Generating loop graphs via Hopf algebra in quantum field theory

Ángela Mestre, Robert Oeckl

Instituto de Matemáticas, UNAM, Campus Morelia,
C. P. 58190, Morelia, Michoacán, Mexico

UNAM-IM-MOR-2006-1
18 July 2006

Abstract

We use the Hopf algebra structure of the time-ordered algebra of field operators to generate all connected weighted Feynman graphs in a recursive and efficient manner. The algebraic representation of the graphs is such that they can be evaluated directly as contributions to the connected n-point functions. The recursion proceeds by loop order and vertex number.

The combinatorics of perturbative quantum field theory is traditionally dealt with via functional methods and generating functions. However, it is possible to use a more intrinsic algebraic approach instead, rooted directly at the level of n-point functions and pioneered in the 1960’s, see [1].

More recently, it was realized that the Hopf algebra structure of the algebra of field operators (with the normal or with the time-ordered product) can be fruitfully exploited. In particular, using the Hopf algebra and its cohomology it was shown (among other things) how different products of the algebra of field operators are related by Drinfeld twists and how interactions correspond to 2-cocycles [2].

Relations between different types of n-point functions and the associated combinatorics of Feynman graphs via the Hopf algebra structure of the time-ordered algebra of field operators was established in [3]. More precisely, the

*email: mestre@matmor.unam.mx
†email: robert@matmor.unam.mx
relations between complete and connected $n$-point functions on the one hand and between connected and 1-particle irreducible $n$-point functions on the other hand were described in this way. At the center of that work stands an algorithm to recursively generate all tree graphs and their values as Feynman graphs. The underlying structure in this is an algebraic representation of graphs in terms of certain generalized monomials in field operators.

In the present paper we extend this algorithm to recursively generate all connected graphs using this algebraic representation. The recursion proceeds by loop number (and by vertex number). The special case of vanishing loop number precisely recovers the algorithm of [3]. Crucially, and as in the special case of tree graphs, the correct weights of graphs are obtained so as to allow for their direct evaluation in terms of the Feynman graph expansion of the connected $n$-point function of a quantum field theory. Note, however, that no type of renormalization procedure is taken into account. In this sense the computed $n$-point functions may be considered as bare ones.

As in the previous work [3] all results apply to bosonic as well as fermionic fields and the algorithm is amenable to direct implementation and should allow efficient calculations.

Section 1 reviews basics about certain graphs and their symmetries, $n$-point functions, Feynman graphs, the algebraic representation of graphs and the Hopf algebra structure of the time-ordered field operator algebra. Section 2 contains the main result with the algorithmic construction of connected graphs and its proof. Section 3 offers some discussion, especially concerning the efficient algorithmic implementation and the inclusion of fermions. The appendix lists all connected graphs without external edges and with up to three internal edges together with their weight factors.

1 Basic concepts and definitions

The basic setup in this paper is substantially similar to that of [3]. Hence, the present section has substantial overlap with Section 2 and a part of Section 4 of that paper. Nevertheless, there are important differences, most importantly a more extensive treatment of abstract graphs and their symmetries.

1.1 Graphs

We introduce certain kinds of graphs and elementary properties of them. The graphs will be later interpreted as Feynman graphs. Here we are only interested in them as abstract graphs.
**Definition 1.** A graph is a finite collection of vertices and edges, such that any end of an edge may be connected to a vertex. Edges that are connected to vertices at both ends are called internal, while edges with at least one free end are called external. Internal edges with both ends connected to the same vertex are also called self-loops. The valence of a vertex is the number of ends of edges connected to the vertex. The loop number of a graph is its number of cycles. A graph is connected if it is connected as a topological space.

We recall the well known relation between vertex number, edge number and loop number of connected graphs:

**Lemma 2.** Consider a connected graph with at least one vertex. Let \(v, e, l\) be its number of vertices, internal edges and loops, respectively. Then,

\[
l = e - v + 1.
\]  

**Definition 3.** A labeled graph is a graph whose ends of external edges are labeled with labels from a label set. Labels on different ends of edges are required to be distinct.

In the following we shall consider only such labeled graphs, i.e., from now on graph means labeled graph. The label set is fixed from the outset and will later be identified with an appropriate set of field operator labels.

**Definition 4.** A graph is said to be vertex ordered if its vertices are ordered. It is said to be edge ordered if the ends of its edges are ordered. A graph is called ordered if it is both vertex ordered and edge ordered.

**Definition 5.** Consider an ordered graph \(\gamma\). A symmetry of \(\gamma\) is a permutation of the ordering of the vertices and of the endpoints of the edges that yields topologically the same ordered (and labeled) graph. The number of symmetries, i.e., the order of the group of permutations leaving the graph invariant, is called the symmetry factor of the graph. It will be denoted by \(S^\gamma\).

Since the symmetry factor is the same for any ordering of the vertices and ends of edges of a graph, the concept is well defined for unordered graphs as well.

**Definition 6.** Consider a vertex ordered graph \(\gamma\). A vertex symmetry of \(\gamma\) is a permutation of the ordering of its vertices, which yields topologically the same vertex ordered (and labeled) graph. The order of the group of vertex symmetries is called the vertex symmetry factor of the graph. It will be denoted by \(S^\gamma_{\text{vertex}}\).
**Definition 7.** Consider an ordered graph $\gamma$. An edge symmetry of $\gamma$ is a permutation of the ordering of the ends of its internal edges that yields topologically the same ordered (and labeled) graph while the order of the vertices is held fixed. The order of the group of edge symmetries is called the edge symmetry factor of the graph. It will be denoted by $S_{\text{edge}}^\gamma$.

Clearly, the concepts of vertex and edge symmetry factors also make sense for unordered graphs as the vertex and edge symmetry factors are the same for any ordering of the vertices and of the ends of the internal edges of a graph, respectively.

**Lemma 8.** Let $\gamma$ denote an ordered graph. The orders of the associated symmetry groups satisfy $S_\gamma = S_{\text{vertex}}^\gamma \cdot S_{\text{edge}}^\gamma$.

**Proof.** Denote the group of symmetries, vertex symmetries and edge symmetries of $\gamma$ by $G_\gamma$, $G_{\text{vertex}}^\gamma$ and $G_{\text{edge}}^\gamma$ respectively. These form an exact sequence of groups

$$0 \rightarrow G_{\text{edge}}^\gamma \rightarrow G_\gamma \rightarrow G_{\text{vertex}}^\gamma \rightarrow 0$$

defined in the obvious way. Hence, as the groups are finite their orders satisfy $S_\gamma = S_{\text{vertex}}^\gamma \cdot S_{\text{edge}}^\gamma$.

**Lemma 9.** Consider a connected graph $\gamma$. Let $v$ be its number of vertices. For each vertex $1 \leq i \leq v$ let $p_i$ be the number of self-loops connected to it. Let $t$ be the number of pairs of vertices which are directly connected through at least one edge. For each pair $1 \leq j \leq t$ of such vertices let $q_j$ be the number of edges connecting it. Then, the edge symmetry factor of $\gamma$ is given by $S_{\text{edge}}^\gamma = (\prod_{i=1}^v 2^{p_i} \cdot p_i!)(\prod_{j=1}^t q_j!)$.

The proof is straightforward combinatorics.

### 1.2 $n$-point functions and Feynman graphs

The physical content of a quantum field theory is usually extracted from its $n$-point functions. In perturbation theory, these are computed as sums of values of Feynman graphs. We briefly review here the essentials. More details can be found in any standard text book on quantum field theory such as [4].

We denote by $G^{(n)}(x_1, \ldots, x_n)$ the complete $n$-point function. This is the vacuum expectation value of the time-ordered product of $n$ field operators, i.e.,

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle.$$
The notation we use here suggests a scalar field theory on Minkowski space-time. In general there would be internal field indices as well and possibly other modifications (other spacetime etc.). The real nature of the fields is completely irrelevant for our treatment as long as the standard perturbative treatment applies. Therefore, we shall continue with our present notation for simplicity. Hence, we denote field operators generically by \( \phi(x) \), where \( x \) is from a label set (here suggestive of points in Minkowski space). Furthermore, we shall assume all fields to be bosonic. The fermionic case is also straightforward, but includes extra factors, see Section 3.

Let \( V \) be the complex vector space of linear combinations of field operators \( \phi(x) \). The algebra generated by the field operators with the time-ordered product is commutative and can be identified with the symmetric algebra \( S(V) \) over \( V \). More precisely, \( S(V) = \bigoplus_{k=0}^{\infty} V^k \), where \( V^k \) is the space of linear combinations of monomials of degree \( k \) in the field operators and \( V^0 \) is the one-dimensional vector space spanned by the identity element \( 1 \). We may now express ensembles of \( n \)-point functions as functions \( S(V) \to \mathbb{C} \). In particular, we may set

\[
\rho(\phi(x_1) \cdots \phi(x_n)) := G^{(n)}(x_1, \ldots, x_n).
\]

In perturbation theory, the \( n \)-point functions can be computed as a sum over values of Feynman graphs. For the complete \( n \)-point functions we may write

\[
G^{(n)}(x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma_n} w_{\gamma} \gamma(x_1, \ldots, x_n). \tag{2}
\]

Here \( \Gamma_n \) is the set of Feynman graphs. These are graphs \( \gamma \) in the sense of Section 1.1 with \( n \) external edges labeled by field operator labels \( x_1, \ldots, x_n \). Indeed, from here onward we fix the label set to be the label set of the field operators. The value of a graph \( \gamma \) labeled by \( x_1, \ldots, x_n \) is denoted above by \( \gamma(x_1, \ldots, x_n) \). The set \( \Gamma_n \) may be taken to be precisely the set of all graphs with \( n \) external legs (up to topological equivalence). The weight factor \( w_{\gamma} \) is precisely the inverse of the symmetry factor \( S^\gamma \) of a graph in the sense of Definition 5.

We should emphasize that the discussion here applies to bare \( n \)-point functions. Renormalization is outside the scope of the present paper.

The type of \( n \)-point functions we shall be interested in in the following are the connected ones, denoted \( G_c^{(n)} \). These may be defined in the same way

\footnote{Note that usually Feynman graphs involve lines of different type depending on particle species. In our treatment lines correspond to sums over all particle species. The information about which particle species can interact resides completely in the vertex functions.}
as (2), but with the restriction that only connected graphs are considered. We define $\sigma : S(V) \to V$ via

$$\sigma(\phi(x_1) \cdots \phi(x_n)) := G_c^{(n)}(x_1, \ldots, x_n).$$

We now turn to the calculation of the value of a Feynman graph. The Feynman propagator $G_F(x, y)$ is the value of the graph that consists of an edge only, its two ends labeled by $x$ and $y$ respectively. The value of a graph that consists of a vertex with external edges labeled by $x_1, \ldots, x_k$ is given by the vertex function $F(x_1, \ldots, x_k)$. Note that we can encode the ensemble of vertex functions in a way analogous to $n$-point functions as a function $\nu : S(V) \to \mathbb{C}$ via

$$\nu(\phi(x_1) \cdots \phi(x_n)) := F(x_1, \ldots, x_n). \quad (3)$$

For more general graphs we also need the inverse Feynman propagator, $G_F^{-1}$, determined by the equation

$$\int dy G_F(x, y) G_F^{-1}(y, z) = \delta(x, z). \quad (4)$$

The value of a general graph may then be computed as follows: Associate a label with each internal edge and form the product over a vertex function associated with each vertex and an inverse Feynman propagator associated to each internal edge. Finally, integrate over all possible assignments of internal labels.

### 1.3 Algebraic representation of graphs

We introduce an algebraic representation of graphs based on the time-ordered operator algebra $S(V)$ and allowing straightforward evaluation of graphs in the above sense. More precisely, we associate a given graph with $v$ vertices with a certain element in $S(V)^{\otimes v}$, the $v$-fold tensor product of $S(V)$.

Each vertex of the graph corresponds to one tensor factor. A product $\phi(x_1) \cdots \phi(x_n)$ in a given tensor factor corresponds to external edges of the associated vertex whose endpoints are labeled by $x_1, \ldots, x_n$. To represent internal edges, we define the formal elements $R_{i,j} \in S(V)^{\otimes v}$ with $1 \leq i \leq j \leq v$ using the inverse Feynman propagator (4) as follows:

$$R_{i,j} := \int dx dy G_F^{-1}(x, y) (1^{\otimes j-1} \otimes \phi(x) \otimes 1^{\otimes j-i-1} \otimes \phi(y) \otimes 1^{\otimes v-j}), \quad (5)$$

$^2$The $R_{i,j}$ is formal insofar as it really lives in a completion of the tensor product $S(V)^{\otimes v}$. However, this fact is largely irrelevant for our purposes.
with the field operators $\phi(x)$ and $\phi(y)$ inserted at the $i^{th}$ and $j^{th}$ positions, respectively. The element $R_{i,j} \in S(V)^{\otimes v}$ corresponds to one internal edge connecting the vertices which occupy the positions $i$ and $j$. In particular, the element $R_{i,i} \in S(V)^{\otimes v}$ for $1 \leq i \leq v$ is interpreted as an internal edge connecting the $i^{th}$ vertex to itself. That is, it corresponds to a self-loop.

Combining several internal edges (which can be self-loops) and their products with external edges by multiplying the respective expressions in $S(V)^{\otimes v}$ allows to build arbitrary graphs with $v$ vertices. Figure 1 shows some examples. It is then obvious that applying the vertex functions $\nu$ defined by (3) to each tensor factor yields precisely the value of the respective graph as a Feynman graph. Thus, the graphs we just discussed are exactly those that are to enter in the $v$-vertex contribution to an $n$-point function.

The ordering of the tensor factors of $S(V)^{\otimes v}$ induces an ordering of the vertices of the graphs in the sense of Definition 4. However, when applying $\nu^{\otimes v}$ the ordering is “forgotten”. Indeed, it is not relevant for the interpretation of graphs as Feynman graphs, but only plays a role at the level of their algebraic representation here. In the following, we will encounter elements of $S(V)^{\otimes v}$ that are linear combinations of expressions corresponding to graphs. In this context, we call the scalar multiplying the expression for a given graph the weight of the graph. Clearly, if we are interested in unordered graphs, the weight of such a graph is the sum of the weights of all vertex ordered graphs that correspond to it upon forgetting the vertex order.
1.4 The field operator algebra as a Hopf algebra

A crucial ingredient of our setting is the fact that the algebra \( S(V) \) of time-ordered field operators is not only an algebra, but a **Hopf algebra**. That is, \( S(V) \) carries a coproduct \( \Delta : S(V) \to S(V) \otimes S(V) \) and a counit \( \epsilon : S(V) \to \mathbb{C} \) that are compatible with its algebra structure and unit. \( S(V) \) also carries an antipode map, but this will not be used in the following.) We refer the reader to [5] for a classical treatment of Hopf algebras and to [6] for the Hopf algebra structure of the symmetric algebra. The significance of this Hopf algebra structure for quantum field theory was developed in [2]. (Note, however, that the product taken there is the normal product and not the time-ordered one.)

At this point we will only mention the explicit form of the coproduct on \( S(V) \). On monomials this takes the form

\[
\Delta(\phi(x_1) \cdots \phi(x_n)) = \sum_{I_1 \cup I_2 = \{\phi(x_1), \ldots, \phi(x_n)\}} T(I_1) \otimes T(I_2),
\]

and is extended to all of \( S(V) \) by linearity. Here the sum runs over partitions of the set of field operators \( \{\phi(x_1), \ldots, \phi(x_n)\} \) into two sets \( I_1 \) and \( I_2 \). \( T \) denotes the time-ordered product of the field operators in the corresponding partition. The coproduct may be extended (by suitable composition with itself) to a map (on monomials and extended by linearity),

\[
\Delta^k(\phi(x_1) \cdots \phi(x_n)) = \sum_{I_1 \cup \cdots \cup I_{k+1} = \{\phi(x_1), \ldots, \phi(x_n)\}} T(I_1) \otimes \cdots \otimes T(I_{k+1}).
\]

The difference to the single coproduct is that the set of field operators is now split into \( k + 1 \) partitions. Note also that the partitions are ordered, i.e., the sets \( I_1, \ldots, I_{k+1} \) are distinguishable. An important property of the coproduct is that it is multiplicative, i.e., it is an algebra map with respect to the algebra structure of \( S(V) \) (and the induced algebra structure on the tensor product). A more extensive discussion of the Hopf algebra structure, adapted to the present context can be found in [3].

2 Generating loop graphs

2.1 Statement of result

The main result of this paper, which is the subject of the present section, may be described as an efficient algorithm that recursively generates all
connected graphs \( \gamma \). The graphs are generated together with the correct weights \( w_\gamma \), explained in Section 1.2. In particular, the recursion is such that it may be organized in ascending loop order. Also, the graphs are generated directly in the algebraic representation introduced in Section 1.3. This allows their direct evaluation as Feynman graphs.

More precisely, we shall construct recursively a set of maps \( \Omega^{l,v} : \mathcal{S}(V) \to \mathcal{S}(V)^{\otimes v} \) indexed by integers \( l \) and \( v \) such that the following theorem holds.

**Theorem 10.** Fix integers \( l, n \geq 0, v \geq 1 \) and operator labels \( x_1, \ldots, x_n \). Then, \( \Omega^{l,v}(\phi(x_1) \cdots \phi(x_n)) \in \mathcal{S}(V)^{\otimes v} \) corresponds to the weighted sum over all connected graphs with \( l \) loops, \( v \) vertices and \( n \) external edges whose endpoints are labeled by \( x_1, \ldots, x_n \), each with weight being the inverse of its symmetry factor.

This specializes for \( l = 0 \) to Lemma 10 of [3], with \( \Lambda^{v-1} = \Omega^{0,v} \).

We may conclude with the interpretation in terms of Feynman graphs and \( n \)-point functions.

**Corollary 11.** The \( l \)-loop and \( v \)-vertex contribution to the ensemble of connected \( n \)-point function \( \sigma \) takes the following form:

\[
\sigma^{l,v} = \nu^{\otimes v} \circ \Omega^{l,v}.
\]

In particular, the \( l \)-loop order contribution \( \sigma^l \) to \( \sigma \) and \( \sigma \) itself are given by

\[
\sigma^l = \sum_{v=1}^{\infty} \sigma^{l,v}, \quad \sigma = \sum_{l=0}^{\infty} \sigma^l.
\]  

(8)

Restricting to the \( l = 0 \) (tree level) contribution recovers Corollary 18 of [3].

The appendix lists all connected graphs without external edges as weighted contributions to \( \Omega^{l,v}(1) \), for edge number \( e = l + v - 1 \leq 3 \).

### 2.2 Construction and proof

The proof proceeds in a manner very analogous to the proof in [3]. Indeed, each intermediate lemma in this section specializes to a corresponding lemma in Section 4 of that paper when restricted to the case \( l = 0 \). We do not point this out explicitly in the following, but refer the reader to that paper for comparison.

In order to construct \( \Omega^{l,v} \) we introduce certain auxiliary maps. Using the component-wise product in \( \mathcal{S}(V)^{\otimes v} \), we may view the elements \( R_{i,j} \), defined by equation (5), as operators on this space by multiplication. In particular, these elements are used to define the following maps:
- \( T_i : S(V)^{\otimes v} \to S(V)^{\otimes v} \), with \( 1 \leq i \leq v \), as the operator \( R_{i,i} \) together with the factor \( 1/2 \):
\[
T_i := \frac{1}{2} R_{i,i}.
\] (9)

- \( Q_i : S(V)^{\otimes v} \to S(V)^{\otimes v+1} \), with \( 1 \leq i \leq v \), given by the composition of \( R_{i,i+1} \) with the coproduct applied to the \( i \)th component of \( S(V)^{\otimes v} \), i.e. \( \Delta_i := \text{id}^{\otimes i-1} \otimes \Delta \otimes \text{id}^{\otimes v-i} : S(V)^{\otimes v} \to S(V)^{\otimes v+1} \), together with a factor of \( 1/2 \):
\[
Q_i := \frac{1}{2} R_{i,i+1} \circ \Delta_i.
\] (10)

The map \( T_i \) given by (9), endows the \( i \)th vertex of a vertex ordered graph with a self-loop together with a factor \( 1/2 \). The latter is the inverse of the edge symmetry factor of a single self-loop (see Lemma 9). The action of the map \( Q_i \) given by (10) is less simple. Consider the coproduct \( \Delta_i \) applied to the \( i \)th component of \( S(V)^{\otimes v} \). Recalling the formula (6), we see that \( \Delta_i \) converts a graph with \( v \) vertices into a sum over graphs with \( v + 1 \) vertices by splitting the \( i \)th vertex into two in all possible ways. That is, the \( i \)th vertex is replaced by two vertices (numbered \( i \) and \( i + 1 \)) and the edges ending on it, (considered as distinguishable) are distributed between the two new vertices in all possible ways. Note that the two new vertices are distinguished due to the ordering of the tensor factors. Thus, to obtain the corresponding operation for unordered graphs we need to divide by a factor of 2. This factor corresponds to the two different relative orderings of the new vertices with which each unordered configuration occurs. The only exception to this is the case when the split vertex has no edges at all. No overcounting happens in this case. The meaning of the map \( Q_i \) given by (10) becomes clear now in terms of graphs. Namely, it splits the \( i \)th vertex into two and subsequently reconnects the two new vertices with an edge. Dividing by two compensates for the double counting as described above if we are interested in unordered graphs (assuming the set of endings of edges of the split vertex is not empty).

We remark that the maps \( T_i \) increase both the loop and edge numbers of a graph by one unit, leaving the vertex number invariant, while the maps \( Q_i \) increase both the edge and vertex numbers by one unit, leaving the loop number invariant.

We use the maps \( T_i \) and \( Q_i \), given by equations (9) and (10), respectively, to define recursively maps \( \Omega^{l,v} : S(V) \to S(V)^{\otimes v} \) for \( l \geq 0 \) and for \( v \geq 1 \) as follows:
\[
\Omega^{0,1} := \text{id},
\]
Figure 2: Recursive dependencies of the maps $\Omega^{l,v}$ up to order $l + v \leq 4$. The right directed arrows correspond to $T_i$ maps while the left directed ones correspond to $Q_i$ maps.

$$\Omega^{l,v} := \frac{1}{l + v - 1} \left( \sum_{i=1}^{v-1} Q_i \circ \Omega^{l,v-1} + \sum_{i=1}^{v} T_i \circ \Omega^{l-1,v} \right).$$

(11)

Note that in the recursion equation above the $T$ and $Q$-summands do not appear when $l = 0$ or when $v = 1$, respectively. Figure 2 shows the recursive dependencies of $\Omega^{l,v}$ for different $l, v$ with $l + v \leq 4$.

We notice that $\Omega^{l,v}$ satisfies the following factorization property:

**Lemma 12.** Fix integers $l, n, m \geq 0$, $v \geq 1$ and field operator labels $x_1, \ldots, x_n$, $y_1, \ldots, y_m$. Then, $\Omega^{l,v}$ satisfies the factorization property

$$\Omega^{l,v}(\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)) = \Omega^{l,v}(\phi(x_1)\cdots\phi(x_n)) \cdot \Delta^{v-1}(\phi(y_1)\cdots\phi(y_m)).$$

(12)

**Proof.** This follows from the multiplicativity of the coproduct and the recursive definition (11), noticing that each time the vertex number increases by one, one coproduct is applied as part of the operator $Q_i$. ☐

We now turn to the proof of Theorem 10. We begin with weaker lemmas, increasing their strength stepwise until reaching the desired result.

**Lemma 13.** Fix integers $l, n \geq 0$, $v \geq 1$ as well as field operator labels $x_1, \ldots, x_n$. (a) $\Omega^{l,v}(\phi(x_1)\cdots\phi(x_n))$ corresponds to a weighted sum of connected graphs with $l$ loops, $v$ vertices and $n$ external edges whose endpoints...
are labeled by $x_1, \ldots, x_n$. (b) Any connected graph with $l$ loops, $v$ vertices and the given external edges occurs in $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ with some positive weight.

Proof. First, it is clear that $\Omega^{0,1}(\phi(x_1) \cdots \phi(x_n))$ corresponds to the connected graph with one single vertex with no self-loops and the external edges whose endpoints are labeled by $x_1, \ldots, x_n$. Moreover, $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ is generated from this by sums of multiple applications of the maps $T_i$ and $Q_i$ with scalar factors according to the recursion formula (11). Both $T_i$ and $Q_i$ convert a term corresponding to a connected graph to a sum over terms corresponding to connected graphs. Thus, $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ is a sum of terms each of which corresponds to a connected graph (with some weight).

Second, the fact that every graph contained in $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ has $l$ loops and $v$ vertices follows from noticing that the maps $T_i$ increase the loop number by one unit, while the vertex number remains fixed, and the maps $Q_i$ increase the vertex number by one unit, leaving the loop number unchanged. This concludes the proof of (a).

To prove (b) we proceed by induction on the internal edge number $e = l + v - 1$ (recall Lemma 2). The result is evidently valid for $e = 0$, corresponding to $l = 0$ and $v = 1$. We assume the result holds for $e - 1$. Let $\gamma$ denote a graph with $l$ loops and $v$ vertices so that $l + v - 1 = e$. We show that it is generated by applying the maps $T_i$ or $Q_i$ to graphs contained in $\Omega^{l-1,v}(\phi(x_1) \cdots \phi(x_n))$ or in $\Omega^{l,v-1}(\phi(x_1) \cdots \phi(x_n))$, respectively. Since both $T_i$ and $Q_i$ produce graphs with positive weight from graphs with positive weight, the weight of a graph $\gamma$ occurring in $\Omega^{l,v}$, being given by a sum over positive contributions according to formula (11), is positive. Now, suppose the graph $\gamma$ has at least one vertex with one or more self-loops. Let this vertex occupy the $i^{th}$ position, for instance. Shrinking one of these self-loops yields a graph that corresponds by assumption to a term in $\Omega^{l-1,v}(\phi(x_1) \cdots \phi(x_n))$ with some positive weight so that applying the map $T_i$ to the vertex $i$ produces the graph $\gamma$ with (positive) weight. Thus, by formula (11) the graph $\gamma$ occurs in $\Omega^{l,v}$. Finally, suppose the graph $\gamma$ does not contain vertices with self-loops. Choose an arbitrary internal edge. Shrinking this edge and fusing the vertices it connects yields a graph that corresponds by assumption to a term in $\Omega^{l,v-1}(\phi(x_1) \cdots \phi(x_n))$. Say, the fused vertex has position $i$. Applying $Q_i$ to this term will yield a sum over terms one of which will correspond to the original. By the recursive definition of $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ it thus contains this term with positive weight. This completes the proof of (b).

What remains in order to prove Theorem 10 is to show that the term
corresponding to each graph has weight given exactly by the inverse of its symmetry factor. We start with a more restricted result.

**Lemma 14.** Fix integers \( l \geq 0, v \geq 1 \) and \( n \geq v \) as well as field operator labels \( x_1, \ldots, x_n \). Consider a connected graph \( \gamma \) with \( l \) loops, \( v \) vertices, \( n \) external edges whose endpoints are labeled by \( x_1, \ldots, x_n \) and the property that each vertex has at least one external edge ending on it. Then, the term in \( \Omega^{l,v}(\phi(x_1) \cdots \phi(x_n)) \) corresponding to that graph has weight given by the inverse of its symmetry factor \( S^{\gamma} \).

**Proof.** We proceed by induction on the number of internal edges \( e \). Clearly, the statement is true for \( e = 0 \) so that we assume it holds for a general number of internal edges \( e - 1 \). Let \( \gamma \) be a connected graph with \( e \) internal edges. Let \( l \) be its loop number and let \( v \) be its vertex number. By Lemma 13, this graph occurs in \( \Omega^{l,v}(\phi(x_1) \cdots \phi(x_n)) \) with positive weight \( \alpha \). We proceed to show that \( \alpha = 1/S^{\gamma} \). Suppose the vertex \( i \) of \( \gamma \) has \( p_i \) self-loops, with \( 1 \leq i \leq v \) and the pair \( j \) of vertices is connected by \( q_j \) edges, with \( 1 \leq j \leq t \), where \( t \) is the total number of connected pairs of vertices. By Lemma 9 the edge symmetry factor of \( \gamma \) is given by

\[
S^{\gamma}_{\text{edge}} = \left( \prod_{i=1}^{v} 2^{p_i} \cdot p_i! \right) \left( \prod_{j=1}^{t} q_j! \right)
\]

with \( e = \sum_{i=1}^{v} p_i + \sum_{j=1}^{t} q_j \). Since the graph \( \gamma \) has the property that each vertex has at least one external edge, its vertices are distinguishable and it has no non-trivial vertex symmetries: \( S^{\gamma}_{\text{vertex}} = 1 \) and \( S^{\gamma}_{\text{edge}} = S^{\gamma} \) (as any symmetry is an edge symmetry). We check from which graphs with \( e - 1 \) internal edges \( \gamma \) is generated by the recursion formula (11) and how many times it is generated. It turns out that we can think of each internal edge of \( \gamma \) as contributing with a factor of \( 1/(e \cdot S^{\gamma}) \) as follows:

(i) Consider the \( i \)th vertex of \( \gamma \) endowed with \( p_i \) self-loops. Shrinking one of these self-loops yields a graph \( \gamma' \) whose \( i \)th vertex has \( p_i - 1 \) self-loops. Consequently, by Lemma 9, the symmetry factor of \( \gamma' \) is related to that of \( \gamma \) via \( S^{\gamma'} = S^{\gamma}/(2p_i) \). By assumption, the graph \( \gamma' \) corresponds to a term in \( \Omega^{l-1,v}(\phi(x_1) \cdots \phi(x_n)) \) which occurs with weight given by the inverse of its symmetry factor: \( 1/S^{\gamma'} = 2p_i/S^{\gamma} \). Applying the map \( T_i \), which carries the factor 1/2, to the vertex \( i \) of \( \gamma' \) produces the graph \( \gamma \) from the graph \( \gamma' \) exactly with factor \( p_i/S^{\gamma} \). Thus, the contribution to (11) is \( p_i/(e \cdot S^{\gamma}) \). Distributing this factor between the \( p_i \) edges considered yields \( 1/(e \cdot S^{\gamma}) \) for each edge considered.

(ii) Consider the \( j \)th pair of vertices of \( \gamma \) connected by \( q_j \) edges. Shrinking one of the edges and fusing the vertices it connects yields a graph \( \gamma'' \) whose fused vertex has \( r := p_j + p_{j+1} + q_j - 1 \) self-loops. Consequently, by Lemma 9,
the symmetry factor of $\gamma''$ is related to that of $\gamma$ as follows:

$$\frac{1}{2^r} \frac{1}{r!} S^{\gamma''} = \frac{1}{2^{p_j} p_j!} \frac{1}{2^{p_{j+1}} p_{j+1}!} \frac{1}{q_j} \frac{1}{S^\gamma}. $$

By assumption, the graph $\gamma''$ corresponds to a term in $\Omega^{l,v-1}(\phi(x_1) \cdots \phi(x_n))$ which occurs with weight given by the inverse of its symmetry factor, i.e.,

$$\frac{1}{S^{\gamma''}} = \frac{1}{p_j!} \frac{1}{p_{j+1}!} \frac{1}{q_j!} \frac{1}{r!} \cdot \frac{1}{2^{q_j-1} S^\gamma}. \quad (13)$$

The map $Q_j$, when applied to the fused vertex, produces a pair of vertices occupying the positions $j$ and $j+1$, distributes the $2r$ endings of edges between the two vertices in all possible ways and attaches them together by an edge. The action of $Q_j$ on the fused vertex $j$ (leaving out external edges) reads explicitly as

$$Q_j R^r_{j,j} = \frac{1}{2} R_{j,j+1} (R_{j,j} + 2R_{j,j+1} + R_{j+1,j+1})^r$$

$$= \sum_{a=0}^{r} \sum_{b=0}^{a} \binom{r}{a} \binom{a}{b} 2^{a-b-1} R_{j,j}^{a-b} R_{j,j+1}^{a-b+1} R_{j+1,j+1}. \quad (14)$$

Taking into account the external edges, there are two terms in equation (15) corresponding to the graph $\gamma$: one with $r-a = p_j$ and $b = p_{j+1}$ and one with $r-a = p_{j+1}$ and $b = p_j$. The sum of the coefficients of these two contributions is

$$2^{q_j-1} \frac{r!}{p_j! p_{j+1}! (q_j - 1)!}. \quad (16)$$

Multiplying (13) with (16), we see that $Q_j$ produces $\gamma$ from $\gamma''$ exactly with a factor $q_j/S^\gamma$ and the contribution to (11) is $q_j/(e \cdot S^\gamma)$. In other words, we get a factor of $1/(e \cdot S^\gamma)$ for each of the $q_j$ edges considered.

Since each of the $e = l + v - 1$ internal edges contributes with a factor of $1/(e \cdot S^\gamma)$ to the weight of the graph $\gamma$, the overall contribution is exactly $1/S^\gamma$. This completes the proof.

To complete the proof of Theorem 10, we show that the term in $\Omega^{l,v}(\phi(x_1) \cdots \phi(x_n))$ corresponding to a connected graph $\gamma$ with $l$ loops, $v$ vertices and external edges whose endpoints are labeled by $x_1, \ldots, x_n$, has weight given by $1/S^\gamma$. If $\gamma$ has external edges attached to every one of its vertices we simply recall Lemma 14. Thus, we may now assume that $\gamma$ has $m$ vertices to which no external leg is attached. Consider a graph $\gamma'$ which is constructed from $\gamma$ by attaching an external edge to every vertex without external edges, choosing
arbitrary but fixed labels \( y_1, \ldots, y_m \) for the endpoints of external edges in the process. By Lemma 14, the graph \( \gamma' \) occurs in the term on the left hand side of equation (12) with weight \( 1/S^\gamma' \). By Lemma 13, the graph \( \gamma \) occurs in the first factor on the right hand side with some non-zero weight, say \( \alpha \). Every summand of \( \Delta^{\gamma-1} (\phi(y_1) \cdots \phi(y_m)) \) (recall formula (7)) which places the endpoints of external edges at the designated vertices of \( \gamma \) to produce \( \gamma' \) contributes to the weight of \( \gamma' \) in terms of that of \( \gamma \). Any different ways this can happen define a vertex symmetry of \( \gamma \). Furthermore, \( \gamma \) can have no more than these vertex symmetries, since its vertices that already carry external edges are distinguishable and thus held fixed under any symmetry. Therefore, using Lemma 12 we obtain the formula \( 1/S^\gamma' = \alpha \cdot S^{\gamma}_{\text{vertex}} \) by extracting the weights from the corresponding terms in equation (12). Moreover, \( S^\gamma = S^\gamma_{\text{edge}} = S^\gamma_{\text{edge}} \). Thus, using \( S^\gamma = S^\gamma_{\text{vertex}} \cdot S^\gamma_{\text{edge}} \) (Lemma 8), we find \( \alpha = 1/S^\gamma \). This completes the proof.

3 Discussion and Conclusion

The results of the present paper can be seen as an extension of those of [3], where only tree graphs were generated. Accordingly, many points in the discussion of the main result in that paper extend to the present setting. In particular, this applies to the algorithmic aspects and to the inclusion of fermions. We refer the reader to Sections 6.3 and 6.4 of [3] for details. Here we shall only touch these points briefly and highlight differences arising through the inclusion of graphs with loops.

The generation of the graphs in their algebraic representation via the recursion formula (11) has the structure of an algorithm. Indeed, this algorithmic structure can be used directly and efficiently in implementing concrete calculations of (loop) graphs. In doing so, external edges may be fixed from the beginning and \( \Omega^{k,\nu} \) as applied to the external edges is calculated recursively rather than as an abstract map. An important aspect for the efficiency of concrete calculations is to discard graphs that do not contribute. In typical quantum field theoretic calculations, the vertex function is such that only vertices with a minimum valence (usually three) contribute. In the case of tree graphs this allows the restriction of the coproduct implicit in the operator \( Q_\nu \) in the recursion formula (11) [3]. Concretely, the coproduct (6) may be replaced by a truncated coproduct \( \Delta_{>k} \) with \( k \geq 1 \). This is defined by removing from the right hand side of (6) all terms where the number of elements in \( I_1 \) or \( I_2 \) is smaller than \( k \). This will prevent graphs from being generated who have vertices with valence smaller than \( k + 1 \). If only tree
graphs are considered this is consistent with the recursion process. More precisely, a graph with all vertices of valence at least \( k + 1 \) cannot be generated by \( Q_i \) from a graph with at least one vertex having valence smaller than \( k + 1 \). The analogous statement is not true for the operator \( T_i \). Hence, considering loop graphs as well (recall that \( T_i \) increases loop number), we can no longer globally restrict the coproduct. However, if we are interested in graphs only up to a maximal loop number \( m \), we may still restrict the coproduct in \( Q_i \) in certain instances. These are precisely the instances when a later application of \( T_i \) to a graph cannot occur, i.e., when the graph has already the maximal loop number \( m \).

The restriction on the valence of vertices to be at least \( a \), where \( a \geq 3 \), leads to another obvious limit we can impose on the algorithm. Namely, for a given number of loops \( m \) and a given number of external edges \( n \) there is an upper bound \( b = (n + 2m - 2)/(a - 2) \) on the number of vertices a graph can have. Thus, in this case we only need to compute \( \Omega^{l,v} \) for \( l \leq m \) and \( v \leq b \).

We now turn to the question of the implementation of fermions. Here the situation is not at all changed by the extension to loop graphs. Namely, the whole formalism is completely functorial and carries over immediately to the case that the vector space \( V \) of field operators is a \( \mathbb{Z}_2 \)-graded space. (Recall that this means that \( V \) is a direct sum of a bosonic and fermionic part.) Concretely, certain field operators will anticommute which introduces minus signs in front the summands in (6) and (7) which correspond to odd permutations of such field operators. In contrast, all formulas appearing in Section 2 remain unchanged as the \( \mathbb{Z}_2 \)-grading is completely implicit there.

The algorithm to generate tree graphs was applied in two contexts in [3]: To relate connected \( n \)-point functions with 1-particle irreducible ones and to generate all tree graphs using the vertex functions. In both cases renormalization does not introduce any alteration. This is different in the present situation where we interpret the algorithm of Section 2 as generating all connected graphs using the vertex functions. Renormalization, via counter-terms, alters this process considerably. Thus, it would be highly desirable to include the renormalization process into the present framework. At this point we have very little to say about this, except to point out that the algorithm of Section 2 is naturally organized as a recursion by loop order, which might facilitate the task.
Acknowledgments

One of the authors (Â. M.) was supported through a fellowship provided by Fundação Calouste Gulbenkian 65709.

A Appendix

This appendix shows all graphs without external edges and with up to three edges computed as contributions to $\Omega^{l,v}(1)$ via (11). The factors in front of the graphs are the inverses of their symmetry factors of Definition 5, see Theorem 10.

$\Omega^{0,1}(1) = \bullet$

$\Omega^{0,2}(1) = \frac{1}{2} \bullet \bullet$

$\Omega^{1,1}(1) = \frac{1}{2} \bigcirc$

$\Omega^{0,3}(1) = \frac{1}{2} \bullet \bullet \bullet$

$\Omega^{1,2}(1) = \frac{1}{2} \bullet \bullet \bullet + \frac{1}{2^2} \bullet \bullet \bullet$

$\Omega^{2,1}(1) = \frac{1}{2^3} \bigcirc$

$\Omega^{0,4}(1) = \frac{1}{2} \bullet \bullet \bullet \bullet + \frac{1}{3!} \bullet \bullet \bullet \bullet$
\[ \Omega^{1,3}(1) = \frac{1}{3!} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \]

\[ \Omega^{2,2}(1) = \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \]

\[ \Omega^{3,1}(1) = \frac{1}{2^{3,1}} \]

References


