THE IDEAL-VALUED INDEX OF FIBRATIONS BETWEEN G_2 FLAG MANIFOLDS

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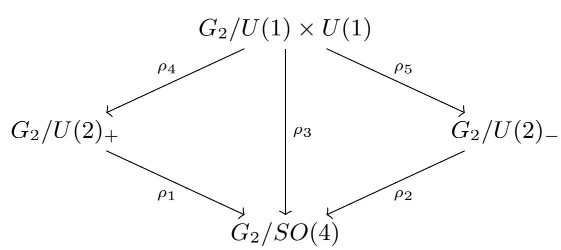
ABSTRACT. Using the cohomology of the G_2-flag manifolds $G_2/U(2)_\pm$, and their structure as a fiber bundle over the homogeneous space $G_2/SO(4)$ with fiber S^2, we compute their Borel cohomology, and the Fadell-Husseini index of such bundles, for the Z_2-action given by complex conjugation of complex structures on associative 3-planes.

Considering the orthogonal complement of the tautological bundle over $\widetilde{G}_3(\mathbb{R}^7)$, we compute the $\mathbb{Z}_2$ Fadell-Husseini index of the pullback bundle induced by the embedding $i : G_2/SO(4) \to \widetilde{G}_3(\mathbb{R}^7)$. We derive a general formula for the pullback, along the inclusion i, of an iterated external product bundle over a product of Grassmann manifolds.

1. Introduction.

A *generalized flag manifold* is an homogeneous space of the form $G/C(T)$, where G is a semisimple, compact and connected Lie group, and $C(T)$ is the cetralizer of a torus $T \subset G$. In case that T is the maximal torus, then $T = C(T)$ and we call G/T a *complete flag manifold*.

The group in which we want to focus is the exceptional Lie group G_2, which is the automorphism group of the $\mathbb{R}$-algebra homomorphisms of the octonions $\mathbb{O}$. From all the possible G_2 flag manifolds, we are particularly interested in the spaces $G_2/SO(4)$, $G_2/U(1) \times U(1)$ and $G_2/U(2)_\pm$. In the following diagram of fiber bundles we appreciate how they are related:



Previous to this work, several authors studied the integral cohomology of these homogeneous spaces. In section 3 and 4 we calculate their cohomology with $\mathbb{Z}_2$ coefficients, and the Stiefel-Whitney classes of ρ_1 and ρ_2. With that information, in section 4.3, we obtain our main first result.

Theorem 4.4. *Consider the action of $\mathbb{Z}_2$ on $G_2/U(2)_\pm$ by complex conjugation. Then the Borel cohomology of $G_2/U(2)_\pm$ is given by*

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm; \mathbb{F}_2) = H^*(B\mathbb{Z}_2 \times G_2/SO(4); \mathbb{F}_2) \Big/ \langle t^3 + u_2 t + u_3 \rangle.$$

Consequently, the Fadell-Husseini index of ρ_1 and ρ_2 is given by

$$\mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_1; \mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_2; \mathbb{F}_2) = \langle t^3 + u_2 t + u_3 \rangle.$$

Consider the orthogonal complement of the tautological bundle $\gamma^\perp$ over $\widetilde{G}_3(\mathbb{R}^7)$. Using the composition of the embedding $i : G_2/SO(4) \hookrightarrow \widetilde{G}_3(\mathbb{R}^7)$ with the fiber bundle ρ_2, we

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construct the pullback bundle

$$\zeta_1 = (\mathcal{S}^1_{\gamma^\perp}, \phi_1, G_2/U(2)_-, S^3)$$

that comes from the sphere bundle associated to $\gamma^\perp$. Notice that this construction can be easily generalized using the n-fold product bundle $(s\gamma^\perp)^n$. In section 4.4 we study some topological properties of the new bundle

$$\zeta_n = (\mathcal{S}^n_{\gamma^\perp}, \phi_n, G_2/U(2)_-, (S^3)^n).$$

This leads us to our next main results.

Theorem 4.6. *The cohomology $H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2)$ is described as follows*

$$H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2) = H^*\big(G_2/U(2)_- \times (S^3)^n;\mathbb{F}_2\big)\Big/(I \cup \{y^2, xy^2\})$$

where

$$I = \langle 1, y^2, x, xy^2\rangle \otimes \left\langle \sum_{j=1}^{m} \bigotimes_{\substack{i\in[m]\setminus\{j\}\\ a_i\in[n]}} z_{a_i} \;\middle|\; a_i \neq a_j \;\forall\, i\neq j \;\; and \;\; 2\leq m\leq n \right\rangle,$$

with $[n]=\{1,2,\dots,n\}$ and $[m]=\{1,2,\dots,m\}$.

Theorem 4.9. *Consider the action of $\mathbb{Z}_2^{n+1}$ on $S^n_{\gamma^\perp}$ where the first summand acts on $G_2/U(2)_-$ by complex conjugation, and the others n summands acts antipodally on the unitary elements. Then the Fadell-Husseini index of $\phi_n\colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_-$ is given by*

$$\mathrm{Index}^{G_2/U(2)_-}_{\mathbb{Z}_2^{n+1}}(\phi_n;\mathbb{F}_2) = \langle y^2 + yt_2^2 + t_2^4, \dots, y^2 + yt_{n+1}^2 + t_{n+1}^4\rangle,$$

where $H^*(\mathbb{Z}_2^{n+1};\mathbb{F}_2) = \mathbb{F}_2[t_1,\dots,t_{n+1}]$ with $|t_1| = \cdots = |t_{n+1}| = 1$.

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2. The exceptional Lie group G_2.

In this section we will recall three equivalent definitions of the real form of the exceptional Lie group G_2. Let us fix now the notation. General references for the upcoming discussion include [2] and [6].

2.1. **Octonions algebra.** Given a normed division algebra A, the Cayley-Dickinson construction creates a new algebra $A^{'}$ with elements $(a,b)\in A^2$ and conjugation $(a,b)^* = (\bar{a},-b)$. The addition in $A^{'}$ is done component-wise, and multiplication goes like

$$(a,b)(c,d) = (ac - d\bar{b}, \bar{a}d + cb),$$

where juxtaposition indicates multiplication in A. An equivalent way to define the new algebra $A^{'}$ is to add an independent square root of -1, i, that multiples the second named element on each pair (a,b). Now the conjugation in $A^{'}$ uses the original conjugation of A and $i^* = -i$. Then the construction becomes an algebra of elements $a+ib$ for some $a,b\in A$.

Starting with the real numbers $\mathbb{R}$, the complex numbers are defined via the Cayley-Dickinson construction to be pairs of reals $a+bi$. Similarly, the quaternions are generated as a real algebra by $\{1,i,j,k\}$, subject to the relations $i^2=j^2=k^2=ijk=-1$. Having this in mind, we get the following definition

Definition 2.1. The octonions are generated as a quaternionic algebra via the Cayley-Dickinson construction as $\mathbb{H}\oplus\mathbb{H}[l]$, where l denotes an independent square root of -1.

Even if the Cayley-Dickinson construction is an excellent method to produce other normed division algebras, we lost very nice properties in the process. The multiplication in $\mathbb{O}$ turns out to be non-commutative and non-associative. As a real vector space, $\mathbb{O}$ is generated by $\{1, i, j, k, l, li, lj, lk\} = \{1, e_1, \dots, e_7\}$, where $e_1, e_2, \dots e_7$ are imaginary units, which square to -1, switch sign under complex conjugation and anticommute. They span the seven dimensional real subspace which we will denote as the purely imaginary part of the octonions, $\operatorname{Im}\mathbb{O}$.

Denote by $\cdot$ the product furnishing the octonions with the structure of a real division algebra. The product as real division algebra determines a cross product in $\mathbb{O}$ expressed as

$$x \times y := \frac{1}{2} x \cdot y - y \cdot x.$$

2.2. **The exceptional Lie group G_2: three equivalent definitions.** We are ready now to introduce the exceptional Lie group G_2.

Definition 2.2. The exceptional Lie group G_2 is the automorphism group of the $\mathbb{R}$-algebra homomorphisms of the octonions $\mathbb{O}$.

Let us consider now an alternative but equivalent definition of G_2: Given two orthogonal and purely imaginary unitary octonions x and y, the cross product $x \times y$ is orthogonal to both of them, and the subalgebra generated by $\{1, x, y, x \times y\}$ is isomorphic to the quaternions. If we consider another purely imaginary unit octonion z orthogonal to the subspace generated by $\{1, x, y, x \times y\} \cong \mathbb{H}$, then the subalgebra over the quaternions $\mathbb{H} \oplus \mathbb{H}[z]$ is isomorphic to the octonions. This has the consequence that an element $g \in G_2$ can be characterized by prescribing three purely imaginary unit octonions x, y, z, with z orthogonal to x, y and $x \times y$. The element g is then the unique automorphism of the imaginary part of the octonions, which sends $\{x, y, z\}$ to $\{i, j, l\}$. Then the exceptional Lie group G_2 is defined to be the group of automorphisms of the algebra $(\operatorname{Im}\mathbb{O}, \times)$.

For the considerations made below, it proves to be useful to consider a third definition of G_2. On $\operatorname{Im}\mathbb{O}$ one can define a three-form on generators e_i, e_j, e_k as

$$\phi(e_i, e_j, e_k) = f^{ijk},$$

where $e_i \times e_j = \sum_k f^{ijk} e_k$. Notice that ϕ encodes the multiplicative structure of cross product in the purely imaginary part of the octonions and then G_2 can be equivalently defined as

$$G_2 = \{g \in GL(\operatorname{Im}\mathbb{O}) \mid g^*\phi = \phi\}.$$

If we consider the dual basis $w^i := e_i^*$ for the generators e_i given above, where the notation w^{ijk} denotes the wedge product $e^i \wedge e^j \wedge e^j$, then the cross product three form is given as

$$\phi = w^{123} - w^{145} - w^{167} - w^{246} + w^{257} - w^{347} - w^{356}.$$

In [4, Theorem 1] Bryant presented some other facts about G_2.

Proposition 2.3. *The Lie group G_2 is a compact subgroup of $SO_7(\mathbb{R})$, of dimension 14, that is connected, simple and simply connected.*

The following result summarizes the discussion of basic facts of the real form of G_2.

Theorem 2.4. *The real form of the group G_2 has dimension 14, is simple and simply connected, and is isomorphic to one and hence to all of the following Lie groups:*

(i) The automorphism group of the real algebra of the octonions.
(ii) The automorphism group of the subalgebra of purely imaginary octonions which preserves cross product.
(iii) The subgroup of SO_7 which preserves the cross product three form.

Also, as a smooth manifold, G_2 is diffeomorphic to the following closed subset of $(\mathbb{R}^7)^3$

$$\big\{(x_1, x_2, x_3) \mid \langle x_i, x_j\rangle = \delta_{i,j} \; \langle x_1 \times x_2, x_3\rangle = 0\big\}$$

3. G_2 FLAG MANIFOLDS

The flag manifolds on G_2 have received recently a lot of attention from several viewpoints. From riemannian geometry ([4], [13]), algebraic topology relevant to global analysis [1], complex and Kähler geometry ([9], [11]), and from the classical computations of their characteristic classes via representation theory of Borel and Hirzebruch ([3], [8]). The combination of the above facts, excellently produced in [9], aroused our interest in the subject and gave us the original idea to calculate the Fadell-Husseini index of some fiber bundles over $G_2/SO(4)$.

We consider now flag manifolds associated to the exceptional Lie group G_2. The examples that we will consider are flags of subspaces in the imaginary part of the octonions, as well as some of the complexifications of $\operatorname{Im}\mathbb{O}$, equipped with a transitive action of the group G_2 as described equivalently in theorem 2.4.

3.1. **The six dimensional sphere $G_2/SU(3)$.** The six dimensional sphere, which can be seen as the unitary vectors in the seven dimensional real subspace $\operatorname{Im}\mathbb{O}$, carries a transitive action of G_2 as follows: Consider z_1 and z_2 unitary vectors in $\operatorname{Im}\mathbb{O}$, and two pair of unitary vector (x_1, y_1) and (x_2, y_2) which generate the orthogonal complemet of z_1 and z_2 respectively. By 2.4 [part ii], there is a unique $g \in G_2$ that sends z_1 to z_2.

To determine the isotropy group of an element $l \in S^6$, consider the subspace V defined as the orthogonal complement of l inside $\operatorname{Im}\mathbb{O}$, and the complex structure on V given by the left multiplication by l. The identification with $\mathbb{C}^3$, equipped with its stardand Hermitian scalar product, induces a scalar product on V defined as

$$\langle v, w\rangle_V = (v, w) + l(v, lw)$$

where $(-,-)$ denote the standard real product. Since $g \in (G_2)_l$ preserves $(-,-)$, it also preserves $\langle -,-\rangle_V$, and $(G_2)_l \subset U(V) \cong U(3)$. Calculating the determinant of $g \in (G_2)_l$, can be proved that actually the isotropy group is isomorphic to $SU(3)$.

We have then the following result.

Proposition 3.1. *There is a transitive action of G_2 on S^6 with isotropy group isomorphic to $SU(3)$, i.e. $S^6 \cong G_2/SU(3)$.*

3.2. **The space of associative subspaces $G_2/SO(4)$.**

Definition 3.2. A 3-dimensional subspace $\xi \subset \operatorname{Im}\mathbb{O}$ is said to be *associative* if it is the imaginary part of the subalgebra isomorphic to the quaternions. The subspace be acquired a canonical orientation from that of the quaternions.

In terms of the three form defined in 2.2, we can also define an associative subspace as the 3-dimensional real subspace of $\mathbb{R}^7 \cong_{\mathbb{R}} \operatorname{Im}\mathbb{O}$ in which the three form ϕ agrees with the volume form $\operatorname{Vol}(\xi)$.

The group G_2 stabilizes 1, and since the scalar product is the real part of the octonian multiplication $\cdot$, it acts by isometries on $\operatorname{Im}\mathbb{O}$. Since also preserves the vector product, it preserves orientation so that

$$G_2 \subset SO(\operatorname{Im}\mathbb{O}) \cong SO(7).$$

Every associative subspace ξ admits an orthonormal basis $\{e_1, e_2, e_3\}$ with $e_1 \times e_2 = e_3$. Given such a triple $\{e_1, e_2, e_3\}$, there exists a unique automorphism in $\operatorname{Im}\mathbb{O}$ taking the ordered triple $\{i, j, l\}$ to it. This induces a transitive action of G_2 on the Grassmannian of associative 3-dimensional subspaces of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$, with stabilizer $SO(4)$. Then $G_2/SO(4)$ is the set of all associative 3-dimensional subspaces of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. Since every associative subspace has a canonical orientation, there is an embedding of $G_2/SO(4)$ in the 3-dimensional oriented Grassmannian manifold $\widetilde{G}_3(\mathbb{R}^7)$, with $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$.

Borel and Hirzebruch studied in [3] the cohomology with $\mathbb{Z}_2$-coefficients and the Stiefel-Whitney classes of $G_2/SO(4)$. They proved the following result.

Proposition 3.3. *The homogeneous space of associative* 3*-dimensional real subspaces, denoted by* G_2/SO_4*, is an* 8*-dimensional manifold for which* $H^*(G_2/SO(4);\mathbb{F}_2)$ *is generated by two elements* u_2, u_3 *of degrees* 2 *and* 3 *respectively, with the relations*

$$u_2^3 = u_3^2 \quad \text{and} \quad u_3 u_2^2 = 0.$$

Also, the Stiefel-Whitney classes of $G_2/SO(4)$ *are non-zero only in dimensions* 0, 4, 6 *and* 8.

On the other hand, Shi and Zhou in [12, Section 10], Thung in [9, Section 2], and Akbulut and Kalafat [1, Theorem 10.3], studied the integral cohomology of $G_2/SO(4)$, and the relations between the generator to write it as a truncated polynomial algebra.

Proposition 3.4. *The integral cohomology of* $G_2/SO(4)$ *is a truncated polynomial algebra generated by two elements* a *and* b *of degree* 3*, respectively* 4*, subject to the relations*

$$\{2b = 0\, b^3 = 0, a^3 = 0, ab = 0\}.$$

3.3. **The space of complex coassociative 2-planes $G_2/U(2)_+$.** We will consider now the complexification of the purely imaginary subspace of the octonions, which is isomorphic as complex vector space $\mathbb{C}^7$, in symbols $\operatorname{Im}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \cong_{\mathbb{C}} \mathbb{C}^7$.

Let J be the orthogonal complex structure on $\xi^\perp$ with $(1,0)$-space W and choose an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\xi^\perp$ with $J(e_1) = e_2$ and $J(e_3) = e_4$. We say that J, or the corresponding W, is positive (resp. negative) according as the basis is positive (resp. negative). This means, W has a real basis given by $u = e_1 - ie_2$ and $v = e_3 - ie_4$ (resp. $u = e_1 + ie_2$ and $v = e_3 + ie_4$).

Definition 3.5. We call a complex 2-dimensional subspace $W \subset \mathbb{C}^7$ *coassociative* if $v \times w = 0$ for all $v, w \in W$

Notice that the subspace W is automatically isotropic. Also, the associative three form $\phi \otimes \mathrm{id}$ defines a complex three-form $\phi_{\mathbb{C}}$ on the complex space $\mathbb{C}^7$. So, in terms of the three form defined in 2.2, a complex vector subspace $W \subset \mathbb{C}^7$ is called coassociative if the complexified form $\phi_{\mathbb{C}}$ vanishes on V. We will consider coassociative complex lines (1-dimensional complex subspaces) and coassociative planes (2-dimensional complex subspaces).

The following result, which connects the notion of complex coassociative and positive subspace, is proved in [13, Lemma 2.2, page 293].

Lemma 3.6. *Let* ξ *be an associative* 3*-dimensional subspace in* $\operatorname{Im}\mathbb{O}$*, and let* $W \subset \xi^\perp \otimes \mathbb{C}$ *be a maximally isotropic subspace. Then* W *is positive if and only if it is complex coassociative.*

Consider the space of all complex coassociative 2-dimensional subspaces of $\mathbb{C}^7$. Since the space has a transitive action of G_2 with stabilizer isomorphic to $U(2)$, we have the following definition.

Definition 3.7. The space of complex coassociative 2-planes in $\operatorname{Im}\mathbb{O} \otimes \mathbb{C}$ will be denoted by $G_2/U(2)_+$.

The reason for the notation comes from lemma 3.6. Moreover, there is a map $\rho_1 : G_2/U(2)_+ \to G_2/SO_4$ which exhibits $G_2/U(2)_+$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. The map ρ_1 is given by $W \mapsto (W \oplus \bar{W})^\perp$. By lemma 3.6 the fiber of any $\xi \in G_2/SO(4)$ is all positive maximally isotropic subspaces of $\xi^\perp \otimes \mathbb{C}$, equivalently, all posible orthogonal complex structures on $\xi^\perp$. Actually, by [13, section 4.4] there is an identification of $G_2/U(2)_+$ with the quaternionic twistor space of G_2/SO_4.

As is exposed in [9, Prop. 3], the cohomology of $G_2/U(2)_+$ with integral coefficients is generated by classes $g_i \in H^{2i}(G_2/U(2)_+, \mathbb{Z})$, for $i = 1, \ldots, 5$, and the multiplicative structure is determined by the relations

$$g_1^2 = 3g_2,\ g_1 g_2 = 2g_3,\ g_2^2 = 2g_4,\ g_1 g_4 = g_5.$$

The cohomology with coefficients in $\mathbb{Z}_2$ will be calculated in subsection 4.3.

3.4. The space of complex non-coassociative 2-planes $\mathbf{G_2/U(2)_-}$.

Definition 3.8. The space of all 2-planes in $\mathrm{Im}\,\mathbb{O}\otimes\mathbb{C}$ which are not complex coassociative will be denoted by $G_2/U(2)_-$.

Similarly to $G_2/U(2)_+$, the reason for the notation comes from lemma 3.6. A 2-dimensional real subspace W, which is a $(1,0)$-space for a negative complex structure in $\xi^\perp$, is said to be negative and is still a maximally isotropic subspace of $\xi^\perp \otimes \mathbb{C}$.

Alternatively, we can think the space $G_2/U(2)_-$ as follows: Let Q^5 be the complex quadric $\{[z_0,\dots,z_5] \in \mathbb{C}P^6 \mid \sum_{i=0}^{5} z_i^2 = 0\}$ consisting of all 1-dimensional isotropic subspaces of $\mathrm{Im}\,\mathbb{O}\otimes\mathbb{C}$. There is a G_2-equivariant isomorphism from Q^5 to $G_2/U(2)_-$ given by $\ell \mapsto \ell^\perp \cap \ell^a$, with inverse $W \mapsto W \times W$, where $\ell^a = \{L \in \mathrm{Im}\,\mathbb{O}\otimes\mathbb{C} \mid L \times \ell = 0\}$.

Since $\widetilde{G}_2(\mathbb{R}^7)$ is also diffeomorphic to the complex quadric Q^5, we can identify every element in $G_2/U(2)_-$ as an oriented plane P with oriented, orthonormal basis $\{x,y\}$. Then there is a well defined map $\rho_6 : G_2/U(2)_- \to S^6$ which sends $P \mapsto x \times y = xy$. Actually, the oriented plane P can be identify, via the almost complex structure of S^6, with a complex line in $T_{xy}S^6$. On the other hand, given an element $k \in S^6$, there is a oriented orthonormal basis $\{x,y\}$ of a complex line in T_kS^6 that satisfies $k = xy$. This means that the fiber over an element $v \in S^6$ is precisely $\mathbb{P}(T_vS^6)$. This exhibits $G_2/U(2)_-$ as $\mathbb{P}(TS^6)$ with ρ_6 as the base point projection. We can also think of ρ_6 as a fibration with fibers diffeomorphic to $\mathbb{C}P^2$.

Finally, the isomorphism $TS^6 \cong T^*S^6$ as real vector bundles induces a diffeomorphism $\mathbb{P}(TS^6) \cong \mathbb{P}(T^*S^6)$. The following result resumes all the diffeomorphic definitions of $G_2/U(2)_-$. See [9, Prop. 8].

Proposition 3.9. *The following* 10*-dimensional manifolds are all diffeomorphic to each other:*

- *(i) The space of complex isotropic lines in* $\mathrm{Im}\,\mathbb{O}\otimes\mathbb{C}$ *has a transitive action of* G_2*, with stabilizer isomorphic to* $U(2)$*.*
- *(ii) The Grassmannian* $\widetilde{G}_2(\mathbb{R}^7)$ *of oriented* 2*-planes in* $\mathbb{R}^7$*.*
- *(iii) The complex quadric*

$$Q^5 = \{(z_1 : z_2 :, \dots :, z_7) \in \mathbb{C}P^6 \mid \sum_{i=1}^{7} z_i^2 = 0\}.$$

- *(iv) The projectivization of the tangent and cotangent bundle* $\mathbb{P}(TS^6)$ *and* $\mathbb{P}(T^*S^6)$ *for any almost complex structure in* S^6*.*

The cohomology with integral coefficients is calculated in [9, Prop. 11]. It is the quotient of a polynomial algebra with generators x in degree 6, and y in degree 2, satisfying the relations $x^2 = 0$ and $y^3 = -2x$. Unlike $G_2/U(2)_+$, using the bundle ρ_6 we can calculate the cohomology ring with $\mathbb{Z}_2$ coefficients of $G_2/U(2)_-$ as follows.

Proposition 3.10. *The cohomology of* $G_2/U(2)_-$ *with coefficients in* $\mathbb{Z}_2$ *is given by*

$$H^*(G_2/U(2)_-, \mathbb{F}_2) = \mathbb{F}_2[x,y]/\langle x^2, y^3\rangle$$

where $deg(y) = 2$*,* $deg(x) = 6$*.*

Proof. Consider the fiber bundle

$$\mathbb{CP}^2 \hookrightarrow G_2/U(2)_- \to S^6$$

and the corresponding Serre spectral sequence. Since S^6 is simply connected, there are no local coefficients and the E_2-term is given by

$$E_2^{p,q} = H^p\big(S^6; H^q(\mathbb{CP}^2; \mathbb{F}_2)\big).$$

Let us denote the cohomology of the fiber $\mathbb{CP}^2$ and the base space S^6 as follows: $H^*(S^6;\mathbb{F}_2) = \mathbb{Z}_2[x]/\langle x^2\rangle$ and $H^*(\mathbb{CP}^2;\mathbb{F}_2) = \mathbb{F}_2[y]/\langle y^3\rangle$. For an illustration of the E_2-term see Figure 1. Then, since all the possible differentials are trivial, $E_2^{p,q} \cong E_\infty^{p,q}$ and

$$H^n(G_2/U(2)_-, \mathbb{F}_2) = \mathbb{F}_2[x,y]/\langle x^2, y^3\rangle,$$

where $deg(y) = 2$ and $deg(x) = 6$. □

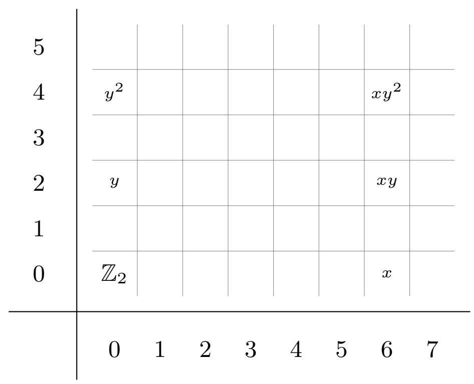


FIGURE 1. $E_2^{p,q} = H^p\big(S^6; H^q(\mathbb{CP}^2;\mathbb{F}_2)\big) \Rightarrow H^*(G_2/U(2)_-;\mathbb{F}_2)$.

Similar to $G_2/U(2)_+$, there exists a map $\rho_2 : G_2/U(2)_- \to G_2/SO_4$ which exhibits $G_2/U(2)_-$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. The map ρ_2 is also given by $W \mapsto (W \oplus \bar{W})^\perp$. By lemma 3.6 the fiber at any $\xi \in G_2/SO(4)$ consist of all negative maximally isotropic subspaces of $\xi^\perp \otimes \mathbb{C}$, or equivalently, all negative orthogonal complex structures on $\xi^\perp$. Let us describe now the induced homomorphism ρ_2^*.

Lemma 3.11. *The induced homomorphism in cohomology ρ_2^*, considering $\mathbb{F}_2$ coefficients, maps the class u_2^2 to y^2.*

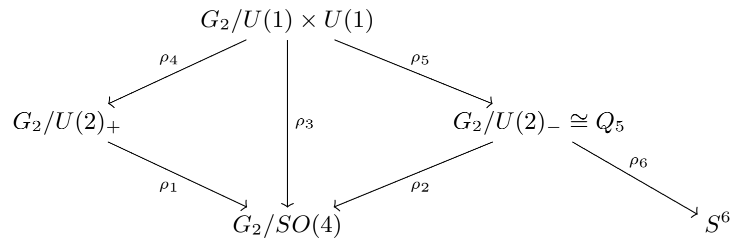
Proof. Analyzing the Gysin sequence

$$\ldots \to H^{i-3}(G_2/SO(4);\mathbb{F}_2) \xrightarrow{\smile w_3} H^i(G_2/SO(4);\mathbb{F}_2) \to$$
$$\xrightarrow{\rho_2^*} H^i(G_2/U(2)_-;\mathbb{F}_2) \to H^{i-2}(G_2/SO(4);\mathbb{F}_2) \to \ldots$$

since $H^1(G_2/SO_4, \mathbb{F}_2) = 0$, by the exactness of the sequence, $\rho_2^* : H^4(G_2/SO(4);\mathbb{F}_2) \to H^i(G_2/U(2)_-;\mathbb{F}_2)$ is a monomorphism. This means that $\rho_2^*(u_2) = y$ and $\rho_2^*(u_2^2) = y^2$. □

3.5. **The full flag manifold $\mathbf{G_2/U(1) \times U(1)}$.** Given a complex isotropic line $\ell \subset \operatorname{Im}\mathbb{O} \otimes_\mathbb{R} \mathbb{C}$, consider the annihilator ℓ^a, that is the subspace of $\operatorname{Im}\mathbb{O} \otimes_\mathbb{R} \mathbb{C}$ described as $\{x \in \operatorname{Im}\mathbb{O} \otimes_\mathbb{R} \mathbb{C} \mid x \times \ell = 0\}$. Notice that this is a complex 3-dimensional isotropic subspace of $\operatorname{Im}\mathbb{O} \otimes_\mathbb{R} \mathbb{C}$. Since the maximal torus in G_2 is of rank 2, the complete flag manifold $G_2/U(1) \times U(1)$ is a smooth complex manifold of dimension 6 that can be described as follows: The space of pair (ℓ, D) where ℓ is a complex isotropic line, D is a 2-plane containing ℓ, and both contained in ℓ^a.

For every pair (ℓ, D) in $G_2/U(1) \times U(1)$, we write $D = \ell \oplus q$, where q is the orthogonal complement of ℓ. Then we get a fibration $\rho_3 : G_2/U(1) \times U(1) \to G_2/SO(4)$ given by $(\ell, D) \mapsto \xi$, where $\xi \otimes_\mathbb{R} \mathbb{C} = q \oplus \bar{q} \oplus (q \times \bar{q})$. The map ρ_3 factors through a fibration over $G_2/U(2)_\pm$, which sends (ℓ, D) to the positive (resp. negative) maximally isotropic subspace of $xi^\perp \otimes_\mathbb{R} \mathbb{C}$ which contains ℓ, and ρ_1 (resp. ρ_2) defined above.



All the maps ρ_i are locally trivial smooth fibrations, with fiber diffeomorphic to $\mathbb{C}P^1 \approx S^2$ (except ρ_3 and ρ_6 which fiber is diffeomorphic to $\mathbb{C}P^2$).This means that, using the map ρ_5, we can also calculate the cohomology of $G_2/U(1)\times U(1)$ with coefficients in $\mathbb{Z}_2$.

Proposition 3.12. *The cohomology of $G_2/U(1)\times U(1)$ with $\mathbb{Z}_2$ coefficients is given by*

$$H^*(G_2/U(1)\times U(1), \mathbb{F}_2) = \mathbb{F}_2[x,y,z]/\langle x^2, y^3, z^2\rangle$$

where $deg(y) = 2$, $deg(x) = 6$ and $deg(z) = 2$.

Proof. Consider the fiber bundle

$$S^2 \hookrightarrow G_2/U(1)\times U(1) \xrightarrow{\rho_5} G_2/U(2)_-,$$

and the corresponding Serre spectral sequence, where the cohomology of the fiber is described as $H^*(S^2;\mathbb{F}_2) = \mathbb{F}_2[z]/z^2$. Since $\pi_1(G_2/U(2)_-)$ is trivial, there are no local coefficients and the E_2-term is given by

$$E_2^{p,q} = H^p\big(G_2/U(2)_-; H^q(S^2;\mathbb{F}_2)\big).$$

For an illustration of the E_2-term see Figure 2. The rest of the proof is similar to the one of 3.10, since all the possible differentials are trivial. ☐

3											
2	z		$z\otimes y$		$z\otimes y^2$		$z\otimes x$		$z\otimes xy$		$z\otimes xy^2$
1											
0	$\mathbb{Z}_2$		y		y^2		x		xy		xy^2
	0	1	2	3	4	5	6	7	8	9	10

FIGURE 2. $E_2^{p,q} = H^p\big(G_2/U(2)_-; H^q(S^2;\mathbb{F}_2)\big) \Rightarrow H^*(G_2/U(1)\times U(1);\mathbb{F}_2)$.

4. CALCULATIONS OF FADELL-HUSSEINI INDEX.

Before we make some calculations, we briefly recall the definition and some basic properties of the Fadell-Husseini index.

4.1. **The Fadell-Husseini Index.** Let G be a finite group, let R be a commutative ring with unit. For a $G-$ equivariant map $p\colon X \longrightarrow B$ and a ring R, the *Fadell-Husseini index* of p is defined to be the kernel ideal of the map in the equivariant cohomology with coefficients in the ring R induced by p,

$$\begin{aligned}\mathrm{Index}_G^B(p;R) &= \ker\big(p*\colon H^*(EG\times_G B;R) \longrightarrow H^*(EG\times_G X;R)\big)\\ &= \ker\big(p*\colon H^*_G(B;R) \longrightarrow H^*_G(X;R)\big).\end{aligned}$$

The equivariant cohomology of a $G-$space X is asumed to be the Cech cohomology of the Borel construction $EG\times_G X$ associated to the space X. The basic properties of the index are:

- *Monotonicity:* If $p\colon X \longrightarrow B$ and $q\colon Y \longrightarrow B$ are $G-$equivariant maps, and $f\colon X \longrightarrow Y$ is a $G-$equivariant map such that $p = q \circ f$, then
$$\text{Index}_G^B(p;R) \supseteq \text{Index}_G^B(q;R).$$
- *Additivity:* If $(X_1 \bigcup X_2, X_1, X_2)$ is an excisive triple of $G-$spaces and $p\colon X_1 \bigcup X_2 \longrightarrow B$ is a $G-$equivariant map, then
$$\text{Index}_G^B(p|_{X_1};R) \cdot \text{Index}_G^B(p|_{X_2};R) \subseteq \text{Index}_G^B(p;R).$$
- *General Borsuk-Ulam-Bourgin-Yang theorem:* Let $p\colon X \longrightarrow B$ and $q\colon Y \longrightarrow B$ be $G-$equivariant maps, and let $f\colon X \longrightarrow Y$ be a $G-$equivariant map such that $p = q \circ f$. If $Z \subseteq Y$ then
$$\text{Index}_G^B(p|_{f^{-1}(Z)};R) \cdot \text{Index}_G^B(q|_{Y\setminus Z};R) \subseteq \text{Index}_G^B(p;R).$$

In the case when B is a point and $p\colon X \longrightarrow B$ is a $G-$equivariant map, we simplifly notation and write $\text{Index}_G^B(p;R) = \text{Index}_G(X;R)$. From here on we will be considering $\mathbb{Z}_2$ coefficients. For more details see [7]

4.2. **Dold's argument.** In [5] Dold presented a very useful tool to deduce some differentials of the Serre spectral sequence associated to a particular kind of fiber bundles. Let us start with the general argument.

Let $E \xrightarrow{\pi} B \xleftarrow{\pi'} E'$ be vector bundles over the same base space B, and $f\colon S(E) \longrightarrow E'$ an odd map, where $S(E)$ is the total space of the sphere bundle associated to π, such that

$$\begin{array}{ccc} S(E) & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & B & \end{array}$$

commutes. Let us define $Z_f = \{x \in S(E) \mid f(x) = 0\}$ and the projection maps
$$S(E) \to \overline{S}(E) = S(E)/\mathbb{Z}_2 \quad \text{and} \quad Z \to \overline{Z} = Z/\mathbb{Z}_2$$
where we are considering the antipodal action. The characteristic classes of the projections are denoted by u, resp. $u \mid \overline{Z}$. Dold proved the following result.

Theorem 4.1. *If $q(t) \in H^*(B)[t]$ is such that $\sigma(q(t)) = q(t) \mid \overline{Z} = 0$, where*
$$\sigma\colon H^*(B)[t] \longrightarrow H^*(\overline{S}(E)) \longrightarrow H^*(\overline{Z})$$
is given by$t \longmapsto u \longmapsto u \mid \overline{Z}$, then
$$q(t)W(E';t) = W(E;t)q'(t)$$
for some $q'(t) \in H^*(B)[t]$ and $W(E;t) = \sum_{j=0}^m w_j(E) \otimes t^{m-j}$.

The last theorem means that $W(E;t)$ divides $q(t)W(E';t)$. We show the effectiveness of this theorem in the following remark.

Remark 4.2. Under the same hypothesis, consider the fiber bundles

$$\begin{array}{ccc} S^3 \hookrightarrow S(E) & \qquad & \{0\} \hookrightarrow B \times \{0\} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle proj_1} \\ B & & B \end{array}$$

where $\mathbb{Z}_2$ acts antipodally on $S(E)$ and trivial on B. Let $f\colon SE \longrightarrow B \times \{0\}$ be the $\mathbb{Z}_2$-equivariant map given by $f(e) = (\pi(e), 0)$, with $Z_f = SE$, such that the following diagram commutes:

$$\begin{array}{ccc} SE & \xrightarrow{\quad f \quad} & B \times \{0\}\ . \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle proj_1} & \\ & B & \end{array}$$

If we consider $q(t)$ as the image of the transgression map d_n of the Serre spectral sequence associated to the sphere bundle

$$S^n \hookrightarrow E\mathbb{Z}_2 \times_{\mathbb{Z}_2} E \to B\mathbb{Z}_2 \times B,$$

then, by theorem 4.1, $q(t) = d_n(z) = \sum_{j=0}^{n} w_j(\pi) \otimes t^{n-j}$.

We are going to use Dold's argument in some of the computations that we present.

4.3. **The Fadell-Husseini index of ρ_1 and ρ_2.** Following the idea presented in section 4.2, we are going to start computing the Stiefel-Whitney classes of ρ_1 and ρ_2 to deduce their corresponding spectral sequences.

Lemma 4.3. *The Stiefel-Whitney classes of the fiber bundle $\rho_2\colon G_2/U(2)_- \longrightarrow G_2/SO(4)$ are non-zero in the dimensions $0, 2$ and 3.*

Proof. Consider the Gysin exact sequence applied to ρ_2

$$\ldots \to H^{i-3}(G_2/SO(4);\mathbb{F}_2) \xrightarrow{\smile w_3} H^i(G_2/SO(4);\mathbb{F}_2) \xrightarrow{\rho_2^*} H^i(G_2/U(2)_-;\mathbb{F}_2) \to$$
$$\to H^{i-2}(G_2/SO(4);\mathbb{F}_2) \to \ldots$$

Since $H^3(G_2/U(2)_-;\mathbb{F}_2) = 0$, we have an epimorphism

$$H^0(G_2/SO(4);\mathbb{F}_2) \twoheadrightarrow H^3(G_2/SO(4);\mathbb{F}_2)$$

that by proposition 3.3 becomes $\mathbb{Z}_2 \twoheadrightarrow \langle u_3 \rangle$. Hence, $w_3(\rho_2) = u_3$. This means that, by the definition of the Stiefel-Whitney classes

$$\phi^{-1} \circ Sq^3(th) \neq 0,$$

where th is the Thom class. Since

$$Sq^3 = Sq^1 \circ Sq^2,$$

$Sq^2(th) \neq 0$ and then $w_2 \neq 0$. Is not hard to see that $w_2 = u_2$. □

Notice that the same arguments used in 4.3 works for $\rho_1\colon G_2/U(2)_+ \longrightarrow G_2/SO(4)$. This means that ρ_1 and ρ_2 have the same Stiefel-Whitney classes.

Now that we know the Stiefel-Whitney classes of ρ_1, we can calculate the cohomology of $G_2/U(2)_+$ as follows: Consider the Serre spectral sequence associated to ρ_1, with E_2-term

$$E_2^{p,q} = H^p\big(G_2/SO(4); H^q(S^2;\mathbb{F}_2)\big).$$

Notice that, for the same reason exposed in 3.10, there are no local coefficients here. For an illustration of the E_2-term see Figure 3. Applying the Gysin sequence to ρ_1 [10, Example 5.C], and by lemma 4.3, we get that

$$d_3^{0,2}(z) = w_3(\rho_1) = u_3,$$

where $H^2(S^2;\mathbb{F}_2) = \langle z \rangle$. Finally, using the Leibniz rule, we can actually verify that $G_2/U(2)_+$ and $G_2/U(2)_-$ have isomorphic cohomologies when we consider $\mathbb{F}_2$ coefficients.

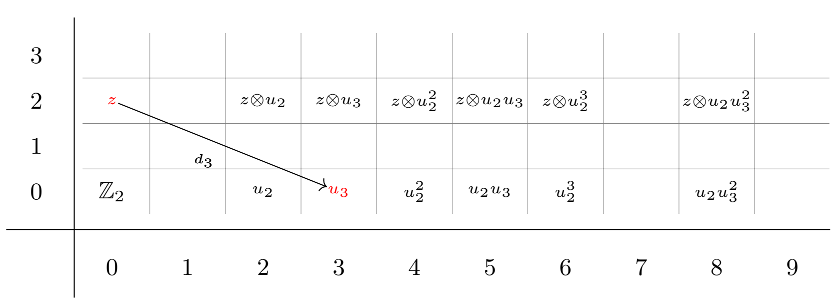


FIGURE 3. $E_2^{p,q} = H^p\big(G_2/SO(4); H^q(S^2;\mathbb{F}_2)\big) \Rightarrow H^*(G_2/U(2)_+;\mathbb{F}_2)$.

Consider the action of $\mathbb{Z}_2$ on $G_2/U(2)_\pm$ by complex conjugation, and trivial on $G_2/SO(4)$. Since ρ_1 and ρ_2 are given by $W \mapsto (W \oplus \bar{W})^\perp$, then both maps are $\mathbb{Z}_2$-equivariant and we can ask for the Fadell-Husseini index of such bundles. Before we prove our first main result, we fix the notation for the cohomology of the group $\mathbb{Z}_2$ as

$$H^*(B\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[t],$$

with $|t| = 1$.

Theorem 4.4. *The Borel cohomology of $G_2/U(2)_\pm$ is given by*

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm;\mathbb{F}_2) = H^*(B\mathbb{Z}_2 \times G_2/SO(4);\mathbb{F}_2)\Big/\langle t^3 + u_2 t + u_3\rangle.$$

Consequently, the Fadell-Husseini index of ρ_1 and ρ_2 with respect to the introduced $\mathbb{Z}_2$ action is given by

$$\mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_1;\mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_2;\mathbb{F}_2) = \langle t^3 + u_2 t + u_3\rangle.$$

Proof. Consider the Borel construction of the bundle ρ_i, for $i = 1, 2$,

$$\begin{array}{ccc} S^2 & \hookrightarrow & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm \\ & & \downarrow \\ & & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/SO(4), \end{array}$$

for which, since $\mathbb{Z}_2$ acts trivially on $G_2/SO(4)$, the base space can be written as $B\mathbb{Z}_2 \times G_2/SO(4)$. Since $\pi_1(B\mathbb{Z}_2 \times G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2) \times \pi_1(G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2)$ acts trivially on the cohomology of S^2 with $\mathbb{F}_2$ coefficients, the E_2-terms of the associated Serre spectral sequence becomes

$$E_2^{p,q} = H^p(B\mathbb{Z}_2 \times G_2/SO(4); H^q(S^2;\mathbb{F}_2)).$$

Using theorem 4.1 and lemma 4.3 we get that

$$d_3^{0,2}(z) = t^3 + u_2 t + u_3,$$

where $H^2(S^2;\mathbb{F}_2) = \langle z\rangle$. By the Leibniz rule $d_3^{0,2}$ defines the complete cohomology of $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm$, and at the same time the Fadell-Husseini index of ρ_1 and ρ_2. □

4.4. **The Fadell-Husseini index of the pullback bundle ζ_n.** Let $\gamma = (E(\gamma_3^7), \pi, \widetilde{G}_3(\mathbb{R}^7), \mathbb{R}^3)$ be the tautological bundle of the oriented Grassmann manifold $\widetilde{G}_3(\mathbb{R}^7)$, and let $\gamma^\perp$ be the orthogonal complement of γ. Consider the sphere bundle associated to $\gamma^\perp$,

$$s\gamma^\perp = \big(E(S\gamma^\perp), s\pi, \widetilde{G}_3(\mathbb{R}^7), S^3\big).$$

Using the embedding $i : G_2/SO(4) \hookrightarrow \widetilde{G}_3(\mathbb{R}^7)$, and the fiber bundle ρ_2 defined in 3.4, we construct the pullback square

$$\begin{array}{ccccc} \mathcal{S}^1_{\gamma^\perp} & \longrightarrow & E\big(i^*(s\gamma^\perp)\big) & \longrightarrow & E(S\gamma^\perp) \\ {\scriptstyle \phi_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle s\pi} \\ G_2/U(2)_- & \xrightarrow{\rho_2} & G_2/SO(4) & \xrightarrow{i} & \widetilde{G}_3(\mathbb{R}^7), \end{array}$$

where the map on the left is the pullback bundle induced by the map $i \circ \rho_2$,

$$\zeta_1 = (\mathcal{S}^1_{\gamma^\perp}, \phi_1, G_2/U(2)_-, S^3),$$

and

$$\begin{aligned} \mathcal{S}^1_{\gamma^\perp} &:= E\big(\rho_2^*(i^*(s\gamma^\perp))\big) \\ &= \{(W;\xi,v) \mid W \in G_2/U(2)_-,\ \rho_2(W) = \xi,\ v \in \xi^\perp \text{ and } |v_1| = 1\}. \end{aligned}$$

We can generalize the construction of the bundle ζ_1 as follows: Consider the the k-fold product bundle $(s\gamma^{\perp})^n$, and the pullback along the diagonal map $\Delta_n : \widetilde{G}_3(\mathbb{R}^7) \to \widetilde{G}_3(\mathbb{R}^7)^n$:

$$\begin{array}{ccc} E\big(\Delta_n^*(s{\gamma^{\perp}}^n)\big) & \longrightarrow & E\big(s\gamma^{\perp}\big)^n \\ \big\downarrow & & \big\downarrow{\scriptstyle (s\pi)^n} \\ \widetilde{G}_3(\mathbb{R}^7) & \xrightarrow{\ \Delta_n\ } & \widetilde{G}_3(\mathbb{R}^7)^n. \end{array}$$

Applying the last construction to $\Delta_n^*(s{\gamma^{\perp}}^n)$ we get a new bundle

$$\zeta_n = (\mathcal{S}^n_{\gamma^\perp}, \phi_n, G_2/U(2)_-, (S^3)^n),$$

where

$$\begin{aligned} \mathcal{S}^n_{\gamma^\perp} &:= E(\zeta_n) \\ &= \{(W;\xi, v_1, \dots, v_n) \mid W \in G_2/U(2)_-,\ \rho_2(W) = \xi, \\ &\qquad v_1, \dots, v_n \in \xi^\perp \text{ and } |v_1| = \cdots = |v_n| = 1\}. \end{aligned}$$

To describe the cohomology of $\mathcal{S}^n_{\gamma^\perp}$ the first step is to calculate the Stiefel-Whitney classes of ζ_1. Notice that since $H^1(G_2/U(2)_-;\mathbb{F}_2) = H^3(G_2/U(2)_-;\mathbb{F}_2) = 0$, we are just looking for $w_2(\zeta_1)$ and $w_4(\zeta_1)$. We have then the following result.

Lemma 4.5. *The Stiefel-Whitney classes of the pullback bundle ζ_1 are non-zero only in the dimensions $0, 2$ and 4, and are given by $w_2(\zeta_1) = y$ and $w_4(\zeta_1) = y^2$.*

Proof. By the naturality of the Stiefel-Whitney classes is enough to describe the image of $w_2(\gamma^\perp)$ and $w_4(\gamma^\perp)$ along the composition

$$G_2/U(2)_- \xrightarrow{\rho_2} G_2/SO(4) \xrightarrow{i} \widetilde{G}_3(\mathbb{R}^7).$$

Akbulut and Kalafat in [1, Theorem 2.11] proved that the embedding $i : G_2/SO_4 \to \widetilde{G}_3(\mathbb{R}^7)$ maps the euler class of $\gamma^\perp$, which reduces to $w_4(\gamma^\perp)$, to the integral generator in dimension 4, which corresponds to u_2^2. Also, since $w_4(\gamma_+^\perp) = w_2(\gamma_+^\perp)^2$, i^* maps $w_2(\gamma_+^\perp)$ to u_2.

On the other hand, by lemma 3.11, $\rho_2^*(u_2) = y$ and $\rho_2^*(u_2^2) = y^2$. This means that

$$w_2(\zeta_1) = (i \circ \rho_2)^*(w_2(\gamma_+^\perp)) = y$$

and

$$w_4(\zeta_1) = (i \circ \rho_2)^*(w_4(\gamma_+^\perp)) = y^2.$$

□

This lead us to our second main result. For the cohomology of $(S^3)^n$ we fix the following notation

$$H^*((S^3)^n;\mathbb{F}_2) \cong \mathbb{F}_2[z_1, \dots, z_n]/\langle z_1^2, \dots, z_n^2\rangle.$$

Theorem 4.6. *The cohomology $H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2)$ is described as follows*

$$H^*(\mathcal{S}^n_{\gamma^\perp};\mathbb{F}_2) = H^*\big(G_2/U(2)_- \times (S^3)^n;\mathbb{F}_2\big)\Big/\big(I \cup \{y^2, xy^2\}\big)$$

where $\langle y^2, xy^2\rangle$

$$I = \langle 1, y^2, x, xy^2\rangle \otimes \left\langle \sum_{j=1}^{m} \bigotimes_{\substack{i\in[m]\setminus\{j\} \\ a_i \in [n]}} z_{a_i} \;\middle|\; a_i \neq a_j\ \forall\ i \neq j \ \text{ and } \ 2 \leq m \leq n \right\rangle,$$

with $[n] = \{1, 2, \dots, n\}$ and $[m] = \{1, 2, \dots, m\}$.

Proof. Consider the sphere bundle

$$S^3 \hookrightarrow \mathcal{S}^1_{\gamma^\perp} \xrightarrow{\phi_1} G_2/U(2)_-,$$

and the corresponding Serre spectral sequence with E_2-term given by

$$E_2^{p,q} = H^p(G_2/U(2)_-; H^q(S^3;\mathbb{F}_2)).$$

Notice that the first non-trivial differential appears on page E_4. Then, as a consecuence of the Gysin sequence applied to ϕ_1 [10, Example 5.C], the trasgression map on E_4 is given by $d_4^{0,3}(z_1) = w_4(\zeta_1) = y^2$.

Let $\mathcal{S}^n_{\gamma^\perp} \to \mathcal{S}^1_{\gamma^\perp}$ be the projection map given by $(W;\xi,v_1,\dots,v_n) \mapsto (W;\xi,v_i)$, for $i = 1,\dots,n$, which induces a morphism of bundles

$$\begin{array}{ccc} \mathcal{S}^n_{\gamma^\perp} & \longrightarrow & \mathcal{S}^1_{\gamma^\perp} \\ \phi_n \big\downarrow & & \big\downarrow \phi_1 \\ G_2/U(2)_- & \xrightarrow{\ \mathrm{id}\ } & G_2/U(2)_-. \end{array}$$

This morphism of bundles induces a morphism between the corresponding Serre spectral sequences, which on the zero column is an monomorphism. Then, by the commutativity of the differentials,

$$d_4^{0,3}(z_1) = \cdots = d_4^{0,3}(z_n) = y^2,$$

and by the the Leibniz rule we can prove that all the differentials are trivial except for

- $d_4^{6,3}(x \otimes z_i) = xy^2$ for every $i \in [n]$.
- $d_4^{0,3m}(\bigotimes_{i=1}^m z_i) = y^2 \otimes \sum_{j=1}^m \bigotimes_{i\in[m]\setminus\{j\}} z_i$ for $2 \le m \le n$.
- $d_4^{6,3m}(x \bigotimes_{i=1}^m z_i) = xy^2 \otimes \sum_{j=1}^m \bigotimes_{i\in[m]\setminus\{j\}} z_i$ for $2 \le m \le n$.

For an ilustration of the page E_4 see Figure 4. Since there are no more differencials, the term E_5 describe the cohomology of $S^n_{\gamma^\perp}$.

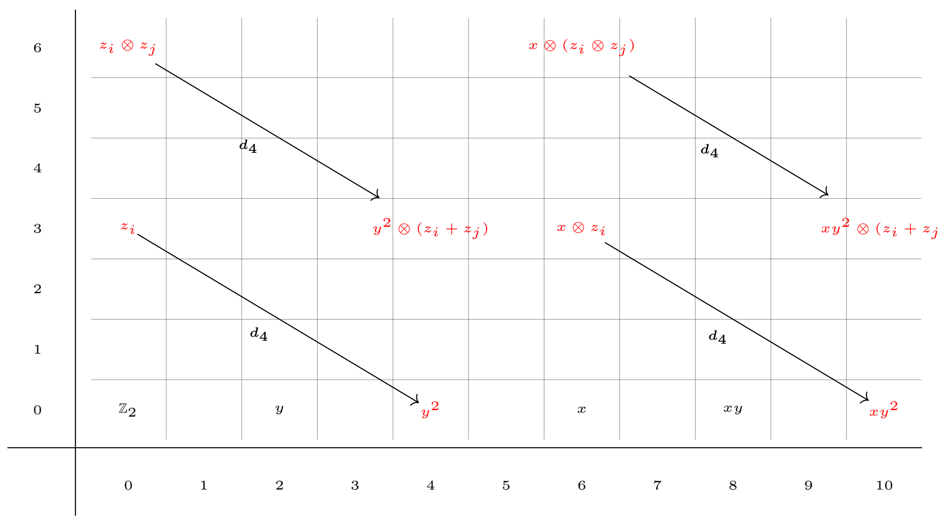


FIGURE 4. $E_4^{p,q} = E_2^{p,q} = H^p\big(G_2/U(2)_-; H^*((S^3)^n;\mathbb{F}_2)\big)$

□

A description of the cohomology of $\mathcal{S}^n_{\gamma^\perp}$ more pleasing to the eye happens in the case $n = 2$.

Corollary 4.7. *The cohomology $H^*(\mathcal{S}^2_{\gamma^\perp};\mathbb{F}_2)$ is described as follows:*

n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
$\mathbb{Z}_2$	0	$\langle y\rangle$	$\langle \xi_1+\xi_2\rangle$	0	$\langle y\xi_1, y\xi_2\rangle$	$\langle x\rangle$	$\langle y^2\xi_1\rangle$	$\langle xy\rangle$	$\langle x\xi_1 + x\xi_2\rangle$

n=10	n=11	n=12	n=13	n=14	n=15	n=16
$\langle y^2(\xi_1\otimes\xi_2)\rangle$	$\langle xy\xi_1, xy\xi_2\rangle$	0	$\langle xy^2\xi_1\rangle$	$\langle xy(\xi_1\otimes\xi_2)\rangle$	0	$\langle xy^2(\xi_1\otimes\xi_2)\rangle$

There is an action of $\mathbb{Z}_2^{n+1}$ on the space $\mathcal{S}^n_{\gamma^\perp}$ defined as follows: The first copy of $\mathbb{Z}_2$ acts on $G_2/U(2)_-$ by complex conjugation, and the next n copies of $\mathbb{Z}_2$ acts antipodally on the points $v_1, \ldots, v_n$. To be more precise, let $(\beta_0, \beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^{n+1}$ and $(W; \xi, v_1, \ldots, v_n) \in \mathcal{S}^n_{\gamma^\perp}$, then

$$(\beta_0, \beta_1, \ldots, \beta_n) \cdot (W; \xi, v_1, \ldots, v_n) = (c^{\beta_0}(W); \xi, (-1)^{\beta_1} v_1, \ldots, (-1)^{\beta_n} v_n),$$

where $c^0 = id$ and c^1 is the complex conjugation. If we consider the action of $\mathbb{Z}_2^{n+1}$ on $G_2/U(2)_-$ by complex conjugation with the first copy of $\mathbb{Z}_2$, and trivial with the rest, then the projection ϕ_n is $\mathbb{Z}_2^{n+1}$-equivariant and we can ask for its Fadell-Husseini index. For the cohomology of the group $\mathbb{Z}_2^{n+1}$ we fix the notation

$$H^*(B\mathbb{Z}_2^{n+1}; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_{n+1}],$$

where $|t_1| = \ldots = |t_{n+1}| = 1$. We will start with the case $n = 1$.

Proposition 4.8. *Consider the action of $\mathbb{Z}_2^2$ on $\mathcal{S}^1_{\gamma^\perp}$ and $G_2/U(2)_-$ introduced before. Then the Borel cohomology of $S^1_{\gamma^\perp}$ is described as*

$$H^*(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} S^1_{\gamma^\perp}; \mathbb{F}_2) = H^*(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-; \mathbb{F}_2) \Big/ \langle y^2 + yt_2^2 + t_2^4 \rangle.$$

Consequently, the Fadell-Husseini index of $\phi_1 \colon \mathcal{S}^1_{\gamma^\perp} \longrightarrow G_2/U(2)_-$ is given by

$$\mathrm{Index}_{\mathbb{Z}_2^2}^{G_2/U(2)_-}(\phi_1; \mathbb{F}_2) = \langle y^2 + yt_2^2 + t_2^4 \rangle.$$

Proof. The inclusion into the second factor $i_2 \colon \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2^2$ induces a morphism of Borel constructions

$$\begin{array}{ccc} E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathcal{S}^1_{\gamma^\perp} & \longrightarrow & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} \mathcal{S}^1_{\gamma^\perp} \\ \downarrow{\scriptstyle \phi_1^{\mathbb{Z}_2}} & & \downarrow{\scriptstyle \phi_1^{\mathbb{Z}_2^2}} \\ E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_- & \longrightarrow & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_- \end{array}$$

that induces a morphism of the corresponding Serre spectral sequences, which on the zero column of the E_2-term is an isomorphism. Notice that, since the base space of $\phi_1^{\mathbb{Z}_2}$ can be seen as $B\mathbb{Z}_2 \times G_2/U(2)_-$, by theorem 4.1 and lemma 4.5, $d_4^{0,3}(z) = y^2 + yt_2^2 + t_2^4$. For an illustration of the E_4-term of the Serre spectral sequence associated to $\phi_1^{\mathbb{Z}_2}$ see Figure 5.

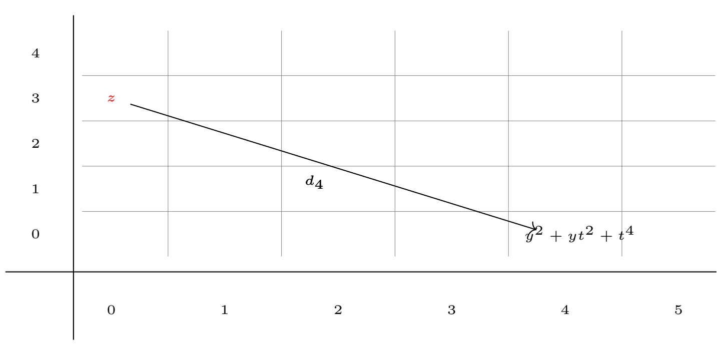


FIGURE 5. $E_4^{p,q} = E_2^{p,q} = H^p(B\mathbb{Z}_2 \times G_2/U(2)_-; H^q(S^3; \mathbb{F}_2))$

Having said that, consider the bundle

$$S^3 \hookrightarrow E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} \mathcal{S}^1_{\gamma^\perp} \xrightarrow{\phi_1^{\mathbb{Z}_2^2}} E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-$$

and the associated Serre spectral sequence with E_2-term

$$E_2^{p,q} = H^p(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-; H^q(S^3; \mathbb{F}_2)).$$

By theorem 4.4, since the second copy of $\mathbb{Z}_2^2$ acts trivially on $G_2/U(2)_-$, and $\mathbb{Z}_2^2$ acts trivially on $G_2/SO(4)$,

$$H^*(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-;\mathbb{F}_2) = H^*(B\mathbb{Z}_2^2 \times G_2/SO(4);\mathbb{F}_2)\Big/\langle t_1^3 + u_2t_1 + u_3\rangle.$$

Now, the first differential appears in the page E_4,

$$d_4^{0,3}\colon \langle z\rangle \longrightarrow H^4(E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-;\mathbb{F}_2) \cong \Big(\langle y^2\rangle \oplus H^4(B\mathbb{Z}_2^2;\mathbb{F}_2) \oplus \big(\langle y\rangle \otimes H^2(B\mathbb{Z}_2^2;\mathbb{F}_2)\big)\Big),$$

and by the commutativity of the differentials is given by $d_4(\xi) = y^2 + yt_2^2 + t_2^4$. Using the Leibniz rule, this describes the complete cohomology of $E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} S^1_{\gamma^\perp}$ and the Fadell-Husseini index of $\phi_1\colon \mathcal{S}^1_{\gamma^\perp} \longrightarrow G_2/U(2)_-$. □

Using the projection of $\mathcal{S}^n_{\gamma^\perp}$ on $\mathcal{S}^1_{\gamma^\perp}$, and lemma 4.8, we can prove out third main result.

Theorem 4.9. *Consider the action of $\mathbb{Z}_2^{n+1}$ on $\mathcal{S}^n_{\gamma^\perp}$ and $G_2/U(2)_-$ introduced before. Then the Fadell-Husseini index of $\phi_n\colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_-$ is given by*

$$\mathrm{Index}_{\mathbb{Z}_2^{n+1}}^{G_2/U(2)_-}(\phi_n;\mathbb{F}_2) = \langle y^2 + yt_2^2 + t_2^4, \ldots, y^2 + yt_{n+1}^2 + t_{n+1}^4\rangle.$$

Proof. Consider the inclusion $i_k\colon \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^{n+1}$ into the first and kth summand, with $2 \le k \le n+1$. The map i_k induces a morphism of Borel constructions

$$\begin{array}{ccc}
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^1_{\gamma^\perp} & \longleftarrow & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} \mathcal{S}^1_{\gamma^\perp} \\
\Big\downarrow \phi_1^{\mathbb{Z}_2^{n+1}} & & \Big\downarrow \phi_1^{\mathbb{Z}_2^2} \\
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_- & \longleftarrow & E\mathbb{Z}_2^2 \times_{\mathbb{Z}_2^2} G_2/U(2)_-,
\end{array}$$

where $\mathbb{Z}_2^{n+1}$ acts on $\mathcal{S}^1_{\gamma^\perp}$ by complex conjugation on $G_2/U(2)_-$ with the first summand, antipodally on the unitary element with the kth summand. The morphism of Borel constructions induces a morphism of the corresponding Serre spectral sequences, which on the zero column of the E_2-term is an isomorphism. Then, by proposition 4.8, the transgression $d_4^{0,3}$ of the spectral sequence associated to $\phi_1^{\mathbb{Z}_2^{n+1}}$ is given by

$$d_4^{0,3}(z) = y^2 + yt_k^2 + t_k^4,$$

for every $2 \le k \le n+1$.

Consider now the projection $p_k\colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow \mathcal{S}^1_{\gamma^\perp}$ given by $p_k(W;\xi,v_1,\ldots v_n) = (W;\xi,v_k)$. The map p_k, with respect to the introduced $\mathbb{Z}_2^{n+1}$ action on $\mathcal{S}^1_{\gamma^\perp}$, induces a morphism of Borel constructions

$$\begin{array}{ccc}
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^n_{\gamma^\perp} & \xrightarrow{E(id)\times_{\mathbb{Z}_2^{n+1}} p_k} & E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} \mathcal{S}^1_{\gamma^\perp} \\
\Big\downarrow \phi_n^{\mathbb{Z}_2^{n+1}} & & \Big\downarrow \phi_1^{\mathbb{Z}_2^{n+1}} \\
E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_- & \xrightarrow{id} & E\mathbb{Z}_2^{n+1} \times_{\mathbb{Z}_2^{n+1}} G_2/U(2)_-
\end{array}$$

that also induces a morphism of the corresponding Serre spectral sequences, which on the zero column of the E_2-term is a monomorphism. Then, by the commutativity of the differentials, the transgression of the spectral sequences associated to $\phi_n^{\mathbb{Z}_2^{n+1}}$ is given by

$$d_4^{0,3}(z_k) = d_4^{0,3}(p_k^*(z)) = \mathrm{id}(d_4^{0,3}(z)) = y^2 + yt_k^2 + t_k^4,$$

for every generator z_k in $H^3((S^3)^n;\mathbb{F}_2)$. Using the Leibnitz rule, this describes the Fadell-Husseini index of $\phi_n\colon \mathcal{S}^n_{\gamma^\perp} \longrightarrow G_2/U(2)_-$. □

Finally, following the same idea in 4.9, we get the Borel cohomology of the total space of the bundle ζ_2.

Corollary 4.10. *The cohomology of* $E\mathbb{Z}_2^3 \times_{\mathbb{Z}_2^3} \mathcal{S}^2_{\gamma^\perp}$*, where the first copy of* $\mathbb{Z}_2$ *acts on* $G_2/U(2)_-$ *by complex conjugation, and the other copies acts antipodally on the unitary element, is given by*

$$H^*_{\mathbb{Z}_2^3}(S^2_{\gamma^\perp};\mathbb{F}_2) = H^*_{\mathbb{Z}_2^3}(G_2/U(2)_-;\mathbb{F}_2)\Big/\langle y^2+yt_2^2+t_2^4, y^2+yt_3^2+t_3^4\rangle$$

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