proj with bounded homology and the complexes of fixed size over Proj (see proposition 4.7).

\((\text{to} H \text{Hom} 3.6 \text{it follows that } \Lambda \text{ is a full and faithful functor. The exactness is immediate.})\)

\(\text{Moreover the functor } \Theta \text{ is an exact functor, i.e., it sends } \xi_{addX} \text{-extensions to } H_{\text{extensions in } \xi_X Y (R_{Y}^X)} \).

\(\text{Proof. } \xi_X Y (R_{Y}^X) \text{ is equivalent to the category of } \Gamma_X \text{-morphisms } t : P_X \rightarrow \text{Hom}_{\Lambda} (X, Y) \otimes \Gamma_Y P_Y, \text{ where } P_X \text{ and } P_Y \text{ are } \Gamma_X \text{-projective and } \Gamma_Y \text{-projective modules respectively.}

\text{For } \alpha \in \text{Hom}_{\Lambda} (W, Z) \text{ in } \text{Morph (addX, addY) put } \Theta (\alpha) = \phi_Z^{-1} G_X (\alpha). \text{ Let } \alpha' \in \text{Hom}_{\Lambda} (W', Z') \text{ and } (f, g) : \alpha \rightarrow \alpha' \text{ be in } \text{Morph (addX, addY) we define } \Theta (f, g) = (G_X (f), 1_{\text{Hom}_{\Lambda} (X, Y)} \otimes G_Y (g)).

\text{The functor } \Theta \text{ is dense by proposition 3.5.3. By proposition 3.5.1 and remark 3.6 it follows that } \Theta \text{ is a full and faithful functor. The exactness is immediate.} \)

\(\Box\)

\(\textbf{4 Complexes and projective resolutions}\)

In this section we see some relations between the homotopy category of upper bounded complexes over Proj with bounded homology and the complexes of fixed size over Proj (see proposition 4.7).

\(\textbf{Notation 4.1} \text{ Let } \Lambda \text{ be an additive category.}\)

1. Denote by \(C(\Lambda)\) the category of complexes over \(\Lambda\), a complex \(X \in C(\Lambda)\) is a sequence \((X^i, d_X^i)_{i \in \mathbb{Z}}\) with \(X^i \in \Lambda\) and \(d_X^i : X^i \rightarrow X^{i+1}\) morphisms in \(\Lambda\) such that \(d_X^{i+1} d_X^i = 0\). If \(X = (X^i, d_X^i)_{i \in \mathbb{Z}}\) and \(Y = (Y^i, d_Y^i)_{i \in \mathbb{Z}}\) are two complexes, a morphism \(f : X \rightarrow Y\) is a sequence of morphisms in \(\Lambda\), \((f^i : X^i \rightarrow Y^i)_{i \in \mathbb{Z}}\) such that \(f^{i+1} d_Y^i = d_Y^i f^i\).

2. If \(X \in C(\Lambda)\) and \(s \in \mathbb{Z}\), the translation functors are defined by \((X[s])^i = X^{i+s}\) and \((d_X[s])^i = (-1)^s (d_X)^{i+s}\).

3. Recall that \(f, g \in \text{Hom}_{C(\Lambda)} (X, Y)\) are homotopic if there are morphisms \(h^i : X^i \rightarrow Y^{i-1}\) such that \(f^i - g^i = h^{i+1} d_X^i + (d_Y)^{i-1} h_i\), for all \(i \in \mathbb{Z}\). For \(X\) and \(Y\) complexes in \(C(\Lambda)\), we denote by \(\text{Hom}_K (X, Y)\) the homomorphisms in the homotopy category.

\(\textbf{Definition 4.2}\)
1. We denote by $C^{\leq m}(A)$ the full subcategory of complexes $X \in C(A)$ such that $X^i = 0$ for $i > m$, and by $C^{\geq m}(A)$ the full subcategory of complexes $X \in C(A)$ such that $X^i = 0$ for $i < m$. For $C^{[m,n]}(A)$ we mean the intersection $C^{\leq n}(A) \cap C^{\geq m}(A)$.

2. Let us denote by $t_m : C(A) \to C^{\leq m}(A)$ the “erase at right” functor, given in objects as $t_m(X) = ((t_m(X))^i, d_{t_m(X)}^i)$:

$$(t_m(X))^i = \begin{cases} X^i & \text{if } m \geq i \\ 0 & \text{otherwise} \end{cases}, \quad (d_{t_m(X)}^i)^i = \begin{cases} dX^i & \text{if } i < m \\ 0 & \text{otherwise} \end{cases}$$

If $f : X \to Y$ is a morphism of complexes then $t_m(f) = ((t_m(f))^i)$ where

$$t_m(f)^i = \begin{cases} f^i & \text{if } m \geq i \\ 0 & \text{otherwise} \end{cases}$$

Dually we define the functor $l_m : C(A) \to C^{\geq m}(A)$ “erase at left”.

Now we denote by $s_m : C(A) \to C^{\geq m}(A)$ the functor “erase and pull”, given in objects as follows:

$$s_m(X)^i = \begin{cases} X^{i+1} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{cases}, \quad (d_{s_m(X)}^i)^i = \begin{cases} dX^{i+1} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{cases}$$

If $f : X \to Y$ is a morphism of complexes then:

$$(s_m(f))^i = \begin{cases} f^{i+1} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{cases}$$

3. We define the m-bending functor

$$\gamma_m : C(A) \to \text{Morph} (C^{\leq m}(A), C^{\geq m}(A))$$

as follows: $\gamma_m(X) = (u^i) : t_m(X) \to s_m(X)$ where $u^m = dX^m$ and $u^i = 0$ for $i \neq m$, and for a morphism $f : X \to Y$ we have $\gamma_m(f) = (t_m(f), s_m(f))$.

**Remark 4.3** In $C(A)$ there is a natural exact structure $E$ given by composable pairs $f : X \to Y$, $g : Y \to Z$ such that $0 \to X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \to 0$ are split exact for all $i \in \mathbb{Z}$. The exact category $(C(A), E)$ has enough projectives and enough injectives, moreover the projectives coincide with the injectives. The stable category $\underline{C}(A)$, which is the category with the same objects as $C(A)$ and morphisms those in $C(A)$ modulo the morphisms which are factorized through projectives, coincides with $K(A)$, the homotopy category of $C(A)$.  

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Remark 4.4 Observe that by proposition 2.3, there are canonical induced exact structures on $C^{\leq n}(A)$, $C^{\geq m}(A)$ and on $C^{\leq n}(A) \cap C^{\geq m}(A)$.

The definitions introduced in this section allow us to describe in a short way the next result.

Lemma 4.5 $\gamma_m : C(A) \to \text{Morph} \left( C^{\leq m}(A), C^{\geq m}(A) \right)$ is an exact isomorphism of categories for any $m \in \mathbb{Z}$.

Since $(C(\text{Proj} A), \mathcal{E})$ is an exact category for $X,Y$ in $(C(\text{Proj} A)$ and $n$ a positive integer we have the extension group $\text{Ext}^n_{C(\text{Proj} A)}(X,Y)$, see 1.2.3 of [8]. For $n = 1$, $\text{Ext}^1_{C(\text{Proj} A)}(X,Y)$ coincides, as in abelian categories, with the set of equivalence classes of sequences in $\mathcal{E}$, $Y \to E \to X$. For $X,Y$ in $\mathcal{U}$, a full subcategory of $(C(\text{Proj} A)$ closed under extensions, we have the extension groups $\text{Ext}^1_{\mathcal{U}}(X,Y)$ corresponding to the induced exact structure on $\mathcal{U}$. Through the paper we say that $W$ an object of $\mathcal{U}$, is projective (respectively injective ) if for all $X \in \mathcal{U}$, $\text{Ext}^1_{\mathcal{U}}(W,X) = 0$ (respectively $\text{Ext}^1_{\mathcal{U}}(X,W) = 0$).

For $Y$ in the category $C(\text{Proj} A)$ there is an exact sequence in $\mathcal{E}$, $Y \to W \to Y[1]$ with $W$ injective. Then for all $X \in C(\text{Proj} A)$ we have $\text{Ext}^1_{C(\text{Proj} A)}(X,Y) \cong \text{Hom}_{C(\text{Proj} A)}(X,Y[1])/I(X,Y[1]) \cong \text{Hom}_K(X,Y[1])$, where $I(X,Y[1])$ is the subspace of morphisms which are factorized through injectives. If $u$ is any integer and $P$ a projective $A$-module we define the complex $J_u(P)$ in $C(\text{Proj} A)$ as follows: $J_u(P)^i = 0$ for $i \neq u$, $i \neq u + 1$, $J_u(P)^u = J_u(P)^{u+1} = P$, $d^u_{J_u(P)} = id_P$. The objects $J_u(P)$ are projectives and injectives in $C(\text{Proj} A)$.

For integers $m,n$ with $n \geq m + 1$ and $P$ a projective $A$-module we define the following complexes in $C^{[m,n]}(\text{Proj} A)$:

$S(P)$ given by $S(P)^i = 0$ for $i \neq m$, $S(P)^m = P$; $T(P)$ defined by $T(P)^i = 0$ for $i \neq n$, $T(P)^n = P$. The projectives in $C^{[m,n]}(\text{Proj} A)$ are the objects $J_u(P)$, $u \in [m,n-1]$ and $T(P)$, the injectives in $C^{[m,n]}(\text{Proj} A)$ are the complexes $J_u(P)$, $u \in [m,n-1]$ and $S(P)$ (see corollary 3.3 of [4]). For $Y \in C^{[m,n]}(\text{Proj} A)$ we have the $\mathcal{E}$-sequence:

$$Y \to \oplus_{u=n-1}^{m-1} J_u(Y^{u+1}) \to Y[1],$$

Taking $l_m$ of the above sequence we obtain the $\mathcal{E}$-sequence in $C^{[m,n]}(\text{Proj} A)$:

$$Y \to S(Y^m) \oplus_{u=m}^{n-1} J_u(Y^{u+1}) \to l_m(Y[1]).$$

Observe that if $W$ is an injective in $C^{[m,n]}(\text{Proj} A)$, any morphism $h : Y \to W$ is the sum of morphisms factorized through $S(Y^m)$ or through some $J_u(Y^{u+1})$, for $u \in [m,n-1]$. For $X,Y \in C^{[m,n]}(\text{Proj} A)$, we denote by $I(X,Y)$ the subspace of morphisms which are factorized through injectives. The space $I(X,Y)$ is generated as $k$-module by the morphisms which are factorized through objects of the form $S(P)$ or $J_u(P)$ for $u \in [m,n-1]$.

We denote by $\text{Coker}_{C^{[m,n]}(\text{Proj} A)}(X,Y)$ the category with the same objects as those of $C^{[m,n]}(\text{Proj} A)$ and morphisms the morphisms in $C^{[m,n]}(\text{Proj} A)$ modulo those which are factorized through injectives. The homomorphisms from $X$ to $Y$ in this category are denoted by $\text{Coker}_{C^{[m,n]}(\text{Proj} A)}(X,Y)$.
Lemma 4.6 Let $Y \in C^{[m,n]}(\text{Proj} \Lambda)$ with $H^{m+1}(Y) = 0$. Take $Y \xrightarrow{\theta} W \xrightarrow{\psi} Y[1]$ be an $E$-sequence in $C^{[m,n]}(\text{Proj} \Lambda)$ with $W$ injective. Then we have:

$$\text{Ext}_{C^{[m,n]}(\text{Proj} \Lambda)}^1(X, Y) \cong \text{Hom}_{C^{[m,n]}(\text{Proj} \Lambda)}(X, Y[1]).$$

Proof. We have the exact sequence of $k$-modules: $\text{Hom}_{C^{[m,n]}(\text{Proj} \Lambda)}(X, W) \xrightarrow{\text{Hom}(1, v)} \text{Hom}_{C^{[m,n]}(\text{Proj} \Lambda)}(X, Y[1]) \rightarrow \text{Ext}_{C^{[m,n]}(\text{Proj} \Lambda)}^1(X, Y) \rightarrow 0$.

For proving our claim we prove that $\text{ImHom}(1, v) = I(X, Y[1])$. For this we only need to prove that a morphism $h : X \rightarrow Y[1]$ which factorizes through $S(P)$ is in the image of $\text{Hom}(1, v)$. But any morphism $h_1 : S(P) \rightarrow Y[1]$ is factorized by $v$ if $\text{Ext}_{C^{[m,n]}(\text{Proj} \Lambda)}^1(S(P), Y) = 0$. Now $\text{Ext}_{C^{[m,n]}(\text{Proj} \Lambda)}^1(S(P), Y) \cong \text{Ext}_{C^{[m,n]}(\text{Proj} \Lambda)}^1(S(P), Y) \cong \text{Hom}_K(S(P), Y[1])$. Take $h : S(P) \rightarrow Y[1]$ a morphism of complexes then $h^m : P \rightarrow Y^{m+1}$ is such that $d_Y^m h^m = 0$. Since $H^{m+1}(Y) = 0$, there is a $g : P \rightarrow Y^m$ with $d_Y^m g = h^m$. This implies that $h$ is homotopic to zero. Consequently $\text{Hom}_K(S(P), Y[1]) = 0$, proving our claim.

□

Proposition 4.7 Let $W$ and $Z$ be complexes in $C^{\leq 0}(\text{Proj} \Lambda)$ with $H^i(W) = 0$ and $H^i(Z) = 0$ for $i \leq -t$ for some positive integer number $t$. Then, for $j > 0$ and $m \geq 0$

$$\text{Ext}_{C(\text{Proj} \Lambda)}^j(W, Z) \cong \text{Ext}_{C(\text{Proj} \Lambda)}^1(l_{(j+t+m)}W, l_{(1+t+m)}Z)[j - 1]$$

as $\text{End}_{C(\text{Proj} \Lambda)}(Z) - \text{End}_{C(\text{Proj} \Lambda)}(W)$-bimodules.

Proof. We denote by $L^{[-s,0]}$ the full subcategory of $K^{\leq 0}(\text{Proj} \Lambda)$, the homotopy category of $C^{\leq 0}(\text{Proj} \Lambda)$, whose objects are those $X$ such that $H^i(X) = 0$ for $i \leq -s$. We recall (see for instance Corollary 5.7 of [4]) that $l_{-s}$ induces an equivalence:

$$l_{-s} : L^{[-s,0]} \rightarrow \overline{C^{[-s,0]}(\text{Proj} \Lambda)}.$$

For $s \geq j$, $l_{-t-s}W, l_{-t-s}(Z[j])$ are in $L^{[-t-s,0]}$, then $\text{Ext}_{C(\text{Proj} \Lambda)}^j(W, Z) \cong \text{Hom}_K(W, Z[j]) \cong \text{Hom}_{C^{[-t-s,0]}(\text{Proj} \Lambda)}(l_{-t-s}W, l_{-t-s}(Z[j]))$. Observe we have $l_{-s-t}(Z[j]) = l_{-t-s}(l_{-t-s}(Z[j])[j])$. Thus:

$$\text{Ext}_{C(\text{Proj} \Lambda)}^j(W, Z) \cong \text{Hom}_{C^{[-t-s,0]}(\text{Proj} \Lambda)}(l_{-t-s}W, l_{-t-s}(l_{-t-s}(Z[j])[j])$$

Now $(l_{-t-s+j-1}Z[j]) = (l_{-t-s+j-1}Z)[j - 1][1]$. We have that the complex $(l_{-t-s+j-1}Z)[j - 1] \in C^{[-s,0]}(\text{Proj} \Lambda)$. Moreover $((l_{-t-s+j-1}Z)[j - 1])^{-t-s} = Z^{-t-s+j-1}$ and $((l_{-t-s+j-1}Z)[j - 1])^{-t-s+1} = Z^{-t-s+j}$. Since $s \geq j$, $-t - s + j = -t - (s - j) \leq -t$. Therefore $H^{-t-s+1}((l_{-t-s+j-1}Z)[j - 1]) = 0$. Then by our previous lemma we have:

$$\text{Ext}_{C(\text{Proj} \Lambda)}^j(W, Z) \cong \text{Hom}_{C^{[-t-s,0]}(\text{Proj} \Lambda)}(l_{-t-s}W, l_{-t-s}(l_{-t-s+j-1}Z)[j - 1][1]) \cong \text{Ext}_{C^{[-t-s,0]}(\text{Proj} \Lambda)}^j(l_{-s-t}W, (l_{-t-s+j-1}Z)[j - 1]) \cong \text{Ext}_{C(\text{Proj} \Lambda)}^1(l_{-s-t}W, (l_{-t-s+j-1}Z)[j - 1]).}$$
Taking \( s = j + m \) we obtain our result.

\[ \square \]

For \( M \in \text{mod } \Lambda \) we choose a minimal projective resolution:

\[ \cdots \rightarrow P_{M}^{-m} \rightarrow \cdots \rightarrow P_{M}^{-m+1} \rightarrow \cdots \rightarrow P_{M}^{-1} \rightarrow \cdots \rightarrow P_{M}^{0} \rightarrow P_{M}^{0} \\
\rightarrow M \rightarrow 0 \]

We denote by \( P_{M} \) the complex in \( C^{\leq 0}(\text{proj } \Lambda) \):

\[ \cdots \rightarrow P_{M}^{-m} \rightarrow \cdots \rightarrow P_{M}^{-m+1} \rightarrow \cdots \rightarrow P_{M}^{-1} \rightarrow P_{M}^{0} \rightarrow 0 \rightarrow 0 \ldots \]

**Corollary 4.8** Let \( L \) and \( N \) be \( \Lambda \)-modules and \( P_{L}, P_{N} \) as above. Then, for \( j \geq 0 \) and \( m \geq 0 \)

\[ \text{Ext}_{\Lambda}^{j}(M, N) \cong \text{Ext}_{C(\text{proj } \Lambda)}^{j}(l_{-(1+j+m)}(P_{M}), l_{-(2+m)}(P_{N})[j-1]) . \]

as \( \text{End}_{\Lambda}(M) - \text{End}_{\Lambda}(N) \)-bimodules.

**Proof.** We know that

\[ \text{Ext}_{\Lambda}^{j}(M, N) \cong \text{Hom}_{K}(P_{M}, P_{N}[j]) \cong \text{Ext}_{C(\text{proj } \Lambda)}^{j}(P_{M}, P_{N}). \]

Now in proposition 4.7 put \( W = P_{M}, Z = P_{N} \), then \( t = 1 \) and we obtain our result.

\[ \square \]

We will need the following results.

**Lemma 4.9** Suppose \( Y \in C^{[-m+1,0]}(\text{proj } \Lambda) \) is such that \( \text{Im}d^{i}_{Y} \subset \text{rad}Y^{i+1} \) for all \( i \in \mathbb{Z} \) and \( \dim_{k}H^{j}(Y) \leq c \) for all \( j \) and for some \( u \in [-m+2, \ldots, 0] \), \( \dim_{k}Y^{u} \leq d_{u} \), then \( \dim_{k}Y^{u-1} \leq (d_{u} + c)L \), with \( L = \dim_{k}\Lambda \).

**Proof.** We have \( \dim_{k}Y^{u-1}/\text{Ker}d^{u-1}_{Y} = \dim_{k}\text{Im}d^{u-1}_{Y} \leq d_{u} \), moreover we know that \( \dim_{k}\text{Ker}d^{u-1}_{Y}/\text{Im}d^{u-2}_{Y} \leq c \). Therefore \( \dim_{k}Y^{u-1}/\text{Im}d^{u-2}_{Y} \leq c + d_{u} \).

Here \( \text{Im}d^{u-2}_{Y} \subset \text{rad}Y^{u-1} \), thus \( \dim_{k}Y^{u-1}/\text{rad}Y^{u-1} \leq \dim_{k}Y^{u-1}/\text{Im}d^{u-2}_{Y} \).

Consequently, \( \dim_{k}Y^{u-1} \leq (c + d_{u})L \).

\[ \square \]

**Lemma 4.10** Let \( Y \in C^{[-m+1,0]}(\text{proj } \Lambda) \), with \( \text{Im}d^{i}_{Y} \subset \text{rad}Y^{i+1} \) for all \( i \in \mathbb{Z} \), such that for all \( j \in \mathbb{Z} \), we have the inequality \( \dim_{k}H^{j}(Y) \leq c \) for some fixed \( c \). Then

\[ \dim_{k}Y \leq c(mL + (m - 1)L^{2} + (m - 2)L^{3} + \ldots + 2L^{m-1} + L^{m}). \]

**Proof.** Here \( Y^{1} = 0 \), then by our previous lemma, \( \dim_{k}Y^{0} \leq cL \). Then again by lemma 4.9 we have, \( \dim_{k}Y^{1} \leq c(L + L^{2}), \dim_{k}Y^{2} \leq c(L + L^{2} + L^{3}), \ldots, \dim_{k}Y^{m+1} \leq c(L + L^{2} + \ldots + L^{m}) \). From here we obtain our result.

\[ \square \]

**Theorem 4.11** (See corollary 9 in [9]) Let \( \Lambda \) be a finite-dimensional algebra over an algebraically closed field \( k \) and \( d = (d_{i})_{i \in \mathbb{Z}} \) be a collection of non-negative
integers with almost all $d_i = 0$. Then the family $U(d)$ of objects $X \in D^b(\Lambda)$ such that $\dim_k H^i(X) = d_i$ for all $i \in \mathbb{Z}$ and $\text{Hom}_{D^b(\Lambda)}(X, X[1]) = 0$ has only a finite number of isomorphism classes in $D^b(\Lambda)$.

**Proof.** We may assume $d_i = 0$ for all $i \leq t$ and $i > 0$. Consider now the family $Y(d)$ of those $P \in C^{\leq 0}(\text{proj} \Lambda)$ such that $\dim_k H^i(P) = d_i$ for all $i \in \mathbb{Z}$, $\text{Hom}_k(P, P[1]) = 0$ and for all $i \in \mathbb{Z}$, $\text{Im}d_i \subset \text{rad} P^{i+1}$. For each $X \in U(d)$ we may choose a quasi-isomorphism $P_X \to X$ with $P_X \in C^{\leq 0}(\text{proj} \Lambda)$.

We have $\text{Hom}_k(P_X, P_X[1]) \cong \text{Hom}_{D^b(\Lambda)}(X, X[1])$. Clearly the assignment $X \mapsto P_X$ establishes a bijection between the isomorphism classes of $U(d)$ and those of $Y(d)$. If $P \in Y(d)$, then $P \in L^{[-t-1, 0]}$.

By proposition 4.7 for $P \in Y(d)$ we have: $0 = \text{Hom}_k(P, P[1]) \cong \text{Ext}^1_{C^{[t, 0]}(\text{proj} \Lambda)}(P, P) \cong \text{Ext}^1_{C^{[-t-1, 0]}(\text{proj} \Lambda)}(l_{-t-1}P, l_{-t-1}P)$.

Using Lemma 4.10 one can prove that there is a number $n(d)$, such that if $P \in Y(d)$ then $\sum_{i \in \mathbb{Z}} \dim_k(l_{-i}P)^i \leq n(d)$. Therefore the functor $l_{-t-1}$ induces a bijection between the isomorphism classes of $Y(d)$ and the isomorphism classes of a subfamily of the family $F(n(d))$ consisting of the complexes $Z \in C^{[t-1, 0]}(\text{proj} \Lambda)$ which have not injectives in this category as direct summands, $\text{Ext}^1_{C^{[t-1, 0]}(\text{proj} \Lambda)}(Z, Z) = 0$ and $\sum_{i \in \mathbb{Z}} \dim_k(Z)^i \leq n(d)$. The category $C^{[t-1, 0]}(\text{Mod} \Lambda)$ is an abelian category with enough projectives, the projectives in this category are the complexes $T(P), J_u(P), u \in [-t-1, -1]$ introduced before. Then taking $H = \oplus_{u=-t-1}^{-1} J_u(\Lambda) \oplus \Gamma$ and $\Gamma = \text{End}_{C^{[-t-1, 0]}(\text{Mod} \Lambda)}(H)$, the functor

$$F = \text{Hom}_{C^{[-t-1, 0]}(\text{Mod} \Lambda)}(H, -) : C^{[t-1, 0]}(\text{Mod} \Lambda) \to \text{Mod} \Gamma$$

is an equivalence of abelian categories.

Now there is a number $m(d)$ such that for all $Z \in F(n(d))$, $\dim_k F(Z) \leq m(d)$.

Since the category $C^{[t-1, 0]}(\text{proj} \Lambda)$ is a full subcategory of the category $C^{[t-1, 0]}(\text{Mod} \Lambda)$, closed under extensions, then for $Z \in F(n(d))$, $0 = \text{Ext}^1_{C^{[-t-1, 0]}(\text{proj} \Lambda)}(Z, Z) = \text{Ext}^1_{C^{[t-1, 0]}(\text{Mod} \Lambda)}(Z, Z) \cong \text{Ext}^1_{\Gamma}(F(Z), F(Z))$. Therefore $F$ gives a bijection between the isomorphism classes of $F(n(d))$ and the isomorphism classes of a family of $\Gamma$-modules $M$ with $\dim_k M \leq m(d)$ and $\text{Ext}_\Gamma(M, M) = 0$. But by a result of D. Voigt ([10]), this last family has only a finite number of isomorphism classes. This implies that our family $U(d)$ has only a finite number of isomorphism classes.

$\square$

## 5 An application to modules

Let $\Lambda$ be an artin algebra over a commutative artinian ring $k$. In this section we study under which conditions two finitely generated $\Lambda$-modules $M$ and $N$ with $\text{Ext}_\Lambda(M, M) = 0$ and $\text{Ext}_\Lambda(N, N) = 0$ are isomorphic. As before for $M \in \text{mod} \Lambda$
we choose a minimal projective resolution:

\[ \cdots \to P^{-m}_M \xrightarrow{d^{-m}_M} P^{-m+1}_M \xrightarrow{d^{-m+1}_M} \cdots P^{-1}_M \xrightarrow{d^{-1}_M} P^0_M \xrightarrow{\eta_M} M \to 0. \]

We denote by \( P_M \) the complex in \( C^{\leq 0}(\text{proj} \Lambda) \):

\[ \cdots \to P^{-m}_M \xrightarrow{d^{-m}_M} P^{-m+1}_M \xrightarrow{d^{-m+1}_M} \cdots P^{-1}_M \xrightarrow{d^{-1}_M} P^0_M \to 0 \to 0 \cdots \]

For \( M \in \text{mod} \Lambda \) we put \( \Omega(M) = \text{Ker}(\eta_M) \).

**Theorem 5.1** Let \( M \) and \( N \) be in \( \text{mod} \Lambda \) such that \( \text{Ext}^1_\Lambda(M, M) = 0 \) and \( \text{Ext}^1_\Lambda(N, N) = 0 \). Then \( M \cong N \) if and only if \( M/\text{rad}M \cong N/\text{rad}N \) and \( \Omega(M) \cong \Omega(N) \).

**Proof.** Necessity is obvious, let us to prove sufficiency.

Let \( P'_M = l_{-2}(P_M) \) and \( P'_N = l_{-2}(P_N) \).

By lemma 4.5 we have an exact isomorphism of categories

\[ \gamma : C(\text{Proj} \Lambda) \to \text{Morph} \left( C^{[-1,0]}(\text{Proj} \Lambda), C^{[-2,1]}(\text{Proj} \Lambda) \right) \]

which induces an exact isomorphism of categories

\[ \gamma : C^{[-2,0]}(\text{Proj} \Lambda) \to \text{Morph} \left( C^{[-2,-1]}(\text{Proj} \Lambda), C^{[-1,-1]}(\text{Proj} \Lambda) \right). \]

We denote by \( \mathcal{M}_2 \) the category \( \text{Morph} \left( C^{[-2,-1]}(\text{Proj} \Lambda), C^{[-1,-1]}(\text{Proj} \Lambda) \right) \).

By corollary 4.8 we have \( \text{Ext}^1_{C(\text{Proj} \Lambda)}(P'_M, P'_N) \cong \text{Ext}^1_\Lambda(M, M) = 0 \). Then

\[ (1) \quad \text{Ext}_{\mathcal{M}_2}(\gamma(P'_M), \gamma(P'_N)) = 0. \]

In a similar way

\[ (2) \quad \text{Ext}_{\mathcal{M}_2}(\gamma(P'_N), \gamma(P'_M)) = 0. \]

The categories of bounded complexes over the finitely generated projective \( \Lambda \)-modules are Krull-Schmidt categories, so using the functors "erase at right," "pull and erase" we get \( t_{-1} (P'_M) \cong \oplus_{i=1}^u a_i X_i \) and \( s_{-1} (P'_M) \cong \oplus_{j=1}^v b_j Y_j \), \( t_{-1} (P'_N) \cong \oplus_{i=1}^u c_i X_i \) and \( s_{-1} (P'_N) \cong \oplus_{j=1}^v h_j Y_j \), where all decompositions are sums of pairwise non-isomorphic indecomposable objects. Now \( t_{-1} (P'_M) \) and \( t_{-1} (P'_N) \) correspond to minimal projective resolutions of \( \Omega(M) \) and \( \Omega(N) \) respectively, since \( \Omega(M) \cong \Omega(N) \) then \( a_i = c_i \) for all \( i \in \{1, \ldots, v\} \). On the other hand the only non zero module in the complexes \( s_{-1} (P'_M) \) and \( s_{-1} (P'_N) \) are \( P^0_M \) and \( P^0_N \) respectively, since \( M/\text{rad}M \cong N/\text{rad}N \), \( P^0_M \cong P^0_N \). Therefore \( s_{-1} (P'_M) \cong s_{-1} (P'_N) \) and this implies that \( b_j = h_j \) for \( j \in \{1, \ldots, u\} \). Let \( X = \oplus_{i=1}^u X_i \) and \( Y = \oplus_{j=1}^v Y_j \).

By proposition 3.7 we have an exact equivalence of categories

\[ \Theta : \text{Morph} (\text{add}X, \text{add}Y) \to \xi^X \left( R^X_Y \right). \]

Then by (1) and (2) we have:
\[
\text{Ext}_{R}^{\vee}(R, \theta \Theta \nabla (P'_M), \theta \Theta \nabla (P'_N)) = 0, \quad \text{Ext}_{R}^{\vee}(R, \theta \Theta \nabla (P'_M), \theta \Theta \nabla (P'_N)) = 0.
\]

It follows by 5.1 of [5] that \( \theta \Theta \nabla (P'_M) \cong \theta \Theta \nabla (P'_N) \). Then \( P'_M \cong P'_N \) and consequently \( M \cong N \).

\[\Box\]

**Theorem 5.2** Assume \( M, N \) in \( \text{mod} \Lambda \) have finite projective dimension and \( \text{Ext}_{\Lambda}^{1}(M, M) \cong \text{Ext}_{\Lambda}^{1}(N, N) = 0 \) and, for all \( j > 1 \) \( \text{Ext}_{\Lambda}^{j}(M, \Omega^{i-1}(M)) \cong \text{Ext}_{\Lambda}^{j}(N, \Omega^{i-1}(N)) = 0 \), then \( M \cong N \) if and only for all \( j; P_{M}^{j} \cong P_{N}^{j} \).

**Proof.** Necessity is obvious, let us to prove sufficiency by induction on \( m = \max (p(M), p(N)) \), where \( p(M) \) and \( p(N) \) are the projective dimension of \( M \) and \( N \) respectively. If \( m = 0 \) our claim is trivial. Suppose our claim proved for \( m-1 \), we will prove it for \( m \). But \( \text{Ext}_{\Lambda}^{1}(\Omega(M), \Omega(M)) \cong \text{Ext}_{\Lambda}^{2}(M, \Omega(M)) = 0 \) and for all \( j > 1 \), \( \text{Ext}_{\Lambda}^{j}(\Omega(M), \Omega^{j-1}(\Omega(M))) \cong \text{Ext}_{\Lambda}^{j+1}(M, \Omega^{j}(M)) = 0 \). Thus \( \Omega(M) \) satisfies the hypothesis of our theorem, similarly \( \Omega(N) \) also satisfies the hypothesis of our theorem, since \( p(\Omega(M)) = p(\Omega(N)) = m-1 \) and \( P_{M}^{j}/\text{rad}P_{M}^{j} \cong \Omega(M)/\text{rad}\Omega(M) \cong P_{N}^{j}/\text{rad}P_{N}^{j} \cong \Omega(N)/\text{rad}\Omega(N) \), by the induction hypothesis, \( \Omega(M) \cong \Omega(N) \). Here \( P_{M}^{0} \cong P_{N}^{0} \), then by theorem 5.1, we obtain \( M \cong N \).

\[\Box\]

**Example 5.3** We present an example of two non-isomorphic modules with both having finite minimal projective resolution and the same projectives. Let \( \Lambda \) be the \( k \)-algebra of the quiver

\[
\begin{array}{ccc}
\circ_1 & \xrightarrow{\alpha_1} & \circ_2 & \xrightarrow{\beta_1} & \circ_3 \\
\alpha_2 & \xrightarrow{} & \beta_2 & \xrightarrow{}
\end{array}
\]

with the relation \( \{\beta_1 \alpha_1, \beta_2 \alpha_2\} \). The representations

\[
\begin{array}{ccc}
k & \xrightarrow{1} & k \\
0 & \xrightarrow{} & 1 \\
1 & \xrightarrow{} & 0 \\
\end{array}
\]

have trivial selfextensions group but both have a minimal projective resolution with data \( (..., 0, ..., 0, \Lambda e_3, \Lambda e_2, \Lambda e_1) \).

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