MOUNTAIN PASS THEOREM WITH INFINITE DISCRETE SYMMETRY

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Abstract. We extend an equivariant Mountain Pass Theorem, due to Bartsch, Clapp and Puppe for compact Lie groups to the setting of infinite discrete groups satisfying a maximality condition on their finite subgroups.

Symmetries play a fundamental role in the analysis of critical points and sets of functionals [2], [20], [12]. The development of Equivariant Algebraic Topology, particularly Equivariant Homotopy Theory, has given a number of tools to conclude the existence of critical points in problems which are invariant under the action of a compact Lie group, as investigated in [11].

In this work we discuss extensions of methods of Equivariant Algebraic Topology to the setting of actions of infinite groups. The main result of this note is the modification of a result by Bartsch, Clapp and Puppe originally proved for actions of compact Lie groups, to infinite discrete groups with appropriate families of finite subgroups inside them.

Theorem 1.1 (Mountain Pass Theorem)**.** *Let G be an infinite discrete group acting by bounded linear operators on a real Banach space E of infinite dimension. Suppose that G satisfies the maximality condition 1.2 and that the linear action is proper outside* 0. *Let $\phi : E \to \mathbb{R}$ be a G-invariant functional of class $\mathcal{C}^{2-}$. For any value $a \in \mathbb{R}$, define the sublevel set $\phi^a = \{x \in E \mid \phi(x) \leq a\}$ and the critical set $K = \cup_{c\in\mathbb{R}} K_c$, where K_c is the critical set at level c, $K_c = \{u \mid \|\phi^{'}(u)\| = 0 \ \phi(u) = c\}$. Suppose that*

- *$\phi(0) \leq a$ and there exists a linear subspace $\hat{E} \subset E$ of finite codimension such that $\hat{E} \cap \phi^a$ is the disjoint union of two closed subspaces, one of which is bounded and contains* 0.
- *The functional ϕ satisfies the Orbitwise Palais-Smale condition 1.3.*
- *The group G satisfies the maximal finite subgroups condition 1.2.*

Then, the equivariant Lusternik-Schnirelmann category of E relative to ϕ^a, $G-cat(E, \phi^a)$ is infinite. If moreover, the critical sets K_c are cocompact under the group action, meaning that the quotient spaces $G\diagdown K_c$ are compact, then $\phi(K)$ is unbounded above.

Recall that given a natural number r, the class $\mathcal{C}^{r-}$ denotes the class of functions whose derivatives up to order r exist and are locally Lipschitz.

Condition 1.2 restricts maximal finite subgroups and their conjugacy relations.

Condition 1.2. Let G be a discrete group and $\mathcal{MAX}$ be a subset of finite subgroups. G satisfies the maximality condition if

- There exists a prime number p such that every nontrivial finite subgroup is contained in a unique maximal p-group $M \in \mathcal{MAX}$.
- $M \in \mathcal{MAX} \implies N_G(M) = M$, where $N_G(M)$ denotes the normalizer of M in G.

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Notice that in particular, the finite subgroups of G are all finite p- groups.

These conditions are satisfied in several cases. Among them:

(i) Extensions $1 \to \mathbb{Z}^n \to G \to K \to 1$ by a finite p-group given by a representation $K \to Gl_n(\mathbb{Z})$ acting freely outside from the origin [30], Lemma 6.3.

(ii) Fuchsian groups, more generally NEC (non-euclidean crystallographic groups) for which the isotropy consists only of p-groups. [30].

(iii) One relator groups $G = \langle q_i \mid r \rangle$ for which the family of finite subgroups consists of p-groups. See [31], Propositions 5.17, 5.18, 5.19. in pages 107 and 108.

The Orbitwise Palais-Smale condition was formulated by Ayala-Lasheras-Quintero in [6] for complete riemannian manifolds with a proper action of a Lie Group. For our purposes, the following notion is more adequate.

Condition 1.3. Let G be a discrete group. Let M be a $\mathcal{C}^{2-}$ Hilbert manifold with a G-action by $\mathcal{C}^{1-}$ diffeomorphisms which is proper. Assume that M has a G-invariant $\mathcal{C}^{1-}$ Riemannian Metric. The G-invariant functional Φ of class $\mathcal{C}^{2-}$ satisfies the orbitwise Palais-Smale condition if given a sequence $\{x_n\} \subset M$ such that $\mid f(x_n) \mid$ is bounded and $\nabla\Phi(x_n)$ converges to 0, then the sequence of orbits Gx_n contains a convergent subsequence in the orbit space M/G.

This paper is organized as follows: in the second section , the usual facts concerning the relation between critical points, Lusternik-Schnirelmann category and equivariant deformation theorems are stated, being modified slightly from [6] and [15].

In the third section, we introduce the notion of Universal Proper Length related to a family of subgroups.

In the third section, we use some algebraic properties of the classifying space for proper actions of groups with an appropriate family of maximal finite subgroups in order to conclude the unboundedness of critical values.

This is done in the fourth section adapting a construction of elements in the Burnside Ring of a finite group, originally due to Bartsch, Clapp and Puppe [12] to the infinite group setting, using the Atiyah-Hirzebruch spectral sequence, as well as a version of the Segal Conjecture for families of finite groups inside discrete groups [23], [7].

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2. Proper Lusternik-Schnirelmann Category and Critical Points

The notion of a proper G-space provides an adequate setting for the study of non-compact transformation groups.

Definition 2.1. Let G be a second countable, Hausdorff locally compact group. Let X be a second countable, locally Hausdorff space. Recall that a G-action is proper if the map

$$\underset{(g,x)\mapsto(x,gx)}{G \times X \xrightarrow[\theta_X]{} X \times X}$$

is proper.

Ayala-Lasheras-Quintero [6] introduced the notion of equivariant Lusternik-Schnirelman category for proper actions of Lie Groups, extending previous work by Marzantowicz [32] for compact Lie groups.

Definition 2.2. Let $X^{'} \subset X$ be paracompact proper G-spaces. The relative G-category of $(X, X^{'})$, denoted by G-$cat(X, X^{'})$ is the smallest number k such that X can be covered by $k+1$ open G-subsets $X_0, X_1, \dots, X_k$ with the following properties:

- $X^{'} \subset X_0$ and there is a homotopy $H : (X_0, X^{'}) \times I \to (X_0, X^{'})$ starting with the inclusion and $H(x, 1) \in X^{'}$.
- For every $i \in \{1, \dots, k\}$ there exist G-maps $\alpha_i : X_i \to A_i$ and $\beta_i : A_i \to Y$ with A_i a G-orbit G/H_i such that the restriction of f to X_i is the is G-homotopic to the composition $\beta_i \circ \alpha_i$

If no such a number exists, then we write G-$cat(X, X^{'}) = \infty$.

The Lusternik-Schnirelman Method can be extended to functionals which are invariant under proper actions.

Lemma 2.3 (Equivariant Deformation)**.** *Let G be a discrete group acting properly on a Hilbert manifold of class $\mathcal{C}^{2-}$. Let $\Phi : X \to \mathbb{R}$ be a G- invariant $\mathcal{C}^{2-}$-functional, $c \in K_c = \{x \in X \mid \Phi^{'}(x) = 0\, \Phi(x) = c\}$. For every $c > a$, every $0 < \delta < c - a$ and every G-neighborhood U of K_c, there is an $\epsilon > 0$ and a homotopy $\eta : \Phi^{c+\epsilon} \times I \to \Phi^{c-\epsilon}$ which is the identity on $\Phi^{c-\delta} \times I$.*

Proof. The Gradient field $-\nabla\Phi$ is locally Lipschitz by assumption. The usual deformation method [35] works G-equivariantly. See [6], lemma 5.4 in page 1130. □

Proposition 2.4. *Let M be a paracompact Hilbert, $\mathcal{C}^{2-}$- manifold. Assume that the discrete group G acts properly by $\mathcal{C}^{1-}$ maps on M. Let $\phi : M \to \mathbb{R}$ be a G-invariant C^{2-}-function satisfying the deformation property with respect to neighbourhoods of critical sets, as in lemma 2.3. Suppose that Φ satisfies the Orbitwise Palais-Smale condition 1.3.*

- *If the function is bounded below, then the number of critical points of ϕ with values $> a$ in M is at least G-$cat(M, \phi^a)$.*
- *If G-$cat(M, \phi^a)$ is greater than the number of critical values of ϕ above a, then there is at least one $c > a$ such that the critical set K_c has positive covering dimension. In particular ϕ has infinitely many critical orbits with values above a.*
- *If G-$cat(M, K) = \infty$, then ϕ has an unbounded sequence of critical values.*

Proof. The proofs given in [15], Theorem 2.3 and Corollary 2.4 , pages 606 and 607, and [16], Theorem 1.1 extend to the proper setting. The point is that the equivariant Lusternik-Schnirelmann Category for proper spaces satisfies subadditivity, deformation monotonicity, and continuity (Proposition 2.3 in [6] in the absolute case, and the obvious modification extends to the relative category). □

3. Universal Cohomology Length

We discuss now cohomology length in the context of equivariant cohomology theories. We use for this the notion of a classifying space for a family of subgroups.

Definition 3.1. Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1 \le n} X_n$, $X_{-1} = \emptyset$,$X_0 = A$ and for which every space X_n is inductively obtained from the previous one by attaching cells in pushout diagrams of the form

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H_i & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i D^n \times G/H_i & \longrightarrow & X_n \end{array}$$

We say that a proper G-CW complex is finite if it consists of a finite number of cells $G/H \times D^n$.

Definition 3.2. Let G be a discrete group. A metrizable proper G-Space X is an Absolute Neighbourhood retract if every G- map $Z \to X$ from a closed subspace Z of a metrizable G-space Y into X has an equivariant extension $U \to X$ to a G-invariant neighbourhood U of Z in Y.

It is proved in [4] , Theorem 1.1 that proper G-ANR are G-homotopy equivalent to G-CW complexes when G is a locally compact Hausdorff group.

We recall the notion of the classifying space for a family of subgroups.

Definition 3.3. Let $\mathcal{F}$ be a collection of subgroups in a discrete group Gwhich is closed under conjugation and intersection. A model for the classifying space for the family $\mathcal{F}$ is a G-CW complex X satisfying

- All isotropy groups of X lie in $\mathcal{F}$.
- For any G-CW complex Y with isotropy in $\mathcal{F}$, there exists up to G-homotopy a unique G-equivariant map $f : Y \to X$.

A model for the classifying space of the family $\mathcal{F}$ will be usually denoted by $E_{\mathcal{F}}(G)$.

Particularly relevant is the classifying space for proper actions, the classifying space for the family $\mathcal{FIN}$ of finite subgroups, denoted by $\underline{E}G$.

The classifying space for proper actions always exists, is unique up to G-homotopy and admits several models. The following list includes some examples. We remit to [27] for further discussion.

- If G is a compact group, then the singleton space is a model for $\underline{E}G$.
- Let G be a group acting properly and co-compactly on a CAT(0) space X, in the sense of [14]. Then X is a model for $\underline{E}G$.
- Let G be a Coxeter group. The Davis Complex is a model for $\underline{E}G$.
- Let G be a mapping class group of an orientable surface. The Teichmüller Space is a model for $\underline{E}G$.

The spaces apearing in applications in analysis are not always G-CW complexes. They satisfy more often numerability conditions.

Definition 3.4. Let $\mathcal{F}$ be family of closed subgroups closed under conjugation and intersection inside the locally compact second countable Hausdorff group G. A G-space X is said to be an $\mathcal{F}$-numerable space if there exists an open covering $\{U_i, \mid i \in I\}$ by G-subspaces such that there is for each $i \in I$ a G-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$ and there is a locally finite partition of unity $\{e_{i|i \in I}\}$ subordinate to $\{U_i\}$ by G-invariant functions. Notice that we do not require that the isotropy groups of X lie in $\mathcal{F}$.

The Slice Theorem 2.3.3, in page 313 of [34] implies that completely regular spaces carrying proper actions of Lie groups are precisely numerable spaces with respect to the family of compact subgroups for which, in addition, the isotropy groups of points are all compact subgroups.

Specializing to Lie groups acting properly on G-CW complexes, the conditions boil down to the fact that all stabilizers are compact, see [24], Theorem 1.23. In particular for a cellular action of a discrete group G on a G-CW complex, a proper action reduces to the finiteness of all stabilizer groups. Notice that any (continuous) action of a compact Lie group or a finite group on a locally compact, Hausdorff space is proper.

The following version of the classifying space for a family extends the notion to $\mathcal{F}$-numerable spaces.

Definition 3.5 (Numerable Version for the Classifying space of a family)**.** Let $\mathcal{F}$ be a family of subgroups. A model $J_{\mathcal{F}}(G)$ for the classifying numerable G-space for the family $\mathcal{F}$ is a G-space which has the following properties:

- $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable
- For any $\mathcal{F}$-numerable space X there is up to G-homotopy precisely one map $X \to J_{\mathcal{F}}(G)$.

Remark 3.6. There exists up to G-homotopy a unique G-equivariant map $\underline{E}G \to J_{\mathcal{F}}(G)$. This map is proved to be a G-homotopy equivalence for a discrete group in Theorem 3.7, part ii of [27].

Recall the notion of an Equivariant Cohomology Theory, [26].

Definition 3.7. Let G be a group and fix an associative ring with unit R. A G-Cohomology Theory with values in R-modules is a collection of contravariant functors $\mathcal{H}^n_G$ indexed by the integer numbers $\mathbb{Z}$ from the category of G-CW pairs together with natural transformations $\partial^n_G : \mathcal{H}^n_G(A) := \mathcal{H}^n_G(A,\emptyset) \to \mathcal{H}^{n+1}_G(X,A)$, such that the following axioms are satisfied:

(i) If f_0 and f_1 are G-homotopic maps $(X,A) \to (Y,B)$ of G-CW pairs, then $\mathcal{H}^n_G(f_0) = \mathcal{H}^n_G(f_1)$ for all n.

(ii) Given a pair (X,A) of G-CW complexes, there is a long exact sequence

$$\dots \overset{\mathcal{H}^{n-1}_G(i)}{\to} \mathcal{H}^{n-1}_G(A) \overset{\partial^{n-1}_G}{\to} \mathcal{H}^n_G(X,A) \overset{\mathcal{H}^n_G(j)}{\to} \mathcal{H}^n_G(X)$$
$$\overset{\mathcal{H}^n_G(i)}{\to} \mathcal{H}^n_G(A) \overset{\partial^n_G}{\to} \mathcal{H}^{n+1}_G(X,A) \overset{\mathcal{H}_{n+1}(j)}{\to} \dots$$

where $i : A \to X$ and $j : X \to (X,A)$ are the inclusions.

(iii) Let (X,A) be a G-CW pair and $f : A \to B$ be a cellular map. The canonical map $(F,f) : (X,A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}^n_G(X \cup_f B, B) \overset{\cong}{\to} \mathcal{H}^n_G(X,A)$$

(iv) Let $\{X_i \mid i \in \mathcal{I}\}$ be a family of G-CW-complexes and denote by $j_i : X_i \to \coprod_{i\in\mathcal{I}} X_i$ the inclusion map. Then the map

$$\Pi_{i\in\mathcal{I}}\mathcal{H}^n_G(j_i) : \mathcal{H}^n_G(\coprod_i X_i) \overset{\cong}{\to} \Pi_{i\in\mathcal{I}}\mathcal{H}^n_G(X_i)$$

is bijective for each $n \in \mathbb{Z}$.

A G-Cohomology Theory is said to have a multiplicative structure if there exist natural, graded commutative $\cup$- products

$$\mathcal{H}^n_G(X,A) \otimes \mathcal{H}^m_G(X,A) \to \mathcal{H}^{n+m}_G(X,A)$$

Let $\alpha : H \to G$ be a group homomorphism and X be a H-CW complex. The induced space $\mathrm{ind}_\alpha X$, is defined to be the G-CW complex defined as the quotient space $G \times X$ by the right H-action given by $(g,x)\cdot h = (g\alpha(h), h^{-1}x)$.

An Equivariant Cohomology Theory consists of a family of G-Cohomology Theories $\mathcal{H}^*_G$ together with an induction structure determined by graded ring homomorphisms

$$\mathcal{H}^n_G(\mathrm{ind}_\alpha(X,A)) \to \mathcal{H}^n_H(X,A)$$

which are isomorphisms for group homomorphisms $\alpha : H \to G$ whose kernel acts freely on X satisfying the following conditions:

(i) For any n, $\partial^n_H \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial^n_G$.

(ii) For any group homomorphism $\beta : G \to K$ such that $\ker \beta \circ \alpha$ acts freely on X, one has

$$\mathrm{ind}_{\alpha\circ\beta} = \mathcal{H}^n_K(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : \mathcal{H}^n_K(\mathrm{ind}_{\beta\circ\alpha}(X, A)) \to \mathcal{H}^n_H(X, A)$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ is the canonical G-homeomorphism.

(iii) For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

$$\mathrm{ind}_{c(g):G\to G} : \mathcal{H}^n_G(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X, A})) \to \mathcal{H}^{\mathrm{n}}_{\mathrm{G}}(\mathrm{X, A})$$

agrees with the map $\mathcal{H}^n_G(f_2)$, where $f_2 : (X, A) \to \mathrm{ind}_{c(g):G\to G}$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G.

Remark 3.8 (Extensions of G-Cohomology theories to more general spaces)**.** Let $\mathcal{H}^*_G$ be a G-cohomology theory defined on proper G-CW complexes. Using a functorial G-CW approximation for proper G-ANR as introduced in [4] for locally compact Hausdorff groups, an equivariant cohomology theory may be extended to the category of proper G-ANR.

More generally, the Čech expansion of [33] provides a Čech extension of a G-cohomology theory to arbitrary pairs of proper G-spaces. That is, a family of R-mod valued functors $\check{\mathcal{H}}^n_G$ defined on pairs of proper G-spaces and natural transformations $\delta^n_{X,A} : \mathcal{H}^n_G(A, \emptyset) \to \mathcal{H}^{n+1}_G(X, A)$ satisfying the axioms:

- G-homotopy invariance.
- Long exact sequences for G- pairs.
- Excision. Let $X_1, X_2 \subset X$ be proper G- invariant spaces such that
$$\overline{X_2 - X_1} \cap X_1 - X_2 = \emptyset = X_2 - X_1 \cap \overline{X_1 - X_2}$$
Then, the inclusion map $(X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1)$ induces a natural isomorphism.
- Axioms i-iii for the Induction structure.

For the purposes of this work we need an extension of a specific cohomology theory to a certain proper G-ANR which is contractible after forgetting the action and is exhausted by finite G-CW complexes. This is done by an ad-hoc construction, see definition 5.1.

Recall [17], [26], that for any Equivariant Cohomology Theory $\mathcal{H}^*$ on finite G-CW complexes there exists a spectral sequence with E^2-term given by Bredon Cohomology

$$E^{p,q}_2 = H^p_{\mathbb{Z}Or(G)}(X, \mathcal{H}^{-q}_G(G/H))$$

converging to $\mathcal{H}^*_G(X)$.

The following result will be used later:

Proposition 3.9. *Let X be an l-dimensional G-CW complex. Suppose that for $r = 2, 3, \dots$ the differential appearing in the Atiyah-Hirzebruch spectral sequence for X and $\mathcal{H}^*_G$ vanishes rationally. Then, for any element*

$$x \in H^0_{\mathbb{Z}Or(G)}(X, \mathcal{H}^0_G(G/?))$$

there exists some positive integer k such that x^k is contained in the image of $\mathcal{H}^0_G(X)$ under the edge homomorphism

$$\mathrm{Edge}_G : \mathcal{H}^0_G(X) \longrightarrow H^0_{\mathbb{Z}Or(G)}(X, \mathcal{H}^0_G(G/?))$$

Proof. Let $x \in H^0_{\mathbb{Z}Or(G)}(X, \mathcal{H}^0(G/?))$. The proof reduces to construct inductively positive integers $k_2, \dots k_{l-1}$ such that the product $x^{\prod_{i=2}^r k_i}$ survives to $E^{0,0}_{r+1}$ for $r = 1, \dots l-1$, in the sense that $k_r d^{0,0}_r(x^{\prod_{i=2}^{r-1} k_i}) = 0$ for $r = 2, \dots, l-1$. Since

$x \in E_2^{0,0}$, we pick k_2 such that $k_2 d_2(x) = d_2(x^{k_2}) = 0$ (this is possible by the rational vanishing of the differentials).

Assume inductively that there are $k_2, \ldots, k_{r-1}$ and $x^{\prod_{i=2}^{r-1} k_i}$ which survive to the $\in E_r^{0,0}$-term. Choose k_r such that $k_r d_r^{0,0}(x^{\prod_{i=2}^{r-1}}) = 0$. This is possible by the rational vanishing of differentials again).

Now, $d_r^{0,0}(x^{\prod_{i=2}^{r}}) = k_r d_r^{0,0}((x^{\prod_{i=2}^{r-1}}))(x^{\prod_{i=2}^{r-1}})^{k_r - 1}$. And since $x^{\prod_{i=2}^{r}} \in E_{r+1}^{0,0}$ for $k = \prod_{i=2}^{l-1} k_i$, the l-dimensionality of X implies $x^k \in E_\infty^{0,0}$ and hence it is on the image under the edge homomorphism. $\square$

Definition 3.10. (Universal Cohomology Length relative to a family of subgroups)

Let $\mathcal{A} = \{G/H_i\}$ be a collection of orbit spaces representing all homogeneous G-spaces with isotropy in some family $\mathcal{F}$ of subgroups of G. Let M be a module over the graded ring $\mathcal{H}_G^*(E_{\mathcal{F}}(G))$. The $\mathcal{H}_{\mathcal{A}}$-length of the module M is the smallest number k such that there exist spaces $A_1, \ldots, A_k \in \mathcal{A}$ such that for any $\gamma \in M$ and ω_i in the kernel of the map

$$\mathcal{H}_G^0(E_{\mathcal{F}}(G)) \to \mathcal{H}_G^0(G/H_i)$$

given by the up to G-equivariant homotopy unique map $G/H \to E_{\mathcal{F}}(G)$, one has

$$\gamma \omega_1 \ldots \omega_k = 0.$$

Given a map $f : X \to Y$, between $\mathcal{A}$-numerable spaces, the $\mathcal{H}_{\mathcal{A}}$-length of f is the $\mathcal{H}_{\mathcal{A}}$ length of the image, considered as $\mathcal{H}_G^*(E_{\mathcal{F}}(G))$- module .

4. Computations in Burnside Rings

We specialize now to equivariant stable cohomotopy for proper actions.

We give a quick summary of important facts involving Equivariant Stable Cohomotopy for finite groups.

Theorem 4.1. *Let G be a finite group. Then*

- *The 0-th equivariant cohomotopy group of a point, $\pi_G^0(\{\bullet\})$ is isomorphic to the* Burnside ring*, denoted by $A(G)$, the Grothendieck ring of isomorphism classes of finite G-sets .*
- *The Burnside ring $A(G)$ is provided with maps $\varphi_H : A(G) \to \mathbb{Z}$, each one for every conjugacy class of subgroups in G. These extend to an injective map $A(G) \to \prod_{H \text{in ccs(G)}} \mathbb{Z}$, where $ccs(G)$ denotes the set of conjugacy classes of subgroups in G.*
- *The prime ideals in $A(G)$ are given by the sets $\mathcal{P}_{K,p} = \{x \mid \varphi_H(x) \cong 0(\mathrm{p})\}$, $\mathcal{P}_{H,0} = \{x \mid \varphi_H(x) = 0\}$, where p is a prime number. The augmentation ideal I_G is defined as the ideal $\{x \mid \varphi_e(x) = 0\}$.*
- *There exists an element, the* Bartsch element *$0 \neq x \in A(G)$ with the property that $\varphi_H(x) = 0$ for every subgroup H.*
- *If p is a prime number and G is a finite p-group, then the completion map $A(G) \to A(G)\hat{_{I_G}}$ is injective and the I_G-adical topology and the p-adical topologies coincide.*

Proof.

- This is well known. See [37], [38].
- See [38], chapter II, section 8 , pages 155-160. The image is characterized by a set of congruences for the number of generators of cyclic subgroups of the *Weyl groups NH/H* for every conjugacy class of subgroups H in G [38], section 5 chapter IV, page 256. Alternatively, Theorem 1.3 in [21], page 41.
- This is proven in [21], page 43, [18].

- This is done in [12]. The element is constructed as follows: let K be a proper subgroup of G. Put $u_K = [G/K] - \mid G/K \mid^K [G/G]$. The element x is defined as the product of all such u_K, each one for every conjugacy class of subgroups in K.
- For a detailed proof see [21]. The first result, Corollary 1.11 in [21], follows from the fact that in this situation the kernel of the completion map, $\cap_n I_G^n$ coincides with $\cap \ker(\varphi_U)$, where U ranges among alll p-sylow groups. The second result follows from Frobenius reciprocity and an analysis of the congruences defining the Burnside ring as subring inside $\underset{H \text{in ccs(G)}}{\Pi} \mathbb{Z}$, proposition 1.12 in [21], page 44 .

$\square$

Equivariant Cohomotopy for proper actions of infinite discrete groups on finite G-CW complexes was defined in [25] via finite dimensional equivariant vector bundles for proper, finite G-CW complexes. Alternative approaches aer given by a construction using nonlinear Fredholm cocycles, which allow actions of noncompact Lie groups on finite G-CW complexes [9], as well as a spectra version [8]. These approaches are compared in [7]. For convenience, we give the definition from [25]:

Definition 4.2. A G-vector bundle over a G-CW-complex X consists of a real vector bundle $\xi : E \to X$ together with a G-action on E such that ξ is equivariant and each $g \in G$ acts on E and X via vector bundle isomorphisms.
Let S^ξ denote its fibrewise one-point compactification.

Definition 4.3. Let X be a proper G-CW-complex. Let $\mathsf{SPHB}^G(X)$ be the category with

- $Ob(\mathsf{SPHB}^G(X)) = \{$G-vector bundles over $X\}$; and
- a morphism from a vector bundle $\xi : E \to X$ to vector bundle $\mu : F \to X$ is given by a bundle map $u : S^\xi \to S^\mu$ which covers the identity $id : X \to X$ and fiberwise preserves the basepoint.
  (It is not required that u is a fiberwise homotopy equivalence.)

Let $\underline{\mathbb{R}^k}$ denote the trivial vector bundle $X \times \mathbb{R}^k \to X$.

Definition 4.4. Fix $n \in \mathbb{Z}$. Let ξ_0, ξ_1 be two G-vector bundles over X, and let k_0 and k_1 be two non-negative integers such that $k_i + n \geq 0$ for $i = 0, 1$. Then two morphisms

$$u_i : S^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} \to S^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$$

are called equivalent, if there are objects μ_i in $\mathsf{SPHB}^G(X)$ for $i = 0, 1$ and isomorphisms of G-vector bundles $v : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$ such that the following diagram in SPHB^G commutes up to homotopy

$$\begin{tikzcd}
S^{\mu_0 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \arrow[r, "id \wedge_X u_0"] \arrow[d] & S^{\mu_0 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0+n}}} \arrow[d] \\
S^{\mu_0 \oplus \xi_0 \oplus \underline{\mathbb{R}^{k_0+k_1}}} \arrow[d] & S^{\mu_0 \oplus \xi_0 \oplus \underline{\mathbb{R}^{k_0+k_1+n}}} \arrow[d] \\
S^{\mu_1 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \arrow[r, "id \wedge_X u_1"] & S^{\mu_1 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1+n}}}
\end{tikzcd}$$

Definition 4.5. For a proper G-CW-complex X define

$$\pi_G^n(X) = \{\text{equivalence classes of morphisms } u \text{ as above}\}$$

By introducing triviality conditions on a G-CW pair, (considering morphisms which are fibrewise constant with the value the point at infinity), equivariant cohomotopy groups are extended to an equivariant cohomology theory with multiplicative structure.

We introduce a Burnside ring for infinite groups, making out of Segal's remark, part 1 in Theorem 4.1, our definition for finite groups:

Definition 4.6. Let G be a group with a finite model for the classifying space for proper actions $\underline{E}(G)$. The Burnside ring for G is the 0-th equivariant cohomotopy ring of the classifying space for proper actions. In symbols

$$A(G) = \pi^0_G(\underline{E}(G))$$

Denote by $A^{\lim}(G) = \lim_{H\in\mathcal{FIN}} A(H)$ the inverse limit of the Burnside rings of the finite subgroups of G. Notice that this agrees with the $0,0$-entry of the E^2-term of the equivariant Atiyah-Hirzebruch spectral sequence. The following relations between the Burnside ring and the inverse-limit Burnside ring are easy consequences of the rational collapse of the Atiyah-Hirzebruch spectral sequence:

Lemma 4.7. *Let G be a discrete group admitting a finite model for the classifying space for proper actions $\underline{E}G$.*

- *(i) The edge Homomorphism $e : A(G) \to A^{\lim}(G)$ has nilpotent kernel and cokernel. Its kernel is the nilradical.*
- *(ii) The edge homomorphism gives an isomorphism between the set of prime ideals in $A(G)$ and $A^{\lim}(G)$ (in fact an homeomorphism in the Zariski topology), by assigning a prime ideal $I \subset A^{\lim}(H)$ its inverse image $e^{-1}(I) \in A(G)$.*
- *(iii) The rationalized Burnside ring $\pi^0_G(\underline{E}(G)) \otimes \mathbb{Q}$ does not contain nilpotent elements.*

In the rest of the section we will describe a completion theorem for families of p-groups inside finite subgroups of discrete groups, which is the main computational tool for the computation of equivariant cohomology lengths needed for the proof of Theorem 1.1. This amounts to a generalization of the Segal Conjecture for families [1]. The result was proved in [7], Theorem 13 in page 58, although similar results have been proved in [28], [29] and [23], from where the crucial ideas and methods come.

Let G be a discrete group and $\mathcal{F}$ be a family of finite subgroups of G, closed under conjugation and under subgroups. Fix a finite proper G-CW complex X and a finite dimensional proper G-CW complex Z whose isotropy subgroups lie in $\mathcal{F}$. Let $f : X \to Z$ be a G-map. Regard $\pi^0_G(X)$ as a module over $\pi^0_G(Z)$.

Definition 4.8. The augmentation ideal with respect to the family $\mathcal{F}$ is defined as the kernel of the homomorphism

$$I = I_{G,\mathcal{F},Z} = (\pi^0_G(Z) \xrightarrow{\mathrm{res}^H_G \circ i^*} \prod_{H\in\mathcal{F}} \pi^0_H(Z^0))$$

Proposition 4.9. *Let $\mathcal{F}$ be a family of finite p-subgroups. Assume that there is an upper bound for the order of subgroups in $\mathcal{F}$.*

Let $\mathcal{P} \subset \pi^0_H(\{\bullet\})$ be a prime ideal.

Then, the ideal

$$I_{H,\mathcal{F}\cap H,\{\bullet\}} := \ker \pi^0_H(\{\bullet\}) \to \prod_{K\in\mathcal{F}} \pi^0_{K\cap H}(\{\bullet\})$$

is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the structure map for H

$$\phi_H : \lim_{K\in\mathcal{F}} \pi^0_K(\{\bullet\}) \to \pi^0_H(\{\bullet\})$$

Proof. Let m be a positive integer number divided by all orders of subgroups in $\mathcal{F}$. For a given subgroup K in the family, Let $u = \{u_1, \dots, u_m\}$ be a finite set of cardinality m with a free K-action. For example, u may be chosen to be a disjoint union of $\frac{m}{|K|}$ copies of K. This gives an injective homomorphism into the symmetric group in m letters, $\rho : K \to S_m$. For a prime p, let Syl_p be the p-Sylow subgroup of S_m.

Let $S_m[\rho]$ be the set S_m with the free K- action given by $k, s \mapsto \rho(h)(s)$ and S_m/Syl_p be the set with the induced K-action. Notice that the fixed point set S_m/Syl_p^L is nonempty if and only if L is a p-subgroup. This construction is compatible with morphisms between subgroups in $\mathcal{F}$ in the sense that an homomorphism $K \to K^{'}$ between groups in the family induces a map taking the free $K^{'}$-set S_m to the free K-set S_m and the same for the homogeneous set S_m/Syl_p.

Consider the elements

$$\{(S_m- \mid S_m \mid K/K)\}_{K\in\mathcal{F}}$$

$$\{(S_m/Syl_p- \mid S_m/Syl_p \mid K/K\}_{K\in\mathcal{F}}$$

Let $\mathcal{P}$ be a a prime ideal containing the image of the structure map under ϕ_H. By the structure of the prime ideal spectrum, $\mathcal{P}$ is of the form $\mathcal{P}(M,p)$, where M is a subgroup of H and p is a prime number or zero. By assumption, $\mathcal{P}$ contains the image under the structure map of the elements above. Since $\varphi^M(S_m- \mid S_m \mid) =\mid S_m \mid$ and $\varphi^M(S_m/Syl_p - S_m/Syl_p) =\mid S_m/Syl_p \mid^M - \mid S_m/Syl_p \mid$ and both elements belong to $p\mathbb{Z}$, because S_m/Syl_p has order prime to p, we conclude that either $p=0$ or M is a p-group.

If M is a p-group, then $\mathcal{P}(M,p) = \mathcal{P}(\{e\},p) \supset \mathcal{P}(\{e\},0) \supset I_{\mathcal{F},H,\{\bullet\}}$. If p=0, then $\mid S^M \mid - \mid S_m \mid= 0$, and hence $M = \{e\}$. For any subgroup $K^{'}$ of every element $K \in \mathcal{F}\cap H$, $\mathcal{P}(K^{'},0) = \mathcal{P}(\{e\},0)$, since $K^{'}$ is a p-group, hence P contains the intersection of all such ideals, which is $I_{\mathcal{F},H,\{\bullet\}}$.

□

Proposition 4.10. *Let L be an n-dimensional G-CW complex with isotropy in the family $\mathcal{F}$ consisting of finite p-subgroups inside the discrete group G. Let $f : G/H \to L$ be a G-map and $\mathcal{P} \subset \pi^0_H(\{\bullet\})$ be a prime ideal. Then $I_{\mathcal{F}\cap H,\{\bullet\}} := \ker \pi^0_H(\{\bullet\}) \to \prod_{K\in F} \pi^0_{K\cap H}(\{\bullet\})$ is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of $I_{\mathcal{F},Z}$ under* $\mathrm{ind}_{H\to G} \circ f^* : \pi^0_G(L) \to \pi^0_H(\{\bullet\})$

Proof. Let $\mathcal{P}$ be a prime ideal containing $I_{\mathcal{F},H,\{\bullet\}}$. By the previous proposition, we can assume that $\mathcal{P}$ contains the image of the structural map ϕ_H.

Let $\psi : H^0_{\mathbb{Z}Or(G)}(E_\mathcal{F}(G), \pi_G(G/?)) \to \lim_K \pi^0_K(\{\bullet\})$ be the isomorphism given by assigning to an element $x \in H^0_{\mathbb{Z}Or(G)}(E_\mathcal{F}(G); \pi^0_K(\{\bullet\}))$ the element whose component under the structural map ϕ_K is the image image under the map induced by the (G-homotopically) unique map $u_K : G/K \to E_\mathcal{F}(G)$, followed by the induction isomorphism

$$\begin{aligned} &H^0_{\mathbb{Z}Or(G)}(E_\mathcal{F}(G); \pi^0_G(G/?)) \to \\ &\qquad H^0_{\mathbb{Z}Or(G)}(G/K, \pi^0_G(G/?)) \to H^0_{\mathbb{Z}Or(K)}(\{\bullet\}, \pi^0_K(K/?)) \cong \pi^0_K(\{\bullet\}) \end{aligned}$$

Given an element $a \in \lim_K I_{\mathcal{F}\cap K,\{\bullet\}}$, denote by x its image under ψ^{-1}. By proposition 3.9, there exist a positive integer k and an element $y \in \pi^0_G(E_\mathcal{F}(G))$ such that $edge(y) = x^k$, which is furthermore an element of $I_{\mathcal{F},G,L}$.

The structure map $\phi_H : \lim \pi^0_K(\{\bullet\}) \to \pi^0_H(\{\bullet\})$ maps a^k to $\mathcal{P}$. Because $\mathcal{P}$ is a prime ideal, the map $\mathrm{ind} \circ f^*$ maps a to $\mathcal{P}$. □

Theorem 4.11 (Segal Conjecture for families of finite p-subgroups)**.** *Let G be a discrete group and $\mathcal{F}$ be a family of subgroups of order p of G closed under conjugation and subgroups. Fix a finite proper G-CW complex X and a finite dimensional proper G-CW complex Z whose isotropy subgroups lie in $\mathcal{F}$ and have bounded order. Let $f : X \to Z$ be a G-map. Regard $\pi_G^0(X)$ as a module over $\pi_G^0(Z)$ and set*

$$I = I_{\mathcal{F},Z} = \ker(\pi_G^0(Z) \xrightarrow{\mathrm{res}_G^H \circ i^*} \prod_{H \in \mathcal{F}} \pi_H^0(Z^0))$$

then

$$\lambda_{X,\mathcal{F},f}^m : \left\{\pi_G^m(X) \diagup I^n \cdot \pi_G^m(X)\right\} \to \left\{\pi_G^m(E_{\mathcal{F}}(G) \times X^{n-1})\right\}$$

is an isomorphism of pro-groups. Also, the inverse system

$$\left\{\pi_G^m((E_{\mathcal{F}}(G) \times X)^n)\right\}_{n \geq 1}$$

satisfies the Mittag-leffler condition. In particular

$$\lim{}^1 \pi_G^m((E_{\mathcal{F}}(G) \times X)^n) = 0$$

and $\lambda_{X,\mathcal{F},f}$ induces an isomorphism

$$\pi_G^m(X)_I \xrightarrow{\cong} \pi_G^m(E_{\mathcal{F}}(G) \times X) \cong \lim_n \pi_G^m((E_{\mathcal{F}}(G) \times X)^n)$$

Proof. Since both functors have Mayer-Vietoris sequences, both of the systems satisfy the Mittag-Leffler condition and in view of the 5-lemma for pro-modules, [5], section 2, an inductive argument can be used to reduce the problem to the situation of $X = G/H$, and where H is a finite group.

In this case, there exists a commutative diagram

$$\begin{tikzcd}
\pi_G^0(Z) \arrow[r, "f^*"] & \pi_G^m(G/H) \arrow[d, "\mathrm{ind}_{H\to G}^{\cong}"'] \\
A(H) \arrow[r, "\cong"'] & \pi_H^0(\{*\})
\end{tikzcd}$$

Hence, the map of pro-modules

$$\lambda_{X,\mathcal{F},f}^m : \left\{\pi_G^m(X) \diagup I^n \cdot \pi_G^m(X)\right\} \to \left\{\pi_G^m(E_{\mathcal{F}}(G) \times X^{n-1})\right\}$$

factorizes as follows

$$\begin{tikzcd}
\left\{\pi_G^m(G/H) \diagup I^n \cdot \pi_G^m(G/H)\right\} \arrow[r] \arrow[d] & \left\{\pi_H^m(\{\bullet\}) \diagup J^n\right\} \arrow[d] \\
\left\{\pi_G^m(E_{\mathcal{F}}(G) \times G/H^{n-1})\right\} & \left\{\pi_H^0(\{\bullet\})/I^n_{\mathcal{F}\cap H,H,\{\bullet\}}\right\} \arrow[l, "\cong"]
\end{tikzcd}$$

Where J is the ideal generated by the image of I under $\mathrm{ind} \circ f^*$ and the lower horizontal map is an isomorphism due to the completion theorem for families inside finite groups of [1], the right vertical map is induced by f. Due to proposition 4.10, the prime ideals containing J and $I_{\mathcal{F}\cap H,h,\{\bullet\}}$ agree and the right vertical map is an isomorphism.

□

Corollary 4.12. *Let p be a prime number. For any group satisfying conditions 1.2 for which the maximal finite subgroups are finite p-groups, the groups $\pi_G^0(\underline{E}G) \otimes \mathbb{Z}_{\hat{p}}$ and $\pi_G^n(\underline{E}G)_{\hat{\mathbb{I}}_{G,\mathcal{MAX}}}$ are isomorphic.*

Proof. The morphism of pro-groups $\{\pi^m_G(X)/p^n\pi^n_G(X)\} \to \{\pi^m_G(X\times E\mathcal{MAX})^{n-1})\}$ is proved to be an isomorphism for $X = G/H$ with H a p-group. The prime ideals in $\pi^0_H(\{\bullet\})$ containing $I_{\mathcal{MAX}\cap H,H,\{\bullet\}}$ and the one generated by the image of $I_{\mathcal{MAX},G,G/H}$ under ind $\circ f^*$ agree by the previous argument. Because H is a p-group, these agree with the ones containing $I_{\mathcal{TR},G,G/H}$ for the trivial family. Due to part 5 of theorem 4.1, these agree with the ones containing p.

Since both functors have Mayer-Vietoris sequences, the result follows by induction on the dimension of X.

□

Proposition 4.13. *Let G be a discrete group satisfying conditions 1.2.*

There exists a " Generalized Bartsch element " $w \in \pi^0_G(\underline{E}G)$ for which the map $\pi^0_G(\underline{E}G) \longrightarrow H^0_{\mathbb{Z}(\mathrm{Or}(G)}(\underline{E}G, \pi^0_?(\{\bullet\}) = \lim_{K\in \mathrm{Sub(G)}} \pi^0_K(\{\bullet\}) \xrightarrow{\psi_M} \pi^0_M(\{\bullet\})$ given by the composition of the edge homomorphism and the structural map for the inverse limit maps w to a power of the element constructed in 4.1 for any maximal subgroup M.

Proof. Let $X_{M_i} \in \pi^0_{M_i}(\{\bullet\})$ be the Bartsch element constructed in Theorem 4.1, part 4. Put $x = \{x_{M_i}\} \in \lim_H \pi^0_H(\{\bullet\})$. Choose an element w and a power k such that w is mapped to x^k under the edge homomorphism. □

5. End of proof

Definition 5.1. Let $\hat{X}$ be a proper and paracompact G-ANR, which is contractible after forgetting the group action. Assume that there is a map $X \to \hat{X}$ from a proper G-CW complex of finite type $X = \cup X_n$ inducing a weak G-homotopy equivalence (a map restricting to weak homotopy equivalences $\hat{X}^H \to X^H$ for all subgroups H). Define

$$\hat{\pi}^*_G(\hat{X}) = \lim_n \pi^*_G(X_n) \otimes \mathbb{Q}_{\hat{p}}$$

Proposition 5.2. *Let G be a discrete group satisfying 1.2. Let X be a paracompact proper G-ANR , which is contractible after forgetting the group action. Assume that there is a map $X \to \hat{X}$ from a proper G-CW complex of finite type $X = \cup X_n$ inducing a weak G-homotopy equivalence.*

The maps $X_n \to \underline{E}G$ together with the G-homotopy equivalence $\underline{E}G \to J_{\mathcal{FIN}}(G)$ induce isomorphisms

$$\hat{\pi}^0_G(J_{\mathcal{FIN}}(G)) \to \hat{\pi}^0_G(\underline{E}G) \xrightarrow{\cong} \lim \pi^0_G(X_n)$$

Proof. The point is the existence of long exact sequences for the functor $\hat{\pi}^*_G(X,A)$, which is guaranteed by the natural equivalence with the Equivariant Cohomology Theory defined by $(X,A) \mapsto \pi^m_G((E_{\mathcal{MAX}}(G), \emptyset) \times (X,A))$ on finite G-CW pairs.

□

Proposition 5.3. *Let G be a group satisfying conditions 1.2. Let $\hat{X}$ be a proper G-ANR as in 5.1. Then, there exists an element $w \in \pi^0_G(\underline{E}G) \otimes \mathbb{Q}$ such that*

- *$w \in \ker \pi^0_G(\underline{E}G) \otimes \mathbb{Q} \to \pi^0_G(G/H) \otimes \mathbb{Q}$ for all finite H.*
- *$w \in \ker \pi^0_G(\underline{E}G) \otimes \mathbb{Q} \to \pi^0_G(X_0) \otimes \mathbb{Q}$.*
- *For every $k > 0$ there exists an $n > 0$ such that the image if w^k under $\hat{\pi}^0_G(\underline{E}G) \to \pi^0_G(X_n) \otimes \mathbb{Q}_{\hat{p}}$ is not zero.*

Proof. Let $v \in \pi^0_G(\underline{E}G) \otimes \mathbb{Q} \cong \Pi_{H\in\mathcal{MAX}} A(H) \otimes \mathbb{Q}$ be the image of the element constructed in proposition 4.13 under the rationalized edge homomorphism.

Let $m = G\text{-}cat(X_0)$ and put $w = v^m$. As in [12], the following diagram commutes:

$$\begin{tikzcd}
\pi^0_G(\underline{E}G)\otimes\mathbb{Q}_{\hat{p}} \arrow[rr] \arrow[d] & & \lim_n \pi^0_G(X_n)\otimes\mathbb{Q}_{\hat{p}} \arrow[d] \\
\pi^0_G(\underline{E}G)_{\hat{\mathbb{I}}_{G,\mathcal{MAX}}}\otimes\mathbb{Q} \arrow[r, "\cong"] & \hat{\pi}^0_G(\underline{E}G) \arrow[r, "\cong"] & \hat{\pi}^0_G(X)
\end{tikzcd}$$

as the left and right vertical maps are isomorphisms, and there are no nilpotent elements in the rationalized Burnside Ring $\pi^0_G(\underline{E}G)\otimes\mathbb{Q}$, there are no nilpotent elements in $\pi^0_G(\underline{E}G)\otimes\mathbb{Q}_{\hat{p}}$, and so there exists a natural number n such that the third condition holds.

□

Let $\hat{E}\subset E$ be a G-invariant linear subspace with a finite dimensional, G-invariant complement F_0 satisfying the mountain pass condition 1 in 1.1 . For any finite dimensional subspace $\hat{F}$, the sum $F = F_0\oplus\hat{F}$ satisfies

$$F - B_r(F)\subset\phi^a$$

Lemma 5.4. *There is a G-map f such that the diagram*

$$\begin{tikzcd}
(F, F - B_r(F)) \arrow[r] \arrow[d, "i_F"] & (E - \{0\}, \phi^a) \arrow[d, "f"] \\
(F, F - S(F_0\oplus F) \arrow[r, "j_F"] & (E - \{0\}, S(\hat{E}))
\end{tikzcd}$$

commutes, where i_F and j_F are given by inclusions.

Proof. Compare lemma 5.2 in [16]. Define a map $f : E\to\hat{E}$ by sending the bounded closed subspace A in theorem 1.1 to 0, mapping $\hat{E}\cap\phi^a$ into $\hat{E} - B_r(\hat{E})$ and extending to all of E , since $\hat{E}$ is a proper, G- absolute retract, Theorem 3.9 in page 1953 of [3]. □

The same argument as in Proposition 5.3, [16], page 17 yields:

Proposition 5.5. *For any Equivariant Cohomology Theory, $\mathcal{H}^*_G$,*

$$G{-}cat(E,\phi^a)\geq\mathcal{H}^*_G\text{lenght }(S(F_0\oplus F)\to S(F_0\oplus\hat{E}, S(F_0))$$

We now finish the proof of Theorem 1.1. This follows the proof of proposition 3.2 in [12].

Proposition 5.6.

$$G{-}cat(E,\phi^a)=\infty$$

Proof. Let F_n be an increasing sequence of finite dimensional linear G-subspaces of $\hat{E}$ such that $\hat{F}=\cup F_n$ is infinite dimensional. as in [12], the $\hat{\pi}^*_{\mathcal{FIN}}$-length of the inclusion

$$S(F_0\oplus F_n)\to S(F_0\oplus\hat{E}, S(F_0))$$

becomes arbitrarily large as n tends to infinity.

The proper G-ANR $S(\hat{E})$ satisfies the hypothesis of lemma 5.2.

Hence there is an element $w\in\pi^0_G(\underline{E}G)$ satisfying conditions 1 to 3 in 5.2. let v and v_n be the images of W along the homomorphism induced by the universal maps $S(F_0\oplus\hat{E})\to\underline{E}G$, respectively $S(F_0\oplus\hat{F}_n)\to\underline{E}G)$. Since the diagram

$$\begin{tikzcd}
\pi^0_G(S(F_0 \oplus \hat{F}_n) \arrow[rr, "j_n^*"] & & \pi^0_G(S(F_0 \oplus \hat{E}), S(F_0)) \arrow[dl, "j^*"] \\
& \pi^0_G(S(F_0 \oplus \hat{E}) \arrow[ul] &
\end{tikzcd}$$

commutes up to homotopy, $v_n \in \mathrm{im(j_n^*)}$, and proposition 5.2 yields that for any k there is an n with $\hat{\pi}^*_{\mathcal{FIN}} -$ lenght $j_n \geq k$.

□

6. Concluding remarks

Paraphrasing Willem, [39], page 3 Minimax-Type Theorems usually consist of different parts:

- Deformation lemma using some pseudo-gradient vector field.
- Construction of Palais-Smale typical sequences, which converge either due to some *a priori* compactness condition, or which give critical points using additional *a posteriori* information, typically *topological intersection properties*, like the intermediate value theorem, the Borsuk-Ulam theorem, degree notions, etc.

In this work, the proof given by Bartsch-Clapp Puppe was adapted using a Borsuk-Ulam-Type Theorem, which may be deduced from 5.5 and 5.3. The problem of classifying the groups satisfying equivariant Borsuk-Ulam-Type theorems has deserved particular attention [10], [22], among others.

Let G be a discrete, linear group which acts properly and linearly on finite dimensional representation spheres S^V. Define the Borsuk-Ulam function $b_G(n)$ as the maximal natural number k such that if there exists a G-map $S^V \to S^W$ where $dimV \geq n$, then $\dim W \geq k$

Problem 6.1. *Classify all linear, discrete groups satisfying*

$$\lim_{n \to \infty} b_G(n) = \infty$$

as in [10], [22], and in this work, condition 1.2, the answer should involve restrictions for the number of primes dividing the cardinality of the isotropy groups.

Remark 6.2 (Topological Noncompact Groups of Symmetry)**.** In the context of Hamiltonian Systems, some proper actions of non-compact Lie groups appear [36]. Equivariant Cohomotopy Theory has been extended in [7], [9] for these class of symmetries. The use of Equivariant Algebraic Topology, particularly Equivariant Cohomotopy may be useful. However, in this context, the Segal Conjecture (which was the main homotopy theoretical input of theorem 1.1, crucially in the proof of the Borsuk-Ulam-type result) is not true, as it is not even true for compact Lie groups, see [19], [13].

Remark 6.3 (Equivariant Degree Notions for Infinite Discrete Groups)**.** In [7], an equivariant degree notion for proper actions of discrete group is defined. This assigns to a quadruple (E, F, T, c) consisting of locally trivial G- Hilbert bundles over a proper, cocompact G-CW complex, a fibrewise Fredholm operator T and a fibrewise compact nonlinearity satisfying the property that the map $T_x + c_x$: $E_x \to F_x$ defined on the fibers E_x, F_x over each point x is proper, an element in the equivariant cohomotopy $\pi^*_G(X)$, as introduced in definition 4.3.

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