ANALYTICAL SIGNATURES AND PROPER ACTIONS

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ABSTRACT. In this short note we compare Mishchenko's definition of noncommutative signature for a manifold with a proper G-action of a discrete, countable group G with the (more analytical) counter part defined by Higson and Roe in the series of articles "Mapping Surgery to analysis". A generalization of the bordism invariance of the coarse index is also adressed.

1. INTRODUCTION

There are different notions of non-commutative signatures that can be applied to oriented proper cocompact G-manifolds for a discrete group G. Higson and Roe studied the relation between a signature of C^*-algebras, an analytic signature and the coarse index of the signature operator, they also show that these signatures are bordism and homotopy invariants.

For these definitions, they consider two types of so-called Hilbert-Poincaré complexes: *algebraic complexes* of finitely generated projective modules over a C^*-algebra C and *analytically controlled complexes* of Hilbert spaces. Both kind of complexes are required to satisfy suitable versions of Poincaré Duality. The *algebraic* signature has values in the K-theory $K_*(C)$ of the algebra C, and the *analytic* signature has values in the Mitchener K-theory of a suitable C^*-category.

All these signatures are defined for the case of a compact smooth manifold X and the authors show that the analytic signature coincides with the K-theoretic index of the signature operator defined on the L^2-completion of the De Rham complex of X. In this case, it is proven that Mitchener K-theory coincides with the K-theory $K_*(C_r(G))$ of the reduced C^*-algebra of the group G.

Their C^*-algebra signature is defined for finitely generated projective Hilbert-Poincaré modules over the algebra $C_0(X)$ of continuous functions vanishing at infinity. In the case of a smooth manifold $\tilde{X}$ with free co-compact action of a discrete group G their definition makes no sense if the quotient $X = \tilde{X}/G$ is not compact, because the complexes considered are not finitely generated over this algebra. The analytic signature does make sense and the proof of its coincidence with the index of the signature operator generalizes to this context.

On the other hand, Mishchenko defined a signature for finitely generated projective algebraic Hilbert-Poincaré complexes over the reduced C^*-algebra $C_r^*(G)$ of the group G. This can be applied to a proper co-compact smooth G-manifold M. The analytic signature of Higson and Roe also makes sense in this context for the L^2-completion of the De Rham complex.

In this paper we show that, with slight modifications to the notion of algebraic Hilbert-Poincaré complex, the C^*-algebra signature defined by Higson and Roe coincides in even dimension with that of Mishchenko. A consequence of this is another proof of the homotopy and bordism invariance of the signature of Mishchenko. The analytic version of the signature can be applied in this context to triangulated *bounded isotropy* proper G-manifolds of even dimension. In this case, the coincidence of the analytic signature with the coarse index of the signature operator is

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a consequence of the results proven by Higson and Roe. Also, another version of bordism invariance due to Wulff is considered in this context. In the last section, we syntetize the relations between the signatures considered.

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3. Algebraic Hilbert-Poincaré complexes and their signature

In [9] a signature for a Hilbert-Poincaré complex was defined. This definition is as follows.

Let C be a C^*-algebra. Recall that an n-dimensional Hilbert-Poincaré complex is a triple (E, b, S) where (E, b) is an n-dimensional chain complex

$$E_0 \xleftarrow[b_1]{} E_1 \xleftarrow[b_2]{} \cdots \longleftarrow E_{n-1} \xleftarrow[b_n]{} E_n \tag{3.1}$$

of finitely generated projective Hilbert modules over a C^*-algebra C, $S : E \to E$ is a self-adjoint operator such that

(i) $S_k : E_{n-k} \to E_k$, where $S_k = S|_{E_{n-k}}$,
(ii) $b_k S_k + S_{k-1} b^*_{n-k+1} = 0$ and
(iii) S induces an isomorphism from the homology of the dual complex (E, b^*) to the homology of the complex (E, b).

The second condition means that $S : (E, -b^*) \to (E, b)$ is a chain map.

We recall the following definition from [1, Def.2.2, p.280]

Definition 3.2. The mapping cone of a chain map $A : (E', b') \to (E, b)$ is the complex

$$E''_0 \xleftarrow[b_1]{} E''_1 \longleftarrow \cdots \xleftarrow[b_n]{} E''_n \xleftarrow[b_{n+1}]{} E''_{n+1} \tag{3.3}$$

where $E''_j = E'_{j-1} \oplus E_j$ and differential $b'' : E'' \to E''$ defined by

$$b''_j = \begin{pmatrix} -b'_{j-1} & 0 \\ A_{j-1} & b_j \end{pmatrix} \tag{3.4}$$

Using the language and notations in [1], the definitions of the signature are as follows.

Definition 3.5. (Mishchenko, [9, sec.3]). Let (E, b, S) be a Hilbert-Poincaré complex of Hilbert C-modules (with S self-adjoint and $bS + Sb^* = 0$) and let $(E \oplus E, b_S)$ the mapping cone of S. Then, the signature of (E, b, S) is the formal difference $[Q_+] - [Q_-]$ in $K_0(C)$ of the positive and negative projection of the restriction of the map B_S to the $+1$ eigenspace of the symmetry which exchanges the two copies of E in $E \oplus E$.

Remark 3.6. In the construction of Mishchenko [9] the summands in the mapping cone are interchanged and this gives a different formula for the operator: $B_S = Tb_S + b_S T$, where T is the symmetry in [9, p.14]. Both constructions give the same operator as a result, and this operator restricts to the operator $G_{ev} = b + b^* + S$ on the $+1$ eigenspace of the symmetry which exchanges the two copies of E in the construction of the mapping cone presented here and borrowed from [1]. More precisely, according to definition 3.5, one has $b_S = \begin{pmatrix} b & 0 \\ S & b^* \end{pmatrix}$ and

$$B_S = \begin{pmatrix} b+b^* & S \\ S & b+b^* \end{pmatrix}$$

Then one has

$$\begin{pmatrix} b+b^* & S \\ S & b+b^* \end{pmatrix}\begin{pmatrix} v \\ v \end{pmatrix} = \begin{pmatrix} (b+b^*+S)v \\ (b+b^*+S)v \end{pmatrix}.$$

But the space $A_{\text{ev}} = \left\{ \begin{pmatrix} v \\ v \end{pmatrix} \mid v \in E \right\}$ is the $+1$ eigenspace of the symmetry $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ interchanging the two summands in $A = E \oplus E$, and B_S commutes with this symmetry. This eigenspace identifies with E and the restriction of B_S to A_{ev} identifies with the operator $G_{ev} = b + b^* + S$

Definition 3.7. (Higson-Roe). Let (E, b, S) be an even dimensional Hilbert-Poincaré complex of Hilbert C-modules. The signature of (E, b, S) is the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$ respectively, where $B = b + b^*$.

Proposition 3.8. *Definitions 3.5 and 3.7 coincide.*

Proof. Definition (3.5) is just the index in $K_0(C)$ of the restriction G_{ev} of the operator B_S to the $+1$ eigenspace A_{ev} of the symmetry which exchanges the two copies of E in $E \oplus E$. In other words, this is the index of the operator G_{ev} in $K_0(C)$. This index comes from a decomposition of the space $A_{ev} = A_{ev}^+ \oplus A_{ev}^-$ with associated projections Q_+ and Q_- respectively.

On the other hand, definition 3.7 associates the positive projections P_+ and P_- of the restrictions $B + S$ and $B - S$ (respectively) of the operator B_S to the $+1$ and -1 eigenspaces A_{ev} and A_{odd} (respectively) of the symmetry interchanging the two copies of E. But the spaces A_{ev} and A_{odd} are isomorphic by the map T in [9, p. 14] and the decomposition $A_{ev} = A_{ev}^+ \oplus A_{ev}^-$ induces a similar decomposition of $A_{odd} = T(A_{ev}^+) \oplus T(A_{ev}^-)$ whose corresponding projections have opposite signs in K-theory. ☐

Remark 3.9. Higson-Roe definition of the index while more elaborated is more suitable for the aim of comparison with the index of the signature operator on the de Rham complex. Mishchenko's definition is a more straightforward generalization of the signature of an algebraic Poincaré complex to the context of C^*-algebras.

4. Analytically controlled Hilbert-Poincaré complexes over C^*-categories and their signature

Here we recall the definition of an analytically controlled Hilbert-Poincaré complex, its signature and other relevant constructions from [1].

Consider a triple (H, b, S), where (H, b) is an n-dimensional chain complex

$$H_0 \xleftarrow[b_1]{} H_1 \xleftarrow[b_2]{} \cdots \longleftarrow H_{n-1} \xleftarrow[b_n]{} H_n \tag{4.1}$$

of Hilbert spaces, the operator b is an unbounded, closed operator such that $b \circ b$ is defined an equal to zero, i.e. $Image(b) \subset Domain(b)$, $b^2 = 0$. The map $S : H \to H$ is an everywhere defined self-adjoint operator such that

(i) $S_k : H_{n-k} \to H_k$, where $S_k = S|_{H_{n-k}}$;
(ii) $S : (H, -b^*) \to (H, b)$ is a chain map, i.e. $S(Domain(b^*)) \subset Domain(b)$ and $(bS + Sb^*)v = 0$ for every $v \in Domain(b^*)$;

(iii) S induces an isomorphism from the homology of the dual complex (H, b^*) to the homology of the complex (H, b).

Such a triple is called an *analytic Hilbert-Poincaré complex*.

Definition 4.2. A C^*-category $\mathfrak{A}$ is a subcategory of all Hilbert spaces and bounded linear maps, i.e. an additive subcategory such that the morphisms sets $Hom_{\mathfrak{A}}(H_1, H_2)$ are Banach subspaces of the set $Hom(H_1, H_2)$ of bounded linear operators from the Hilbert space H_1 to the Hilbert space H_2.

Definition 4.3. A C^*-category ideal $\mathfrak{J}$ of the C^*-category $\mathfrak{A}$ is a C^*-subcategory possibly without identity morphisms such that any composition of a morphism in $\mathfrak{A}$ with a morphism in $\mathfrak{J}$ is a moprhism in $\mathfrak{J}$.

Remark 4.4. In the case of a C^*-category with a single object, this definition of ideal coincides with that of a (bilateral) ideal of a C^*-algebra of bounded operators on a fixed Hilbert space. In all of the following constructions, we will fix this Hilbert space.

Definition 4.5. An unbounded, self-adjoint Hilbert space operator $D : H \to H$ is said to be analytically controlled over the pair $(\mathfrak{A}, \mathfrak{J})$ if

(i) H is an object of $\mathfrak{J}$,
(ii) the operators $(D \pm iI)^{-1}$ are morphisms of $\mathfrak{J}$, and
(iii) the operator $D(1 + D^2)^{\frac{1}{2}}$ is a morphism of $\mathfrak{A}$.

This definition means that $f(D)$ is a morphism of $\mathfrak{J}$ for every $f \in C_0(\mathbb{R})$ and $f(D)$ is a morphism of $\mathfrak{A}$ for every $f \in C_0[-\infty, \infty]$.

Definition 4.6. A complex (H, b) of Hilbert spaces is said to be analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if the self-adjoint operator $B = b + b^*$ is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ according to definition 4.5.

Definition 4.7. An analytic Hilbert-Poincaré complex (H, b, S) is said to be analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if the complex (H, b) is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ in the sense of the previous definition, i.e. if $B = b + b^*$ is analytically controlled, and the duality operator T is a morphism in $\mathfrak{A}$.

It is shown in [1, lemma 5.8 and the discussion on p.291] that for a Hilbert-Poincaré complex analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ the difference $P_+ - P_-$ of the positive projections of the operators $B + S = b + b^* + S$ and $B - S = b + b^* - S$ belongs to the ideal J of A, where A is the C^*-algebra of $\mathfrak{A}$-endomorphisms of the space H and J is the C^*-algebra of $\mathfrak{J}$-endomorphisms of the same space. This means that the formal difference $[P_+] - [P_-]$ is an element of the group $K_n(J)$. There is a natural map $K_n(J) \to K_n(\mathfrak{J})$ so there is a class in $K_n(\mathfrak{J})$ determined by the difference $[P_+] - [P_-]$.

Definition 4.8. Let (H, b, S) be a Hilbert-Poincaré complex analytically controlled over $(\mathfrak{A}, \mathfrak{J})$. Its analytical signature is the class determined by the formal difference $[P_+] - [P_-]$ in $K_n(\mathfrak{J})$.

5. Signatures of a G-manifold

In this section we modify some the notions in [2] to extend the main results there to include proper, not necessarily free actions. Namely, we extend the definitions of the control categories. Also, we take into account the additional structure on the complex $C_*^{l^2}(M)$ of a co-compact G-manifold M needed to make it an algebraic Hilbert-Poincaré complex over the reduced C^*-algebra $C_r^*(G)$. This complex also is interpreted by Higson and Roe as an analytically controlled Hilbert-Poincaré complex and, therefore, it has two signatures. The relation between this signatures is addressed.

5.1. **The algebraic signature of a triangulated smooth G-manifold.** In [4], [8] it is shown that a smooth manifold M with proper action of a discrete group G admits G-invariant triangulations. They also show the uniqueness of this piecewise linear structure up to barycentric subdivision. In this case one shall choose a triangulation such that every simplex is either fixed point-wise or permuted by the action.

Following [2, p.306-310] we denote by $C_*(M)$ the space of finitely supported chains on M with complex coefficients, then for each p the complex vector space $C_p(M)$ has a basis comprised of the p-simplices. Define an inner product on $C_p(M)$ such that this basis is orthonormal. The completion of this space is denoted by $C_p^{l^2}(M)$, in other words, this is the Hilbert space of square summable p-chains on M. The differentials $\partial_p : C_p(M) \to C_{p-1}(M)$ extend to operators $b_p : C_p^{l^2}(M) \to C_{p-1}^{l^2}(M)$.

The operators b_p are bounded if the number of simplices in the triangulated space M with a common boundary is bounded, and this assumption can in turn by reduced to requiring that the number of simplices containing a point in the space M is bounded. Such a space M is called of *bounded geometry*.

Also, the adjoint operators $b_p^* : C_{p-1}^{l^2}(M) \to C_p^{l^2}(M)$ identify with the extension of the co-boundary maps. This makes $(C_*^{l^2}(M), b)$ a complex of Hilbert spaces.

Denote by $C_0(M)$ the algebra of continuous functions vanishing at infinity. Define a representation of $C_0(M)$ on $C_*^{l^2}(M)$ as follows: for every $f \in C_0(M)$ and chain $c = \sum_\sigma c_\sigma[\sigma]$,

$$f \cdot c = \sum_\sigma f(b_\sigma) c_\sigma[\sigma],$$

where b_σ is the barycenter of the simplex σ. With this and the bounded geometry assumption, one might interpret $(C_*^{l^2}(M), b)$ as a complex of Hilbert $C_0(M)$-modules, but this spaces are not in general finitely generated over this algebra.

On the other hand, as the action of $M \times G \to M$ is simplicial, the complex $C_*(M)$ has a natural action of this group defined by the formula

$$c \cdot g = \sum_\sigma c_\sigma[\sigma]g = \sum_\sigma c_\sigma[\sigma g],$$

for $g \in G$. The action is simplicial, so it commutes with the boundary map. As the action either fixes simplices or permutes them, this action is by unitaries, and it extends to a representation of the reduced C^*-algebra $C_r^*(G)$ of the group G. This means that $(C_*^{l^2}(M), b)$ is a complex of $C_r^*(G)$-modules. If the quotient M/G is compact, then the modules $C_p^{l^2}(M)$ are finitely generated: one may assume that there is a finite number of simplices in the triangulation of the compact quotient $X = M/G$ induced by the map $M \to M/G$, and this means that there is a finite number of G-orbits of simplices in M.

In order to analyze Poincaré duality in this context one shall first give some explicit expression of the action of G on cochains. If $u : C^p(M) \to \mathbb{C}$ a p-cochain, this is defined by the rule

$$(u \cdot g)[\sigma] = u([\sigma g^{-1}]),$$

for $\sigma \in C^p(M)$.

The Poincaré duality homomorphism of an oriented, possibly non-compact manifold M is given by the intersection $[M] \cap u$ of the fundamental class of the manifold with a finitely supported cochain u. More precisely, let $u : C^{n-p}(M) \to \mathbb{C}$ a finitely supported $(n-p)$-cochain and $[M] = \sum_\sigma (-1)^{\epsilon(\sigma)}[\sigma]$ the fundamental class, where $\epsilon(\sigma)$ denotes the orientation of the simplex σ induced by the orientation of the manifold M, and the sum runs over all n-simplices in the triangulation of M. Then, the Poincaré duality homomorphism $T_p : C^{n-p}(M) \to C^p(M)$ is defined by the formula

$$T_p(u) = [M] \cap u = \sum_{\sigma=[v_0\cdots v_n]} (-1)^{\epsilon(\sigma)} u([v_0 \cdots v_{n-p}])[v_{n-p} \cdots v_n].$$

This map is G-equivariant, i.e. satisfies the identity

$$T_p(u \cdot g) = (T_p(u)) \cdot g.$$

Indeed,

$$\begin{aligned}
(T_p(u)) \cdot g &= \left(\sum_{\sigma=[v_0\cdots v_n]} (-1)^{\epsilon(\sigma)} u([v_0 \cdots v_{n-p}])[v_{n-p} \cdots v_n] \right) \cdot g = \\
&= \sum_{\sigma=[v_0\cdots v_n]} (-1)^{\epsilon(\sigma)} u([v_0 \cdots v_{n-p}])[v_{n-p} \cdots v_n]g = \\
&= \sum_{\sigma=[v_0\cdots v_n]} (-1)^{\epsilon(\sigma)} u([v_0 \cdots v_{n-p}]g^{-1}g)[v_{n-p} \cdots v_n]g = \\
&= \sum_{\sigma=[v_0\cdots v_n]} (-1)^{\epsilon(\sigma)} (u \cdot g)[v_0 \cdots v_{n-p}]g[v_{n-p} \cdots v_n]g = \\
&= \sum_{\sigma g=[v_0\cdots v_n]g} (-1)^{\epsilon(\sigma)} (u \cdot g)[v_0 \cdots v_{n-p}][v_{n-p} \cdots v_n] = \\
&= T_p(u \cdot g).
\end{aligned}$$

where in the last step one must require that g preserves orientation. The equivariant map $T : C^*(M) \to C_*p(M)$ satisfies the classic Poincaré duality identities and extends to a G-linear map $T : C^{l^2}_*(M) \to C^{l^2}_*(M)$. Then, if the dimension of M is even, the operator $S : C^{l^2}_*(M) \to C^{l^2}_*(M)$ defined by the rule

$$S_p = i^{p(p-1)} T_p : C^{l^2}_{n-p}(M) \to C^{l^2}_p(M)$$

satisfies the identities

(i) S is self-adjoint,
(ii) $bS + Sb^* = 0$ and
(iii) S induces an isomorphism from the homology of the dual complex $(C^*(M), b^*)$ to the homology of the complex $(C^*(M), b)$.

Therefore, $(C^*(M), b, S)$ is an algebraic Hilbert-Poincaré complex over $C^*_r(G)$ and has algebraic signature in $K_0(C^*_r(G))$ as in definitions 3.5 or 3.7.

With this structure, one obtains another proof of the following:

Proposition 5.1. *The signature of Mishchenko is a homotopy invariant.*

Proof. This is theorem 4.3 of [1] applied to the signature defined there, but using the algebraic Hilbert-Poincaré complex we have just defined. These signatures coincide by proposition 3.8. □

Remark 5.2. The construction of this Hilbert-Poincaré complex has been presented by Mishchenko in several conference talks before 2010, so the authors claim no originality. We refer to [9, sec.3] and check that this complex satisfies the definition given there.

5.2. **The analytic signatures of a smooth G-manifold.** Here we generalize the C^*-categories considered in [2] and reinterpret the complex $(C^*(M), b, S)$ as an equivariant analytically controlled Hilbert-Poincaré complex. Then we show that the results about invariance of the analytic signature can be applied to bounded geometry spaces with *bounded isotropy* action, and that this is the case for proper spaces with bounded geometry quotient.

Definition 5.3. Let M be a proper metric space. An M-module H is a separable Hilbert space equipped with a non-degenerate representation of the C^*-algebra $C_0(M)$ of continuous, complex-valued functions on M vanishing at infinity.

Definition 5.4. Let G a finitely generated discrete group. A G-presented space X is a proper geodesic metric space presented as the quotient $X = M/G$ of a proper geodesic metric space M by an isometric proper action $\mu : G \times M \to M$ of the group G. The pair (M, μ) is called a G-presentation of X.

For fixed discrete group G and space X, the presentations of X together with equivariant maps form a category. We avoid the action in the notation and say that M is a G-presentation of X. We shall assume in the following that all such presentations have an invariant open set where the action of the group G is free.

Definition 5.5. An equivariant G-X-module is an M-module H, where M is a G-presentation of X equipped with a compatible unitary representation of G.

Given a locally compact, separable and metrizable space, together with a non-degenerate representation on the Hilbert space H, that is, a nondegenerate continuous $*$-homomorphism

$$\rho : C_0(X) \to B(H),$$

we define the suppport of $\nu \in H$ to be the complement in X of the union of all open subsets $U \subset X$ such that $\rho(f)(\nu) = 0$ for all $f \in C_0(U)$. An operator $T \in B(H)$ is locally compact on X if fT and Tf are compact operators for all functions $f \in C_0(X)$.

Definition 5.6. The support of an operator $T \in B(H)$, denoted by $Supp(T)$, is the complement in $X \times X$ of the union of all open subsets $U \times V \subset X \times X$ such that $\rho(f)T\rho(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$. More generally , if $C_0(X)$ and $C_0(Y)$ are represented no degnerately on Hilbert spaces H_X and H_Y, then the support of a bounded operator $T : H_X \to H_Y$ is the complement in $Y \times X$ of the union of all open subsets $U \times V \subset Y \times X$ such that $\rho_Y(f)T\rho_X(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$.

Definition 5.7. Let X be a locally compact separable and metrizable space, proper in the sense of metric geometry, meaning that closed balls are compact. Let $\rho : C_0(X) \to B(H)$ be a nondegenerate representation on the Hilbert space X.

An operator $T \in B(H)$ is boundedly controlled if the support $Supp(T)$ is at bounded distance of the diagonal in $X \times X$, that means

$$\sup_{y \in Supp(T), x \in \Delta(X)} \{(d_{X \times X}(y, x)\} < \infty.$$

An operator T is locally compact on X if fT and Tf are compact for all functions $f \in C_0(X)$.

Given an operator $T \in B(H)$, we define its propagation $Prop(T)$, to be the following extended real number:

$$Prop(T) = sup\{d_{X \times X}(d(x, y) \mid x, y \in Supp(T)\},$$

and will say that an operator is of finite propagation if this number is finite.

Definition 5.8. Let M be an equivariant G-X-module. The category $\mathfrak{A}(G,M) = \mathfrak{A}(X,G,M)$ is the category where the objects are equivariant G-X-modules for a fixed presentation M, and the morphisms are norm limits of G-equivariant, bounded, finite propagation operators between G-X-modules. The ideal $\mathfrak{C}(G,M) = \mathfrak{C}(X,G,M)$ is the category with the same objects as $\mathfrak{A}(G,M)$, and morphisms given by norm limits of G-equivariant, bounded, compactly supported operators.

The category $\mathfrak{A}(G,M)$ and its ideal $\mathfrak{C}(G,M)$, are defined in an analogous way to the categories $\mathfrak{A}(X)$ and $\mathfrak{C}(X)$, compare [2, p. 304].

Definition 5.9. Let X be a proper (both in the sense of group actions and metric geometry), locally compact and metrizable G-space. $D : H \to H$ be a bounded selfadjoint operator . We will say that T is analytically controlled if it is controlled over $(\mathfrak{A}(X,G,M), \mathfrak{C}(X,G,M))$ in the sense of definition 4.5.

We will now include for the sake of completness the following notion of geometrically controlled operator, which will be relevant for the comparison with Hilbert-Poincaré complexes (see Def. 5.3 and 5.5 in [1] for more details on geometric control):

Definition 5.10. Let X be a geodesic, proper space (in the sense of metric geometry, meaning that closed balls are compact). A complex based vector space V is called geometrically controlled over X if it is provided with a basis $B \subset V$, and a function $c : V \to X$ with the following property: for every $r > 0$, there is an $N < \infty$ such that if $S \subset X$ has diameter less than R, then $c^{-1}(S)$ has cardinality less than N.

A linear transformation $T : V \to W$ between geometrically controlled spaces is geometrically controlled if

- The matrix coefficients with respect to the basis are uniformly bounded.
- There exists some $C > 0$ such that the (v,w)-matrix coefficient is zero whenever $d(c(v), c(w)) > C$.

For X compact, one can now proof the analogous of lemma 2.12 in [2]:

Lemma 5.11 (2.12 in [2])**.** *The C^*-algebra of endomorphisms of a non-trivial object in $\mathfrak{C}(X,G,M)$ is Morita equivalent to $C^*_r(G)$ and, therefore, their K-theories are isomorphic.*

Proof. Actually, the arguments for its proof in [13] are given for a proper cocompact action of G. □

Let M be a simplicial complex, and let $G \times M \longrightarrow M$ be a proper simplicial action of a discrete group G. Assume that the quotient M/G is compact. Let $\mathfrak{F}_M$ the family generated by the (finite) subgroups of G having non empty fixed point set in M, i.e.

$$\mathfrak{F}_M = \{H < G \mid M^H \neq \emptyset\},$$

where

$$M^H = \{x \in M \mid hx = x, h \in H\}.$$

Definition 5.12. The action $G \times M \longrightarrow M$ is said to be of bounded isotropy if the order of the elements in $\mathfrak{F}_M$ is uniformly bounded, i.e. there is a constant c_M such that $|H| < c_M$ for every $H \in \mathfrak{F}_M$.

Lemma 5.13. *If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \longrightarrow M$ is of bounded isotropy, then M is of bounded geometry.*

Proof. Take a point $x \in M$ and let $S(x)$ the set of simplices containing x. Denote by $p : M \longrightarrow M/G$ the projection on the quotient. Then $p(S(x)) = S(p(x))$ and, therefore, $\#S(x) \leqslant \#S(p(x)) \cdot c_M \leqslant N \cdot c_M$, where N is the bound on the number of simplices containing a point in M/G. □

Lemma 5.14. *A proper co-compact space M, with action $G \times M \longrightarrow M$ of a discrete group G is of bounded isotropy.*

Proof. Choose finite family (U_i, G_i), $i = 1, \ldots, N'$ such that $U_i \subset M$ are open subsets and $G_i < G$ are finite subgroups such that, if a point $x \in gU_i$ for some $g \in G$, then one has that $G_x < gG_ig^{-1}$. Therefore $\beta = \max_i |G_i|$ is a bound on the orders of isotropy groups of points in the space M. □

We will not discuss the functorial properties of the C^*-alebras associated to coarse structures of a proper metric space, called in the literature "morphism covering a coarse map". However, we will need a restriction map for the inclusion of a boundary component into a a bordism satisfying some additional assumptions, see the coments preceeding section 6.

Definition 5.15. Let X be a proper space and M a G-presentation of X. A Hilbert-Poincaré complex is equivariantly analytically controlled if it is analytically controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$, i.e. the modules in the complex are objects of these categories, the operator $B = b + b^*$ is controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$ and the duality operator S is a morphism in the category $\mathfrak{A}(X, G, M)$.

In the following, by controlled in the case of a complex of Hilbert modules we mean equivariantly analytically controlled and in the case of an operator we mean controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$.

Theorem 5.16. *If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \longrightarrow M$ is of bounded isotropy and orientation preserving, then its Higson-Roe non-commutative signature is a homotopy and bordism invariant in the controlled category.*

Proof. As M is of bounded geometry, its simplicial chain and cochain complexes are geometrically controlled. The action either permutes or fixes simplices and is therefore unitary, and the fundamental cycle of such a triangulation is invariant. By theorem 3.14 in [2, p.309] geometric control implies analytic control. The comment before section 3.2 on [2, p.310] ensure that this is true also in the equivariant setting. This means that the l^2-chain complex $C_*^{l^2}(M)$ of M is an example of an analytically controlled Hilbert-Poincare complex.

In the case of bordism invariance, one shall assume that one has a triangulated bordism such that the simplices in the boundary coincide with the given triangulation of M.

The result now follows as a corollary of theorems 5.12 and 7.9 of [1]. □

6. Directed bordism invariance

In this section, we review the approach to bordism invariance of the coarse index due to C. Wulff [14] and extend it to the context of manifolds with proper actions of a discrete group.

We recall that given a smooth manifold with a proper, smooth G-action M, the existence of G- invariant Riemannian metrics due to Palais [12] implies the existence of a G-invariant geodesic length metric on M. Recall that this geodesic metric is proper in the sense of metric geometry, meaning that closed balls are compact.

In order to define adequately the (coarse index) boundary maps and the functoriality properties after K-treory, certain remarks on the bounded coarse structure on a proper geodesic manifold are pertinent within an equivariant setting. References for the bounded coarse bounded structure, and other ones on a geodesic metric space include [3], chapter 6, although we specialize here to the Riemannian manifold case. We recall however that, as noticed by Wulff [14], the proof of bordism invariance of the coarse index is independent of the coarse structure considered on the manifolds.

Definition 6.1 (Coarse map in the bounded metric structure)**.** Let M and N be proper Riemannian G-manifolds equipped with G-invariant geodesic length metrics d_M and d_N. A map $f : N \to M$ is a coarse map is

- The inverse image of every closed ball is compact.
- For every $R > 0$, there exists $\delta > 0$, such that $d_N(x, x^{'}) < R$ implies $d_M(f(x), f(x^{'})) < \delta$.

A coarse map induces a C^*-homomorphism of the alebras of locally compact and finite propagation operators by lemma 6.3.13 in [3].

Definition 6.2 (Coarse equivalence)**.** A coarse equivalence is a map $f : N \to M$ for which there exists another map $g : M \to N$, called a coarse inverse, such that the compositions $f \circ g$ and $g \circ f$ are at bounded distance to the identity, meaning that there exists a real number $M > 0$ that satisfies the following inequalities:

$$d_N(g \circ f(x), g \circ f(x^{'})) < M$$
$$d_M(f \circ g(y), f \circ g(y^{'})) < N$$

Definition 6.3 (Directed bordism)**.** Let N_1 and N_2 be proper, oriented G-manifolds of dimension n. Assume that they are furnished with the bounded coarse structure associated to the geodesic length metric.

A directed bordism from N_1 to N_2 is a proper G-manifold W, such that $\partial W = N_1 \coprod N_2$, the inclusions $i_1 : N_1 \to W$, $i_2 : N_2 \to W$ are coarse maps, and the coarse map i_2 is a coarse equivalence.

Definition 6.4. [Analytical c- bordism groups] Let M be a proper G-space with an action of bounded isotropy. The analytical bordism group $\Omega_n^{\mathrm{an,eq}}(M)$ is the group with generators $(N, f, E, b,)$, such that N is a proper, oriented manifold of bounded isotropy with an equivariant coarse map $f : N \to M$. E is G-X-Hilbert module with presentation N, i.e. an equivariant N-module with $X = N/G$, and $b : E \to E$ is a controlled operator.

Two of such generators (N_1, f_1, E_1, b_1) and (N_2, E_2, f_2, b_2) are said to be c-bordant if there exist directed bordisms W from N_1 to N_2, and $\bar{W}$ from N_2 to N_1, together with coarse maps $F : W \to M$, $\bar{F} : \bar{W} \to M$, maps $E_i \to E$, covering the inclusions $N_i \to W$, $N_i \to \bar{W}$ and a pair of controlled operators B , $\bar{B}$ restricting to f_i, respectively b_i .

If the space M is a proper oriented manifold of bounded isotropy, then one defines the signature representative in the group $\Omega_n^{\mathrm{an,eq}}(M)$ by taking $f = \mathrm{id}$ and, for example, $E = \Omega^*_{L^2}(M)$, the L^2-completion of the de Rham complex of M and b as the signature operator. Although this is an unbounded operator, the generalized conditions of analytical control meet (meaning that the Cayley transform is locally compact and of finite propagation and the resolvent has finite propagation).

One can also, take $E' = C_*^{l^2}(M) \oplus C_*^{l^2}(M)$ and $b = B_S$ as in (3.5), where S is the Poincaré duality homomorphism completion. Both choices coincide in terms of index by theorems 5.5 and 5.12 in [2], using the version of analytic control defined in here.

Definition 6.5 (Coarse Index)**.** The coarse index of a bordism class $\alpha \in \Omega_n^{\mathrm{an,eq}}(M)$ with representative $(M, f, E, b)$is the class in $K^{n-1}(\mathfrak{A}(X, G, M)/\mathfrak{C}(X, G, M))$ of the boundedly controlled operator b associated to the Hilbert-Poincaré complex E. We will denote this group homomorphism by I, denoting the coarse index.

Definition 6.6 (Analytical signature)**.** The analytical signature of a bordism class $\alpha \in \Omega_n^{\text{an,eq}}(M)$ with representative (M, f, E, b) is given as the composition of the coarse index map I together with the coarse assembly map . (The coarse assembly map for X is the homomorphism

$$\mu : K_i(M) \cong K_{i+1}(D_G^*(M)/C_G^*(M)) \to K_i(C_G^*(M)),$$

where the first isomorphism is given by Paschke duality and the second is the boundary map in the long exact sequence of K-groups associated to the ideal $C^*(M)$ in $D^*(M)$.)

In the following we shorten the notation $\mathfrak{A}(X, G, M)$, $\mathfrak{C}(X, G, M)$ by $\mathfrak{A}(M)$, $\mathfrak{C}(M)$ respectively.

We interpret now the main result of [14] in an equivariant setting:

Theorem 6.7. *The analytical G-signature is a directed bordism invariant.*

Proof. The situation is completely analogous to [14], where the invariance is seen to be a consequence of the naturality of the assembly map. Consider the diagram of G-equivariant inclusions, which are assumed to give coarse maps.

$$\begin{array}{ccccc} N_1 & \longrightarrow & \partial W & \longleftarrow & N_2 \\ & & \big\downarrow & & \\ & & W & & \end{array}$$

The long exact sequence in K-theory of C^*-algebras gives:

$$\begin{array}{ccccc} K_{p+1}(\mathfrak{A}(W \diagup \partial W)/\mathfrak{C}(W \diagup \partial W)) & \xrightarrow{\partial} & K_p(\mathfrak{A}(\partial W)/\mathfrak{C}(\partial W) & \longrightarrow & K_p(\mathfrak{A}(W)/\mathfrak{C}(W) \\ & & \big\downarrow A_{\partial W} & & \big\downarrow A_W \\ & & K_p(C^*(\partial W)) & \xrightarrow[i_*]{} & K_p(C^*(W)) \end{array}$$

Where the upper morphism ∂ is the connecting homomorphism, and the vertical morphisms are coarse assembly maps.

The functoriality of the index morphism, assembly map gives thus that

$$A_W(i_1([b_1]) = A_W(i_2([b_2]).$$

□

7. Mapping surgery to analysis

In this section, we will state the main theorem of this note:

Theorem 7.1. *Let M be a proper G manifold with a bounded isotropy action. Assume that the quotient M/G has bounded geometry. Then, the following diagram is commutative*

$$\begin{array}{ccccccc} \Omega_n^{\text{an,eq}}(M) & & & \xrightarrow{\text{Alg Signature}} & & & \\ \big\downarrow \text{I} & & & & & & \searrow \\ K_{n-1}(\mathfrak{A}(M)/\mathfrak{C}(M)) & \xrightarrow[\omega_1]{} & KK_G^n(C_0(M), \mathbb{C}) & \xrightarrow[\omega_2]{} & KK_G^n(C_0(\underline{E}G), \mathbb{C}) & \xrightarrow[\mu]{} & K_n(C_*^r(G)) \end{array} ,$$

where the maps are definded as follows: the map I is the coarse index, ω_1 is the isomorphism constructed in [13]*(denoted by ω_4 in page 242), the group homomorphism ω_2 is induced by the up to G-equivariant homotopy unique map $M \to \underline{E}G$, and μ denotes the analytical Baum-Connes assembly map in KK-theory. We will call the composition*

$$\mu \circ w_2 \circ w_1 \circ I : \Omega_n^{\mathrm{an,eq}}(M) \longrightarrow K_n(C_*^r(G))$$

the analytical signature.

Proof. The analytical Assembly map $\mu : KK_G^n(C_0(\underline{E}G), \mathbb{C}) \to K_n(C_*^r(G))$ is given by the composite of the *descent homomorphism*

$$KK_G^n(C_0(\underline{E}G), \mathbb{C}) \to KK^n(C_0(\underline{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G)$$

followed by composing with the map given by the Kasparov product with the Mishchenko-Fomenko line bundle for $\underline{E}G$,

$$KK^n(C_0(\underline{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G) \to KK^n(\mathbb{C}, C_r^*(G)).$$

By KK-theoretical homotopy invariance, the composite map

$$KK_G^n(C_0(M), \mathbb{C}) \underset{\omega_2}{\longrightarrow} KK_G^n(C_0(\underline{E}G), \mathbb{C}) \underset{\mu}{\longrightarrow} K_n(C_*^r(G))$$

agrees with the composite

$$KK_G^n(C_0(M), \mathbb{C}) \to KK^n(C_0(M) \rtimes_r G, \mathbb{C} \rtimes_r G) \to KK^n(\mathbb{C}, C_r^*(G)),$$

which consists of the descent homomorphism followed by the Kasparov product with a Mishchenko-Fomenko element for $C_0(M)$ (called w_5 and w_6 in [13], p. 242, respectively.) □

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