

# Twisted Equivariant $K$ -Theory and $K$ -Homology of $Sl_3\mathbb{Z}$

NOÉ BÁRCENAS  
MARIO VELÁSQUEZ

We use a spectral sequence to compute twisted equivariant  $K$ -Theory groups for the classifying space of proper actions of discrete groups. We study a form of Poincaré Duality for twisted equivariant  $K$ -theory studied by Echterhoff, Emerson and Kim in the context of the Baum-Connes Conjecture with coefficients and verify it for the Group  $Sl_3\mathbb{Z}$ .

In this work, we examine computational aspects relevant to the computation of twisted equivariant  $K$ -theory and  $K$ -homology groups for proper actions of discrete groups.

Twisted  $K$ -theory was introduced by Donovan and Karoubi [DK70] assigning to a torsion element  $\alpha \in H^3(X, \mathbb{Z})$  abelian groups  ${}^\alpha K^*(X)$  defined on a space by using finite dimensional matrix bundles. After the growing interest by physicists in the 1990s and 2000s, Atiyah and Segal [AS04] introduced a notion of twisted equivariant  $K$ -theory for actions of compact Lie Groups. In another direction, orbifold versions of twisted  $K$ -theory were introduced by Adem and Ruan [AR03], and progress was made to develop computational tools for Twisted Equivariant  $K$ -Theory with the construction of a spectral sequence in [BEUV13].

The paper [BEJU12] introduces Twisted equivariant  $K$ -theory for proper actions, allowing a more general class of twists, classified by the third integral Borel cohomology group  $H^3(X \times_G EG, \mathbb{Z})$ .

We concentrate in the case of twistings given by discrete torsion, which is given by cocycles

$$\alpha \in Z^2(G, S^1)$$

representing classes in the image of the projection map

$$H^2(G, S^1) \xrightarrow{\cong} H^3(BG, \mathbb{Z}) \rightarrow H^3(X \times_G EG, \mathbb{Z}).$$

Under this assumption on the twist, a version of Bredon cohomology with coefficients in twisted representations can be used to approximate twisted equivariant  $K$ -Theory, by means of a spectral sequence studied in [BEUV13] and [Dwy08].

The Bredon (co)-homology groups relevant to the computation of twisted equivariant  $K$ -theory, and its homological version, twisted equivariant  $K$ -homology satisfy a Universal Coefficient Theorem, 1.13. We state it more generally for a pair of coefficient systems satisfying conditions 1.12.

**Theorem** (Universal Coefficient Theorem) Let  $X$  be a proper, finite  $G$ -CW complex. Let  $M^?$  and  $M_?$  be a pair of functors satisfying Conditions 1.12. Then, there exists a short exact sequence of abelian groups

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}(H_{n-1}^G(X, M_?), \mathbb{Z}) \rightarrow H_G^n(X, M^?) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n^G(X, M_?), \mathbb{Z}) \rightarrow 0$$

The main application for the cohomological methods described in this note are computations verifying a form of Poincaré Duality in twisted equivariant  $K$ -Theory for discrete groups, as studied by Echterhoff, Emerson and Kim [EEK08] in the context of the Baum-Connes Conjecture with coefficients.

Given a  $G$ - $C^*$ -algebra  $A$ , the Baum-Connes conjecture with coefficients in  $A$  for  $G$  predicts that an analytical assembly map

$$K_*^G(\underline{EG}, A) = KK_G(C_0(\underline{EG}), A) \longrightarrow K_*(A \rtimes G)$$

is an isomorphism, where  $A \rtimes G$  is the crossed product  $C^*$ -algebra.

A discrete torsion twist defines an action of  $G$  on the  $C^*$ -algebra of compact operators on  $\ell^2(G)$ , which only depends on the cohomology class of  $\omega$  and not on particular representing cocycles. We will denote this  $G$ - $C^*$ -algebra by  $K_\omega$ . Given a discrete torsion twist  $\omega$ , there exist analytic versions of twisted equivariant  $K$ -homology and  $K$ -theory groups,  $K_G^*(X, \omega)$ ,  $K_*^G(X, \omega)$ , which are defined in terms of the equivariant Kasparov  $KK$ -theory groups  $KK_G^*(C_0(X), K_\omega)$ , respectively  $KK_G^*(K_\omega, C_0(X))$ . See [EEK08] for more details.

Our computational methods describe a duality relation, corollary 7.3 between these groups as follows:

**Corollary** (Duality for twisted equivariant  $K$ -theory) Let  $G$  be a discrete group with a finite model for  $\underline{EG}$ . Let  $\omega \in Z^2(G, S^1)$ . Assume that the Bredon cohomology groups  $H_G^*(X, R^{-\alpha})$  relevant to the computation of the twisted equivariant  $K$ -theory are all free abelian and are concentrated in degree 0 and 1. Then, there exists a duality isomorphism

$$K_G^*(\underline{EG}, K_{-\omega}) \longrightarrow K_G^*(K_\omega, \underline{EG})$$

The family of groups  $G$  satisfying the previous assumptions on both the twist and the classifying space  $\underline{E}G$  includes several examples. We pay attention to  $Sl_3\mathbb{Z}$ .

We use the computation of the cohomology of  $Sl_3\mathbb{Z}$  due to Soulé [Sou78], previous work by Sánchez-García [SG08], as well as the theory of projective representations of finite groups to compute the twisted equivariant  $K$ -theory on the classifying space  $\underline{E}Sl_3\mathbb{Z}$ , by computing the Bredon cohomology group associated to a specific torsion twist. Using the spectral sequence we verify that the computation of twisted equivariant  $K$ -theory reduces to Bredon cohomology in Theorem 6.3.

**Theorem** (Calculation of twisted equivariant  $K$ -homology of  $Sl_3\mathbb{Z}$ ) The equivariant  $K$ -homology groups of  $Sl_3\mathbb{Z}$  with coefficients in the  $Sl_3\mathbb{Z}$ - $C^*$  algebra  $\mathcal{K}_{u_1}$  are given as follows:

$$K_p^{Sl_3\mathbb{Z}}(\underline{E}Sl_3\mathbb{Z}, \mathcal{K}_{u_1}) = 0 \text{ p odd, } K_p^{Sl_3\mathbb{Z}}(\underline{E}Sl_3\mathbb{Z}, \mathcal{K}_{u_1}) \cong \mathbb{Z}^{\oplus 13} \text{ p even}$$

## 0.1 Acknowledgments

This first author was supported by the Hausdorff Center for Mathematics, Wolfgang Lück's Leibnizpreis and a CONACYT postdoctoral fellowship. The second author was supported by COLCIENCIAS through the grant "Becas Generación del Bicentenario" number 494, the Fundación Mazda para el Arte y la Ciencia, Wolfgang Lück's Leibnizpreis and a UNAM postdoctoral fellowship. The authors thank an anonymous referee for valuable suggestions concerning the presentation of the material exposed in this note.

## 1 Bredon cohomology

We recall briefly some definitions relevant to Bredon homology and cohomology, see [MV03] for more details. Let  $G$  be a discrete group. A  $G$ -CW-complex is a CW-complex with a  $G$ -action permuting the cells and such that if a cell is sent to itself, this is done by the identity map. We call the  $G$ -action proper if all cell stabilizers are finite subgroups of  $G$ .

**Definition 1.1** A model for  $\underline{E}G$  is a proper  $G$ -CW-complex  $X$  such that for any proper  $G$ -CW-complex  $Y$  there is a unique  $G$ -map  $Y \rightarrow X$ , up to  $G$ -homotopy equivalence.

One can prove that a proper  $G$ -CW-complex  $X$  is a model of  $\underline{EG}$  if and only if the subcomplex of fixed points  $X^H$  is contractible for each finite subgroup  $H \subseteq G$ . It can be shown that classifying spaces for proper actions always exist. They are clearly unique up to  $G$ -homotopy equivalence.

Let  $\mathcal{O}_G$  be the orbit category of  $G$ ; a category with one object  $G/H$  for each subgroup  $H \subseteq G$  and where morphisms are given by  $G$ -equivariant maps. There exists a morphism  $\phi : G/H \rightarrow G/K$  if and only if  $H$  is conjugate in  $G$  to a subgroup of  $K$ .

**Definition 1.2** (Cellular Chain complex associated to a  $G$ -CW complex) Let  $X$  be a  $G$ -CW-complex. The contravariant functor  $\underline{C}_*(X) : \mathcal{O}_G \rightarrow \mathbb{Z} - \text{CHCOM}$  assigns to every object  $G/H$  the cellular  $\mathbb{Z}$ -chain complex of the  $H$ -fixed point complex  $\underline{C}_*(X^H) \cong C_*(\text{Map}_G(G/H, X))$  with respect to the cellular boundary maps  $\underline{\partial}_*$ .

We will use homological algebra to define Bredon homology and cohomology functors.

A contravariant coefficient system is a contravariant functor

$$M : \mathcal{O}_G \rightarrow \mathbb{Z} - \text{MODULES}$$

Given a contravariant coefficient system  $M$ , the Bredon cochain complex  $C_G^*(X; M)$  is defined as the abelian group of natural transformations of functors defined on the orbit category  $\underline{C}_*(X) \rightarrow M$ . In symbols,

$$C_G^n(X; M) = \text{Hom}_{\mathcal{O}_{\mathcal{F}_G}}(\underline{C}_n(X), M)$$

Where  $\mathcal{F}_G$  is a family containing the isotropy groups of  $X$ .

Given a set  $\{e_\lambda\}$  of orbit representatives of the  $n$ -cells of the  $G$ -CW complex  $X$ , and isotropy subgroups  $S_\lambda$  of the cells  $e_\lambda$ , the abelian groups  $C_G^n(X, M)$  satisfy:

$$C_G^n(X, M) = \bigoplus_{\lambda} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/S_\lambda))$$

with one summand for each orbit representative  $e_\lambda$ . They afford a differential  $\delta^n : C_G^n(X, M) \rightarrow C_G^{n-1}(X, M)$  determined by  $\underline{\partial}_*$  and maps  $M(\phi) : M(G/S_\mu) \rightarrow M(G/S_\lambda)$  for morphisms  $\phi : G/S_\lambda \rightarrow G/S_\mu$ .

**Definition 1.3** (Bredon cohomology) The Bredon cohomology groups with coefficients in  $M$ , denoted by  $H_G^*(X, M)$  are the cohomology groups of the cochain complex  $(C_G^*(X, M), \delta^*)$ .

Dually to the cohomological situation, given a covariant functor

$$N : \mathcal{O}_G \rightarrow \mathbb{Z} - \text{MODULES},$$

the chain complex

$$C_*^G = \underline{C}^n(X) \otimes_{\mathcal{O}_G} N = \bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes N(G/S_{\lambda})$$

admits differentials  $\delta_* = \partial_* \otimes N(\phi)$  for morphisms  $\phi : G/S_{\lambda} \rightarrow G/S_{\mu}$  in  $\mathcal{O}_G$ .

**Definition 1.4** (Bredon homology) The Bredon homology groups with coefficients in  $N$ , denoted by  $H_*^G(X, N)$ , are defined as the homology groups of the chain complex  $(C_*^G(X, N), \delta_*)$

**Remark 1.5** (Determination of Bredon (co)-homology in practice) The coefficient systems considered in this note yield chain complexes, respectively cochain complexes of free abelian groups with preferred bases to compute both Bredon homology and cohomology.

Notice that if we have a complex of free abelian groups

$$\dots \rightarrow \mathbb{Z}^{\oplus n} \xrightarrow{f} \mathbb{Z}^{\oplus m} \xrightarrow{g} \mathbb{Z}^{\oplus k} \rightarrow \dots$$

with  $f$  and  $g$  represented by matrices  $A$  and  $B$  for some fixed basis, then the homology at  $\mathbb{Z}^{\oplus m}$  is

$$\ker(g)/\text{im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_s\mathbb{Z} \oplus \mathbb{Z}^{\oplus(m-s-r)},$$

where  $r = \text{rank}(B)$  and  $d_1, \dots, d_s$  are the elementary divisors of  $A$ .

Given a discrete group  $G$ , the complex representation ring defines two functors defined on the subcategory generated by objects  $G/H$  with  $H$  finite.

$$\mathcal{R}^? \quad \mathcal{R}_?$$

For every object  $G/H$ , the groups  $\mathcal{R}^?(G/H), \mathcal{R}_?(G/H)$  agree with the Grothendieck group of isomorphism classes of complex representations  $R_{\mathbb{C}}(H)$  of the finite subgroup  $H$ .

The contravariant functor  $\mathcal{R}^?$  assigns to a  $G$ -map  $\phi : G/H \rightarrow G/K$  the restriction map  $R_{\mathbb{C}}(K) \rightarrow R_{\mathbb{C}}(gHg^{-1}) \cong R_{\mathbb{C}}(H)$  to the subgroup  $gKg^{-1}$  of  $H$  determined by the morphism  $\phi : G/H \rightarrow G/K$ , where  $g \in G$  is such that  $\phi(eH) = gK$ .

The covariant functor  $\mathcal{R}_?$  assigns to the morphism  $\phi$  the induction map  $R_{\mathbb{C}}(H) \cong R_{\mathbb{C}}(gHg^{-1}) \rightarrow R_{\mathbb{C}}(K)$ .

Given a finite group  $H$  the group  $R_{\mathbb{C}}(H)$  is free abelian, isomorphic to the free abelian group generated by the set  $\rho_1, \dots, \rho_s$  of irreducible characters. For any representation  $\rho$ , there exists a unique expression  $\rho = n_1\rho_1, \dots, n_s\rho_s$ , where  $n_i = (\rho \mid \rho_i)$ , and  $(\mid)$  is the scalar product of characters.

Recall that due to Frobenius reciprocity, given a subgroup  $K$  of  $H$ , a representation  $\tau$  of  $K$  and a representation  $\rho$  of  $H$ , the equation  $(\tau \uparrow \mid \rho)_H = (\tau \mid \rho \downarrow)_K$  holds, where  $\downarrow$  denotes restriction and  $\uparrow$  denotes induction.

### 1.1 Bredon (co)-homology with coefficients in twisted representations.

**Definition 1.6** Let  $H$  be a finite group and  $V$  be a complex vector space. Given a cocycle  $\alpha : H \times H \rightarrow S^1$  representing a class in  $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$ , an  $\alpha$ -twisted representation is a function  $P : H \rightarrow Gl(V)$  satisfying:

$$\begin{aligned} P(e) &= 1 \\ P(x)P(y) &= \alpha(x, y)P(xy) \end{aligned}$$

The isomorphism type of an  $\alpha$ -twisted representation only depends on the cohomology class in  $H^2(H, S^1)$ .

**Definition 1.7** Let  $H$  be a finite group and  $\alpha : H \times H \rightarrow S^1$  be a cocycle representing a class in  $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$ . The  $\alpha$ -twisted representation group of  $H$ , denoted by  ${}^{\alpha}\mathcal{R}(H)$  is the Grothendieck group of isomorphism classes of complex,  $\alpha$ -twisted representations with direct sum as binary operation.

Let  $H$  be a finite group. Given a cocycle  $\alpha \in H^2(H, S^1)$  representing a torsion class of order  $n$ , the normalization procedure gives a cocycle  $\beta$  cohomologous to  $\alpha$  such that  $\beta : H \times H \rightarrow S^1$  takes values in the subgroup  $\mathbb{Z}/n \subset S^1$  generated by a primitive  $n$ -th root of unity  $\eta$ . Associated to a normalized cocycle, there exists a central extension

$$1 \rightarrow \mathbb{Z}/n \rightarrow H^* \rightarrow H \rightarrow 1$$

with the property that any twisted representation of  $H$  is a linear representation of  $H^*$ , with the additional property that  $\mathbb{Z}/n$  acts by multiplication with  $\eta$ . Such a group is called a Schur covering group for  $H$ .

**Definition 1.8** Let  $\rho : H \rightarrow Gl(V)$  be an  $\alpha$ -twisted representation. The character of  $\rho$  is the map  $H \rightarrow \mathbb{C}$  given by  $\chi(h) = \text{trace}(\rho(h))$ .

Given a cocycle  $\alpha$ , an element  $h \in H$  is said to be  $\alpha$ -regular if  $\alpha(h, x) = \alpha(x, h)$  for all  $x \in H$ . For a choice of representatives of conjugacy classes of  $\alpha$ -regular elements,  $\{x_1, \dots, x_k\}$  in  $H$ , an  $\alpha$ -character table gives the values of the character of irreducible  $\alpha$ -twisted representations by evaluating  $\chi(x_i)$ . An  $\alpha$ -character is not constant in conjugacy classes of elements in  $H$ ; it depends on the choice of representatives of the conjugacy classes of  $\alpha$ -regular elements. The following result shows the relation to different choices of a set of representatives by comparing it to the linear characters of a Schur covering group  $H^*$ . It is proven in Theorem 1.1, Part i Chapter 5, page 205 in [Kar94].

**Theorem 1.9** (Rectification procedure for characters of a Schur Covering group) *Let  $1 \rightarrow A \rightarrow H^* \xrightarrow{f} H \rightarrow 1$  be a finite central group extension and let  $\mu : H \rightarrow H^*$  be a fixed section of  $f$ . For any given  $\xi \in \text{Hom}(A, \mathbb{C}^*)$ , let  $\alpha = \alpha_\xi \in Z^2(H, \mathbb{C}^*)$  be defined by*

$$\alpha(x, y) = \xi(\mu(x)\mu(y)\mu(xy)^{-1}) \text{ for all } x, y \in H$$

*Then if  $\lambda_1^*, \dots, \lambda_n^*$  are all distinct irreducible  $\mathbb{C}$ -characters of  $H^*$  whose restriction to  $A$  has  $\xi$  as an irreducible constituent and if  $\lambda_i : H \rightarrow \mathbb{C}$  is defined by*

$$\lambda_i(g) := \lambda_i^*(\mu(g)) \text{ for all } g \in H$$

*then  $\lambda_1, \dots, \lambda_r$  are all distinct irreducible  $\alpha$ -characters of  $H$ . Moreover, if  $\mu$  is a conjugation preserving section, then each  $\lambda_i$  is a class function.*

We can define contravariant and covariant coefficient systems for the family  $\mathcal{F}_G = \mathcal{FLN}$  of finite subgroups agreeing on objects by using the  $\alpha$ -twisted representation group functor  $\mathcal{R}_\alpha(\quad)$ .

**Definition 1.10** Let  $G$  be a discrete group and let  $\alpha \in Z^2(G, S^1)$  be a cocycle. Define  $\mathcal{R}_\alpha$  on objects by

$$\mathcal{R}_{\alpha?}(G/H) = \mathcal{R}_\alpha^?(G/H) := {}^{i^*(\alpha)} \mathcal{R}(H)$$

where  $i : H \rightarrow G$  is the inclusion.

And induction of  $\alpha$ -twisted, representations for the covariant, part  $\mathcal{R}_{\alpha?}$ , respectively restriction of  $\alpha$ -representations for the contravariant part  $\mathcal{R}_\alpha^?$ .

Orthogonality relations between irreducible  $\alpha$ -characters and Frobenius reciprocity go over the the setting of  $\alpha$ -twisted representations, compare Proposition 11.7 in [Kar94], page 73 and Theorem 11.8 in page 73.

**Definition 1.11** Let  $G$  be a discrete group, let  $X$  be a proper  $G$ -CW complex, and let  $\alpha \in Z^2(G, S^1)$  be a cocycle. The  $\alpha$ -twisted Bredon cohomology, respectively  $\alpha$ -twisted homology groups of  $X$  are the Bredon cohomology, respectively homology groups with respect to the functors described in definition 1.10.

We will consider coefficient systems which consist of abelian groups with preferred bases. This condition produces a convenient duality situation, and produces based (co)-chain complexes as input for the computation of Bredon (co)-homology groups.

**Conditions 1.12** Let  $G$  be a discrete group, Let  $M_\gamma$  and  $M^\gamma$  be covariant, respectively contravariant functors defined on a subcategory  $\mathcal{O}_G$  of the orbit category  $\mathcal{O}$  agreeing on objects. Suppose that

- There exists for every object  $G/H$  a choice of a finite basis  $\{\beta_{iH}\}$  expressing  $M_\gamma(G/H) = M^\gamma(G/H)$  as the finitely generated, free abelian group on  $\{\beta_{iH}\}$  and isomorphisms  $a_H : M^\gamma(G/H) \xrightarrow{\cong} \mathbb{Z}[\{\beta_{iH}\}] \xleftarrow{\cong} M_\gamma(G/H) : b_H$ .
- For the covariant functor  $\widehat{M} := \text{Hom}_{\mathbb{Z}}(M^\gamma(\ ), \mathbb{Z})$ , the dual basis  $\{\widehat{\beta}_{iH}\}$  of  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\{\beta_{iH}\}], \mathbb{Z})$  and the isomorphisms  $a_H$  and  $b_H$ , the following diagram is commutative:

$$\begin{array}{ccccccc}
 M_\gamma(G/H) & \xrightarrow{b_H} & \mathbb{Z}[\{\beta_{iH}\}] & \xrightarrow{D_H} & \mathbb{Z}[\{\widehat{\beta}_{iH}\}] & \xrightarrow{\widehat{a}_H} & \widehat{M}(G/H) \\
 \downarrow M_\gamma(\phi) & & & & & & \downarrow \widehat{M}(\phi) \\
 M_\gamma(G/K) & \xrightarrow{b_K} & \mathbb{Z}[\{\beta_{jK}\}] & \xrightarrow{D_K} & \mathbb{Z}[\{\widehat{\beta}_{jK}\}] & \xrightarrow{\widehat{a}_K} & \widehat{M}(G/K)
 \end{array}$$

Where  $D_H$ ,  $D_K$  are the duality isomorphisms associated to the bases and  $\phi : G/H \rightarrow G/K$  is a morphism in the orbit category.

Conditions 1.12 are satisfied in some cases:

- Constant coefficients  $\mathbb{Z}$ .
- The complex representation ring functors defined on the family  $\mathcal{FIN}$  of finite subgroups,  $\mathcal{R}^\gamma$ ,  $\mathcal{R}_\gamma$ . A computation using characters as bases and Frobenius reciprocity yields conditions 1.12.



- Consider a discrete group  $G$  and a normalized torsion cocycle

$$\alpha \in Z^2(G, S^1),$$

take the  $\alpha$ - and  $\alpha^{-1}$  twisted representation ring functors  $\mathcal{R}_{-\alpha}, \mathcal{R}_\alpha$  defined on the objects  $G/H$ , where  $H$  belongs to the family  $\mathcal{FIN}$  of finite subgroups. Consider for every object  $G/H$  the cocycles  $i_H^*(\alpha)$ , where  $i_H : H \rightarrow G$  is the inclusion, and assume without loss of generality that they are normalized and correspond to a family of Schur covering groups in central extensions  $1 \rightarrow \mathbb{Z}/n_H \rightarrow H^* \rightarrow H \rightarrow 1$ .

We select the set  $\{\beta_H\}$  given as the set of characters of irreducible representations of  $H^*$  where  $\mathbb{Z}/n_H$  acts by multiplication with a primitive  $n_H$ -th root of unity. Given a choice of sections for the quotient maps  $H^* \rightarrow H$ , one can construct isomorphisms  $i^{*(\alpha)}\mathcal{R}(G/H) \xrightarrow{\cong} \mathbb{Z}[\{\beta_H\}]$ . The orthogonality relations and Frobenius reciprocity for their twisted characters guarantee that conditions 1.12 yield.

**Theorem 1.13** *Let  $X$  be a proper, finite  $G$ -CW complex. Let  $M^\flat$  and  $M^\flat$  be a pair of functors satisfying conditions 1.12. Then, there exists a short exact sequence of abelian groups*

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}(H_{n-1}^G(X, M^\flat), \mathbb{Z}) \rightarrow H_G^n(X, M^\flat) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n^G(X, M^\flat), \mathbb{Z}) \rightarrow 0$$

**Proof** The proof consists of two steps:

- Construction of chain homotopy equivalences

$$\begin{aligned} (C_n^G(X, M^\flat), \delta_n) &\rightarrow (C_n^G(X, \widehat{M}), \widehat{\delta}_n) \\ (C_G^n(X, M^\flat), \delta^n) &\rightarrow ((\text{Hom}_{\mathbb{Z}}(C_n^G(X, \widehat{M}), \mathbb{Z}); \text{Hom}_{\mathbb{Z}}(\delta_n, \mathbb{Z})) \end{aligned}$$

- Construction of an exact sequence

$$0 \rightarrow \ker h \rightarrow H_G^n(X, M^\flat) \xrightarrow{h} \text{Hom}_{\mathbb{Z}}(H_n^G(X, \widehat{M}), \mathbb{Z}) \rightarrow 0$$

and identification of the kernel as  $\text{Ext}_{\mathbb{Z}}(H_{n-1}^G(X, M^\flat), \mathbb{Z})$ .

For the first chain homotopy equivalence, notice that given a set  $\{e_\lambda\}$  of orbit representatives of the  $n$ -cells of the  $G$ -CW complex  $X$ , and isotropy subgroups  $S_\lambda$  of the cells  $e_\lambda$ , the chain complex for computing the  $n$ -th Bredon homology with coefficients in

$M_?$  is given by  $\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes M(G/S_{\lambda})$ . Condition 1.12 gives for every  $\lambda$  an isomorphism of free abelian groups  $M_?(G/S_{\lambda}) \rightarrow \widehat{M}(G/S_{\lambda})$  which gives a chain map

$$C_n^G(X, M_?) \cong \bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes M_?(G/S_{\lambda}) \longrightarrow \bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes \widehat{M}(G/S_{\lambda}) \cong C_n^G(X, \widehat{M})$$

giving a chain homotopy equivalence and subsequently an isomorphism in homology groups  $H_n^G(X, M_?) \xrightarrow{\cong} H_n^G(X, \widehat{M})$ .

For the second isomorphism, consider the chain map

$$A : C_n^G(X, M^?) \rightarrow \text{Hom}_{\mathbb{Z}}(C_n^G(X, \widehat{M}), \mathbb{Z})$$

$$A : \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}], M^?(G/S_{\lambda})\right) \rightarrow \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes \widehat{M}(G/S_{\lambda}), \mathbb{Z}\right)$$

assigning to  $\varphi \in C_n^G(X, M^?)$  the homomorphism  $\psi$  defined on basis elements  $[e_{\lambda}] \otimes \widehat{\beta}_{is_{\lambda}}$  as  $\widehat{\beta}_{is_{\lambda}}(\varphi([e_{\lambda}])) \in \mathbb{Z}$ . A chain homotopy inverse for this map is given by the chain map

$$B : \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes \widehat{M}(G/S_{\lambda}), \mathbb{Z}\right) \rightarrow \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}], M^?(G/S_{\lambda})\right)$$

assigning to a homomorphism  $\psi \in \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{\lambda} \mathbb{Z}[e_{\lambda}] \otimes \widehat{M}(G/S_{\lambda}), \mathbb{Z}\right)$  the homomorphism defined on the basis  $[e_{\lambda}]$  as  $\psi([e_{\lambda}]) = \sum_{is_{\lambda}} \psi([e_{\lambda}] \otimes \widehat{\beta}_{is_{\lambda}}) \beta_{is_{\lambda}}$ .

For the second part, consider the cochain complex given in degree  $n$  by

$$C_n^G(X; M^?) = (\text{Hom}_{\mathcal{F}_G}(\underline{C}_n(X), M), \delta^n)$$

and the chain complex  $C_n^G(X, \widehat{M}) = (\underline{C}_n(X) \otimes_{\mathcal{O}_G} \widehat{M}, \delta_n)$ . Denote by  $Z^n = \ker \delta^n$ ,  $B^n = \text{im } \delta^{n-1}$  and  $Z_n = \ker \delta_n$ ,  $B_n = \text{im } \delta_{n+1}$ .

A class in  $H_n^G(X, \widehat{M})$  is represented by a homomorphism  $\varphi : C_n^G(X, \widehat{M}) \rightarrow \mathbb{Z}$  such that  $\text{Hom}_{\mathbb{Z}}(\delta_n, \mathbb{Z})(\varphi) = 0$ , that means  $\varphi | B_n = 0$ . Then if  $\varphi_0 = \varphi | Z_n$  we have a map defined in the quotient  $\overline{\varphi}_0 : Z_n/B_n \rightarrow \mathbb{Z}$ , or in others words an element of  $\text{Hom}_{\mathbb{Z}}(H_n^G(X, \widehat{M}), \mathbb{Z})$ . We have defined a map

$$h : H_n^G(X, \widehat{M}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n^G(X, \widehat{M}), \mathbb{Z}).$$

This map is surjective. Now we proceed to identify  $\ker(h)$ .

As we have a projective resolution of  $H_n^G(X, \widehat{M})$

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n^G(X, \widehat{M}),$$

the group  $\text{Ext}(H_n^G(X, \widehat{M}), \mathbb{Z})$  can be calculated using the exact sequence

$$(1.14) \quad \text{Hom}_{\mathbb{Z}}(Z_n, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(B_n, \mathbb{Z}) \rightarrow \text{Ext}(H_n^G(X, \widehat{M}), \mathbb{Z}) \rightarrow 0.$$

On the other hand, we have the following long exact sequences

$$(1.15) \quad 0 \rightarrow Z_n \rightarrow C_n^G(X, \widehat{M}) \xrightarrow{\delta} B_{n-1} \rightarrow 0$$

$$(1.16) \quad 0 \rightarrow H_n^G(X, \widehat{M}) \rightarrow C_n^G(X, \widehat{M})/B_n \xrightarrow{\delta} B_{n-1} \rightarrow 0$$

The sequences 1.15 and 1.16 are split because  $B_{n-1}$  is free abelian.

We can decompose  $\delta$  as the map

$$(1.17) \quad \begin{aligned} 0 &\longrightarrow C_n^G(X, \widehat{M}) \\ &\longrightarrow C_n^G(X, \widehat{M})/B_n \longrightarrow C_n^G(X, \widehat{M})/Z_n \xrightarrow{\cong} \\ &B_{n-1} \subset Z_{n-1} \subset C_{n-1}^G(X, \widehat{M}) \end{aligned}$$

We have the following diagram of exact sequences (here  $H_n = H_n^G(X, \widehat{M})$  and  $C_n = C_n^G(X, \widehat{M})$ ).

$$(1.18) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ & & & & \text{Hom}_{\mathbb{Z}}(H_n, \mathbb{Z}) & & \\ & & & & \uparrow & & \\ \text{Hom}_{\mathbb{Z}}(B_n, \mathbb{Z}) & \longleftarrow & \text{Hom}_{\mathbb{Z}}(C_n, \mathbb{Z}) & \longleftarrow & \text{Hom}_{\mathbb{Z}}(C_n/B_n, \mathbb{Z}) & \longleftarrow & 0 \\ & & & & \uparrow & & \uparrow \\ 0 & \longleftarrow & \text{Ext}(H_n, \mathbb{Z}) & \longleftarrow & \text{Hom}_{\mathbb{Z}}(B_{n-1}, \mathbb{Z}) & \longleftarrow & \text{Hom}_{\mathbb{Z}}(Z_{n-1}, \mathbb{Z}) \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & \text{Hom}_{\mathbb{Z}}(C_{n-1}, \mathbb{Z}) \end{array}$$

The right vertical sequence is exact because 1.15 splits, the left vertical sequence is split exact because 1.16 is. The upper horizontal sequence is obtained from applying  $Hom$  to the projective resolution of  $H_n^G(X, \widehat{M})$  and the lower horizontal sequence is 1.14.

Analyzing the diagram 1.1 and decomposition 1.17 we obtain

$$Z^n \cong Hom_{\mathbb{Z}}(C_n(X, \widehat{M})/B_n, \mathbb{Z}),$$

on the other hand

$$B^n \cong im(Hom_{\mathbb{Z}}(Z_{n-1}, \mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(C_n(X, \widehat{M})/B_n, \mathbb{Z})).$$

Then

$$H_G^n(X, M^?) \cong coker(Hom_{\mathbb{Z}}(Z_{n-1}, \mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(C_n(X, \widehat{M})/B_n, \mathbb{Z})).$$

The maps  $Hom_{\mathbb{Z}}(Z_{n-1}, \mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(B_{n-1}, \mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(C_n(X, \widehat{M})/B_n, \mathbb{Z})$  together with the section  $Hom_{\mathbb{Z}}(C_n(X, \widehat{M})/B_n, \mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(B_{n-1}, \mathbb{Z})$  induce an exact sequence of cokernels identifying  $\ker h$  with  $Ext_{\mathbb{Z}}^G(H_{n-1}^G(X, M^?), \mathbb{Z})$ .

□

## 2 Spectral sequences for Twisted Equivariant $K$ -Theory.

Twisted equivariant  $K$ -theory for proper and discrete actions has been defined in a variety of ways. For a torsion cocycle  $\alpha \in Z^2(G, S^1)$ , it is possible to define twisted equivariant  $K$ -theory in terms of finite dimensional, so called  $\alpha$ -twisted vector bundles, an  $\alpha$ -twisted equivariant  $K$ -theory for proper actions of discrete groups on finite, proper  $G$ -CW complexes.

**Definition 2.1** Let  $\alpha \in Z^2(G, S^1)$  be a normalized torsion cocycle of order  $n$  for the discrete group  $G$ , with associated central extension  $0 \rightarrow \mathbb{Z}/n \rightarrow G_\alpha \rightarrow G$ . An  $\alpha$ -twisted vector bundle is a finite dimensional  $G_\alpha$ -equivariant complex vector bundle such that  $\mathbb{Z}/n$  acts by multiplication with a primitive root of unity. The groups  ${}^\alpha \mathbb{K}_{G_\alpha}^0(X)$  are defined as the Grothendieck groups of the isomorphism classes of  $\alpha$ -twisted vector bundles over  $X$ .

Given a proper  $G$ -CW complex  $X$ , define the  $\alpha$ -twisted equivariant  $K$ -theory groups  ${}^\alpha K_G^{-n}(X)$  as the kernel of the induced map

$${}^\alpha \mathbb{K}_{G_\alpha}^0(X \times S^n) \xrightarrow{\text{incl}^*} {}^\alpha \mathbb{K}_{G_\alpha}^0(X)$$

The  $\alpha$ -twisted twisted equivariant  $K$ -theory catches relevant information to the class of twistings coming from the torsion part of the group cohomology of the group, in the sense that the  $K$ -groups trivialize for cocycles representing non-torsion classes.

As noted in [BEUV13], there is a spectral sequence connecting the  $\alpha$ -twisted Bredon cohomology and the  $\alpha$ -twisted equivariant  $K$ -theory of finite proper  $G$ -CW complexes. When the twisting is discrete this spectral sequence is a special case of the Atiyah-Hirzebruch spectral sequence for *untwisted*  $G$ -cohomology theories constructed by Davis and Lück [DL98]. In particular, it collapses rationally.

**Theorem 2.2** *Let  $X$  be a finite proper  $G$ -CW complex for a discrete group  $G$ , and let  $\alpha \in Z^2(G, S^1)$  be a normalized torsion cocycle. Then there is a spectral sequence with*

$$E_2^{p,q} = \begin{cases} H_G^p(X, \mathcal{R}_\alpha^q) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases}$$

so that  $E_\infty^{p,q} \Rightarrow {}^\alpha K_G^{p+q}(X)$ .

In [BEJU12], a definition for twisted equivariant  $K$ -Theory is proposed, where the class of twistings is extended to include arbitrary elements in the third Borel cohomology group with integer coefficients  $H^3(X \times_G EG, \mathbb{Z})$ . Extending the work by [Dwy08], this theory gives non trivial twisted equivariant  $K$ -theory groups for cocycles which are non torsion. In the work [BEUV13], a spectral sequence is developed to compute the twisted equivariant  $K$ -theory under these conditions, in terms of generalizations of Bredon cohomology which capture more general twisting data. Specializing to the trivial group  $\{e\}$ , the spectral sequence of [BEUV13] yields explicit descriptions of both the  $E_2$  term and the differentials of the non-equivariant Atiyah-Hirzebruch spectral sequence described in [AS06]. In contrast to the spectral sequence constructed in [Dwy08], the spectral sequence in [BEUV13] does not collapse rationally in general.

### 3 Cohomology of $Sl_3\mathbb{Z}$ and twists

We concentrate now in the example of the group  $Sl_3\mathbb{Z}$ . This group is particularly accessible due to the existence of a convenient model for the classifying space of proper

actions and the fact that the group cohomology relevant to the twists is completely determined by finite subgroups.

### 3.1 Twist Data

We will describe the twist data, which reduce completely to torsion classes. After the work of Soulé, Theorem 4, page 14 in [Sou78] the integral cohomology of  $Sl_3\mathbb{Z}$  only consists of 2 and 3-torsion. The 3-primary part is isomorphic to the graded algebra

$$\mathbb{Z}[x_1, x_2] \mid 3x_1 = 3x_2 = 0$$

with both generators in degree 4.

The two-primary component is isomorphic to the graded algebra

$$\mathbb{Z}[u_1, \dots, u_7]$$

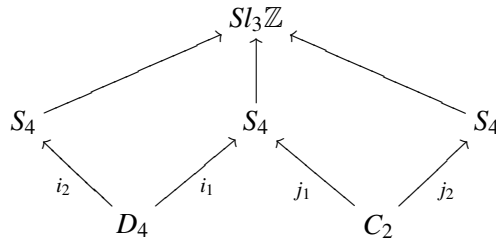
with respective degrees 3,3,4, 4,5, 6,6, subject to the relations

$$\begin{aligned} 2u_1 &= 2u_3 = 4u_3 = 4u_4 = 2u_5 = 2u_6 = 2u_7 = 0 \\ u_7u_1 &= u_7u_4 = u_7u_5 = u_7u_6 = u_2u_5 = u_2u_6 = 0 \\ u_7^2 + u_7u_2^2 &= u_3u_4 + u_1u_5 = u_3u_6 + u_3u_1^2 = u_3u_6 + u_5^2 = 0 \\ u_1u_6 + u_4u_5 &= u_3^0u_4^2 + u_6^2 = u_5u_6 + u_5u_1^2 = 0 \end{aligned}$$

The twists in equivariant  $K$ -theory are given by the classes in  $H^3(Sl_3\mathbb{Z}, \mathbb{Z})$ . For this reason we shall restrict to the two-primary component in the integral cohomology. In order to have a local description of these classes, we describe the cohomology of some finite subgroups inside  $Sl_3\mathbb{Z}$ . Again Theorem 4 in page 14 of [Sou78], gives the following result: There exists an exact sequence of abelian groups ( $n \in \mathbb{N}$ )

$$0 \rightarrow H^n(Sl_3\mathbb{Z})_{(2)} \xrightarrow{\phi} H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \xrightarrow{\delta} H^n(D_4) \oplus H^n(\mathbb{Z}_2) \rightarrow 0$$

where  $\phi$  and  $\delta$  (see Corollary 2.1.b in page 9 of [Sou78]) are determined by the system of inclusions



Denote  $R = i_1^*(H^*(S_4)) \cap i_2^*(H^*(S_4))$ . Then, the image of the morphism  $\phi : H^*(Sl_3\mathbb{Z})_{(2)} \rightarrow H^*(S_4)_{(2)} \oplus (i_1^*)^{-1}(R)$ , is the set of elements  $(y, z)$  such that  $j_2^*(y) = j_1^*(z)$ . From Soulé's work, we know that  $H^*(S_4)_{(2)} = \mathbb{Z}[y_1, y_2, y_3]$ , with  $2y_1 = 2y_2 = 4y_3 = y_1^4 + y_2^2y_1 + y_3y_1^2 = 0$ , and, if  $R$  is as above, then  $(i_1^*)^{-1}(R) = \mathbb{Z}[z_1, z_2, z_3]$ , with  $2z_1 = 4z_2 = 2z_3 = z_3^2 + z_3z_1^2 = 0$ . Furthermore  $j_2^*(y_1) = t$ ,  $j_2^*(y_2) = 0$ ,  $j_2^*(y_3) = t^2$ ,  $j_1^*(z_1) = 0$ ,  $j_1^*(z_2) = t^2$ , and  $j_1^*(z_3) = 0$ . Then the elements  $u_1 = y_2$ ,  $u_2 = z_1$ ,  $u_3 = y_1^2 + z_2$ ,  $u_4 = y_1^2 + y_3$ ,  $u_5 = y_1y_2$ ,  $u_6 = y_1y_3 + y_1^3$  and  $u_7 = z_3$  generate  $H^*(Sl_3\mathbb{Z})_{(2)}$ .

In  $H^3(\ )$  the above discussion can be summarized in the following diagram

(3.1)

$$\begin{array}{ccccccc}
 & & \langle u_1, u_2 \rangle = H^3(Sl_3\mathbb{Z}) & & & & \\
 & \swarrow i^* & \downarrow i^* & \searrow i^* & \swarrow i^* & & \\
 \langle z_1 \rangle \subseteq H^3(S_4) & & \langle z_1 \rangle \subseteq H^3(S_4) & & \langle y_2 \rangle \subseteq H^3(S_4) & & \langle y_2 \rangle \subseteq H^3(D_6) \\
 \downarrow i_1^* & \swarrow i_2^* & \downarrow j_1^* & \swarrow j_2^* & \downarrow & \swarrow & \\
 \langle x_3 \rangle \subseteq H^3(D_4) & & 0 & & \langle y_2 \rangle \subseteq H^3(D_2) & & \\
 \downarrow & & & & & & \\
 \langle x_3 \rangle \subseteq H^3(D_2) & & & & & & 
 \end{array}$$

There are four twists up to cohomology, namely  $0, u_1, u_2, u_1 + u_2$ , we will work with  $u_1$ .

### 3.2 A model for the classifying space of proper actions

We recall the model for the classifying space for proper actions of  $Sl_3\mathbb{Z}$ , as described in [Sou78], but also in [SG08]. Let  $Q$  be the space of real, positive definite  $3 \times 3$ -square matrices. Multiplication by positive scalars gives an action whose quotient space  $Q/\mathbb{R}^+$  is homotopy equivalent to  $Sl_3\mathbb{Z}/\underline{ESl_3\mathbb{Z}}$ .

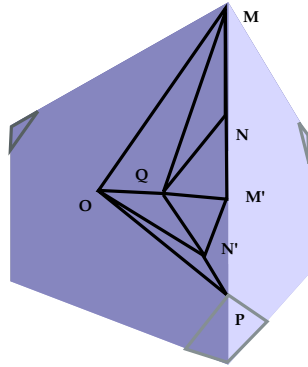
We describe its orbit space. Let  $C$  be the truncated cube of  $\mathbb{R}^3$  with centre  $(0, 0, 0)$  and side length 2, truncated at the vertices  $(1, 1, -1), (1, -1, 1), (-1, 1, 1)$  and  $(-1, -1, -1)$ , through the mid-points of the corresponding sides. As stated in [Sou78], every matrix  $A$  admits a representative of the form

$$\begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix}$$

which may be identified with the corresponding point  $(x, y, z)$  inside the truncated cube. We introduce the following notation for the vertices of the cube:

$$\begin{aligned} O &= (0, 0, 0) & Q &= (1, 0, 0) \\ M &= (1, 1, 1) & N &= (1, 1, 1/2) \\ M' &= (1, 1, 0) & N' &= (1, 1/2, -1/2) \\ P &= (2/3, 2/3, -2/3) \end{aligned}$$

Figure 1: Triangulation for the fundamental region.



Note that the elements of  $Sl_3\mathbb{Z}$

$$q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad q_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

send the triangle  $(M, N, Q)$  to the triangle  $(M', N', Q)$  and the quadrilateral

$(N, N', M', Q)$  to  $(N', N, M', Q)$ . Thus, the following identification must be performed in the quotient:  $M \cong M'$ ,  $N \cong N'$ ,  $QM \cong QM'$ ,  $QN \cong QN'$ ,  $MN \cong M'N' \cong M'N$  and  $QMN \cong QM'N \cong QM'N'$ .



Following [SG08] we now describe the orbits of cells and corresponding stabilizers. This can be found also in Theorem 2 of Soulé's article [Sou78] (although we use a cellular structure instead of a simplicial one). We have changed the chosen generators so that they agree with the presentations on section 4. We summarize the information on Table 1. We use the following notations:  $\{1\}$  denotes the trivial group,  $C_n$  the cyclic group of  $n$  elements,  $D_n$  the dihedral group with  $2n$  elements and  $S_n$  the Symmetric group of permutations on  $n$  objects.

vertices				2-cells			
$v_1$	$O$	$g_2, g_3$	$S_4$	$t_1$	$OQM$	$g_2$	$C_2$
$v_2$	$Q$	$g_4, g_5$	$D_6$	$t_2$	$QM'N$	$g_1$	$\{1\}$
$v_3$	$M$	$g_6, g_7$	$S_4$	$t_3$	$MN'P$	$g_{12}, g_{14}$	$C_2 \times C_2$
$v_4$	$N$	$g_6, g_8$	$D_4$	$t_4$	$OQN'P$	$g_5$	$C_2$
$v_5$	$P$	$g_5, g_9$	$S_4$	$t_5$	$OMM'P$	$g_6$	$C_2$
edges				3-cells			
$e_1$	$OQ$	$g_2, g_5$	$C_2 \times C_2$	$T_1$	$g_1$	$\{1\}$	
$e_2$	$OM$	$g_6, g_{10}$	$D_3$				
$e_3$	$OP$	$g_6, g_5$	$D_3$				
$e_4$	$QM$	$g_2$	$C_2$				
$e_5$	$QN'$	$g_5$	$C_2$				
$e_6$	$MN$	$g_6, g_{11}$	$C_2 \times C_2$				
$e_7$	$M'P$	$g_6, g_{12}$	$D_4$				
$e_8$	$N'P$	$g_5, g_{13}$	$D_4$				

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The generating elements referred to above are

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad g_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad g_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad g_6 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{aligned}
g_7 &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} & g_8 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} & g_9 &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\
g_{10} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & g_{11} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} & g_{12} &= \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
g_{13} &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} & g_{14} &= \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}
\end{aligned}$$

Finally we describe the cells coboundary, we fix an orientation; namely, the ordering of the vertices  $O < Q < M < M' < N < N' < P$  induces an orientation in  $E$  and also in  $\underline{BSl}_3\mathbb{Z} = E/\cong$ . We resume this in Figure 1.

## 4 Representation theoretical input

### 4.1 Cyclic group $C_2$

The cyclic group has no third dimensional integer cohomology, hence the twisted representation ring coincides with the usual one. The character table is as follows:

$$\begin{array}{ccc}
C_2 & 1 & g_i \\
\rho_1 & 1 & -1 \\
\rho_2 & 1 & -1
\end{array}$$

### 4.2 Dihedral group $D_n = \langle g_i, g_j \rangle = \langle g_i, g_j \mid g_i^2 = g_j^2 = (g_i g_j)^n = 1 \rangle$

The dihedral group of order six has only the trivial class in 3 dimensional integer cohomology. Thus the projective representations do agree with the linear ones. In the even case, the subgroups inside  $Sl_3\mathbb{Z}$  are  $C_2 \times C_2 = D_2$  and  $D_4$ .

The following is the linear character table for  $D_n$ :

$$\begin{array}{lcl}
D_n & \langle (g_i, g_j)^k \rangle & \langle g_j (g_i g_j)^k \rangle \\
\xi_1 & 1 & 1 \\
\xi_2 & 1 & -1 \\
\hat{\xi}_3 & -1^k & -1^k \\
\hat{\xi}_4 & -1^k & -1^{k+1} \\
\hat{\phi}_p & 2 \cos(2\pi p k / n) & 0
\end{array}$$

where  $0 \leq k \leq n-1$ ,  $p$  varies from 1 to  $(n/2) - 1$  ( $n$  even) or  $(n-1)/2$  ( $n$  odd) and the hat denotes a representation which only appears in the case  $n$  even. Most of the information is taken from [Kar94], Chapter 5 Section 7, specially Theorem 7.1 and Corollary 7.2 in pages 258-261. The group  $H^2(D_n, S^1)$  is isomorphic to  $\mathbb{Z}/2$  if  $n$  is even and 0 if  $n$  is odd. From now on we concentrate in the case  $n$  even. For simplicity consider the following presentation

$$D_n = \langle a, b \mid a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$$

Given a primitive  $n$ th root of unity  $\epsilon \in S^1$ , one can normalize a nontrivial cocycle  $\alpha : D_n \times D_n \rightarrow S^1$  to one satisfying  $\alpha(a^i, a^j b^k) = 1$  and  $\alpha(a^i b, a^j b^k) = \epsilon^j$ . For  $r \in \{1, 2, \dots, n/2\}$  let

$$A_r = \begin{pmatrix} \epsilon^r & 0 \\ 0 & \epsilon^{1-r} \end{pmatrix}$$

and

$$B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then, the irreducible  $\alpha$ -twisted representations of  $D_n$  are given by  $\rho_r(a^i b_j) = A_r^i B_r^j$  for  $i \in \{0, \dots, n-1\}$   $j = 0, 1$ , and  $r \in \{1, \dots, n/2\}$ . The projective representations  $\rho_j$  for  $j \in \{1, \dots, \frac{n}{2}\}$  are nonequivalent irreducible projective representations.

Consider the group

$$D_2^* = \langle h_1, h_3, z \rangle$$

which is isomorphic to the quaternions. A linear character table is given by

$D_2^*$	1	$z$	$\{h_1, h_1^{-1}\}$	$\{h_3, h_3^{-1}\}$	$\{h_1 h_3, (h_1 h_3)^{-1}\}$
$\eta_1$	1	1	1	1	1
$\eta_2$	1	1	1	-1	-1
$\eta_3$	1	1	-1	1	-1
$\eta_4$	1	1	-1	-1	1
$\eta_5$	2	-2	0	0	0

The linear character table for a Schur covering group of  $D_6$  is

$D_6^*$	1	$z$	$a$	$za$	$a^2$	$za^2$	$a^3$	$b$	$ab$
$\gamma_1$	1	1	1	1	1	1	1	1	1
$\gamma_2$	1	1	1	1	1	1	1	-1	1
$\gamma_3$	1	1	-1	-1	1	1	-1	1	-1
$\gamma_4$	1	1	-1	-1	1	1	-1	-1	1
$\gamma_5$	2	2	-1	-1	-1	-1	-2	0	0
$\gamma_6$	2	2	-1	-1	-1	-1	2	0	0
$\gamma_7$	2	-2	$\frac{3+i\sqrt{3}}{2}$	$-\frac{3+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$-\frac{1+i\sqrt{3}}{2}$	0	0	0
$\gamma_8$	2	-2	0	0	$-1 - i\sqrt{3}$	$1 + i\sqrt{3}$	0	0	0
$\gamma_9$	2	-2	$-\frac{3+i\sqrt{3}}{2}$	$\frac{3+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$-\frac{1+i\sqrt{3}}{2}$	0	0	0

### 4.3 Symmetric group $S_4$

The projective representation theory of the Symmetric groups goes back to the foundational work of Schur [Sch11]. The information concerning the representation theory of the Symmetric group in four letters is taken from [HH92], p. 46 and [Kar94], pages 215-243. Recall that the conjugacy classes inside the group  $S_4$  are determined by their cycle type. The cycle type of a permutation  $\pi$  is a sequence  $(1^{a_1}, 2^{a_2}, \dots, k^{a_k})$ , where the cycle factorization of  $\pi$  contains  $a_i$ -cycles of length  $i$ .

The symmetric group admits the presentation

$$S_4 = \langle g_1, g_2, g_3 \mid g_i^2 = (g_j g_{j+1})^3 = (g_k g_l)^2 = 1 \rangle$$

$$1 \leq i \leq 2, j = 1, k \leq l - 2$$

where  $g_i$  is the transposition given by  $(i - 1, i)$ .

The linear character table of  $S_4$  is as follows:

$S_4$	$(1^4)$	$(2, 1^2)$	$(3, 1)$	$(4)$	$(2^2)$
$\theta_1$	1	1	1	1	1
$\theta_2$	1	-1	1	-1	1
$\theta_3$	2	0	-1	0	2
$\theta_4$	3	1	0	-1	-1
$\theta_5$	3	-1	0	1	-1

The representations  $\theta_1$  and  $\theta_2$  are induced from the 1 dimensional trivial, respectively the sign representation.  $\theta_3$  is obtained from the 2-dimensional representation of the quotient group  $S_3$  the character  $\theta_4$  is given as the  $\xi - \theta_1$ , where  $\xi$  is induced from the  $S_3$ -trivial representation, and the character  $\theta_5$  is the character assigned to the representation  $g \mapsto \theta_2(g)V_4(g)$ , where  $V_4$  is the irreducible representation associated with the character  $\theta_4$ . The linear character table of a Schur covering group  $S_4^*$  is obtained in page 254 of [Kar94] by considering the group with the presentation

$$S_4^* = \langle h_1, h_2, h_3, z \mid h_i^2 = (h_j h_{j+1})^3 = (h_k h_l)^2 = z, z^2 = [z, h_i] = 1 \rangle$$

$$1 \leq i \leq 3, j = 1, k \leq l - 2$$

and the central extension

$$1 \rightarrow \langle z \rangle \rightarrow S_4^* \xrightarrow{f} S_4 \rightarrow 1$$

given by  $f(h_i) = g_i$ , as well as the choice of representatives of regular conjugacy classes as below.

$S_4^*$	$(1^4)$	$(1^4)'$	$(2, 1^2)$	$(2^2)$	$(3, 1)$	$(3, 1)'$	$(4)$	$(4)'$
$\epsilon_1$	1	1	1	1	1	1	1	1
$\epsilon_2$	1	1	-1	1	1	1	-1	-1
$\epsilon_3$	2	2	0	2	-1	-1	0	0
$\epsilon_4$	3	3	1	-1	0	0	-1	-1
$\epsilon_5$	3	3	-1	-1	0	0	1	1
$\epsilon_6$	2	-2	0	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$
$\epsilon_7$	2	-2	0	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$
$\epsilon_8$	4	-4	0	0	-1	1	0	0

where the first five lines are characters associated to  $S_4$ ,  $\epsilon_6$  is the *Spin* representation.

## 5 Untwisted Equivariant K-Theory.

In this section we use the Atiyah-Hirzebruch spectral sequence to calculate the equivariant K-theory groups  $K_{SL_3\mathbb{Z}}^*(\underline{E}SL_3\mathbb{Z})$ . The cochain complex associated to the  $SL_3\mathbb{Z}$ -

CW-complex structure of  $\underline{E}Sl_3\mathbb{Z}$  described in section 3 is:

$$0 \rightarrow \bigoplus_{k=1}^5 R(stab(v_k)) \xrightarrow{\Phi_1} \bigoplus_{j=1}^8 R(stab(e_j)) \xrightarrow{\Phi_2} \bigoplus_{i=1}^5 R(stab(t_i)) \xrightarrow{\Phi_3} R(stab(T)) \rightarrow 0.$$

Where the  $\Phi_i$  is the coboundary given by restriction over representations rings.

Frobenius reciprocity implies that if  $H'$  is a subgroup of  $H$  and

$$\uparrow_H^H: R(H') \rightarrow R(H)$$

is represented by a matrix  $A$  (in the basis given by irreducible representations), then

$$\downarrow_H^H: R(H) \rightarrow R(H')$$

is represented by the transpose matrix  $A^T$  (in the same basis). Given  $A_i$  are the matrix representing the morphism for  $K$ -homology calculated in [SG08], one has that  $A_i^T$  are the matrices representing the morphism  $\Phi_i$ .

We calculate the elementary divisors of the matrices representing the morphism  $\Phi_i$ . Using that we obtain

$$H^p(\underline{E}Sl_3\mathbb{Z}, \mathcal{R}) = 0, \text{ if } p > 0, \text{ and}$$

$$H^0(\underline{E}Sl_3\mathbb{Z}, \mathcal{R}) \cong \mathbb{Z}^{\oplus 8}.$$

As the Bredon cohomology concentrates at low degree, using the same argument that in [SG08] we conclude that

$$K_{Sl_3\mathbb{Z}}^0(\underline{E}Sl_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 8},$$

$$K_{Sl_3\mathbb{Z}}^1(\underline{E}Sl_3\mathbb{Z}) = 0.$$

This agrees with the result predicted by Theorem 1.13

## 6 Twisted Equivariant K-Theory.

In this section we use the spectral described in Theorem 2.2 to calculate the equivariant twisted K-theory groups  ${}^{u_1}K_{Sl_3\mathbb{Z}}(\underline{ESl_3\mathbb{Z}})$ .

The cochain complex associated to the  $Sl_3\mathbb{Z}$ -CW-complex structure of  $\underline{ESl_3\mathbb{Z}}$  described in section 3 is:

$$\begin{aligned}
 0 \rightarrow \bigoplus_{k=1}^5 R_{u_1}(\text{stab}(v_k)) \\
 \xrightarrow{\Phi_1} \bigoplus_{j=1}^8 R_{u_1}(\text{stab}(e_j)) \xrightarrow{\Phi_2} \bigoplus_{i=1}^5 R_{u_1}(\text{stab}(t_i)) \\
 \xrightarrow{\Phi_3} R_{u_1}(\text{stab}(T)) \rightarrow 0
 \end{aligned}$$

As the class  $u_1$  restricts to 0 at the level of 2-cells and 3-cells, we have the following result

**Proposition 6.1** For  $* = 3$ , we have a natural isomorphism

$$H^*(\underline{ESl_3\mathbb{Z}}; \mathcal{R}_{u_1}) \cong H^*(\underline{ESl_3\mathbb{Z}}; \mathcal{R}).$$

We only have to determine  $\Phi_1$  and  $\Phi_2$ , because  $\Phi_3$  was determined in section 5.

### 6.1 Determination of $\Phi_1$

We know from section 3 that the class  $u_1$  comes from the the subgroup  $\text{stab}(v_1) = \langle g_2, g_3 \rangle \cong S_4$ . We have an inclusion

$$\begin{aligned}
 \text{stab}(e_1) &\xrightarrow{i} \text{stab}(v_1) \\
 \langle g_2, g_5 \rangle &\rightarrow \langle g_2, g_3 \rangle \\
 g_2 &\mapsto g_2 \\
 g_5 &\mapsto g_3 g_2 g_3^{-1} g_2 g_3,
 \end{aligned}$$

consider the central extension associated to a representing cocycle the unique non-trivial 3-cohomology class in  $S_4$  (say  $u_{1|}$ ) and its restriction to  $C_2 \times C_2$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & S_4^* & \longrightarrow & S_4 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow j & & \uparrow i & & \\
0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & G^* & \longrightarrow & C_2 \times C_2 & \longrightarrow & 0
\end{array}$$

To evaluate the morphism  $i^* : R_{u_1}(S_4) \rightarrow R_{u_1}(C_2 \times C_2)$  is equivalent to determine the morphism  $j^* : R(S_4^*) \rightarrow R(G^*)$ .

We have a description of  $S_4^*$  in terms of generators and relations given in section 4. In terms of these generators the group  $G^*$  can be described as follows

$$D_2^* = \langle h_1, h_3, z \rangle$$

and the homomorphism  $j : D_2^* \rightarrow S_4^*$  is the inclusion.

Using the character tables from section 4, the following fact is straightforward

$$\begin{aligned}
j^*(\epsilon_1) &= \eta_1, \\
j^*(\epsilon_2) &= \eta_4, \\
j^*(\epsilon_3) &= \eta_1 + \eta_4, \\
j^*(\epsilon_4) &= \eta_1 + \eta_2 + \eta_3, \\
j^*(\epsilon_5) &= \eta_2 + \eta_3 + \eta_4, \\
j^*(\epsilon_6) &= \eta_5, \\
j^*(\epsilon_7) &= \eta_5, \\
j^*(\epsilon_8) &= 2\eta_5.
\end{aligned}$$

The matrix corresponding to the morphism  $i^* : R_{u_1}(stab(v_1)) \rightarrow R_{u_1}(stab(e_1))$  is given by

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}$$



As the class  $u_1$  restricts non-trivially to  $stab(e_1)$ , we have to determine the morphism  $i^* : R_{u_1}(stab(v_2)) \rightarrow R_{u_1}(stab(e_1))$ .

Using the above character table one has:

$$\begin{aligned}
 i^*(\gamma_1) &= \eta_1, \\
 i^*(\gamma_2) &= \eta_4, \\
 i^*(\gamma_3) &= \eta_3, \\
 i^*(\gamma_4) &= \eta_2, \\
 i^*(\gamma_5) &= \eta_2 + \eta_3, \\
 i^*(\gamma_6) &= \eta_1 + \eta_4, \\
 i^*(\gamma_7) &= \eta_5, \\
 i^*(\gamma_8) &= \eta_5 \\
 i^*(\gamma_9) &= \eta_5.
 \end{aligned}$$

The matrix corresponding to the morphism  $i^* : R_{u_1}(stab(v_2)) \rightarrow R_{u_1}(stab(e_1))$  is given by

$$\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1
 \end{pmatrix}$$

Now, we have to determine the morphisms

$$R_{u_1}(stab(v_1)) \rightarrow R_{u_1}(stab(e_i))$$

with  $i = 2, 3$ , note that  $R_{u_1}(stab(e_i)) \cong R(stab(e_i))$  because  $H^3(D_3; \mathbb{Z})$  is trivial.

The inclusion  $stab(e_2) \rightarrow stab(v_1)$  is given by

$$\langle g_6, g_{10} \rangle \rightarrow \langle g_2, g_3 \rangle$$

$$\begin{aligned} g_6 &\mapsto g_3 g_2 g_3^{-1} \\ g_{10} &\mapsto g_2 g_3 g_2 g_3^{-1}. \end{aligned}$$

This map induces a map  $i : \text{stab}(e_2)^* \rightarrow \text{stab}(v_1)^*$ , where  $G^*$  denotes the inverse image of  $G \subseteq S_4$  by the covering map  $S_4^* \rightarrow S_4$ . It is easy to see that  $i(g_6) \sim g_2$ ,  $i(g_{10}) \sim z g_2$  and  $i(g_6 g_{10}) \sim z g_2 g_3$ . Note that  $S_4$  has three non isomorphic irreducible projective characters, namely  $\epsilon_6$ ,  $\epsilon_7$  and  $\epsilon_8$ . We have that

$$i^*(\epsilon_6)(1) = 2, i^*(\epsilon_6)(g_6) = 0, \text{ and } i^*(\epsilon_6)(g_6 g_{10}) = -1,$$

and then  $i^*(\epsilon_6) = \lambda_3$ . In the same way we obtain  $i^*(\epsilon_7) = \lambda_3$ . On the other hand

$$i^*(\epsilon_8)(1) = 4, i^*(\epsilon_8)(g_6) = 0, \text{ and } i^*(\epsilon_8)(g_6 g_{10}) = 1,$$

and then  $i^*(\epsilon_8) = \lambda_1 + \lambda_2 + \lambda_3$ . Combining the above results with the obtained in [SG08] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(\text{stab}(v_1)) \rightarrow R(\text{stab}(e_2))$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Now we determine the morphism

$$i^* R_{u_1|}(\text{stab}(v_1)) \rightarrow R(\text{stab}(e_3))$$

The inclusion  $\text{stab}(e_3) \rightarrow \text{stab}(v_1)$  is given by

$$\begin{aligned} \langle g_6, g_5 \rangle &\rightarrow \langle g_2, g_3 \rangle \\ g_6 &\mapsto g_3 g_2 g_3^{-1} \\ g_5 &\mapsto g_3 g_2 g_3^{-1} g_2 g_3. \end{aligned}$$

this map induces a map  $i : \text{stab}(e_3)^* \rightarrow \text{stab}(v_1)^*$ . It is easy to see that  $i(g_6) \sim g_2$ , and  $i(g_6 g_5) \sim z g_2 g_3$ , we have that

$$i^*(\epsilon_6)(1) = 2, i^*(\epsilon_6)(g_6) = 0, \text{ and } i^*(\epsilon_6)(g_6 g_5) = -1,$$

and then  $i^*(\epsilon_6) = \lambda_3$ . In the same way we obtain  $i^*(\epsilon_7) = \lambda_3$ . On the other hand

$$i^*(\epsilon_8)(1) = 4, i^*(\epsilon_8)(g_6) = 0, \text{ and } i^*(\epsilon_8)(g_6g_{10}) = 1,$$

and then  $i^*(\epsilon_8) = \lambda_1 + \lambda_2 + \lambda_3$ . Combining the above results with the obtained in [SG08] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(stab(v_1)) \rightarrow R(stab(e_3))$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We have to determine now the morphism

$$R_{u_1|}(stab(v_2)) \rightarrow R(stab(e_i)),$$

for  $i = 4, 5$ , note that  $R_{u_1|}(stab(e_i)) \cong R(stab(e_i))$  because  $H^3(C_2; \mathbb{Z})$  is trivial.

The inclusion  $stab(e_4) \rightarrow stab(v_2)$  is given by

$$\begin{aligned} \langle g_6 \rangle &\rightarrow \langle g_4, g_5 \rangle \\ g_6 &\mapsto g_5(g_4g_5)^3 \end{aligned}$$

this map induces a map  $i : stab(e_4)^* \rightarrow stab(v_2)^*$ . It is clear that  $i(g_6) \sim ab$ , in the notation used for character table of  $D_6^*$ ,

$$i^*(\gamma_7)(1) = 2 \text{ and } i^*(\gamma_7)(g_6) = 0,$$

and then  $i^*(\gamma_7) = \rho_1 + \rho_2$ , in the same way we obtain  $i^*(\gamma_8) = \rho_1 + \rho_2$  and  $i^*(\gamma_9) = \rho_1 + \rho_2$ . Combining the above results with the obtained in [SG08] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(stab(v_2)) \rightarrow R(stab(e_4))$$

is given by



## 6.2 Determination of $\Phi_2$

In order to determine  $\Phi_2$  notice that the unique morphisms that differ from the untwisted case are  $i^* : R_{u_1|}(stab(e_1)) \rightarrow R_{u_1|}(stab(t_1))$  and  $i^* : R_{u_1|}(stab(e_1)) \rightarrow R_{u_1|}(stab(t_4))$  in both cases we have that  $i^*(\eta_5) = \rho_1 + \rho_2$ . We obtain that the elementary divisors of the matrix representing the morphism  $\phi$  are 1, 1, 1, 1, 1, 1, 1, 1. The rank of this matrix is 10.

As the cochain complexes involved are free, the computation of ranks and elementary divisors yield

**Theorem 6.2** *The Bredon cohomology with coefficients in twisted representations satisfies*

$$H^p(\underline{ESl}_3\mathbb{Z}, \mathcal{R}_{u_1}) = 0 \text{ if } p > 0, \quad H^0(\underline{ESl}_3\mathbb{Z}, \mathcal{R}_{u_1}) \cong \mathbb{Z}^{\oplus 13}.$$

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in section 2.2 collapses at level 2 and we conclude

**Theorem 6.3**

$${}^{u_1}K_{Sl_3\mathbb{Z}}^0(\underline{ESl}_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 13},$$

$${}^{u_1}K_{Sl_3\mathbb{Z}}^1(\underline{ESl}_3\mathbb{Z}) = 0.$$

## 7 Twisted equivariant K-Homology and relation to the Baum-Connes Conjecture with coefficients

The Baum-Connes Conjecture [BCH94], [MV03] predicts for a discrete group  $G$  the existence of an isomorphism

$$\mu_i : K_i^G(\underline{EG}) \rightarrow K_i(C_r^*(G))$$

given by the (analytical) assembly map, where  $C_r^*(G)$  is the reduced  $C^*$ -algebra of the group  $G$ .

More generally, given any  $G$ - $C^*$ -Algebra, the Baum-Connes conjecture with coefficients predicts that a map

$$\mu_i : K_i^G(\underline{EG}, A) \rightarrow K_i(A \rtimes G)$$

Where  $K_t^G(\underline{EG}, A)$  is defined in terms of equivariant and bivariant  $KK$ -groups ,

$$K_*^G(\underline{EG}, A) = \operatorname{colim}_{G\text{-compact } X \subset \underline{EG}} KK_*(C_0(X), A)$$

and  $A \rtimes G$  denotes the crossed product  $C^*$ -algebra. See [CE01], [Ech08] for more details.

The class of twists considered in the example of  $Sl_3\mathbb{Z}$  let define a particular choice of coefficients for this assembly map.

**Definition 7.1** Let  $G$  be a discrete group. Given a cocycle  $\omega \in Z^2(G, S^1)$ , an  $\omega$ -representation on a Hilbert space  $\mathcal{H}$  is a map  $V : G \rightarrow \mathcal{U}(\mathcal{H})$  satisfying  $v(s)V(t) = \omega(s, t)V(st)$ .

Composing with the quotient map  $\mathcal{U}(\mathcal{H}) \rightarrow PU(\mathcal{H}) = \mathcal{U}(\mathcal{H})/S^1$ , together with the identification of the group  $PU(\mathcal{H})$  as the automorphisms of the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ , denoted by  $\mathcal{K}$  gives an action of  $G$  on  $\mathcal{K}$ . This algebra is denoted as  $\mathcal{K}_\omega$ . Theorem 1.13 together with the Atiyah-Hirzebruch spectral sequence yield:

**Corollary 7.2** Let  $G$  be a discrete group with a finite model for  $\underline{EG}$ . Let  $\omega \in Z^2(G, S^1)$ . Assume that the Bredon homology groups  $H_*^G(X, R^{-\alpha})$  are all free abelian and are concentrated in degree 0 and 1. Then there exists a duality isomorphism

$$K_G^*(C_0(\underline{EG}), K_{-\omega}) \longrightarrow K_G^*(K_\omega, C_0(\underline{EG}))$$

Related duality isomorphisms have been deduced for almost connected groups by Echterhoff [Ech08] using methods from equivariant Kasparov  $KK$ -theory (particularly the Dirac-Dual Dirac method) and positive results for the Baum-Connes Conjecture with coefficients [EEK08].

As a consequence of Theorem 1.13, and Theorem 6.2, the  $\alpha$ -twisted Bredon homology of  $\underline{ESl}_3\mathbb{Z}$  is given by

$$H_p(\underline{ESl}_3\mathbb{Z}, \mathcal{R}_{u_1}) = 0, \text{ if } p > 0, \quad H_0(\underline{ESl}_3\mathbb{Z}, \mathcal{R}_{u_1}) \cong \mathbb{Z}^{\oplus 13}.$$

A spectral sequence argument and the particular shape of the twists let us conclude that these groups agree with the equivariant  $K$ -homology groups with coefficients in  $\mathcal{K}_{u_1}$ , appearing in the left hand side of the conjecture with coefficients.

**Corollary 7.3** The equivariant  $K$ -homology groups of  $Sl_3\mathbb{Z}$  with coefficients in the  $Sl_3\mathbb{Z}$ - $C^*$  algebra  $\mathcal{K}_{u_1}$  are given as follows:

$$K_p^{Sl_3\mathbb{Z}}(\underline{ESl}_3\mathbb{Z}, \mathcal{K}_{u_1}) = 0 \text{ } p \text{ odd}, \quad K_p^{Sl_3\mathbb{Z}}(\underline{ESl}_3\mathbb{Z}, \mathcal{K}_{u_1}) \cong \mathbb{Z}^{\oplus 13} \text{ } p \text{ even}$$

## References

- [AR03] Alejandro Adem and Yongbin Ruan. Twisted orbifold  $K$ -theory. *Comm. Math. Phys.*, 237(3):533–556, 2003.
- [AS04] Michael Atiyah and Graeme Segal. Twisted  $K$ -theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004.
- [AS06] Michael Atiyah and Graeme Segal. Twisted  $K$ -theory and cohomology. In *Inspired by S. S. Chern*, volume 11 of *Nankai Tracts Math.*, pages 5–43. World Sci. Publ., Hackensack, NJ, 2006.
- [BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and  $K$ -theory of group  $C^*$ -algebras. In  *$C^*$ -algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.
- [BEJU12] N. Barcenás, J. Espinoza, M. Joachim, and B. Uribe. Classification of twists in equivariant  $K$ -theory for proper and discrete group actions. preprint, arXiv:1202.1880 [math.AT], 2012.
- [BEUV13] N. Barcenás, Espinoza, B. Uribe, and M. Velásquez. Segal’s spectral sequence in twisted equivariant  $K$  theory for proper actions. preprint, arXiv:1307.1003 [math.AT], 2013.
- [CE01] Jérôme Chabert and Siegfried Echterhoff. Twisted equivariant  $KK$ -theory and the Baum-Connes conjecture for group extensions. *K-Theory*, 23(2):157–200, 2001.
- [DK70] P. Donovan and M. Karoubi. Graded Brauer groups and  $K$ -theory with local coefficients. *Inst. Hautes Études Sci. Publ. Math.*, (38):5–25, 1970.
- [DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in  $K$ - and  $L$ -theory. *K-Theory*, 15(3):201–252, 1998.
- [Dwy08] Christopher Dwyer. Twisted equivariant  $K$ -theory for proper actions of discrete groups. *K-Theory*, 38(2):95–111, 2008.
- [Ech08] Siegfried Echterhoff. The  $K$ -theory of twisted group algebras. In  *$C^*$ -algebras and elliptic theory II*, Trends Math., pages 67–86. Birkhäuser, Basel, 2008.
- [EEK08] Siegfried Echterhoff, Heath Emerson, and Hyun Jeong Kim.  $KK$ -theoretic duality for proper twisted actions. *Math. Ann.*, 340(4):839–873, 2008.
- [HH92] P. N. Hoffman and J. F. Humphreys. *Projective representations of the symmetric groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1992. *Q*-functions and shifted tableaux, Oxford Science Publications.
- [Kar94] Gregory Karpilovsky. *Group representations. Vol. 3*, volume 180 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1994.
- [MV03] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

- [Sch11] I. Schur. Über die darstellungen der symmetrischen und der alternierenden gruppen durch gebrochene lineare substitutionen. *J. Reine Angew. Math.*, 139:155–250, 1911.
- [SG08] Rubén Sánchez-García. Bredon homology and equivariant  $K$ -homology of  $SL(3, \mathbb{Z})$ . *J. Pure Appl. Algebra*, 212(5):1046–1059, 2008.
- [Sou78] Christophe Soulé. The cohomology of  $SL_3(\mathbb{Z})$ . *Topology*, 17(1):1–22, 1978.

*Centro de Ciencias Matemáticas., UNAM Campus Morelia, Michoacán., Ap. Postal 61-3 Xangari MEXICO 58089*

*Centro de Ciencias Matemáticas., UNAM Campus Morelia, Michoacán., Ap. Postal 61-3 Xangari MEXICO 58089*

[barcenasm@matmor.unam.mx](mailto:barcenasm@matmor.unam.mx), [mavelasquezm@gmail.com](mailto:mavelasquezm@gmail.com)

<http://www.matmor.unam.mx/~barcenasm>