Math. J. Okayama Univ. **56** (2014), 91–115

EQUIVARIANT STABLE HOMOTOPY THEORY FOR PROPER ACTIONS OF DISCRETE GROUPS

Noé Bárcenas

Abstract. Following ideas of Graeme Segal [Segal(1973)], [Segal(1968)], Christian Schlichtkrull [Schlichtkrull(2007)] and Kazuhisa Shimakawa [Shimakawa(1989)] we construct equivariant stable homotopy groups for proper equivariant CW complexes with an action of a discrete group. [1]

1. Introduction

In this work we extend the notion of equivariant stable homotopy groups to the context of proper actions of discrete groups on G-CW complexes. This is done by using an adequate generalization of Γ-spaces.

This paper is organized as follows: in section 1, after a quick review of equivariant algebraic topology, we come over to the definition of equivariant homotopy groups for actions of finite groups on G-CW complexes. We recall facts related to Γ-spaces in section 2, and we introduce adequate equivariant generalizations of them. We also prove in this section that equivariant homotopy groups generalize naively graded equivariant homotopy groups for finite groups.

In section 3, we recall the axiomatic description of an Equivariant Homology Theory and prove in section 4.1 the main theorem of this note, which states that the definition of equivariant stable homotopy groups provide an equivariant homology theory. In section 5 we extend the notion of equivariant stable homotopy theory to a bivariant theory. We finally state some problems and results related to stable maps between classifying spaces.

This work was supported by the SFB 878 “Groups, Geometry and Actions” at the University of Münster, the Hausdorff Center for Mathematics in Bonn and Wolfgang Lück's Leibnizpreis.

Mathematics Subject Classification. 55Q91,55N91 Secondary: 22F99 .

Key words and phrases. proper actions, equivariant homotopy theory, configuration spaces.

Contents

2. Γ-spaces, configuration spaces and equivariant stable homotopy groups for finite groups

2.1. **Equivariant stable homotopy for finite groups.** For completeness, we include the following elementary facts about G-CW complexes, which will be needed later:

Definition 1. Let G be a discrete group. A G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup X_n$ $\phi = X_{-1} \subset X_0 = A \subset \ldots \subset X_n \subset X_{n+1} \subset X$ for which every space is inductively obtained from the previous one by attaching cells in pushout diagrams of the type

$$\begin{tikzcd} \coprod_i S^{n-1} \times G/H_i \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_i D^n \times G/H_i \arrow[r] & X_n \end{tikzcd}$$

A G-CW complex is called proper if all isotropy groups are finite. In particular, any G-CW complex is proper if the acting group is finite.

For any G-CW pair, the inclusion $A \to X$ is a G-cofibration and (X, A) is a G-equivariant neighborhood deformation pair.

We will need for technical purposes the following lemma about homotopy classes of maps between G-CW complexes:

Lemma 2.1 (Lemma 1.1 in [Lück and Oliver(2001)]). *Let $\mathcal{F}$ be a family of subgroups of G, which is closed under taking subgroups and conjugation. Let $f : Y \to Y^{'}$ be a map such that the restriction to fixed points in $\mathcal{F}$ is a homotopy equivalence. Then, for any G-CW complex all of which isotropy subgroups are in $\mathcal{F}$, the map $f_* : [X, Y]_G \to [X, Y^{'}]_G$ is a bijection.*

Paralell to G-CW complexes, we will use also spaces over the orbit category.

We refer the reader to [Davis and Lück(1998)] for further reference and for the proof of the results in this section. All spaces have the compactly generated topology, in the sense of [McCord(1969)].

Definition 2. Let $\mathcal{C}$ be a small category. A covariant (contravariant) $\mathcal{C}$-space over the category $\mathcal{C}$ is a covariant (contravariant) functor $\mathcal{C} \longrightarrow Spaces$ to the category of compactly generated spaces.

Example 1. Let $\mathcal{F}$ be a family of subgroups of the discrete group G closed under intersection and conjugation. The orbit categoy $\mathrm{Or}(G, \mathcal{F})$ has as objects are homogeneous spaces G/H for $H \in \mathcal{F}$, a morphism is a G-equivariant map $F : G/H \longrightarrow G/K$. If X is a G-space, we define the contravariant $\mathrm{Or}(G, \mathcal{F})$-space associated to X to be the functor $G/H \longmapsto X^H$. The covariant $\mathrm{Or}(G, \mathcal{F})$-space associated to X is the functor $G/H \mapsto G/H \times_G X$. We will deote by $\mathcal{FIN}$ the family of finite subgroups of G, and we will use the notation $\mathrm{Or}(G)$ for the family of all subgroups of G.

We present two useful constructions for spaces over a category. They are an instance of ends and coends in category theory. Well known constructions like geometric realizations and mapping spaces give examples of coends.

Definition 3. Let X be a contravariant $\mathcal{C}$-space over $\mathcal{C}$ and let Y be a covariant $\mathcal{C}$-space over $\mathcal{C}$. Their tensor product $X \otimes_{\mathcal{C}} Y$ is the space defined by

$$\coprod_{C \in Obj(\mathcal{C})} X(C) \times Y(C)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$.

Definition 4. Let X and Y be $\mathcal{C}$-spaces of the same variance. Their map space $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is the space of natural transformations between the functors X and Y, which is topologized as subspace of the product $\Pi_{C \in Ob(\mathcal{C})} \mathrm{Map}(X(C), Y(C))$

In the case of discrete groups the orbit category provides a convenient tool for the study of transformation groups. There exists a pair of adjoint functors between a certain category of $\mathrm{Or}(G)$-CW complexes and G-CW complexes.

Definition 5. Let $\mathcal{C}$ be a small category. A contravariant, free $\mathcal{C}$-CW complex is a contravariant $\mathcal{C}$-space together with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset \ldots = \cup_n X_n$$

such that X has the colimit topology with respect to this filtration and each X_n is obtained from the previous one by a pushout of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} mor_{\mathcal{C}}(?, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} mor_{\mathcal{C}}(?, c_i) \times D^n & \longrightarrow & X_n \end{array}$$

where I_n is an index family and c_i is an object in $\mathcal{C}$.

Definition 6. Let ∇ be the covariant $\mathrm{Or}(G)$-space given by sending G/H to itself. Given a contravariant $\mathrm{Or}(G, \mathcal{F})$-space X, we define the associated left G-space by

$$\hat{X} = X \otimes_{\mathrm{Or}(G,\mathcal{F})} \nabla$$

The left action of an element $g \in G$ is given by $\mathrm{id} \otimes L_g$, where $L_g : G/H \to G/H$ is the map of covariant $\mathrm{Or}(G, \mathcal{F})$-spaces given by left multiplication with g. The construction $\hat{\ }$ is functorial and is refered to as the coalescence functor.

The following results are stated and proved on [Davis and Lück(1998)], page 249 and 250:

Lemma 2.2. *The map $f : X(G/1) \to \hat{X}$ given by the composition*

$$X(G/1) \cong X(G/1) \times \{1\} \subset X(G/1) \times G/1 \to \hat{X}$$

is a G-homeomorphism.

Lemma 2.3. *The coalescence functor $\hat{\ }$ and $Map_G(\ \ , X)$ constructed in Example 1 and Definition 6 are adjoint, i.e. for a contravariant $\mathrm{Or}(G, \mathcal{F})$-space X and a left G-space Y there is a natural homeomorphism*

$$Map_G(\hat{X}, Y) \to \mathrm{Mor}_{\mathrm{Or}(G,\mathcal{F})}(X, Map_G(\ \ , Y))$$

Theorem 2.4. *Let Y, X be left $\mathrm{Or}(G, \mathcal{F})$-spaces.*

1. *The adjoint of the identity on $Map_G(\ \ , Y)$ under the adjunction of proposition 2.2 is a natural homeomorphism.*
2. *Given a contravariant $\mathrm{Or}(G, \mathcal{F})$-space, the adjoint of the identity on $\hat{X}$ under the adjunction of proposition 2.2 is a natural map of $\mathrm{Or}(G, \mathcal{F})$-spaces*

$$S(X) : X \to Map_G(\ \ , \hat{X})$$

It is an isomorphism if and only if for each $H \in \mathcal{F}$, the projection $G/1 \to G/H$ induces an homeomorphism $X(G/H) \to X(G/1)^H$. This condition is satisfied if X is a free $\mathrm{Or}(G, \mathcal{F})$*-CW complex.*

(3) *If Y is a left G-CW complex with isotropy in the family $\mathcal{F}$, then $Map_G(\quad, Y)$ is a free* $\mathrm{Or}(G, \mathcal{F})$*- CW complex.*

In the case of a finite group, equivariant stable homotopy is an $RO(G)$-graded equivariant homology theory, in the sense of [May(1996)].

Definition 7. Let G be a finite group and X a G-CW complex. For any representation W, form the one-point compactification S^W and define the set of arbitrary pointed maps $\Omega^W S^W \wedge X = \mathrm{Map}(S^W, S^W \wedge X_+)$, on which G acts by conjugation. The equivariant stable homotopy group in degree $V = V_1 - V_2$, where V_i are finite dimensional real representations, is defined to be the abelian group constructed as the colimit of the system of homotopy classes of maps

$$\pi_V^G(X) = \mathrm{colim}_W [S^{V_1}, \Omega^W S^{W \oplus V_2} \wedge X_+]_G$$

where the systems runs along a complete G-universe, that is a Hilbert space containing as subspaces all irreducible representations, where the trivial representation appears infinitely often. Given any finite group G and any G-CW complex X, one defines the equivariant infinite loop space associated to X, $Q_G(X)$ as the colimit of the directed diagram associating a representation W the G space $\Omega^W S^W \wedge X = \mathrm{Map}(S^W, S^W \wedge X_+)$, and where the order is given by the inclusion of representations as direct summand.

In particular, for any integer n, the n-th-equivariant stable homotopy groups of an G-CW complex X, $\pi_n^H(X)$ are defined as

$$[S^n, Q_G(X)]_G \quad n \geq 0$$

$$[S^0, Q_G(S^{-n} \wedge X_+)]_G \quad n < 0$$

where S^n is the n-dimensional sphere with the trivial G-action.

Finite dimensional representations are involved in this definition in a crucial way and this is the main handicap to extend equivariant homotopy to more general settings. In fact, pathological examples from group theory [Ol′shanskiĭ(1982)] provide finitely generated, discrete groups for which all finite dimensional representations over $\mathbb{R}$, or $\mathbb{C}$ are trivial. Precisely:

Example 2. Let G be a finitely presented group. G is said to be residually finite if for every element $g \neq 1$, there exists a homomorphism φ to a finite group mapping g to an element different from 1. The maximal residually finite quotient G_{mrf} of G is the quotient by the normal subgroup consisting of the intersection of all subgroups of finite index. In symbols:

$$G_{mrf} = \frac{G}{\bigcap_{(G:H)<\infty} H}$$

This is a residually finite group, characterized by the property that every group homomorphism to a residually finite group factorizes trough the quotient map. Recall [Malcev(1940)], that if G is a finitely generated subgroup of $Gl_n(F)$ for some field, then G is residually finite. This means that in this situation, every finite dimensional representation of G is induced from one of G_{mrf}. An example of Olshanskii, [Ol′shanskiĭ(1982)] gives for every prime $p > 10^{75}$ a finitely generated, infinite group all of whose proper subgroups are finite of order p. That means that G does not contain proper subgroups of finite index, hence $G_{mrf} = \{e\}$ and consequently every finite dimensional representation of G is trivial.

2.2. **Γ-spaces.** We use in a fundamental way a generalization of certain configuration spaces with labels in (unpointed) spaces [Segal(1973)], see also [Schlichtkrull(2007)] in order to represent equivariant stable homotopy theory. This construction goes back to Graeme Segal und lies in a close interaction with infinite loop space Theory and Γ-spaces in particular. In this section we recall the needed definitions and facts related to these constructions.

Let Γ be the category whose objects are finite sets, and where a morphism $\theta : S \to T$ consists of a map which sends elements $s \in S$ to subsets $\theta(s) \subset T$ in such a way that for two different elements $s, s' \in S$ one has $\theta(s) \cap \theta(s') = \emptyset$. Such maps correspond to a map between power sets $P(S) \to P(T)$ which sends disjoint unions in S to disjoint unions in T.

The simplicial category Δ is the category whose objects are the the sets $\underline{n} = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}_0$, and whose morphisms consist of the order preserving maps $\varphi : \underline{m} \to \underline{n}$. Note that there is a canonical functor $\Delta \to \Gamma$. It is the identity on objects, and on morphisms it is given by sending $\varphi : \underline{m} \to \underline{n}$ to the map $\theta(\varphi)$ with $\theta(\varphi)(i) = \{j \in \underline{n} \mid \varphi(i-1) < j < \varphi(i)\}$.

Let $\mathcal{TOP}$ denote the category of compactly generated weak Hausdorff spaces.

Definition 8. A Γ-space is a functor $A : \Gamma^{op} \to \mathcal{TOP}$ with the property that

(1) $A(\underline{0})$ is a point[2]; and
(2) for each $n > 0$ the canonical map $A(\underline{n}) \to \prod_{i=1}^{n} A(\underline{1})$, induced by the i maps $\kappa_i : \underline{1} \to \underline{n}$, $\kappa_i(1) = \{i\}$, is a homotopy equivalence.

[2]Segal in his original article [Segal(1974)] requires $A(\underline{0})$ to be contractible

From the definition it follows that the spaces $A(S)$ are pointed, the base points being the image of $A(\underline{0})$ under the map induced by the unique morphims $S \to \underline{0}$ in Γ, which sends every subset of S to the empty set. Through precomposition with the functor $\Delta^{op} \to \Gamma^{op}$ one can regard any Γ-space as a simplicial space, and we define $|A|$ to be its fat (topological) realization. Here and in the following realization always stands for the so-called fat realization (as described in [Segal(1974)], appendix A), i.e., without taking degeneracies.

For a Γ-space A we can define its delooping BA as the Γ-space given by $BA(S) = |A(S \times \_)|$. The latter can be iterated, and we define $B^n A = B(B^{n-1}A)$ for $n \in \mathbb{N}$. By construction of $B^n A(\underline{1})$, it naturally comes equipped with a map $S^1 \wedge B^{n-1}A \to B^n A$. Hence the sequence of spaces $BA^n(\underline{1})$ together with the maps $S^1 \wedge B^n A \to B^{n+1}A$ yield a spectrum, which we shall denote $\mathcal{A}$. By construction $\mathcal{A}$ is an Ω-spectrum in the sense that the adjoints of the structure maps $BA^n \to \Omega BA^{n+1}$ are homotopy equivalences for $n \geq 1$, see Theorem in [Puppe(1974)]. In particular, $\mathcal{A}$ defines a (naively graded) cohomology theory on the category of CW-complexes.

We state now the universal property of Γ-spaces, the group completion theorem:

Lemma 2.5. *Let A be a Γ-space such that $\pi_0(A(\underline{1}))$ contains a cofinal free abelian monoid. Denote by K_A the contravariant functor defined on the category of compact spaces by $X \mapsto [X, \Omega \mathrm{B}A(\underline{1})]$. Then the natural transformation $[X, A(\underline{1})] \to K_A$, induced as adjoint to the map $\Sigma A(\underline{1}) \to \mathrm{B}A(\underline{1})$ is universal among transformations $\Theta : [\quad, A(\underline{1})] \to F$, where F is a representable abelian-group valued functor on compact spaces and Θ is a transformation of monoid valued functors.*

Proof. See [Segal(1974)], proposition 4.1 □

A relevant example for Γ-spaces comes from categories with sums. This is an instance of a more general construction stated in the following

Definition 9. Let $\mathcal{CAT}$ be the category of small categories. A Γ-category is a functor $C : \Gamma^{op} \to \mathcal{CAT}$, which satisfies

(1) $C(\underline{0})$ is equivalent to the category with one object and one morphism; and
(2) for each $n > 0$ the canonical functor $C(\underline{n}) \to \prod_{i=1}^{n} C(\underline{1})$, induced by the i maps $\kappa_i : \underline{1} \to \underline{n}$, $\kappa_i(1) = \{i\}$, is an equivalence of categories.

Example 3. Let $\mathcal{C}$ be a topological category where finite sums exist. For every object $\underline{n}$ in Γ^{op}, consider the category $\mathcal{C}(n)$ of functors $\mathcal{P}(\underline{n}) \to \mathcal{C}$

which send disjoint unions to finite sums, and where a morphism is an isomorphism of functors.

In the previous example, notice that

- $\mathcal{C}(0)$ is equivalent to the category with one object and one morphism
- For each n, there is a functor $p_n : \mathcal{C}(n) \to \underbrace{\mathcal{C}(1) \times \ldots \times \mathcal{C}(1)}_{\text{n-times}}$ induced by the morphisms in Γ, $i_k : 1 \to n$ which gives an equivalence of categories.

Hence, this determines in a natural way a Γ-category. We obtain straightforwardly:

Lemma 2.6 (Corollary 2.2 in [Segal(1974)]). *If C is a Γ-category, then the functor $|C| : S \mapsto |C(S)|$ is a Γ-space.*

We turn our attention to topological categories. All spaces to be considered have the compactly generated topology.

We focus in the following construction, which goes back to Graeme Segal [Segal(1974)]:

Definition 10. Let Y be an (unpointed) space. The category of finite sets labelled on Y, $\mathcal{C}_Y$ has the following description. Objects are pairs (S, φ), where S is a finite set and $\varphi : S \to Y$ is an injective function. Morphisms from (S, φ) to (T, ψ) consist of an *isomorphism* $\Theta : S \to T$ inducing an isomorphism $\varphi^*(S) \to \varphi^*(T)$ [3]. This is a topological category, where the space of morphisms carries the discrete topology, and the space of objects is topologized as the configuration space $C_*(Y)$, see [Segal(1973)], that is, as a subspace of the symmetric product

$$\mathrm{Sp(Y)} = \coprod_{\mathrm{k}} \mathrm{Y^k/\Sigma_k}$$

where Σ_k is the symmetric group in k letters.

The category of finite sets labelled in a space admits a symmetric monoidal structure determined essentially by the disjoint union of finite sets and the fact that $Y^{S \coprod T} \approx Y^S \times Y^T$ if the spaces are given the compactly generated topology.

Associated to this structure there is a Γ-space $M_{\mathcal{C}}(Y)$ such that $\Omega \mathrm{B} M_{\mathcal{C}(Y)}(\underline{1})$ is homotopy equivalent to $\Omega^\infty \Sigma^\infty Y_+$. If Y is the singleton space, then the construction above is related to the Barrat-Priddy-Quillen-Segal theorem [Segal(1974)]. We remit the reader to the article [Schlichtkrull(2007)] for a further, more recent reference of these facts, as well as their relationship to

[3] Segal in his original paper, [Segal(1974)] requires the morphisms to be injective maps $\theta : S \to T$ such that $\theta^*(\psi) = \varphi$

structured ring spectra. We point out that altough the construction does not take pointed spaces as input, the space of objects of the topological category of sets on Y has a basis point: the empty configuration.

In order to deal with the cohomological aspects of these constructions, and their generalizations to the equivariant setting, we need some facts related to the notion of a homotopy fibration. The author thanks an anonymous referee for the suggestion of methods related to homotopy fibrations.

Given a pointed map $p : X \to Y$ between compact generated spaces and $y \in Y$, the homotopy fiber at y, $\mathrm{hofib}(p)_y$ is defined as the space of maps beginning on the base point and ending on a preimage of y under p. In symbols,

$$\{(x, \alpha) \mid \alpha : I \to X \quad p(\alpha(0)) = y, \alpha(1) = \{\bullet\}\}$$

Definition 11. A map $f : E \to X$ is a homotopy fibration with fiber F if the homotopy fiber at every point $x \in X$ is homotopy equivalent to F.

Homotopy fibrations are well behaved with respect to (homotopy) pushouts and sequentially directed (homotopy) colimits, as the following results show, Lemmas 2 and 3 in [Puppe(1974)], pages 3 and 4 .

Lemma 2.7. (1) *Let*

$$\begin{tikzcd}
X_1 \arrow[d, "p_1"'] & X_0 \arrow[l, "\tilde{f}_1"'] \arrow[r, "\tilde{f}_2"] \arrow[d, "p_0"'] & X_2 \arrow[d, "p_2"'] \\
A_1 & A_0 \arrow[l, "f_1"] \arrow[r, "f_2"'] & A_2
\end{tikzcd}$$

be a diagram where the maps $X_0 \to {f_i}^*\mathrm{hofib}(p_i)_b$ are homotopy equivalences for all points b. Suppose that the map $X_Z \to A_Z$ between the row-wise mapping cylinders is a homotopy fibration with fiber F, and the canonical pairs of inclusions $X_i \to X_Z$ and $A_i \to A_Z$ are homotopy fibrations. If the projections $X_Z \to X$ and $A_Z \to A$ of the double mapping cylinders to the pushouts are homotopy equivalences (which is the case if at least one horizontal map in each row is a cofibration), then the induced map between the row-wise colimits is a homotopy fibration with fiber F.

(2) *Suppose that in the diagram*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "p_0"'] & X_1 \arrow[r] \arrow[d, "p_1"'] & X_2 \arrow[r] \arrow[d, "p_2"'] & \cdots \\
A_0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots
\end{tikzcd}$$

the induced map $X_T \to A_T$ between the row-wise telescopes is a homotopy fibration with fiber F and the canonical pairs of inclusions $X_i \to X_T$, $A_i \to A_T$ are homotopy fibrations with fiber F. If the projections of the row-wise telescopes to the row-wise colimits are homotopy equivalences, (e.g if all horizontal maps are cofibrations), then the induced map between the row-wise colimits is a homotopy fibration with fiber F.

Recall that a semisimplicial object in the category of spaces is a simplicial space with no degeneracies and the fat geometric realization is the space obtained by ignoring the degeneracy maps in the identification of the quotient space giving the geometric realization. As a consequence of the previous results, Puppe [Puppe(1974)] obtains:

Theorem 2.8. *Let $p : X^\bullet \to A^\bullet$ be a semisimplicial map between semisimplicial objects in the category of compactly generated spaces which is given levelwise by homotopy fibrations with fiber F. Then, the map between the fat geometric realizations is a homotopy fibration with fiber F.*

2.3. **Introducing G-equivariance.** We implement equivariance on these constructions. In the case of finite groups, two definitions of equivariant Γ-spaces are available and both of them are equivalent [Shimakawa(1991)]. For the purposes of studying proper actions of discrete (infinite) groups on G-CW complexes, the lack of finite G-sets and finite dimensional respresentations makes the usual techniques unavailable. The same applies for methods and results coming from the context of $RO(G)$-graded homology theories, much more suitable for compact Lie groups. The definition below seems to be the most adequate to address proper actions of discrete groups on G-CW complexes. We stress that this approach is not adequate to handle actions of lie groups (even compact) and remit the reader to [Blumberg(2006)], Appendix C for more on this point.

Definition 12. Let $G-\mathcal{TOP}$ be the category of compactly generated, weak Hausdorff G-spaces. A G-equivariant Γ-space is a contravariant functor $A : \Gamma^{op} \to G - \mathcal{TOP}$ which satisfies

(1) $A(\underline{0})$ is a point; and
(2) for each $n > 0$ and any subgroup $H \subset G$ the canonical map $A(\underline{n})^H \to \prod_{i=1}^n A(\underline{1})^H$, induced by the n maps $\kappa_i : \underline{1} \to \underline{n}$, $\kappa_i(1) = \{i\}$, is a homotopy equivalence.

Note, if $A_G : \Gamma \to G - \mathcal{TOP}$ is a G-equivariant Γ-space then the spaces $BA^n_G(\underline{1})$ come equipped with continuous G-actions and the maps $S^1 \wedge B^n A_G \to B^{1+n} A_G$ are G-equivariant. Thus the corresponding spectrum $\mathcal{A}_G$ has the structure of a (naive) G-spectrum.

We need the following definitions for our construction of an equivariant Γ-space.

Definition 13 (Categories associated to a group)**.** Let G be a discrete group. The transport category $\mathcal{E}G$ has as objects the element of the group, with exactly one morhism between every pair of elements. If H is a subgroup of G, the groupoid category $\mathcal{B}H$ is defined as the category with only one object and morphisms all of H.

Remark (Relevant category equivalences). There is a equivalence of categories $\mathcal{E}G/H \cong \mathcal{B}H$, which is given by the composition of functors given by the inclusion $\mathcal{E}H/H \to \mathcal{E}G/H$ and the the equivalence $\mathcal{B}H \cong \mathcal{E}H/H$.

Definition 14 (Configuration Model)**.** Let Y be a proper G-CW complex. The category of G-sets on Y is the category of functors from the transport category to the category of sets labelled on Y. The Γ-space of G-sets on Y is the Γ-space associated to the symmetric monoidal structure of $\mathcal{C}_Y$. In symbols

$$M_{\mathcal{C}_Y}(\bar{n}) =| \operatorname{Fun}(\mathcal{E}G, \mathcal{C}_Y(\bar{n})) |$$

there is a simplicial action of G on this space determined by the action on the category of functors which is described as follows. Let $f : \mathcal{E}G \to \mathcal{C}_Y(\bar{n})$ be such a functor. Denote by $f(h) = (S, \varphi(h) : S \to X)$ the object associated to the object h. $g \cdot f$ is the object in the functor category which assigns to the object h the object $(S, g \cdot \varphi(g^{-1}(h))$, where $g \cdot \varphi$ stands for the action on the space Y. Note that this Γ-space also admits deloopings, denoted by B and that $\Omega \mathrm{B} M_{\mathcal{C}_Y}(\underline{1})$ is an infinite loop space with an action of G.

Lemma 2.9. *The* Γ*-space* $M_{\mathcal{C}_{X,G}}(\underline{1})$ *is an equivariant* Γ*-space.*

Proof. Let H be any subgroup. Consider the map

$$\mathcal{C}_Y(\underline{n})^H \to \prod_{i=1}^{n} \mathcal{C}_Y(\underline{1})^H$$

induced by assigning an additive functor the direct sum of its values at $\underline{1}$. Notice that the target of the map can be identified with the realization of the category of functors from $\mathcal{E}G/H$ to $\mathcal{C}_Y^n$, while the source can be thought of as the realization of the category of functors from $\mathcal{E}G/H$ to a category of direct sum decompositions in $\mathcal{C}_Y$. Choosing a model for the direct sum for each n-tuple of objects in $\mathcal{C}_Y$ yields a functor from $\mathcal{C}_Y^n$ to $\mathcal{C}_Y(\underline{n})$ which can be used to show that the canonical functor $\mathcal{C}_Y(\underline{n}) \to \mathcal{C}_Y(\underline{1})^n$ is an equivalence. Hence for any $H \subset G$ the induced functor $\operatorname{Fun}(\mathcal{E}G/H, \mathcal{C}_Y(\underline{n})) \to \operatorname{Fun}(\mathcal{E}G/H, \mathcal{C}_Y^n)$ is an equivalence of topological categories, and hence yields a homotopy equivalence after realization. $\square$

We examinate the equivariant homotopy type of the Γ-space $\Omega \mathrm{B} M_{\mathcal{C}_{X,G}}(\underline{1})$.

Lemma 2.10. *Let H be a finite subgroup of G. Let X be a space with trivial action by H. Then, for every G-CW complex Y, there is an isomorphism*

$$[X \times G/H, \Omega \mathrm{B} M_{\mathcal{C}_Y}(\underline{1})]_G \cong [X, Q_H(Y_+ |_H)]_H$$

Proof. We first recall that $\mathcal{E}G/H$ is naturally equivalent to $\mathcal{B}(H)$ We will study the set $[X, |\operatorname{Fun}(\mathcal{B}H, \mathcal{C}_Y)\ |]_H$. We note that giving a functor out of the groupoid category amounts to give an object of $\mathcal{C}_Y$ with a conveniently defined action of the finite group H. That is, there is a group homomorphism $H \to \operatorname{Auto}(S, \varphi)$. It follows that we can identify such a functor with a finite H-set and a map to Y -defined on the finite set- which is compatible with the action. That is to say, the map takes values into the H-fixed point set. We shall call such objects a finite H-set on Y. The geometric realization has the homotopy type of

$$\coprod_{S,\varphi} \mathrm{BAuto}_{S,\varphi}$$

where S, φ denote the isomorphism classes of finite H-sets on Y. Given an H-set S, we look at its decomposition into orbits of the type H/K_i and write $S = \coprod n_i H/K_i$, where $n_i H/K_i$ denotes the disjoint union of the orbit H/K_i iterated n_i times. We consider now an H- universe, $V^\infty = \oplus_{\mathbb{N}} \mathbb{C}(H)$.

Due to an equivariant version of the Whitney embedding theorems, see [Wasserman(1969)], the space of equivariant embeddings $\mathrm{Emb}_H(H/K, V^\infty)$ is a model for the classifying space of the Weyl group of K in H, in symbols

$$\mathrm{E}W_{K,H} \simeq_H Emb_H(H/K, V^\infty)$$

we choose such a fixed homotopy equivalence for the rest of the proof. On the other hand, for every map $\varphi : S \to Y$, evaluation at the identity coset gives a map

$$\operatorname{Map}(n_i H/K_i, V^\infty) \approx \prod_i \operatorname{Map}(H/K_i, V^\infty) \xrightarrow{\varphi} \prod_i Y^{K_i}$$

which we denote by φ_i. By means of this we can identify the spaces

$$\coprod_{S,\varphi} \mathrm{BAuto}_{S,\varphi} \simeq \coprod_n \coprod_K \mathrm{Emb}(nH/K, V^\infty) \underset{(W_{K,H})^n \wr \Sigma_n}{\times} (Y^K)^n$$

Now, up to homotopy, given an embedding of $\coprod n_i H/K_i$ in V^∞, we can choose finite dimensional, $W_{K_i,H}$-invariant subspaces V_i such that $n_i H/K_i \subset U_i \subset V_i$ and a tubular neighborhood $U = \cup U_i$ suitable for constructing a collapse map $S_i : S^{\mathrm{V_i}} \to S^{V_i}$ sending the tubular neigborhood homeomorphically into V_i and collapsing the exterior to ∞. We now form the $W_{K,H}$ -equivariant map $\tau_i : S^{V_i} \to S^{V_i} \wedge Y_+^{K_i}$ defined by

$$x \mapsto \begin{cases} \infty & v \in V^i - \bar{U} \text{ or } v = \infty \\ S(v) \wedge \varphi_i(eH) & v \in U_i \end{cases}$$

And we pass to the quotient

$$\mathrm{Emb}_H(H/K, V^\infty) \times Y^K \to \mathrm{Emb}(H/K, V^\infty) \underset{W_{K,H}}{\times} Y^K$$

This defines a map to the equivariant stable $W_{K,H}$-maps as follows, where the product is meant to contain all subgroups $K \subset H$.

$$|\operatorname{Fun}(\mathcal{B}H, \mathcal{C}_Y)| \to \prod_K \{\{*\}, \mathrm{E}W_{K,H} \underset{W_{H,K}}{\times} Y^K\}_{W_{K,H}}$$

To resume, we get a map

$$[X, |\operatorname{Fun}(\mathcal{B}H, \mathcal{C}_Y)|]_H \xrightarrow{\alpha} \prod_K [X, Q_{W_{H,K}}(\mathrm{E}W_{K,H} \underset{W_{K,H}}{\times} Y^K)]$$

The map α determines a transformation of contravariant functors between a monoid valued and a group valued functor, thus factoring out trough the group completion. As both functors of X are representable, the universal property of the group completion implies that the map $\tilde{\alpha}$ in the following diagram is an isomorphism

$$\begin{tikzcd}
{[X, |\operatorname{Fun}(\mathcal{B}H, \mathcal{C}_Y)|]_H} \arrow[r] \arrow[d] & \prod_K [X, Q_{W_{K,H}}(\mathrm{E}W_{K.H} \underset{W_{K,H}}{\times} Y^K)] \cong [X, Q_H Y]_H \\
{[X, \Omega \mathrm{B} M_{\mathcal{C}_Y}]_H} \arrow[ur, "\tilde{\alpha}"'] &
\end{tikzcd}$$

Where the right upper isomorphism is obtained by a splitting result for finite groups, for instance Theorem 2.1 p. 206 in [May(1996)]. □

We obtain from proposition 2.10

Corollary 1. For any finite subgroup H of G, and any number $n \in \mathbb{N} \cup \{0\}$, there exist group isomorphisms

$$[S^n \times G/H, \Omega \mathrm{B} M_{\mathcal{C}_Y}]_G \cong \pi_n^H(Y)$$

$$\pi_0(\mathrm{Map}_G(G/H, \Omega \mathrm{B}^n M_{\mathcal{C}_Y})_G) \cong \pi_{-n}^H(Y)$$

2.4. **The associated contravariant** Or(G)**-space.** Although the configuration model introduced in 14 is useful for some purposes, particularly the proof of exactness, lemma 4.2 below, the configuration model is not functorial.

We overcome this difficulty by describing an Or(G)-space which does depend functorially on X and which represents the same homotopy type with respect to spaces with finite isotropy. Related constructions have been considered in [Shimakawa(1991)], section 1 page 243.

Let set be the category of finite sets, which has a symmetric monoidal structure coming from disjoint union. For any discrete group G, define the equivariant Γ-space $\mid \mathrm{set}_G \mid$ as the functor which assigns to an object $\underline{n}$ in Γ the geometric realization of the category of functors from the transport category of G to the category set($\underline{n}$) of disjoint unions of finite sets indexed by $\underline{n}$ and coherent set isomorphisms.

In Symbols:

$$\mid \mathrm{set}_G \mid = \mid \mathrm{Fun}(\mathcal{E}G, \mathrm{set}(\underline{n})) \mid$$

as described in example 1, this G- space has an associated Or(G)-space, which we denote by $\mathrm{set}^?$.

The following is an ad-hoc adaptation of [Shimakawa(1989)].

Definition 15. Let X be a proper G-CW complex.

The associated Or(G)-Γ- space associated to X is the contravariant Or(G) space given by assigning to an orbit G/H the functor which to an object G/H assigns the functor defined on Γ

given by the geometric realization of the topological category Σ_X^H whose space of objects is the space

$$X^{H_+^{\underline{n}}} \wedge \mathrm{set}^H(\underline{n})$$

and where a morphism between the objects $x(\underline{n}) \wedge y(\underline{n})$ and $x(\underline{m}) \wedge y(\underline{m})$ is a morphism in Γ, $\theta : \underline{n} \to \underline{m}$ for which $x(\theta(\underline{n})) \wedge y(\theta(\underline{n})) = x(\underline{m}) \wedge y(\underline{m})$.

Lemma 2.11. *Let X and Y be proper G-CW complexes. There exists a natural isomorphism*

$$[Y^?, \Omega\mathrm{B}(X_+^? \wedge \mathrm{set}^?)]_{\mathrm{Or}(G,\mathcal{FIN})} \longrightarrow [Y^?, Q_?(X_+)]_{\mathrm{Or(G},\mathcal{FIN})}$$

Proof. The map is given as the composition of two functors, the Γ-category functors $\mathcal{C}_Y(\underline{n}) \to \Sigma_Y^{\{e\}}(\underline{n})$ which assigns to the object (S, φ) the object $(S, \varphi(S))$. The second functor involved in the definition of the left hand side map is the associated space functor $Z \mapsto Z^?$. After this composition, it is easy to verify that the obtained Or(G)-space asigns to an object (G/H) the category

$\mathrm{Fun}(\mathcal{E}G, \Sigma_Y)^H$, which is equivalent to $\mathrm{Fun}(\mathcal{B}H, \Sigma_Y)$. Using an analogous argument to proposition 2.10 for Σ_X, one checks that for any object G/H

in $\mathrm{Or}(G,\mathcal{FIN})$, the group completions agree with $Q_H(Y_+)$ up to weak homotopy equivalence.

□

3. Equivariant Homology Theories

3.1. **Axiomatic description of Equivariant Homology Theories.** In this section, we define equivariant stable homotopy groups for proper G-CW complexes and prove that this definition yields an equivariant homology theory on the category of proper G-CW complexes.

We recall the axioms of an Equivariant Homology Theory, as introduced in [Lück and Reich(2005)].

Definition 16. Let G be a group and fix an associative ring with unit R. A G-homology theory with values in R-modules is a collection of covariant functors $\mathcal{H}_n^G$ indexed by the integer numbers $\mathbb{Z}$ from the category of G-CW pairs together with natural transformations $\partial_n^G : \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}^G(A) = \mathcal{H}_{n-1}(A,\emptyset)$, such that the following axioms are satisfied:

(1) If f_0 and f_1 are G-homotopic maps $(X,A) \to (Y,B)$ of G-CW pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for all n.

(2) Given a pair (X,A) of G-CW complexes, there is a long exact sequence

$$\dots \overset{\mathcal{H}_{n+1}^G(j)}{\to} \mathcal{H}_{n+1}^G(X,A) \overset{\partial_{n+1}^G}{\to} \mathcal{H}_n^G(A) \overset{\mathcal{H}_n^G(i)}{\to} \mathcal{H}_n^G(X)$$
$$\overset{\mathcal{H}_n(j)}{\to} \mathcal{H}_n^G(X,A) \overset{\partial_n^G}{\to} \mathcal{H}_{n-1}^G(A) \overset{\mathcal{H}_{n-1}^G(i)}{\to} \dots$$

where $i : A \to X$ and $j : X \to (X,A)$ are the inclusions.

(3) Let (X,A) be a G-CW pair and $f : A \to B$ be a cellular map. The canonical map $(F,f) : (X,A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(X,A) \overset{\cong}{\to} \mathcal{H}_n^G(X \cup_f B, B)$$

(4) Let $\{X_i \mid i \in \mathcal{I}\}$ be a family of G-CW-complexes and denote by $j_i : X_i \to \coprod_{i\in\mathcal{I}} X_i$ the inclusion map. Then the map

$$\oplus_{i\in\mathcal{I}}\mathcal{H}_n^G(j_i) : \oplus_{i\in\mathcal{I}}\mathcal{H}_n^G(X_i) \overset{\cong}{\to} \mathcal{H}_n^G(\coprod_i X_i)$$

is bijective for each $n \in \mathbb{Z}$.

Let $\alpha : H \to G$ be a group homomorphism and X be an H-CW complex. The induced space $\mathrm{ind}_\alpha X$, is defined to be the G-CW complex defined as the quotient space obtained from $G \times X$ by the right H-action given by $(g,x)\cdot h = (g\alpha(h), h^{-1}x)$.

An equivariant homology theory consists of a family of G-homology theories $\mathcal{H}_*^G$ together with natural group homomorphisms

$$\mathrm{ind}_\alpha : \mathcal{H}_n^H(X,A) \longrightarrow \mathcal{H}_n^G(\mathrm{ind}_\alpha(X,A))$$

associated with a homomorphism $\alpha : H \to G$ which satisfy the following conditions

(1) ind_α is an isomorphism whenever $\ker\alpha$ acts freely on X.
(2) For any n, $\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H$.
(3) For any group homomorphism $\beta : G \to K$ such that $\ker\beta\circ\alpha$ acts freely on X, one has

$$\mathrm{ind}_{\alpha\circ\beta} = \mathcal{H}_n^K(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X,A))$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ is the canonical G-homeomorphism.
(4) For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

$$\mathrm{ind}_{c(g):G\to G}\mathcal{H}_n^G(X,A) \to \mathcal{H}_n^G(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X,A}))$$

agrees with the map $\mathcal{H}_n^G(f_2)$, where $f_2 : (X,A) \to \mathrm{ind}_{c(g):G\to G}(X,A)$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G.

4. Definition and verification of the axioms

4.1. **G-homological structure.** We introduce now the equivariant stable homotopy groups between proper G-CW pairs

Definition 17. Let (X,A) be a proper G-CW pair. The equivariant stable homotopy groups of (X,A) are defined as

$$\pi_n^G(X,A) = \begin{cases} \pi_n(\Omega \mathrm{B} M_{\mathcal{C}_{X\cup \mathrm{Cone} A, G}}(\underline{1})), n \geq 0 \\ \pi_1(\Omega \mathrm{B}^{1-n} M_{\mathcal{C}_{X\cup \mathrm{Cone} A, G}}(\underline{1})), n < 0 \end{cases}$$

The main result of the rest of this section is

Theorem 4.1. *Definition 17 gives an example of a functor satifying exactness and excision. By means of proposition 2.11, it can be associated with an Equivariant Homology Theory.*

Remark. It is known, [Lück and Reich(2005)], proposition 6.8 in page 94, that in order to provide a Homology Theory, it suffices to construct a functor from the category of small groupoids to the category of spectra sending equivalences to weak equivalences. Whereas this approach can be used for the equivariant Homology Theory associated to algebraic K-theory and (topological) K-Homology, as for example in [Davis and Lück(1998)], it presents some difficulties when trying to define equivariant stable homotopy theory. Our proof of exactness uses the concrete model from definition 14.

The following lemma deals with the exactness property:

Lemma 4.2. *Let (X, A) be a G-CW pair. The inclusions $j : X \to X \cup_A$* Cone(A) *and $i : A \to X$ induce a sequence of abelian groups*

$$\pi_*^G(A) \xrightarrow{i_*} \pi_*^G(X) \xrightarrow{j_*} \pi_*^G(X, A)$$

which is exact in the middle.

Proof. The question is reduced to analize the behaviour of the map

$$M_{\mathcal{C}_{X,G}}(\underline{1}) \to M_{\mathcal{C}_{X\cup\mathrm{Cone}A,G}}(\underline{1})$$

induced by the map j. Using Remark 2.3, and Lemma 2.1, as well as Theorem 2.8, the question is reduced to prove that for any subgroup $H \subset G$, any natural number n, the natural transformation j_*, defined between the simplicial objects Ch_H^n in the category of compactly generated spaces whose objects are given by length n chains of natural transformations

$$\begin{array}{ccc} f_1 : \mathcal{B}H & \longrightarrow & \mathcal{C}_X \\ & \downarrow & \\ f_2 : \mathcal{B}H & \longrightarrow & \mathcal{C}_X \\ & \downarrow & \\ \dots & & \dots \\ & \downarrow & \\ f_n : \mathcal{B}H & \longrightarrow & \mathcal{C}_X \end{array}$$

between functors f_i, induces a homotopy fibration with fiber A.

We will examinate the behaviour of the natural transformation induced by j when restricted to Ch_H^n, beginning with Ch_H^0, the space of 0-simplices. We will introduce the following notation

Given an equivalence class of a functor $f : \mathcal{B}H \to \mathrm{set}$, a model for f is a finite set $f(\{\bullet\}) = S \subset \mathbb{N}$ together with an group homomorphism $H \to \Sigma_{|S|}$ implementing an action given by the functor. We call a model S irreducible if it has no proper subfunctors. Let $\mathcal{C}_S^H(X)$ be the full subcategory of functors equivalent to those defined by finite disjoint unions $\coprod_{i=1}^k f$, in the sense that they admit a model given by the disjoint union of $\coprod_i^k S$, where S is an irreducible model for the functor, and a map $\coprod \varphi_i : \coprod_i S \to$ Objects $\mathcal{C}_{X^H}^*$ to the space of objects.

Specializing to $\{\bullet\}$, the functor admiting a model S with only one point and trivial action, the category $\mathcal{C}_{\{\bullet\}}^H(X)$ consists of the category of finite sets labelled in X^H. Topologized as subspace of the configuration space, the space of functors given by an irreducible model agrees with X^H_+. The natural transformation induced by j agrees up to homeomorphism with the map $X^H_+ \to (X \cup_A \mathrm{ConeA}^H)_+$, which is homotopy equivalent to the quotient map $X^H_+ \to X/A^H$. Lemma 2.7 allows to conclude that the natural transformation induced by j induces a homotopy fibration with fiber A^H at the level of morphism spaces in the category $\mathcal{C}^H_{\{\bullet\}}(X)$. In particular, for any natural number, the map defined on the reducible functors given as n-th disjoint union of maps defined in $\{\bullet\}$, up to homeomorphism the map induced on the configuration spaces $C^n(X^H_+) \to C^n(X \cup_A \mathrm{Cone}A^H)$, is a homotopy fibration:

For any irreducible model S, the irreducible functors in $\mathcal{C}^H_{\{\bullet\}}(X)$ consist of a point in the configuration space $C^{|S|}(X^H)$ and a permutation of the elements induced by the action on the model S. Hence, the natural transformation induced by j restricts to the induced map $C^{|S|}(X^H) \to C^{|S|}(X \cup_A \mathrm{Cone}A^H)$ equivariantly with respect to the actions permuting the coordinates labelled on S.

Since the space of objects decomposes as the wedge of $\mathcal{C}^H_S(X)$ for all equivalence classes of models S, the result follows for Ch^H_0. Since morphisms are discrete, Theorem 2.8 finishes the proof

□

Lemma 4.3. *Let $X = \coprod X_i$. Then $M_{\mathcal{C}_{X,G}}(\underline{1}) \simeq \vee_i M_{\mathcal{C}_{X_i}}(\underline{1})$*

Proof. This boils down to the fact that $C_*(\coprod_i X_i) \approx \vee_i C_*(X_i)$ □

Lemma 4.4. *Let (X, A) be a proper G-CW pair and $f : A \to B$ be a cellular map. The canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism*

$$\pi^G_n(X, A) \xrightarrow{\cong} \pi^G_n(X \cup_f B, B)$$

Proof. The result is equivalent to the fact that for any G-CW complexes, and any subcomplexes X_1, X_2 such that $X = X_1 \cup X_2$, the groups $\pi^G_n(X, X_1)$ and $\pi^G_n(X_1, X_1 \cap X_2)$ are isomorphic. But this follows since the inclusion $(X_1, X_1 \cap X_2) \to (X, X_1)$ induces an equivariant homotopy equivalence.

□

4.2. **Induction Structure.** The following proposition resumes the results concerning induction structure.

Lemma 4.5. *Let X be an H-CW complex and $\alpha : H \to G$ be an homomorphism of discrete groups. Suppose that* $\ker \alpha$ *is a finite group and acts freely on the H- space X. Then there exists a map*

$$M_{\mathcal{C}_{X,H}} \to M_{\mathcal{C}_{\mathrm{ind}_\alpha \mathrm{X,G}}}$$

inducing an isomorphism at the level of homotopy groups.

Proof. We will construct the map as a functor ind_α between the relevant categories. Let $f : \mathcal{E}H \to \mathcal{C}_X$ be a functor and denote by $f(e) = S, \varphi : S \to X$ the value at the object $e \in H$. Denote by $\coprod_{|\ker\alpha|} f(e)$ the object in $\mathcal{C}_X$ given by the set $S_{\ker\alpha}$ obtained as the iterated disjoint union of copies of S, as many times as the cardinality of $\ker\alpha$, and the function $\coprod \varphi$. Since $\ker\alpha$ acts freely, the map $i : X \to \mathrm{ind}_\alpha X$ is injective, and hence the constant functor $\mathcal{E}G \to \mathcal{C}_{\mathrm{ind}\, X}$ with value $S_{\ker\alpha}, i \circ \coprod_{\ker\alpha} \varphi$ is well defined. We will analyze the behaviour of the functor $\mathrm{ind}\,\alpha$. Let $K \leq H$ be any subgroup. Using remark 2.3, the behaviour of the map on the K, respectively $\alpha(K)$- fixed point set reduces to the study of the functor categories $\mathrm{Fun}(\mathcal{E}H/K, \mathcal{C}_X)$, $\mathrm{Fun}(\mathcal{E}G/\alpha(K), \mathcal{C}_{\mathrm{ind}_\alpha X})$, and ultimately to the behaviour of the functor ind_α in the functor categories $\mathrm{Fun}(\mathcal{B}K, \mathcal{C}_X)$, $\mathrm{Fun}(\mathcal{B}\alpha(K), \mathcal{C}_{\mathrm{ind}_\alpha X})$. That is the functor which assigns to a functor f defined on the unique object with value $S, \varphi : S \to X$, the object with value $i \circ \coprod S, \coprod \varphi$, and where an automorphism given by $k \in K$ gives the block isomorphism given by the iterated union of the isomorphisms. We will use the usual fibering Theorem [Quillen(1973)], Theorem A in page 93 to conclude the (weak) homotopy equivalence of these spaces.

Let $f \in \mathrm{Fun}(\mathcal{B}\alpha(K), \mathcal{C}_{\mathrm{ind}_\alpha X})$ be a functor. Given Any pair of objects c_1, c_2 in $\mathrm{Fun}(\mathcal{B}K, \mathcal{C}_X)$ with isomorphisms $\mathrm{ind}_\alpha ci \to f$ is isomorphic to f via an isomorphism induced from one of finite sets. The freeness of the action of $\ker\alpha$ implies that there exists isomorphisms $c_1 \to c_2$. Hence, the functor induces a homotopy equivalence.

□

Remark. The previous functor defines a kind of "transfer map"

$$\mathrm{Fun}(\mathcal{E}H/K, \mathcal{C}_X) \to \mathrm{Fun}(\mathcal{E}G/\alpha(K), \mathcal{C}_{\mathrm{ind}_\alpha X})$$

in the sense that it gives a map in the opposite direction to the one induced by the projection $H/K \to G/\alpha(K)$. Induction structures are closely related to collapse maps or appropriate transfer constructions in $RO(G)$-graded cohomology built up out of them. See [Wirthmüller(1974)], and also [Lewis et al.(1986)Lewis, May, Steinberger, and McClure]. Although the hypotesis of the finiteness of $\ker\alpha$ seems to be annoying, the reason for this lies perhaps in this fact. Transfer Maps are usually defined under the assumption of compact fibers.

5. The Bivariant Theory and the Bivariant Burnside Module for infinite groups.

A naive spectrum E with an action of a group G yields a $\mathbb{Z}$-graded G-Cohomology Theory, defined by

$$\mathcal{H}^n_G(X,A) = \pi_*(\mathrm{Map}_G(X/A, E(n+*))$$

We extend the covariant notion given in the previous section to a bifunctor defined in the category of proper G-CW complexes, depending covariantly and contravariantly, consisting in the covariant part of an equivariant homology theory and in the contravariant part of an equivariant cohomology theory. We will describe a multiplicative structure in the cohomological part.

Definition 18. Let (X,A) and (Y,B) be proper G-CW complexes. We define the stable bivariant theory to be the groups

$$\omega^{-n}_G((X,A);(Y,B)) = \pi_n(\mathrm{Map}_G(X/A, \Omega \mathrm{B} M_{\mathcal{C}_{Y\cup \mathrm{Cone} B}})(\underline{1})), n \leq 0$$

respectively

$$\omega^{n}_G((X,A);(Y,B)) = \pi_1(\mathrm{Map}_G(X/A, \Omega \mathrm{B}^{1-n} M_{\mathcal{C}_{Y\cup \mathrm{Cone} B}})(\underline{1})), n < 0$$

We have straightforwardly from theorem 4.1, and from the fact that equivariant Γ-spaces as introduced in definition 8 produce $\mathbb{Z}$-graded equivariant cohomology theories:

Lemma 5.1. *The stable bivariant theory restricts to a $\mathbb{Z}$-graded equivariant homology, respectively, cohomology theory for a given proper G-CW complex X, respectively Y.*

The $\mathbb{Z}$-graded equivariant cohomoloogy theories associated to equivariant Γ-spaces in the sense of definition 8 come with an (internal) multiplicative structure.

Let $\coprod_{s\in S} X_s$, $\coprod_{t\in T} Y_t$ be objects in the Γ-category of finite sets and disjoint union. By means of an isomorphism $\alpha : \mathbb{N}\times\mathbb{N} \cong \mathbb{N}$, the set $X_s \times Y_t \subset \mathbb{N}\times\mathbb{N}$ can be handled as a finite set in $\mathbb{N}$. Hence, for every pair of finite sets S, T we get a functor $\alpha_* : \mathrm{set}_S \times \mathrm{set}_T \to \mathrm{set}_{S\times T}$ given by $(\coprod_{s\in S} X_s, \coprod_{t\in T} Y_t) \mapsto \coprod_{(s,t)\in S\times T} \alpha_*(X_s \times Y_t)$.

For any pair of groups G, H, this extends to a functor $\alpha_* : M_{\mathcal{C}_{X,G}}(S) \times M_{\mathcal{C}_{X,H}}(T) \to M_{\mathcal{C}_{X,G\times H}}(S\times T)$.

After geometric realization, this gives a $G\times H$- equivariant map $M_{\mathcal{C}_{X,G}}(S)\times M_{\mathcal{C}_{X,H}}(T) \to M_{\mathcal{C}_{X,G\times H}}(S\times T)$. Since the restriction of the map to $M_{\mathcal{C}_{X,G}}(S)\vee M_{\mathcal{C}_{X,H}}(T)$ is constant with value on the basis point (the product with the empty set is empty), the map α_* determines one $\beta_* : M_{\mathcal{C}_{X,G}}(S)\wedge M_{\mathcal{C}_{X,H}}(T) \to M_{\mathcal{C}_{X,G\times H}}(S\times T)$. Notice that both the target and the source of this map have two compatible structures of Γ-spaces, namely, those which are given by letting S, respectively T, run over the category of finite sets. After considering the separate realization of this Γ-spaces as simplicial spaces we get a map $\mathrm{B}M_{\mathcal{C}_{X,G}}(\underline{1}) \wedge \mathrm{B}M_{\mathcal{C}_{X,H}}(\underline{1}) \to \mathrm{B}^2 M_{\mathcal{C}_{X,G\times H}}(\underline{1})$. And at the level of loop spaces, $\Omega^2(\mathrm{B}M_{\mathcal{C}_{X,G}}(\underline{1}) \wedge \mathrm{B}M_{\mathcal{C}_{X,H}}(\underline{1})) \stackrel{\beta_*}{\to} \Omega^2\mathrm{B}^2 M_{\mathcal{C}_{X,G\times H}}(\underline{1})$.

Now define the pairing

$$\begin{aligned}&\alpha_X : \Omega\mathrm{B}M_{\mathcal{C}_{X,G}}(\underline{1}) \wedge \Omega\mathrm{B}M_{\mathcal{C}_{X,H}}(\underline{1}) \to \\ &\Omega^2(\mathrm{B}M_{\mathcal{C}_{X,G}}(\underline{1}) \wedge \mathrm{B}M_{\mathcal{C}_{X,H}}(\underline{1})) \stackrel{\Omega^2|\beta_*|}{\to} \Omega^2\mathrm{B}^2 M_{\mathcal{C}_{X,G\times H}}(\underline{1}) \simeq \Omega\mathrm{B}M_{\mathcal{C}_{X,G\times H}}(\underline{1})\end{aligned}$$

This map does not depend on the choice of the isomorphism α as a consequence of the following

Lemma 5.2. *Let $\alpha : \mathbb{N}^\infty \to \mathbb{N}^\infty$ be an injective map. Then, the induced map $\alpha_* : M_{\mathcal{C}_{X,G}}(\underline{1}) \to M_{\mathcal{C}_{X,G}}(\underline{1})$ defined by composition with* set $\stackrel{\alpha}{\to}$ set *is G-homotopic to the identity.*

Proof. There is a natural transformation of the functor α_* to the identity. □

Notice that there is more than one homotopy equivalence $\Omega^2\mathrm{B}^2 M_{\mathcal{C}_{X,G}}(\underline{1}) \to \Omega\mathrm{B}M_{\mathcal{C}_{X,G}}(\underline{1})$. The following technical result denies a role of this choice in our discussion.

Lemma 5.3. *Let A be any equivariant Γ-space. Then, for any n and $0 \le k \le n$, the maps $i_n^k : \Omega^{n+1}\mathrm{B}^{n+1}A(\underline{1}) \to \Omega^n\mathrm{B}^n A(\underline{1})$ induced by the inclusions $\{1\} \to \{1,\dots,n\}$ differ up to weak G-equivariant stable equivalence.*

Proof. Note that all i_n^k differ by permutation of the coordinates of $\mathrm{B}^n A$ as a simplicial set and a switch of the looping. Write $\sigma_* : \Omega^n\mathrm{B}^n A \to \Omega^n\mathrm{B}^n A$ for the map induced by such a permutation $\sigma \in \Sigma_n \subset \Sigma_{n+1}$. It suffices to show that all σ_* are homotopic to the identity. Consider for this the following diagram

$$\begin{tikzcd} \Omega\mathrm{B}A(\underline{1}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"'] & \Omega^{n+1}\mathrm{B}^{n+1}A(\underline{1}) \arrow[d, "1\times\sigma_*"'] & \Omega^n\mathrm{B}^n A(\underline{1}) \arrow[l, "i_n^n"'] \arrow[d, "\sigma_*"] \\ \Omega\mathrm{B}A(\underline{1}) \arrow[r, "\varphi"'] & \Omega^{n+1}\mathrm{B}^{n+1}A(\underline{1}) & \Omega^n\mathrm{B}^n A(\underline{1}) \arrow[l, "i_n^n"] \end{tikzcd}$$

where $\varphi = i_n^0 \circ \ldots \circ i_n^n$ is induced by identifying $A(S)$ with $A(S,1,\dots,1)$. All maps are weak G-homotopy equivalences. Hence, $(1\times\sigma_*)$ and $(\sigma)_*$ are weakly G-homotopic to the identity. □

Consider the pairing $\alpha_{(X,A);(Y,B)}$ defined on $\omega^*_G((X,A),(Y,B))$ given by the composite maps

$$\omega^q_G((X,A),(Y,B)) \times \omega^p_G((X,A),(Y,B)) \to \omega^{p+q}_{G\times G}((X,A),(Y,B)) \\ \overset{\Delta^*}{\to} \omega^{p+q}_G((X,A),(Y,B))$$

Where the first map is the pairing $\alpha_{X\cup \mathrm{Cone} A}$, and Δ^* is induced by restriction to the diagonal subcategory $\mathcal{E}G$ of $\mathcal{E}G \times G$. Then, we have the following:

Lemma 5.4. *For any discrete group G and any proper G-CW pair (Y,B) the pairings $\alpha_{(X,A);(Y,B)}$ define a structure of graded ring on $\omega^*_G((X,A)(Y,B))$*

We will now introduce a bivariant Burnside module. This has been described for finite groups by [Lewis et al.(1982)Lewis, May, and McClure] and generalized for pairs of a finite and a compact Lie group in [Lee(1994)]. We recall for this the notion of the classifying space for a family of subgroups.

Definition 19. Let $\mathcal{F}$ be a collection of subgroups in a discrete group G closed under conjugation and intersection. A model for the classifying space for the family $\mathcal{F}$ is a G-CW complex X whose isotropy subgroups all lie in $\mathcal{F}$ and for every subgroup $H \in \mathcal{F}$, the fixed point set X^H is either contractible if $H \in \mathcal{F}$ or empty otherwise.

Particularly relevant is the classifying space for proper actions, the classifying space for the family $\mathcal{FIN}$ of finite subgroups, denoted by $\underline{E}G$.

The classifying space for proper actions always exists, is unique up to G-homotopy and admits several models [Lück(2005)]. The following list includes some examples. We remit to [Lück(2005)] for further discussion.

- If G is a finite group, then the singleton space is a model for $\underline{E}G$.
- Let G be a group acting properly and cocompactly on a Cat(0) space X. Then X is a model for $\underline{E}G$.
- Let G be a Coxeter group. The Davis complex is a model for $\underline{E}G$.
- Let G be a mapping class group of a surface. The Teichmüller space is a model for $\underline{E}G$.
- Let G be a Gromov hyperbolic group. The Rips complex is a model for the classifying space for $\underline{E}G$.

Remark. The classifying space for proper actions, $\underline{E}G$ can be characterized by the universal property of being a proper G-CW-complex $\underline{E}G$ such that for any proper G-CW complex X, there exists up to G-homotopy a unique G-equivariant map

$$X \to \underline{E}G$$

Definition 20. Given a pair of discrete groups Π, G , the classifying space for proper, Π-free actions is the classifying space for the family of finite subgroups $H \subset G \times \Pi$ for which $H \cap e \times \Pi = \{e\}$. We will denote this space as

$$E_{\mathcal{FIN},\Pi}(G \times \Pi)$$

this is a G-space via the projection $G \times \Pi \to \Pi$

Definition 21. [Bivariant Burnside module for infinite groups] Let G and Π be discrete groups. The bivariant Burnside module $A(G, \Pi)$ is defined as the 0-th equivariant cohomology associated to $E_{\mathcal{FIN},\Pi}(G \times \Pi)$ evaluated in the classifying space for proper actions $\underline{E}G$. In symbols:

$$A(G, \Pi) = \omega_G^*(\underline{E}G, E_{\mathcal{FIN},\Pi}(G \times \Pi)) = [\underline{E}G, \Omega \mathrm{B} A_{\mathcal{C}_{E_{\mathcal{FIN},\Pi}(G\times\Pi)}}(\underline{1})]$$

Problem 1. Let G be a group accepting a finite model for $\underline{E}G$. Relate the bivariant Burnside ring to the set of stable maps between classifying spaces

$$[\mathrm{B}G, \Omega^\infty \Sigma^\infty (\mathrm{B}\pi_+)]$$

When both G and Π are finite groups, with the additional hypothesis that G is a finite p-group, a quite complete description was given by [Lewis et al.(1982)Lewis, May, and McClure]. Some progress has been made as to give an explicit decomposition of the set of stable maps between the p-completed classifying spaces.

The following result was proven in [Lee(1994)], Theorem 1.2 page 722:

Theorem 5.5. *Let Π be a virtually torsion-free group vith finite virtual cohomological dimension and Q a finite p-group. For each conjugacy class of subgroup of Q, the group $W_Q(H)$ acts on the set of Π-conjugacy classes of homomorphisms $\varphi : H \to \Pi$. For each representative (φ) of the $W_Q(H)$-orbit, define*

$$\Delta\varphi = \{(u, \varphi(u) \mid u \in H\} \subset Q \times H \quad N_\varphi = N_{Q\times\pi}(\Delta\varphi) \quad W_\varphi = (N\varphi/\Delta\varphi)$$

Then there is an equivalence map of spectra

$$\Psi : \vee_\varphi \mathrm{B}W_\varphi \longrightarrow F[\mathrm{B}Q_+, \mathrm{B}\Pi_+]$$

which is a stable homotopy equivalence after completion at p, where F denotes the function spectrum.

The relation to the bivariant Burnside ring is as follows: Since Π is a virtually torsion free group it may be assumed that there exists a finite group quotient $K = \Gamma/\Gamma'$. Then, there exists a finite model for the classifying space for proper Γ- actions denoted by X in [Lee(1994)]. In this case, the space $E_{Q\times K}(K, \mathcal{FIN}) \underset{\Pi}{\times} X$ is homotopy equivalent to BΠ. After the application of the splitting result cited in [May(1996)], page 206 Theorem 2.1, and a consequence of the Segal Conjecture, Theorem C , page 198 in [Carlsson(1984)], the result follows.

References

[Blumberg(2006)] Andrew J. Blumberg. Continuous functors as a model for the equivariant stable homotopy category. *Algebr. Geom. Topol.*, 6:2257–2295, 2006. ISSN 1472-2747.

[Carlsson(1984)] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984. ISSN 0003-486X. URL http://dx.doi.org/10.2307/2006940.

[Davis and Lück(1998)] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998. ISSN 0920-3036. URL http://dx.doi.org/10.1023/A:1007784106877.

[Lee(1994)] Chun-Nip Lee. Stable homotopy type of the classifying space of virtually torsion-free groups. *Topology*, 33(4):721–728, 1994. ISSN 0040-9383. URL http://dx.doi.org/10.1016/0040-9383(94)90005-1.

[Lewis et al.(1982)Lewis, May, and McClure] L. G. Lewis, J. P. May, and J. E. McClure. Classifying G-spaces and the Segal conjecture. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 165–179. Amer. Math. Soc., Providence, R.I., 1982.

[Lewis et al.(1986)Lewis, May, Steinberger, and McClure] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. ISBN 3-540-16820-6. With contributions by J. E. McClure.

[Lück(2005)] Wolfgang Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005. URL http://dx.doi.org/10.1007/3-7643-7447-0-7.

[Lück and Oliver(2001)] Wolfgang Lück and Bob Oliver. Chern characters for the equivariant K-theory of proper G-CW-complexes. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 217–247. Birkhäuser, Basel, 2001.

[Lück and Reich(2005)] Wolfgang Lück and Holger Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[Malcev(1940)] A. Malcev. On isomorphic matrix representations of infinite groups. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):405–422, 1940.

[May(1996)] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. ISBN 0-8218-0319-0. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[McCord(1969)] M. C. McCord. Classifying spaces and infinite symmetric products. *Trans. Amer. Math. Soc.*, 146:273–298, 1969. ISSN 0002-9947.

[Olʹshanskiĭ(1982)] A. Yu. Olʹshanskiĭ. Groups of bounded period with subgroups of prime order. *Algebra i Logika*, 21(5):553–618, 1982. ISSN 0373-9252.

[Puppe(1974)] Volker Puppe. A remark on "homotopy fibrations". *Manuscripta Math.*, 12:113–120, 1974. ISSN 0025-2611.

[Quillen(1973)] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Schlichtkrull(2007)] Christian Schlichtkrull. The homotopy infinite symmetric product represents stable homotopy. *Algebr. Geom. Topol.*, 7:1963–1977, 2007.

[Segal(1968)] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968. ISSN 0073-8301.

[Segal(1973)] Graeme Segal. Configuration-spaces and iterated loop-spaces. *Invent. Math.*, 21:213–221, 1973. ISSN 0020-9910.

[Segal(1974)] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974. ISSN 0040-9383.

[Shimakawa(1989)] Kazuhisa Shimakawa. Infinite loop G-spaces associated to monoidal G-graded categories. *Publ. Res. Inst. Math. Sci.*, 25(2):239–262, 1989. ISSN 0034-5318. URL `http://dx.doi.org/10.2977/prims/1195173610`.

[Shimakawa(1991)] Kazuhisa Shimakawa. A note on Γ_G-spaces. *Osaka J. Math.*, 28(2):223–228, 1991. ISSN 0030-6126. URL `http://projecteuclid.org/getRecord?id=euclid.ojm/1200783054`.

[Wasserman(1969)] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8: 127–150, 1969. ISSN 0040-9383.

[Wirthmüller(1974)] Klaus Wirthmüller. Equivariant homology and duality. *Manuscripta Math.*, 11:373–390, 1974. ISSN 0025-2611.

Noé Bárcenas
Hausdorff Center for Mathematics
Mathematisches Institut
Endenicherallee, 62,
D-53115, Germany

e-mail address: barcenas@math.uni-bonn.de

(*Received August 2, 2011*)
(*Revised March 7, 2013*)