Abstract: The Teichmüller polynomial of a fibered 3-manifold plays a useful role in the construction of mapping classes with small entropy (small stretch factor). We provide an algorithm that computes the Teichmüller polynomial of the fibered face associated to a pseudo-Anosov mapping class of a disc homeomorphism. We also derive all the relevant information on the topology of the different fibers that belong to the face.

1. Introduction

The Teichmüller polynomial of a fibered 3-manifold plays a useful role in the construction of mapping classes with small entropy (small stretch factor). It appears in the late 90’s in the work of McMullen [McM00]. Then it was used latter as a natural source of pseudo-Anosov homeomorphism having a small normalized stretch factors: infinite families of pseudo-Anosov homeomorphism $[\psi] \in \text{Mod}(\Sigma_g)$ satisfying $\lambda(\psi)^g = O(1)$ as $g \to \infty$.

These stretch factors $\lambda(\psi)$ are packaged in a single polynomial viewed as a Laurent polynomial: $\Theta_F \in \mathbb{Z}[H_1(M,\mathbb{Z})/\text{Torsion}]$ associated to a fibered face $F \subset H^1(M,\mathbb{R})$. Each integral cohomology class in the cone $\mathbb{R}^+ \cdot F \subset H^1(M,\mathbb{R})$ one obtains a monodromy $\psi$ (see below) of a fibration $M \to S^1$ representing the map $\pi_1(M) \to \mathbb{Z}$. The expanding factor of $\psi$ is then the largest root of the polynomial $\Theta_F$ evaluated at $t^g$.

So far there are two different ways to defined the Teichmüller polynomial. One can define it by using the fitting ideal of module (defined by a lamination) over $\mathbb{Z}[H_1(M,\mathbb{Z})/\text{Torsion}]$. The Teichmüller polynomial can be also defined in terms of the transition matrix for a train track associated to a fibration. We will use the latter formulation in the present paper.

As mentioned above, this aspect has been initiated in [McM00] and then it has been intensively used in the papers by Hironaka [Hir10], Hironaka-Kin [Hir06], Kin-Takasawa [Kin13a], Kin-Takasawa and Mitsuhiko [Kin11], Kin-Kojima-Takasawa [Kin13b]. Most of the pseudo-Anosov having a small normalized stretch factor are coming from fibrations of two very particular hyperbolic manifolds (the manifold coming from the simplest hyperbolic braid and the “magic manifold” see [Kin11]).

One goal of our paper is to present an algorithm to compute explicitly the Teichmüller polynomial. More precisely we will denote the mapping torus of $[\psi] \in \text{Mod}(S)$ by $M_\psi := S \times [0,1]/(x,1) \sim (\psi(x),0)$

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Theorem 1.1. For any pseudo-Anosov class $f \in \text{Mod}(D_n)$ represented by a path in some automaton $M$ and we will suppose that the first Betti number of $\Sigma$ is at least 2.

Based on the results of Penner and Papadopoulos [Pap87] on train tracks and elementary operations (folding operations in the present paper), we provide an algorithm that determines uniquely in term of the path in the automaton.

\begin{align*}
\tau_n &\rightarrow T_{\tau} T_{\tau-1} \cdots T_2 T_1 \rightarrow \tau_0,
\end{align*}

with transition matrices $M_i = M(T_i)$, the Teichmüller polynomial $\Theta_F(t,u)$ of the associated fibered face $F$ determined by $[f_\beta]$ is:

$$\Theta_F(t,u) = \det (u \cdot \text{Id} - M_\epsilon D_\epsilon \cdot M_{\epsilon-1} D_{\epsilon-1} \cdots M_1 D_1)$$

where the diagonal matrices $D_i$ are determined uniquely.

We shall show:

Theorem 2.1. Let $M$ be an irreducible and atoroidal manifold. Then the unit ball of the Thurston norm has a very special geometry.

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2. Thurston’s theory of fibered faces and the Teichmüller polynomial

In this section we recall Thurston’s theory of fibered faces. We also review the construction of the Teichmüller polynomial and its relation with the stretch factor associated to the monodromy of a fibration $\Sigma \to M \to S^1$.

We begin by fixing some notation. Let $S$ be a surface (for which one might have $\partial S \neq \emptyset$). Let $[\psi]$ be a class in $\text{Mod}(S)$. A deep result by Thurston (see e.g. [Far12, §13, Thm. 13.4]) tell us that $M_\psi$ admits a hyperbolic metric if and only if the mapping class $[\psi]$ is pseudo-Anosov. By Mostow’s rigidity theorem [Mos68], the isometry class of $M_\psi$ does not depend on the choice of the representative or the conjugacy class of $[\psi] \in \text{Mod}(S)$.

2.1. Thurston norm and fibered faces. Thurston introduced a very effective tool for studying essential surfaces in 3-manifolds: a norm on $H_2(M,\mathbb{R})$. For practical reasons, we will define this norm on $H^1(M,\mathbb{R})$. For nice references see e.g. [Cal07], [Thu86], [FLP79] exposé 14.

For a compact connected surface $S$, let $\chi(S) = \min\{0,\chi(S)\}$. In general, if a surface $S$ has $r$ connected components $S_1, \ldots, S_r$, we naturally define $\chi(S) = \sum_{i=1}^r \chi_i(S_i)$. This defines a function $||| \cdot |||_r : H^1(M,\mathbb{R}) \to \mathbb{N} \cup \{0\}$ as follows:

$$|\alpha||_r := \inf \{\chi(S) \mid [S] \text{ is a properly embedded oriented surface s.t. } [S] \text{ is dual to } [\alpha] \},$$

where $[S] \in H_2(M,\mathbb{Z})$ (or $H_2(M,\partial M,\mathbb{Z})$ if $\partial M \neq \emptyset$). So far this function just measures the minimal topological complexity of a surface dual to $[\alpha]$. However, if $M$ is irreducible (i.e. if every embedded sphere bounds a ball) then $|\cdot||_r$ satisfies the pseudo-norm properties. Therefore it has a unique continuous extension to a pseudo norm on $H^1(M,\mathbb{R})$. If in addition $M$ is atoroidal and $\chi(\partial M) = 0$, this continuous extension is a norm. This is the so called Thurston norm. The unit ball of this norm has a very special geometry.

Theorem 2.1. [Thu86] Let $M$ be an irreducible and atoroidal manifold. Then the unit ball of the Thurston norm is a convex finite sided polytope.
An avid reader can consult the proof on the preceding theorem on Calegari’s book (see [Cal07, Theorem 5.10]). The most striking aspect of the Thurston norm is that it provides a very nice picture for homology classes representing fibrations of $M$ over the circle.

2.2. From homology classes to fibrations. Let $[M, S^1]$ denote the set of homotopy classes of maps from $M$ to $S^1$. Given a class $[f] \in [M, S^1]$ one can choose a smooth representative $f : M \to S^1$ and $df$ the angle form on $S^1$. The pullback defines a class $[f^* df]$ in $H^1(M, \mathbb{R})$. This correspondence defines a bijection between $H^1(M, \mathbb{Z})$ and $[M, S^1]$. We will call $[\alpha] \in H^1(M, \mathbb{Z})$ a fibration if the corresponding class in $[M, S^1]$ is a fibration. Let us define:

$$\Phi(M) := \{[\alpha] \in H^1(M, \mathbb{Z}) \mid [\alpha] \text{ is a fibration}\}$$

and for every face $F$ of the Thurston norm ball let $\mathbb{R}^+ \cdot F$ denote the positive cone in $H^1(M, \mathbb{R})$ whose basis is $F$.

**Theorem 2.2.** [Thu86] Suppose that $b_1(M) \geq 2$. If $\Phi(M) \cap \mathbb{R}^+ \cdot F \neq \emptyset$ for some top-dimensional face $F$ of the Thurston norm unit ball, then $\Phi(M) \cap \mathbb{R}^+ \cdot F = H^1(M, \mathbb{Z}) \cap \Phi(M)$. When $\Phi(M) \cap \mathbb{R}^+ \cdot F \neq \emptyset$ we call $F$ a fibered face and $\mathbb{R}^+ \cdot F$ a fibered cone. A fiber of a fibration minimizes the Thurston norm in its homology class (see [Cal07, Corollary 5.13]).

2.3. Hyperbolic manifolds. If the manifold $M$ is hyperbolic, then the monodromy of each fibration $\Sigma \to M \to S^1$ defines a pseudo-Anosov class in $\text{Mod}(\Sigma)$. Hence we can think of each integer point in a fibered cone $\mathbb{R}^+ \cdot F$ as a pseudo-Anosov class. We want to compute, for a fixed fibered face $F$, the stretch factors of all pseudo-Anosov maps arising as monodromies of fibrations in the fibered cone $\mathbb{R}^+ \cdot F$. This can be done by using an invariant of the fibered face called the Teichmüller polynomial. Roughly speaking, this polynomial invariant is an element of the group ring $\mathbb{Z}[G]$, where $G = H_1(M, \mathbb{Z})/\text{Tor}$ and $\text{Tor}$ is the torsion subgroup of $H_1(M, \mathbb{Z})$. Following McMullen, let us denote it by $\Theta_F$. We will now explain how $\Theta_F$ is used to calculate stretching factors of pseudo-Anosov monodromies and we will later deal with its definition. Since $H^1(M, \mathbb{Z}) \cong \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$ we can associate to each $[\alpha] \in H^1(M, \mathbb{Z})$ a morphism $\xi_\alpha : H_1(M, \mathbb{Z}) \to \mathbb{Z}$. Given that $\Theta_F$ is an element of the group ring $\mathbb{Z}[G]$, it can be written as a formal sum:

$$\Theta_F = \sum_{g \in G} a_g g, \quad a_g \in \mathbb{Z} \text{ for all } g \in G$$

where at most a finite number of coefficients $a_g$ are different from zero. The valuation of $\Theta_F$ at $[\alpha]$ is defined as follows:

$$\Theta_F(\alpha) := \sum_{g \in G} a_g \cdot t^{\xi_\alpha(g)} \in \mathbb{Z}[t, t^{-1}],$$

Remark that $\Theta_F(\alpha)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$. Let $\lambda(\alpha)$ be the expansion factor of the pseudo-Anosov class in $\text{Mod}(\Sigma)$ defined by the monodromy of the fibration corresponding to $[\alpha]$. The following theorem tells us that the Laurent polynomial $\Theta_F(\alpha)$ can be used to compute $\lambda(\alpha)$.

**Theorem 2.3.** [McM00] For any fibration $[\alpha] \in \mathbb{R}^+ \cdot F$ we have that

$$\lambda(\alpha) = \sup \{t > 1 \mid \sum_{g \in G} a_g \cdot t^{\xi_\alpha(g)} = 0\}$$

3. Teichmüller polynomial and train tracks

In this section we describe a general strategy to compute the Teichmüller polynomial $\Theta_F$ of the fibered face $F$ of $M$ naturally associated to a pseudo-Anosov homeomorphism. As a byproduct of the algorithm, we also derive the Thurston norm on $H^1(M, \mathbb{R})$ and the topology (genus, number and type of singularities) of any fiber of a fibration in the convex cone $\mathbb{R}^+ \cdot F$. 

3.1. The Teichmüller polynomial of a fibered face. In the sequel $G$ will denote $H_1(M;\mathbb{Z})/\text{Tor}$, where Tor denotes the torsion subgroup of $H_1(M;\mathbb{Z})$. As before we assume that $b_1(M)\geq 2$. The pseudo-Anosov monodromy $\varphi$ of any fibration $\pi:\Sigma\to F$ with fiber $\Sigma$ has an expanding invariant lamination $\lambda\subset \Sigma$ which is unique up to isotopy. Let $L$ be the mapping torus of $\varsigma: \lambda \to \lambda$ and $\tilde{L}$ be the preimage of the lamination $L$ on the covering space
\[
\pi: \tilde{M} \to M
\]
corresponding to $\pi_1(M) \to G$. As Fried explains (see exposée 14 [FLP79]), $L$ is a compact lamination which, up to isotopy, depends only on the fibered face $F$. Using this fact, McMullen [McM00] defines the Teichmüller polynomial of the fibered face as
\[
\Theta_F = \text{gcd}(f : f \in I) \in \mathbb{Z}[G]
\]
where $I$ is the Fitting ideal of the finitely presented $\mathbb{Z}[G]$-module of transversals of the lamination $\tilde{L}$. Remark that $\Theta_F$ is well defined up to multiplication by a unit $\pm g \in \mathbb{Z}[G]$. One of the main results of [McM00] that we exploit in this article is a formula that allows to compute explicitly $\Theta_F$. We recall how to derive this formula in the next sections.

3.2. The setting. The formula that allows us to compute explicitly $\Theta_F$ needs a particular splitting of the group $G$ that we explain in the following paragraphs. Fix a fiber $\Sigma \hookrightarrow M$ and let $\varsigma: \Sigma \to \Sigma$ be the corresponding pseudo-Anosov monodromy. We will denote by $H = \text{Hom}(H^1(\Sigma,\mathbb{Z})^\rho,\mathbb{Z}) = \mathbb{Z}^{b_1(M)}$ the dual of the $\varsigma$-invariant cohomology of $\Sigma$. The natural map $\pi_1(S) \to H^1(S,\mathbb{Z}) \to H$ determines an infinite $\mathbb{Z}^{b_1(M)}$-covering:
\[
\rho: \tilde{\Sigma} \to \Sigma
\]
with deck transformation group $H$. We can think of $\tilde{\Sigma}$ as a component of the preimage of a fixed fiber $\Sigma$ in the covering $\pi: \tilde{M} \to M$ and $H$ as the subgroup of Deck($\pi$) = $G$ fixing $\tilde{\Sigma}$. For every lift
\[
(3.4) \quad \tilde{\varsigma}: \tilde{\Sigma} \to \tilde{\Sigma}
\]
of $\varsigma$ the three manifold $\tilde{M}$ can be easily described in terms of $\tilde{\Sigma}$ and $\tilde{\varsigma}$ as follows. For every $k \in \mathbb{Z}$ let $A_k$ denote a copy of $\tilde{\Sigma} \times [0,1]$. Then $\tilde{M}$ is obtained from $\bigsqcup_{k \in \mathbb{Z}} A_k$ by identifying $(s,1) \in A_k$ with $(\tilde{\varsigma}(s),0) \in A_{k+1}$, for every $k \in \mathbb{Z}$. In this setting, the deck transformation group of $M$ splits as:
\[
G = H \oplus \mathbb{Z}[\tilde{\varsigma}]
\]
where the map $\tilde{\varsigma}$ acts on $\tilde{M}$ as $\tilde{\varsigma}(s,t) = (\tilde{\varsigma}(s), t-1)$. Equipped with these coordinates, if $F \subset H^1(M,\mathbb{R})$ is the fibered face with $[\Sigma] \in \mathbb{R}^\rho \cdot F$, then we can regard $\Theta_F$ as a Laurent polynomial $\Theta_F(t,u) \in \mathbb{Z}[G] = \mathbb{Z}[H] \oplus \mathbb{Z}[u]$ where $t = (t_1, \ldots, t_n)$ is a basis of $H$ and $u = \tilde{\varsigma}$.

Remark 3.1. If $\tilde{\varsigma}_1$ and $\tilde{\varsigma}_2$ are two different lifts of $\varsigma$ to $\tilde{\Sigma}$ then $\tilde{\varsigma}_1 = t \cdot \tilde{\varsigma}_2$ for some $t \in H = \text{Deck}(\rho)$. Hence, taking a different lift in (3.4) is trasnformed into a change of variables of the form $u' = tu$. On the other hand, since the topology of $\tilde{M}$ is independent of $\varsigma$, the topology of the infinite surface $\tilde{\Sigma}$ is also independent of $\varsigma$.

3.3. Train tracks. A train track is a connected graph with an additional “smooth” structure. More precisely let $\tau$ be a graph and let $h: \tau \to \Sigma$ be an embedding such that the branches are tangent at the vertices. Since the vertices are smooth, at each vertex the edges can be partitioned into two sets, called ingoing and outgoing for convenience (the choice of the partition is arbitrary). We will also assume that at each vertex of $\tau$ we have a cyclic order (given by $h$). This gives the notion of cusps at a vertex: this is a region formed by a consecutive pair (in terms of the cyclic order) of either two ingoing or two outgoing edges.

The pair $(\tau, h)$ (often called simply $\tau$ if there is no confusion) is a train track if the following properties are fulfilled:

1. $\tau$ has no vertex of valence 1 or 2;
2. The connected components of $\Sigma \setminus h(\tau)$ are either polygons with at least 1 cusp or annuli with one boundary contained in $\partial \Sigma$ and the other boundary with 1 cusp.

A (transverse) measure $\mu$ on a train track is an assignment of a positive real number $\mu(e) \geq 0$ to each edge $e$ of $\tau$ which satisfy the switch condition at each vertex: the sum of measures of edges in the ingoing set is the same as that for the outgoing set. The train track $\tau$ equipped with a measure $\mu$ will be called a measured train track, and will be denoted $(\tau, \mu)$. 
3.4. Measured foliations and train tracks. We can construct a measured foliation $F$ from a measured train track $(h, \tau, \mu)$ as follows. We replace each edge $e$ of $h(\tau)$ by a rectangle foliated of arbitrary width and height $\mu(e)$, foliated by horizontal leaves. According to the switch condition, the rectangles glued together give a subsurface $\tilde{S} \subset S$ (with boundaries) and a measured foliation $\tilde{F}$ on $\tilde{S}$. Now to define the foliation on the whole surface, one has to collapse the complementary regions. By assumption, no complementary components of $\tilde{S}$ into $S$ are smooth annuli, so that we can contract each boundary component in order to obtain a well-defined measured foliation on $S$ (see [Pap87], [LP79] for details). We will denote the sides of the polygons of $S \setminus \tau$ around the punctures the infinitesimal edges.

Remark 3.2. Note that there are many arbitrary parameters in the above construction, but the equivalence classes $[F]$ of $F$ is well defined.

There is a converse to the above construction. Let $F'$ be a representative measured foliation of $[F]$ and let $p \in F'$ be a singularity. Consider a polygon $\Delta_p$ embedded into the surface $S$, where the each side $\Delta_p$ is contained in a leaf of $F'$. We will say that the subsurface $S \setminus \cup_{p \in \text{sing}}. \Delta_p$ has a partial foliation (still denoted $F'$) induced by $F'$. If $[F]$ is an element of $MF$, we will usually represent $[F]$ by a partial measured foliation. If no complementary region of a partial measured foliation is a disc with zero or one cusp, then this partial foliation has a well-defined class in $MF$. We can now collapse the leaves of the foliation to a measured train track $(\tau, \mu)$.

The above procedure prompts to define (from $\tau$) is a fibered neighborhood $N(\tau) \subset S$ equipped with a retraction $N(\tau) \rightarrow \tau$. $N(\tau)$ has cusps on its boundary, and the fibers of the retraction are called ties. The train track $\tau$ can be recovered from $N(\tau)$ by collapsing every tie to a point. We will say that $F'$ is carried by $\tau$, $F' \prec \tau$, if $F'$ can be represented by a partial foliation contained in $N(\tau)$ and transverse to the ties. If in addition no leaves of $F'$ connect cusps of $N(\tau)$, we say that $\tau$ is suited to $F'$.

The next sections are intended to make explicit some well-known relations between pseudo-Anosov homeomorphisms and train track maps.

3.5. Invariant train tracks. By definition, any (class of) pseudo-Anosov map $[\psi] \in \text{Mod}(\Sigma)$ leaves invariant a pair of transverse measured foliations $(F^t, F^u)$. However the action of $\psi$ on these foliations is rather difficult to describe. A good tool to understand this action is given by train tracks (see e.g. [Pap87], §4). Let $h : \tau \mapsto \Sigma$ which is suited to $F^u$. Since $F^a$ is invariant by $\psi$ it follows that $\tau$ is invariant by $\psi$, namely:

1. The foliation $F^u$ can be represented by a partial measured foliation $F$ whose support is a fibered neighborhood $N(\tau)$ of $h(\tau)$.
2. The image $\psi(h(\tau))$ can be isotoped to a train track $h'(\tau')$ which is contained in a fibered neighborhood $N(h(\tau))$ of $h(\tau)$, is transversal to the tie foliation of $h(\tau)$ and has switches that are disjoint from the collection of central ties of $h(\tau)$.

If (2) holds, we will say that $\psi(\tau)$ is carried by $\tau$ and use the notation $\psi(\tau) \prec \tau$.

Remark 3.3. In this paper, we will work with labeled train tracks, that is, triples $(\tau, h, \varepsilon)$, where $\varepsilon : E(\tau) \rightarrow A$ is a labeling map from the set of edges of $\tau$ into a fixed finite alphabet $A$. In the sequel we will abuse of the notation $\tau$ for $(\tau, h, \varepsilon)$ whenever the embedding of the graph $h : \tau \mapsto \Sigma$ and the labelling are clear from the context.

3.6. Incidence matrix. In the above situation, if $\psi(\tau) = \sigma \prec \tau$ we naturally associate an incidence matrix $M(\psi)$ to $\psi$ (having entries in $\mathbb{Z}$) in the following way. For any edge $e$ of $\tau$ we make a choice of a tie above an interior point $e$ (called the central tie associated to the edge $e$). Let $\sigma'$ isotopic to $\sigma$ be such that $\sigma' \subset N(\sigma)$. For any edge $f$ of $\sigma$ we have a corresponding edge $f'$ of $\sigma'$ given by the isotopy. We can furthermore isotop $\sigma'$ slightly so that it is in general position with respect to the central ties of $\tau$. Now for any pair $(e, f)$ we define $M_{e,f}(\psi)$ to be the geometric intersection between the edge $f'$ of $\tau'$ with the central tie associated to the edge $e$ of $\tau$. Observe that $M(\psi)$ depends on the choice of an ordering of the edges.

A classical theorem (see [Pap87], Theorem 4.1) asserts that this matrix is a Perron-Frobenius matrix and the stretch factor of the pseudo-Anosov class $[\psi]$ equals the Perron-Frobenius eigenvalue of $M(\psi)$.

Now consider $\tilde{\tau} \subset \tilde{\Sigma}$ a component of $\rho^{-1}(\tau)$ lying in $\tilde{\Sigma}$. This is an infinite train track whose set of edges and vertices can be identified with $E \times H$ and $V \times H$ respectively. Since $\tau$ is $\psi$-invariant, $\tilde{\tau}$ is $\tilde{\psi}$-invariant. This means that $\tilde{\psi}(\tilde{\tau})$ can be isotoped to a train track $\tilde{\tau}'$ which lies in a tie neighborhood of $N(\tilde{\tau})$ of $\tilde{\tau}$, is
transverse to $\tilde{\tau}$'s ties and whose switches are disjoint from the collection of central ties of $\tilde{\tau}$. As with the train track $\tau$ and the map $\psi$, we can associate to $\tilde{\tau}$ and $\tilde{\psi}$ a $(n(E) \times n(E))$ incidence matrix $P_E(t)$. Given that we are dealing with an infinite train track in the covering $\tilde{\Sigma}$, this matrix will have entries in $\mathbb{Z}[H]$. Analogously, there is a matrix $P_V(t)$ with entries in $\mathbb{Z}[H]$ associated to the set of vertices of $\tilde{\tau}$. The next theorem states that the Teichmüller polynomial associated to $F$ can be recovered from the matrices $P_E(t)$ and $P_V(t)$.

**Theorem 3.4 ([McM00]).** The Teichmüller polynomial of the fibered face $F$ is given by:

$$\Theta_F(t, u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))}$$

when $b_1(M_f) \geq 2$.

### 3.7. Train track maps.

**Definition 3.5** (After [Los10]). A map $T$ between two train tracks $(\tau, h)$ and $(\tau', h')$ is a train-track morphism if it is cellular and preserves the smooth structure. If in addition $(\tau, h)$ and $(\tau', h')$ are isomorphic as train tracks we call $T$ a train track map.

A train-track map $T : \tau \rightarrow \tau'$ is a representative of $[f] \in \text{Mod}(\Sigma)$ given that:

1. The diagram

$$
\begin{array}{ccc}
\tau & \xrightarrow{h} & \Sigma \\
\downarrow{r} & & \downarrow{f} \\
\tau' & \xrightarrow{h'} & \Sigma 
\end{array}
$$

commutes, up to isotopy, and

2. $f \circ h(\tau) \subset N(h'(\tau'))$ and $f \circ h(\tau)$ is transverse to the tie foliation of $h'(\tau')$.

To any train track map $T$ one can associate an incidence matrix $M(T)$ in the following way: for any pair $(e, e')$ one defines $M(T)_{e,e'}$ is the number of occurrences of $e'$ in the edge path $T(e)$.

It is clear from the definitions that if $T : \tau \rightarrow \tau$ is the representative map of a homeomorphism $\psi : \Sigma \rightarrow \Sigma$ and if $\psi(\tau) < \tau$ then the incidence matrix $M(\psi)$ defined in the preceding section and the incidence matrix $M(T)$ are equal.

Observe that changing the labelling of a train track will change the train track map and the incidence matrix.

If $T$ is a representative of a pseudo-Anosov class, we have the following result:

**Theorem 3.6.** ([Pap87]) Let $\psi$ be pseudo-Anosov homeomorphism and let $\mathcal{F}^u$ be in the class of its unstable foliation. There exists a train track $\tau$ suited to $\mathcal{F}^u$ so that $\psi(\tau) < \tau$. Furthermore $\psi(\tau)$ is isotopic to a train track $\tau' \subset N(\tau)$ which is transverse to the ties so that the matrix describing the linear map from the space of weights on the edges of $\tau$ to the space of weights on the edges of $\tau$ is primitive irreducible (i.e., has some iterate all of whose entries are strictly positive).

### 3.8. Elementary operations.

One of the main difficulties to use the aforementioned formulas (computing matrix $M(\psi)$ and Formula 3.5) is that $\psi(\tau)$ (or $\tilde{\psi}(\tau)$) might look very complicated so that the isotopy needed to embed this train track in a tie neighborhood of $\tau$ transverse to the ties might be difficult to find. There is a general strategy that will simplify calculation, involving two natural (dual) operations defined on a train track called folding and splitting.

Roughly speaking, they are defined by folding or splitting a fibered neighborhood $N(\tau)$ along a cusp. For a more precise definition, see [Los10, Pap87] (for splitting operation) and [SKL02] (for folding operation).

In this paper, we shall make use of folding operation which will produce from a train track $\tau$ a new train track $\tau'$ with the property that $\tau < \tau'$. We now describe briefly the combinatorial folding operations. Observe that these operations first appear as (dual) right/left splits described in [Pap87]. We refer to [SKL02] for more details.

Let $\tau$ be a train track. Let $e_1, e_2$ be two edges of $\tau$ that are issued from the same vertex $v_1$ and that form a cusp $C$. We assume that one of the two edges is not infinitesimal (say it is $e_1$). We describe the folding where edge $e_1$ is folded onto edge $e_2$ (the other case being similar). The edge $e_2$ (respectively, $e_1$) has two
end vertices $v_1$ and $v_2$ (respectively, $v_1$ and $v_2$). The orientation around $v_1$ determines an edge $e$ attached to $v_2$ (see Figure 2). If the cup determined by $e$ is on the same side that the cusp $C$ then we can not fold $e_1$ onto $e_2$. In the other case we form a new graph $\tau'$ in the following way: we identify the edges $e_2$ and $e_1$ so that the new graph we obtain has a new edge: $e_1'$. Then $e$ is an infinitesimal edge we then fold again $e_1'$ on $e$.

The new train track $\tau'$ naturally inherits a labelling $e'$ induced from the one of $\tau$.

**Definition 3.7.** We will say that a train track $\tau$ refines to a train track $\sigma$ if there exists a sequence

$$\sigma = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = \tau$$

where $\tau_i$ is obtained from $\tau_{i-1}$ by a folding operation.

**Proposition 3.8.** Suppose that $\mathcal{F} < \sigma < \tau$ where $\sigma$ is contained in a fibred neighborhood $N(\tau)$ and $\tau$ is suited to $\mathcal{F}$. Then $\tau$ refines to $\sigma$.

**Sketch of the proof.** We prove the proposition by using splitting instead of folding. From that it is easy to get the sequence of foldings. Up to making an isotopy, one can find a fibred neighborhood $N(\sigma)$ contained in the interior of $N(\tau)$ whose tie foliation is formed by sub arcs of the tie foliation of $N(\tau)$. The number of cusps of $N(\sigma)$ and $N(\tau)$ is the same and one can define a pairing between these two sets of cusps with a family of disjoints embedded arcs $[\Gamma_i]_{i=1}^n$ contained in $N(\tau) \setminus \text{Int}(N(\sigma))$ which are transverse to the ties (Lemma 2.1, [Pap87]). The sequence of splittings is obtained by cutting $N(\sigma)$ along $\Gamma_i$, $i = 1, \ldots, n$. Each time the arc $\Gamma_i$ crosses a singular tie of $N(\tau)$, the cutting along $\Gamma_i$ defines a splitting operation on $\tau$. The concatenation of these operations produces the sequence (3.6).

The previous proposition has a simple but important consequence: the refinement of $\tau$ to $\psi(\tau)$ allows us to factorize the incidence matrix $M(\psi)$ as a product of matrices associated to folding operations. In the sequel we explain how this can be done.

**3.9. Folding operations and train track maps.** Each folding operation from $(\tau, h, e)$ to $(\tau', h', e')$ produces a train track map that represents some $[f] \in \text{Mod}(\Sigma)$ such that $f(h(\tau)) < h(\tau')$. Hence our preceding discussions can be reformulated as follows

**Lemma 3.9** (Penner-Papadoupoulos [Pap87]). Every (labeled) train track map representing a class $[f] \in \text{Mod}(\Sigma)$ is obtained by a finite sequence of train track maps induced by folding operations and then followed by a relabeling operation.

Hence to any pseudo-Anosov class $[\psi]$ with an invariant train track $\tau$, one can define a (non unique) sequence of folding operations defined by the refinement sequence

$$\psi(\tau) = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = \tau$$

The sequence of folding operations defines a sequence of train track maps:

$$\psi(\tau) = (\tau_n, e_n) \xrightarrow{T_n} (\tau_{n-1}, e_{n-1}) \xrightarrow{T_{n-1}} \cdots \xrightarrow{T_1} (\tau_1, e_1) \xrightarrow{T_0} (\tau_0, e_0)$$

Therefore lemma 3.9 in this context implies that $T = R \circ T_1 \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n$. We draw the incidence matrix $M(\psi)$ as

$$M(T) = M(R \circ T_1 \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n) = M(T_n)M(T_{n-1}) \cdots M(T_1)M(R).$$

**Remark 3.10.** Observe that since we work with **labelled train tracks**, all the transition matrices $M(T_i)$ have the form $\text{Id} + E$ where $E$ is a non negative matrix.

To summarize, to each pseudo-Anosov homeomorphisms, one can associate a train track and a sequence of folding operations such that the corresponding product of matrices is irreducible (Theorem 3.6). In general the converse is not true, however under mild assumption one has:

**Theorem 3.11.** Let $\tau = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = \tau$ be a refinement sequence defined by folding operations such that the corresponding incidence matrix is irreducible and the Perron-Frobenius eigenvector satisfies the switch condition of the train track $\tau$. Then the train track map $T$ associated to this sequence is the representative of a pseudo-Anosov homeomorphism.

This fundamental result leads to the following construction of the automaton.
3.10. **Elementary operations and standardness.** As we have seen in the preceding paragraphs, every train track map $\tau \xrightarrow{T} \tau$ representing a class in $\text{Mod}(\Sigma)$ is the product of train track maps defined by elementary operations. When $\Sigma$ is the $n$-punctured disc $D_n$, this product can be refined by asking that every train track in $(3.6)$ is *standardly embedded*. Since all the calculations that we present in this article happen in $D_n$, we will discuss these notions in detail.

We consider $D_n$ to be modeled on the unit disc in $C$ with $n$ punctures along the real line $R$. Let $l_i$ be a vertical segment joining the $i-\text{th}$ puncture to the boundary of the disc. Now consider a train track $h : \tau \leftrightarrow D_n$. As we have seen each puncture in $D_n$ is enclosed in a multigon of $h(\tau)$ whose edges are usually called infinitesimal. For each puncture we choose an infinitesimal edge of the multigon enclosing the puncture. If all the edges except these $n$ infinitesimal edges are embedded in the upper (or lower) half disc, then we say that $(\tau, h)$ is *standard*. If one open edge of $h(\tau)$ (i.e. an edge incident to different vertices) intersects only once $\cup l_i$, $n$ infinitesimal edges (one from each multigon enclosing a puncture) intersect $\cup l_i$, and all the other edges are embedded in the upper (or lower) half disc, then we say that $(\tau, h)$ is *almost standard*. These notions were first introduced in [SKL02].

If $(\tau, h)$ is standard and we perform a folding or splitting operation on $h(\tau)$, then the resulting train track $(\tau_1, h_1)$ is, in the worst case scenario, almost standard. But we can easily turn $(\tau_1, h_1)$ into a standard marking. Indeed, let $B_n$ be the $n$-th braid group with standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and consider the natural map $B_n \rightarrow \text{Mod}(D_n)$ which associates to each braid $\beta$ the mapping class $f_\beta$. From [SKL02] lemma 1 we know that if $(\tau_1, h_1)$ is almost standard, there exists a braid $\beta$ (called standardizing braid) of the form $\delta_{[m]}^\pm = (\sigma_{m-1} \sigma_{m-2} \cdots \sigma_1)^\pm$ such that $(\tau_1, f_\beta \circ h_1)$ is standard. In this case $f_\beta$ is called a standardizing homeomorphism for $(\tau_1, h_1)$.

**Definition 3.12.** Any standardizing homeomorphism $f_\beta$ permutes the punctures and is acting as a rotation in the neighborhood near punctures. We encode this action by a permutation plus the rotation number of the corresponding rotation.

**Example 3.13.** In Figure 2 the two standardizing homeomorphisms corresponding to $\sigma_1^{-1}$ and $\sigma_2$ act on the punctures as follows:

$$
\pi(\sigma_2) : (A, B, C) \rightarrow (A, C^+, B) \quad \text{and} \quad \pi(\sigma_1^{-1}) : (A, B, C) \rightarrow (B, A^-, C).
$$

3.11. **Construction of the automaton.** In this section, for simplicity, we specify to the case of the punctured disc. However all the discussion can be done for surfaces of higher genera. Let us fix $n$, the number of punctures, and the singularity data of train track (i.e. the number and type of prongs).

Obviously the number of labelled train tracks $(\tau, h, \varepsilon)$ of $D_n$ where

- $(\tau, h)$ is standard,
- $\tau$ has prescribed singularity data and labelling $\varepsilon$

is finite. Moreover this set is also (set-wise) invariant by folding operations. Finally, given a tuple $(\tau, h, \varepsilon)$ into this finite set, since the number of cusps and edges is finite, there is only finitely many possible folding operations on $h(\tau)$. These three finiteness ingredients allow us to construct a graph in the following way:

1. Vertices are tuples $(\tau, h, \varepsilon)$ where $h : \tau \rightarrow D_n$ is standard.
2. There is an edge between $(\tau, h, \varepsilon)$ and $(\tau', h', \varepsilon')$ if and only if there is a folding operation from $\tau$ to $\tau'$, $h'$ is induced by the folding and eventually a standardizing braid $i.e. h' = f_\beta \circ h$, and $\varepsilon' = R \circ \varepsilon$, where $R$ is a relabeling map, that is, $R \in \text{Sym}(\Delta)$, $\Delta$ being the alphabet used to label $\tau$.

The resulting directed graph is called the automaton associated to the number of marked points of $D_n$ and the type of singularities. Observe that this graph is not necessarily strongly connected (even not connected). It would be nice to have a description of the topology of these graph in general.

For any train track $(\tau, h)$ we will denote by $N(\tau, h)$ the corresponding connected component containing $(\tau, h, \varepsilon)$ where $\varepsilon$ is any labelling. One can perform the same construction without labelling. In this case the connected components of the automaton are denoted by $N(\tau, h)$.

**Example 3.14.** See Figure 3 and Figure 7 for examples of automata. See also Figure 10 for a more complicated example.
3.12. Closed loops in $\mathcal{N}(\tau, h)$ and pseudo-Anosov homeomorphisms. The labelling allows us to define for each edge of $\mathcal{N}^{lab}(\tau, h)$ a train track map and its transition matrix. Hence given a path $\eta$ in $\mathcal{N}^{lab}(\tau, h)$ (not necessarily closed) represented by
\[ (\tau_n, e_n) \xrightarrow{T_1} (\tau_{n-1}, e_{n-1}) \xrightarrow{T_2} \cdots \xrightarrow{T_1} (\tau_1, e_1) \xrightarrow{T_1} (\tau_0, e_0) \]
one defines the matrix $M(\eta)$ by using the formula (3.7):
\[ M(\eta) = M(T_1 \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n) = M(T_n)M(T_{n-1}) \cdots M(T_1). \]
Now if $\gamma$ is a loop in $\mathcal{N}(\tau, h)$ starting at some point $(\sigma, f)$ we can lift $\gamma$ to some path $\tilde{\gamma}$ in $\mathcal{N}^{lab}(\tau, h)$ starting at $(\sigma, f, e)$. The end point of $\tilde{\gamma}$ (that is $(\sigma, f, e')$) defines a new labelling $e' : E(\tau) \to \hat{\Lambda}$ and thus a relabeling map $R \in \text{Sym}(\Lambda)$ by $R = e \circ e'^{-1}$. We define
\[ M(\gamma) = M(\tilde{\gamma}) \cdot M(R). \]
Observe that the matrix $M(\gamma)$ is well defined up to conjugacy.

This construction allows us to reformulate Theorem 3.6 and Lemma 3.9 as follows: each pseudo-Anosov can be constructed from a closed loop in some graph $\mathcal{N}(\tau, h)$ by using the above construction. The converse is almost true: this is Theorem 3.11.

3.13. Lifting elementary operations. We denote by $\rho : \tilde{\mathcal{D}}_n \to \mathcal{D}_n$ the $H = \mathbb{Z}^{h(M)}$-covering of the punctured disc (see Section 3.1). The infinite surface $\tilde{\mathcal{D}}_n$ can be constructed by gluing $H$ copies of the simply connected domain obtained by cutting the base $\mathcal{D}_n$ along $n$ disjoint segments going from the $i$-th puncture to the exterior boundary. The way one should glue is dictated by the monodromy of the covering. We call each of these simply connected domains a leaf of the covering $\rho : \tilde{\mathcal{D}}_n \to \mathcal{D}_n$. Henceforth, we choose a leaf in $\tilde{\mathcal{D}}_n$ and we label it with $e_H$, the identity element in $H$. We call it the leaf at level zero.

For each $(\tau, h, e)$, there is a natural way to define an infinite train track $\tilde{h} : \tilde{\tau} \to \tilde{\mathcal{D}}_n$. Indeed, let $\tilde{h}(\tau) := \rho^{-1}(h(\tau))$. The edges and vertices of $\tilde{\tau}$ are in bijection with $E(\tau) \times H$ and $V(\tau) \times H$ respectively. We label the edges of $\tilde{\tau}$ as follows. For every edge $e$ of $\tilde{\tau}$ whose image under $\tilde{h}$ is properly contained in the leaf at level zero we define $\tilde{e}(e) = e_1(e)$, where $\rho(e) = e$. By the way we defined the leaves of the covering, there are exactly $2n$ edges of $\tilde{\tau}$ whose image under $\tilde{h}$ are not properly contained in the leaf of level zero. Moreover, these edges can be grouped in pairs $\{f_1, f_2^i\}_{i=1}^n$, where $\rho(f_1^i) = \rho(f_2^i) = e_i$, and $e_i$ is an infinitesimal edge of $\tau_i$.

We define $\tilde{e}(f_i^1) = e_i(e)$ for every $i = 1, \ldots, n$. Finally, we extend $\tilde{e}_i$ to the remaining edges of $\tilde{\tau}$ using the $H$-monodromy action of the covering.

In short, for every vertex $(\tau_i, h_i, e_i)$ of $\mathcal{N}^{lab}(\tau, h)$, we can define an infinite labeled train track $(\tilde{\tau}_i, \tilde{h}_i, \tilde{e}_i)$ embedded in the infinite surface $\tilde{\mathcal{D}}_n$.

Let $\{f_{\beta}\}$ be a pseudo-Anosov class in $\text{Mod}(\mathcal{D}_n)$ with invariant train track $(\tau, h)$. In this section we explain step by step how to calculate the Teichmüller polynomial associated to $\{f_{\beta}\}$ using elementary operations.

By assumption $\{f_{\beta}\}$ is represented (in a non unique way) by
\[ (\tau_n, e_n) \xrightarrow{T_1} (\tau_{n-1}, e_{n-1}) \xrightarrow{T_2} \cdots \xrightarrow{T_1} (\tau_1, e_1) \xrightarrow{T_1} (\tau_0, e_0) \xrightarrow{R} (\tau_0, e_0') \]
Recall that each each edge
\[ (3.8) \]
is a representative of some homeomorphism $[f]$ that is either the identity map, or a standardizing map.

1. For each vertex $(\tau_i, h_i, e_i)$, there is a natural way to define an infinite train track $\tilde{h}_i : \tilde{\tau}_i \to \tilde{\mathcal{D}}_n$. Indeed, let $\tilde{h}(\tilde{\tau}_i) := \rho^{-1}(\tau_i)$. The edges and vertices of $\tilde{\tau}_i$ are in bijection with $E(\tau_i) \times H$ and $V(\tau_i) \times H$ respectively. We label the edges of $\tilde{\tau}_i$ as follows. For every edge $e$ of $\tilde{\tau}_i$ whose image under $\tilde{h}_i$ is properly contained in the leaf at level zero we define $\tilde{e}_i(e) = e_i(e)$, where $\rho(e) = e$. By the way we defined the leaves of the covering, there are exactly $2n$ edges of $\tilde{\tau}_i$ whose image under $\tilde{h}_i$ are not properly contained in the leaf of level zero. Moreover, these edges can be grouped in pairs $\{f_1^i, f_2^i\}_{i=1}^n$, where $\rho(f_1^i) = \rho(f_2^i) = e_i$, and $e_i$ is an infinitesimal edge of $\tau_i$. We define $\tilde{e}_i(f_1^i) = e_i(e)$ for every $i = 1, \ldots, n$. Finally, we extend $\tilde{e}_i$ to the remaining edges of $\tilde{\tau}_i$ using the $H$-monodromy action of the covering.
In short, for every vertex \((\tau_i, h_i, e_i)\) of \(N^{\text{lab}}(r, h)\), we can define an infinite labeled train track \((\tilde{\tau}_i, h_i, e_i)\) embedded in the infinite surface \(D_i\).

(2) For each standardizing mapping \([f]\) appearing in the path that represent \([f_0]\), we choose the lift \(\tilde{f} : \tilde{D}_n \rightarrow D_n\) of \(f\) that fixes the leaf of level zero.

(3) Each \(\tilde{f}\) induces train track maps:

\[
\begin{align*}
\tilde{\tau}_i : \tilde{T}_i & \rightarrow \tilde{T}_{i-1} \\
\end{align*}
\]

where \(\tilde{T}_i\) are lifts of the maps in \([3.8]\). By construction \(M(\tilde{T}_i) = M(T_i) \cdot \text{Diag}(h_1, \ldots, h_r)\) for suitable \(h_j \in H\).

(4) By construction, the composition of the maps \((3.9)\), then followed by the relabeling operation \(R\), defines a train track map \(\tilde{T} : \tilde{T} \rightarrow \tilde{T}\) which represents a lift \(\tilde{f}_0\) of \(f_0\) to \(D_n\).

(5) All train tracks used in the computation of Teichmüller polynomials in this article can be chosen in such a way that the contribution of infinitesimal edges to the numerator in the determinant formula \((3.5)\) cancels out with the numerator. Because of this phenomenon, for each \(i\) we will compute the matrices \(M(\tilde{T}_i)\) with entries in \(\mathbb{Z}[H]\) but taking into account only non infinitesimal edges of the involved infinite train tracks.

We summarize the above discussion in the following:

**Theorem 3.15.** Let \([f_0] \in \text{Mod}(D_n)\) be a pseudo-Anosov class represented by a path in \(N^{\text{lab}}(r, h)\) given by

\[
(\tau_n, e_n) \xrightarrow{T_r} (\tau_{n-1}, e_{n-1}) \xrightarrow{T_{r-1}} \cdots \xrightarrow{T_1} (\tau_1, e_1) \xrightarrow{T_0} (\tau_0, e_0) \xrightarrow{R} (\tau_0, e'_0)
\]

Then the Teichmüller polynomial \(\Theta_F(t, u)\) of the associated fibered face \(F\) determined by \([f_0]\) is:

\[
\Theta_F(t, u) = \text{det}(u \cdot \text{Id} - M)
\]

with

\[
M = M(T_n)D_n \cdot M(T_{n-1})D_{n-1} \cdots M(T_1)D_1 \cdot M(R)
\]

for suitable diagonal matrices \(D_i\) with coefficients in \(\mathbb{Z}[H]\).

**Proof of Theorem 3.15.** [ERWAN: TO DO] \(\Box\)

**Remark 3.16.** Let \(t\) be the variable of \(H\) and \(u\) corresponds to the lift \(\tilde{u}\). Then

\[
\Theta_F(t, u) \in \mathbb{Z}[G] = \mathbb{Z}[t] \oplus \mathbb{Z}[u].
\]

From remark \((3.1)\) we conclude that picking different lifts of the standardizing maps in (3) above results in multiplying \(M(\tilde{T})\) by some \(t_0 \in H\). This does not affect the expression for the Teichmüller polynomial since \(\Theta_F(t, u) = \text{det}(uI - t_0M(\tilde{T})) = t_0^{-1}t^{n-1}\text{det}(t_0^{-1}uI - M(\tilde{T})) = t_0^{-1}\Theta_F(u', t)\), where \(u' = t_0^{-1}u\) is the coordinate corresponding to the other lift.

4. COMPUTING THE TEICHMÜLLER POLYNOMIAL

In this section we shall prove our main result:

**Theorem 4.1.** ...

Recall that \(\rho : \tilde{D}_n \rightarrow D_n\) is the \(H = \mathbb{Z}^{h(M)}\)-covering of the punctured disc.

We begin with an elementary lemma. Let \([f]\) be a standardizing map that represent an edge

\[
(\tau, e) \xrightarrow{T} (\tau', e')
\]

of the folding automaton. We consider the lift \(\tilde{f} : \tilde{D}_n \rightarrow \tilde{D}_n\) of \(f\) that fixes the leaf of level zero. It induces a train track map of the infinite track tracks:

\[
(\tilde{\tau}, \tilde{e}) \xrightarrow{\tilde{T}} (\tilde{\tau}', \tilde{e}').
\]

The two matrices \(M(T) \in M_r(\mathbb{Z})\) and \(M(\tilde{T}) \in M_r(\mathbb{Z}[H])\) are related as follows:

**Lemma 4.2.** One has

\[
M(\tilde{T}) = M(T) \cdot \text{Diag}(h_1, \ldots, h_r)
\]


5. Topology of a Fiber

In this section we provide a way to compute the topology (genus, number and type of singularities) of a fiber. We begin by introducing some notation that will be used throughout the sequel. As usual we will consider a mapping torus $M_\Phi = \mathbb{D}_r \times [0, 1]/(x, 1) \sim (\psi(x), 0)$ induced by a pseudo-Anosov braid $\beta \in B_n$.

It turns out that there is a natural model for $M_\Phi$ as a link complement $S^3 \setminus N(L)$ where $N(L)$ is a regular neighborhood of a link $L$ in the 3-sphere. To construct the link $L = L_\Phi$, simply close the braid $\beta$ representing $\psi$ while passing it through an unknot $\alpha$ (representing the boundary of the disc $\mathbb{D}_r$). Let $\Sigma \to M \to S^1$ be a fibration with monodromy $\phi : \Sigma \to \Sigma$. Recall that, if $\Sigma$ has genus $g$ and $b$ boundary components, then $\chi_{\Sigma}(\Sigma) = 2g + b - 2$. Hence the Thurston norm does not determine completely the topology of $\Sigma$. To achieve this we have to determine one of the numbers $g$ or $b$ (the surface $\Sigma$ is orientable).

5.1. Computing the number of boundary components. Since $M_\Phi$ is homeomorphic to the link complement $S^3 \setminus N(L)$ we can describe its homology group easily. First there is an embedding $i : D_r \hookrightarrow M$ such that the image of the exterior boundary of $D_r$ spans $\alpha$ and $i(D_r)$ is punctured by the $n$ strands of $\beta$.

The boundary of $M$ is a union of tori $T_1, \ldots, T_r$, where $r = b_1(M)$ (viewed as the boundary of a regular neighborhood of link components $\partial N(L_i)$). Let $[S_1], \ldots, [S_r]$ be a basis of $H_2(M, \partial M; \mathbb{R})$ (e.g. take Seifert surfaces whose boundary is $T_j$). By convention we normalize so that $S_r = i(D_r)$. This normalization implies that $T_r$ comes from the unknot $\alpha$. The meridians of components of $L_\Phi$ give a natural basis for $H_1(M, \mathbb{Z})$ \cite{Hilliker}.

Now let $[m_1], [l_i]$ be a meridian and longitude basis for $H_1(T_i; \mathbb{R})$, where the orientation of $l_i \subset \partial S_i$ is induced by the orientation of $[S_i]$. We consider the long exact sequence of the homology groups of the pair $(M, \partial M)$. We write the boundary map:

$$\partial_* : H_2(M, \partial M; \mathbb{R}) \to H_1(\partial M; \mathbb{R}).$$

On the chosen basis, for any $i = 1, \ldots, r$, one has

$$\partial_*([S_i]) = \sum_{j=1}^{r} (a_{ij}[m_j] + b_{ij}[l_j])$$

with $a_{ij}, b_{ij} \in \mathbb{Z}$. We set $A = (a_{ij})_{i,j=1,\ldots,r}$ and $B = (b_{ij})_{i,j=1,\ldots,r}$.

**Proposition 5.1.** Let $\kappa = \sum_{i=1}^{r} c_i[S_i]$, where $c = (c_1, \ldots, c_r) \in \mathbb{Z}^r$, be an integral homology class (not necessarily primitive). Then for any embedded surface $S \subset M_\Phi$ (not necessarily minimal representative) such that $[S] = \kappa$, and for any $j = 1, \ldots, r$, the number of connected components of $S \cap T_j$ is $\gcd(a_{ij}, b_{ij})$ where $(a_1, \ldots, a_r) = cA$ and $(b_1, \ldots, b_r) = cB$.

**Proof of Proposition 5.1.** Writing $[S] = \sum_{i=1}^{r} c_i[S_i] \in H_2(M, \partial M; \mathbb{R})$, elementary linear algebra gives

$$\partial_*([S]) = \sum_{i=1}^{r} \left( \sum_{j=1}^{r} c_i a_{ij}[m_j] + c_i b_{ij}[l_j] \right).$$

Now $S \cap T_j \subset \partial S$ is a union of oriented parallel simple closed curves. Hence its homology class is given by

$$\left( \sum_{i=1}^{r} c_i a_{ij}[m_j] + \sum_{i=1}^{r} c_i b_{ij}[l_j] \right) \in H_1(T_i; \mathbb{R})$$

Thus the number of connected components of $S \cap T_j$ is given by

$$\gcd\left( \sum_{i=1}^{r} c_i a_{ij}, \sum_{i=1}^{r} c_i b_{ij} \right).$$

The proposition is proved. \hfill $\Box$

**Remark 5.2.** In our situation, since $S_i$ is a Seifert surface whose boundary is the torus $T_i$ one has:

$$\partial_*([S_i]) = [l_i] - \sum_{j=1}^{r} \text{Lk}(L_i, L_j)[m_j].$$
where \( \text{Lk}(L_i, L_j) \) is the linking number of the two closed curves \( L_i \) and \( L_j \) with orientations given by the orientations of \([l_i]\) and \([l_j]\). In other words: \( B = \text{Id} \) and \( A = (\text{Lk}(L_i, L_j)))_{i,j=1,...,r} \).

We end this section with the following corollary on the connected components of \( \Sigma \cap T_i \).

**Corollary 5.3.** For any \( \Sigma = \sum_{i=1}^{r} c_i[S_i] \in H_2(M, \partial M; \mathbb{R}) \) where \( c = (c_1, \ldots, c_r) \in \mathbb{Z}^r \) we let \( a = cA \) and \( b = cB \) as above. Then each connected component of \( \Sigma \cap T_i \) is identified to a curve (well defined up to isotopy):

\[
c_i = p[m_i] + q[l_i] \in H_1(T_i; \mathbb{R}), \quad \text{with} \quad p = \frac{a_i}{\gcd(a_i, b_i)}, \quad q = \frac{b_i}{\gcd(a_i, b_i)}.
\]

From the last corollary we make the following definition. If \( T \) is a torus, and if \( H_1(T, \mathbb{Z}) \) is equipped with its preferred basis given by meridian and longitude (denoted by \(基本\) \( m \) and \( l \) then the slope of an essential simple closed curve \( [c] = p[m] + q[l] \) (with \( \gcd(p, q) = 1 \) is \( \frac{p}{q} \). Conversely for any slope \( r \in \mathbb{Q} \cup \{\infty\} \) one defines the (isotopy class) \( c \), of the corresponding simple closed curve.

### 5.2. Computing the number and type of singularities of the fiber.

Let \( \Sigma \hookrightarrow M_\phi \rightarrow S^1 \) be a fibration in \( \mathbb{R}^+ \times F \) with pseudo-Anosov monodromy \( \phi \). In this section we explain how to compute the singularity data of the stable measured foliation of \( \Sigma \) that is invariant by \( \phi \) using the singularity data of the stable measured foliation of \( \psi \). The arguments are based on work by Fried.

In the sequel we denote by \( \mathcal{F} \) the measured foliation invariant by \( \psi \). Up to isotopy one can assume that \( \psi(\mathcal{F}) = \mathcal{F} \). This determines a 2-dimensional lamination \( \mathcal{L}_\phi = \mathcal{F} \times \mathbb{R} / (s, t) \sim (\psi(s), t - 1) \) obtained as the mapping torus of \( \psi : \mathcal{F} \rightarrow \mathcal{F} \). The “vertical flow lines” \( \{s\} \times \mathbb{R} \subset \Sigma \times \mathbb{R} \) descend to the leaves of a 1-dimensional foliation whose leaves will be called the flow lines of \( \psi \). Hence \( \mathcal{L}_\phi \) is swept out by the leaves of the flow lines passing through \( \mathcal{F} \).

We distinguish two cases: \( \mathcal{F} \) has no singularities in the interior of \( D_0 \) (see Proposition 5.1) or \( \mathcal{F} \) has some singularities in the interior of \( D_0 \) (see Proposition 5.6). Any singularity \( s \) of \( \mathcal{F} \) in the boundary of \( D_0 \) determines a closed curve \( \gamma_s \) of slope \( \frac{p}{q} \) on the torus \( T_s \subset \partial M \), given by the flow line passing throughout \( s \). See for example figures 4 and 5 in [Kin13a].

**Proposition 5.4 (\( \mathcal{F} \) has no singularities in the interior of \( D_0 \)).** For any \( \Sigma \in \mathbb{R}^+ \times F \subset H_2(M, \partial M; \mathbb{R}) \) with monodromy \( \phi : \Sigma \rightarrow \Sigma \), the singularity data of the stable foliation \( \mathcal{F}_\phi \) of the pseudo-Anosov map \( \phi \) is given by:

1. \( \mathcal{F}_\phi \) has no singularities in the interior of \( \Sigma \).
2. At each connected component of \( \partial \Sigma \cap T_i \) (of slope \( \frac{p}{q} \) given by Corollary 5.3), there is a singularity of \( \mathcal{F}_\phi \) of type \( k \cdot |p, q-p| \)-prong if \( \Sigma \) is a \( k \)-prong singularity.

**Remark 5.5.** In all examples that we will be treating in this article, every singularity of \( \mathcal{F} \) in a boundary of \( D_0 \), that does not intersect \( T_i \), is a 1-prong. Hence \( \mathcal{F}_\phi \) has no singularities in the interior of \( \Sigma \) and for each \( 1 \leq i < r \), writing \( \partial \Sigma \cap T_i = a[m_i] + b[l_i] \in H_1(T_i; \mathbb{R}) \), \( \mathcal{F}_\phi \) has \( \gcd(a, b) \) singularities in \( T_i \), each of them being is a \( |p, q-p| \)-prong, where \( p = \frac{a}{\gcd(a, b)} \) and \( q = \frac{b}{\gcd(a, b)} \). The Euler-Poincaré formula.

**Proof of Proposition 5.4.** Let \( \Sigma \in \mathbb{R}^+ \times F \subset H_2(M_\phi, \partial M_\phi; \mathbb{R}) \) be a fiber of \( M_\phi \), with monodromy \( \phi : \Sigma \rightarrow \Sigma \). We will use the following result of Fried (see [Fri82, McM00]): After an isotopy we have that

1. The fiber \( \Sigma \) is transverse to the flow lines of \( \psi \), and
2. The monodromy of the fibration determined by \( \Sigma \) coincides with the first return map of the foliation \( \mathcal{F} \).

Hence the monodromies of any two points in \( \mathbb{R}^+ \times F \subset H_2(M_\phi, \partial M_\phi; \mathbb{R}) \) determine, up to isotopy, the same lamination \( \mathcal{L}_\phi \). Let \( \tau : D_0 \rightarrow M_\phi \) be a train track invariant by our given pseudo-Anosov homeomorphism \( \psi \). Up to isotopy we assume that \( \psi(\tau) \) is contained a fibered neighborhood of \( \tau \) and transverse to the tie foliation. We assume that \( \tau \) carries the measured foliation \( \mathcal{F} \). Let \( \mathcal{L}_\tau \) be the mapping torus of \( \tau \), namely \( \mathcal{L}_\tau = \mathcal{F} \times [0, 1] / (\psi(x), y) \sim (x, t) \).

The aforementioned result of Fried implies that the intersection \( \mathcal{F}_\phi = \Sigma \cap \mathcal{L}_\phi \) defines an invariant measured foliation for \( \phi \) and \( \mathcal{F}_\phi \) is carried by the train track \( \tau_\phi = \Sigma \cap \mathcal{L}_\tau \). By construction \( \mathcal{L}_\phi \) is carried by...
the branched surface $L_s$. For any singularity $s$ of $F$, one obtains a simple closed curve $\gamma_s \subset M$ which is the closed orbit of the flow line passing throughout $s$. Notice that the union of all $\gamma_s$ is the branched loci of $L_s$. Since $F$ has no singularities in the interior of $D_n$, all curves $\gamma_s$ lies in $T_j$ for some $i$. Hence all the singularities of $F_\phi$ lies in $\partial \Sigma \cap T_j$ which proves the first point of the proposition.

Now we determine the number and type of prongs of $\tau_s$. For that we consider the number of prongs of $F_\phi$ at each component of $\partial \Sigma \cap T_j$ for each $j$ (clearly for a given $j$ the type of the singularity at each component is the same). Let $c_{[i]} = p[m_i] + q[l_i] \in H_1(T_j; \mathbb{R})$ be the corresponding curve representing a connected component of $\partial \Sigma \cap T_j$ (see Corollary 5.3). By the aforementioned result of Fried, each intersection between $c_{[i]}$ and $\gamma_s$ contributes to $k$--infinitesimal edges (if $s$ is a $k$--prong singularity). Hence total number of prongs of $F_\phi$ at $\partial \Sigma \cap T_j$ is equal to $k \cdot i(c_{[i]}, \gamma_s)$.

Since the slope of $\gamma_s$ is $p_s/q_s$ one draws:

$$i(c_{[i]}, \gamma_s) = |p_sq - q_sp|.$$

The ends the proof of Proposition 5.4.

We now address the case when $F$ has singularities in the interior of $D_n$. Roughly speaking, the idea is to remove the interior singularities of $F$ to be in the context of the preceding case.

Note that in the definition of pseudo-Anosov homeomorphism we can remove or add punctures while keeping the ‘same’ map $\psi : S \to S$. More precisely when $\psi'(s)$ is a periodic orbit of unpunctured points, puncturing at $\psi'(s)$ refers to adding them to the puncture set $\{p_i\}$. Conversely, when $\psi'(p)$ is a periodic orbit of $k$-prong punctured singularities for $k > 1$, capping them off refers to removing them from the puncture set. For pseudo-Anosov braids, puncturing or capping off corresponds to adding or removing some strands.

**Proposition 5.6** ($F$ has singularities in the interior of $D_n$). Puncturing at $\psi'(s)$ for any singularity $s$ of $F$ in the interior of $D_n$ gives rise to a pseudo-Anosov $\tilde{\psi} : D_m \to D_m$ where $m > n$. By construction $F_{\tilde{\psi}}$ has no interior singularities. Moreover the injection $D_n \to D_m$ induces a map $M_{\phi} \to M_{\tilde{\psi}} =: \tilde{\Sigma}$ and each class $[\Sigma] \in \mathbb{R}^+ \cdot F \subset H_2(M, \partial M; \mathbb{R})$ (with monodromy $\phi$) determines a class $[\tilde{\Sigma}] \in \mathbb{R}^+ \cdot F \subset H_2(\tilde{\Sigma}, \partial \tilde{\Sigma}; \mathbb{R})$ with monodromy $\tilde{\phi}$.

The map $\phi$ is obtained by capping the singularities of $\tilde{\phi}$ that lie in the interior of $D_n$. In particular $F_{\tilde{\phi}}$ and $F_{\phi}$ shares the same number and type of singularities.

**Proof of Proposition 5.6** The proof is clear from the definition of capping off and puncturing at singularities.

5.3. **Orientability of singular foliation.** In this section we determine whether or not the measured foliation $F_{\phi}$ is orientable. For that we will use the following well known theorem of Thurston:

**Theorem.** For any pseudo-Anosov homeomorphism $\phi$ on a surface $\Sigma$ the following are equivalent:

1. The stretch factor of $\phi$ is an eigenvalue of the linear map $\phi_*$ defined on $H_1(\Sigma, \mathbb{Z})$.
2. The invariant measured foliation $F_{\phi}$ of $\phi$ is orientable.

To compute the homological dilatation we will make use of the Alexander polynomial. As the Teichmüller polynomial, the Alexander polynomial of $M$:

$$\Delta_M = \sum_{g \in G} b_g \cdot g$$

is an element of the group ring $\mathbb{Z}[G]$, where $G = H_1(M, \mathbb{Z})/\text{Tor}$. For a precise definition see [McM02, §2]. The Alexander polynomial can be evaluated in an homology class $[\Sigma] \in H_2(M, \partial M; \mathbb{R})$ using Poincaré-Lefschtez duality and then proceeding as with the Teichmüller polynomial in the corresponding dual cohomology class.

**Theorem 5.7.** Let $[\alpha] \in \mathbb{R}^+ \cdot F \subset H^1(M, \mathbb{R})$ with monodromy $\phi : \Sigma \to \Sigma$. Then the characteristic polynomial of $\phi$, acting on $H_1(\Sigma, \mathbb{Z})$ is given by the Alexander polynomial $\Delta_M$ evaluated in $[\alpha]$. 


6. First Examples in \(B_3\)

The goal of this section is to revisit classical examples, first studied by Hironaka [Hir06], Kin-Takasawa [Kin13a] and McMullen [McM00]. We stress that the novelty in this section are the methods presented to perform calculations, and not the results of these. In the next sections we will address examples in \(B_4\) and \(B_5\).

**Convention.** We will use the the standard generators \(\sigma_1, \ldots, \sigma_{n-1}\) induced by left Dehn half-twists around loops enclosing the punctures \(p_i\) and \(p_{i+1}\) (up to homotopy). Geometrically the map \(f_{\sigma_i} \in \text{Mod}(D_n)\) acts counter-clockwise for the standard orientation of \(D_n\).

6.1. The simplest’s pseudo-Anosov braid. We consider first the homeomorphism \(\psi = f_{\sigma_1^{-1}\sigma_2}\) and treat this example in detail.

6.1.1. Invariant train track. It is well known that the isotopy class of \(\psi\) is pseudo-Anosov. Indeed the homeomorphism \(\psi\) leaves invariant the train track \(\tau_0\) presented in Figure 1. The map \(f_{\sigma_1^{-1}\sigma_2}\) is then represented by the train track map \(T: \tau_0 \to \tau_0\), defined by \(a \to aab\) and \(b \to ab\). The incidence matrix, \(([2, 1] [1, 1])\), is irreducible.

![Figure 1. An invariant train track \(\tau_0\) for \(f_{\sigma_1^{-1}\sigma_2}\).](image)

We now quickly review how one can find a sequence of foldings discussed in the previous sections. For this purpose, consider the folding automaton and the two folding maps \(F_{ba}\), \(F_{ab}\) corresponding to the two standardizing homeomorphisms \(f_{\sigma_1^{-1}}\) and \(f_{\sigma_2}\) depicted in Figure 2.

![Figure 2. The folding automaton for \(B_3\). The map \(f_{tot}\) is an isotopy (rotation in the neighborhood near punctures).](image)

**Remark 6.1.** In the sequel, we will encode the folding automaton by representing the isotopy near the punctures by a permutation (see Definition 3.12 and Example 3.13). This defines a simpler automaton: see Figure 3.
The two foldings $F_{ba}$ and $F_{ab}$ define two train track maps $T_{ba}$ and $T_{ab}$ (representing the two homeomorphisms $f_{r_{b1}}$ and $f_{r_{b2}}$, respectively):

$$\begin{align*}
\tau_0 &\mapsto \tau_0 \\
a &\mapsto a \\
b &\mapsto ba
\end{align*}$$

whose incidence matrices are $M_{ba} := M(T_{ba}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_{ab} := M(T_{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

To be more precise, one sees that in this very particular example $\tau_0$ is both invariant by $f_{r_{b}}$ and $f_{r_{b1}}$ and the associated train track maps are given by $T_{ab}$ and $T_{ba}$. Hence the path in the automaton representing $f_{r_{b1}r_{b2}} = f_{r_{b1}} \circ f_{r_{b2}}$ has the train track map given by $T_{ba} \circ T_{ab}$. Therefore the incidence matrix is

$$M(T_{ba} \circ T_{ab}) = M_{ab} \cdot M_{ba} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

Observe that in this case the relabeling map involved is equal to the identity.

**Remark 6.2.** If one works with train track with labelled punctured, namely each puncture $p_i$ has a label $t_i$, then one needs to apply a relabeling map. More precisely

$$\begin{align*}
\tau_0 &\mapsto \tau_0 \\
a &\mapsto a \\
b &\mapsto ba
\end{align*}$$

6.1.2. The Teichmüller polynomial. We now compute the Teichmüller polynomial of the fibered face containing the fibration defined by the suspension of $f_{r_{b1}r_{b}}$. Recall that $D_3$ is the complement of 3 round discs $D_1$, $D_2$ and $D_3$ lying along a diameter of the closed unit disc. The rank of the group $H$ is given by the number of cycles of the permutation induced by the action of $f_{r_{b1}r_{b}}$ on the boundary $\{\partial D_i\}_{i=1}^3$. Since the braid $\beta = r_{b}^{-1}r_{b}$ permutes 3 strands cyclically, $H$ is isomorphic to $\mathbb{Z}$. Therefore $\pi : \widetilde{D_3} \to D_3$ is a $\mathbb{Z}$-covering. The infinite surface $\widetilde{D_3}$ can be constructed by glueing $\mathbb{Z}$ copies of the simply connected domain obtained by cutting $D_3$ along three disjoint segments going from $D_i$ to the exterior boundary of $D_3$. These are the so-called leaves of the covering $\pi : \widetilde{D_3} \to D_3$ (see §5.13). For our computations, we fix a labeling by $t \in \mathbb{Z}$ of the set of leaves forming $\widetilde{D_3}$ that is coherent with the action of Deck($\pi$). This labeling induces a labeling for the edges and vertices of the infinite train track $(\tau_0, \beta)$.

**Remark 6.3.** As stated in §5.13(6), the contribution of infinitesimal edges in the formula to $\det(uI - P_E(t))$ cancels the contribution of $\det(uI - P_F(t))$. Hence:

$$\Theta_F(u, t) = \det(u \cdot \text{Id} - M(\bar{T})),
$$

where $M(\bar{T})$ is the incidence matrix of the train track map $\bar{T} : \tau_0 \to \tau_0$ representing some lift $f_{r_{b1}r_{b2}}$, where only edges that are not infinitesimal are considered. We stress again that $\Theta_F(u, t)$, up to multiplication by a unit in $\mathbb{Z}[G]$, does not depend on the lift.

As noted before, the path in the automaton $N(h, \tau_0)$ representing $f_{r_{b1}r_{b2}}$ is $T_{ba} \circ T_{ab}$. In Figure 3, we depict the lift of each factor in this path to $\widetilde{D_3}$.
The first train track map $T_{ab}$ corresponds to the homeomorphism $f_{r_2}$. We choose the lift $\tilde{f}_{r_2}$ of $f_{r_2}$ that fixes the zero leaf. Equipped with this choice we get a train track map $\tilde{T}_{ab} : \tilde{\tau}_0 \to \tilde{\tau}_0$ induced by $\tilde{f}_{r_2} : \tilde{\tau}_0 < \tilde{\tau}_0$ (see Figure 4). Similarly the train track map $T_{ba}$ corresponds to the homeomorphism $f_{r_1^{-1}}$.

We choose the lift $\tilde{f}_{r_1^{-1}}$ of $f_{r_1^{-1}}$ that fixes the zero leaf so that we get train track map $\tilde{T}_{ba} : \tilde{\tau}_0 \to \tilde{\tau}_0$. The corresponding incidence matrices are given below:

$$M(\tilde{T}_{ab}) = \begin{pmatrix} 1 & t \\ 0 & t \end{pmatrix} \quad \text{and} \quad M(\tilde{T}_{ba}) = \begin{pmatrix} t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix}.$$  

Hence, the incidence matrix of the train track map $\tilde{T}$ representing $f_{r_1^{-1}r_2}$ is

$$M(\tilde{T}) = M(\tilde{T}_{ba} \circ \tilde{T}_{ab}) = \begin{pmatrix} 1 & t \\ 0 & t \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 + t^{-1} & t \\ t^{-1} & t \end{pmatrix}.$$  

Therefore the characteristic polynomial is

$$\Theta_T(t,u) = u^2 - (1 + t + t^{-1})u + 1$$

**Remark 6.4.** From the preceding calculations it is easy to compute the Teichmüller polynomials associated to braids in $B_3$ that permute the strands cyclically (by considering products of $M(\tilde{T}_{ba})$ and $M(\tilde{T}_{ab})$). Compare with [McM90] §11.

**Remark 6.5.** For each folding, one can easily encode how the lift permutes the edges of $\tilde{T}$. This information can be deduced directly from the automaton on the disc. Indeed it is sufficient to keep track of the infinitesimal edges when performing the folding procedure. For instance, when the edge $b$ is folded onto edge $a$, it is also folded onto the infinitesimal edge labelled $B$ (in the negative way). Hence the matrix decomposes as $M(\tilde{T}_{ba}) = M(\tilde{T}_{ab}) \cdot \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.

For the other case, when the edge $a$ is folded onto edge $b$, it is also folded onto the infinitesimal edge labelled $B$ (in the positive way). Hence the matrix decomposes as $M(\tilde{T}_{ab}) = M(\tilde{T}_{ba}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$.

6.1.3. **Evaluating the Teichmüller polynomial of $\sigma_1^{-1}\sigma_2$.** First given a class $f_\beta \in \text{Mod}(D_4)$ we explain how to assign coordinates on $H^1(M; \mathbb{Z})$ such that the cohomology class corresponding to the fibre defined by $f_\beta$ is $(0, \ldots, 0, 1)$. Following Section 5.1, we choose an ordered basis $B = \{[m_1], \ldots, [m_r]\} \subset H_1(M; \mathbb{Z})$ formed by the meridians of the tori $T_1, \ldots, T_r$ respectively. Given that $H_1(M; \mathbb{Z})$ is torsion free, the base $B$ defines a base $B^* = \{[s_1], \ldots, [s_r], [\gamma]\}$ for $H^1(M; \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z})$ by duality. Here $s_i = m_i^*$ for $i = 1, \ldots, r$ and $m_r = y$. Let $[s_j] = i(D_a)$ denote the class of the fibre of the fibre defined by $f_\beta$. The intersection of $[s_j]$ with $[m_i]$ is given by $\delta_{ij}$ and hence, using the Universal Coefficient Theorem and Poincaré duality, we deduce that the coordinates of $[s_j]^* \in H^1(M; \mathbb{Z}) = H_2(M, \partial M; \mathbb{Z})$ for the basis $B^*$ are precisely $(0, \ldots, 0, 1)$. In the rest of the examples presented in this text we always choose the ordered basis $B^*$. Remark that $\{[m_1], \ldots, [m_{r-1}]\}$ generate the $f_\beta$-invariant homology of the $r - 1$ punctured disc and $[m_r]$ corresponds to the natural lifting $f_\beta$ of $f_\beta$, hence we can identify $\{[m_1], \ldots, [m_{r-1}], [m_r]\}$ with the variables $\{t_1, \ldots, t_{r-1}, u\}$ of the Teichmüller polynomial (see Section 3.1). In the following figure we depict the link complement defined by $\sigma_1^{-1}\sigma_2$ and the choices made to obtain the basis $B^*$ (we will simply write $(s,y)$ for the coordinates of $H^1(M; \mathbb{Z})$).

We now determine the Thurston norm for the case $\beta = \sigma_1^{-1}\sigma_2$. We achieve this by computing first the Alexander norm of $M = M_{f_\beta}$. Direct computation (using e.g. Morton’s algorithm [Mor99]) shows that the
Alexander polynomial of $\beta$ is

$$\Delta_M(t, u) = u + u^{-1} - (t^{-1} + 1 - t)$$

(6.11)

(Well defined up to multiplication by a unit in $\mathbb{Z}[G]$ (see [McM02]). The Newton polygon $N(\Delta_M)$ of this polynomial is the symmetric diamond forming the convex hull of the points $\{(0, \pm 1), (\pm 1, 0), (0, 0)\}$; its Newton polytope is the square of vertices $\{(\pm \frac{1}{2}, \pm \frac{1}{2})\}$). By Remark 5.3 the unit ball of the Alexander norm is the square of vertices $\{(\pm \frac{1}{2}, \pm \frac{1}{2})\}$ (in $H^1(M; \mathbb{Z})$). Hence

$$\|(s, y)\|_A = \max(\{2s, 2y\}) \quad \text{for all } (s, y) \in H^1(M; \mathbb{Z}).$$

By theorems 5.1 and 5.2 we conclude that the segment joining the points $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$ is the fibered face $F$ of the Thurston norm ball whose cone $\mathbb{R}^+ \cdot F$ contains the fiberation defined by $\sigma_1^{-1}\sigma_2$. Hence

$$\|(s, y)\|_F = \max(\{2s, 2y\}) \quad \text{for all } (s, y) \in \mathbb{R}^+ \cdot F \cap H^1(M; \mathbb{Z}).$$

We finally explain how to evaluate $\Theta_F$ in a point $(s, y) \in H^1(M; \mathbb{Z})$. Let $f_s, f_y : H_1(M; \mathbb{Z}) \to \mathbb{Z}$ be the duals of $[x]$ and $[y]$ respectively. Hence, the dual of a point $(s, y) \in H^1(M; \mathbb{Z})$ is given by $f_{(s,y)} := sf_s + yf_y$. Since by definition $f_{(s, y)}(u) = y$ and $f_{(s, y)}(t) = s$ one has

$$\Theta_F(s, y) = X^{f_{(s, y)}(u)} - (1 + X^{f_{(s, y)}(t)})X^{f_{(s, y)}(u)} + 1 = X^{2y} - (1 + X^s + X^{-s})X^y + 1$$

6.1.4. The topology of the fiber. Let $\Sigma$ be the fiber of the fibration determined by the point $(s, y) \in \mathbb{R}^+ \cdot F$, where $F$ is the segment joining the points $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$. Since every fiber is (Thurston) norm minimizing in its homology class, we have:

$$\|(s, y)\|_F = \|y\| = -\chi(\Sigma) = 2\text{genus}(\Sigma) - 2 + \#\text{boundary components of }\Sigma.$$ 

We calculate now the number of boundary components of $\Sigma$ as follows. We choose a basis $\{[S_1], [S_2]\}$ of $H_2(M, \partial M; \mathbb{Z})$ by taking two Seifert surfaces of the components of the $6_2$ knot shown in Figure 5. By Remark 5.2 we have that

$$\partial_i[S_1] = l_i - \text{Lk}(L_1, L_2)m_2 \quad \partial_i[S_2] = l_2 - \text{Lk}(L_2, L_1)m_1.$$

A straightforward computation shows that $|\text{Lk}(L_1, L_2)| = |\text{Lk}(L_2, L_1)| = 3$. Letting $A = \left( \begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix} \right)$, Proposition 5.1 implies that the number of connected components of $\partial \Sigma \cap T_j$ is $gcd(a_j, b_j)$ where $a = (s, y)A = (3y, 3s)$ and $b = (s, y)$. Therefore the total number of connected components of $\partial \Sigma$ is $gcd(s, 3y) + gcd(3s, y) = gcd(3, s) + gcd(3, y)$. Plugging this data into (6.12) we get

$$\text{genus}(\Sigma) = \|y\| + 1 - \frac{gcd(3, s) + gcd(3, y)}{2}.$$ 

With notations of Corollary 5.3 the slope of the boundary components are $3/s$ and $3/y$. 

\[\text{Figure 5. The link } 6_2^2 \text{ and the fiber of the fibration defined by } \sigma_1^{-1}\sigma_2.\]
6.1.5. The singularities of the fiber. We already observed that $\Sigma$ has $\gcd(3, s)$ boundary components at $T_1$ and $\gcd(3, y)$ boundary components at $T_2$.

For any singularity $s$ of $\mathcal{F}$ one needs to determine the slope of $\gamma_s \subset M$ where $\gamma_s$ is the closed orbit of the flow line passing throughout $s$. Since $\mathcal{F}$ has no singularities in the interior of $D_n$ all curves $\gamma_s$ lies in $T_i$ for some $i$. We label the prong $A, B, C$. One sees that the braid $\beta$ permutes the prongs $(A, B, C)$ to $(C, A, B)$. We denote the corresponding permutation $\pi(\beta)$. Since $\pi(\beta)$ has only one cycle, there is only one torus component (see Figure 6). Now when performing the pseudo-Anosov braid and isotopy, one needs to understand the rotation in the neighborhood near punctures (see Definition 3.12 and Example 3.12).

As usual one can obtain the permutation $\beta$ as follows: for each elementary step we have a permutation encoding how the isotopy (rotation) acts in the neighborhood near punctures. More precisely:

$$\pi(\sigma_2) : (A, B, C) \to (A, C^+, B) \quad \text{and} \quad \pi(\sigma_1^{-1}) : (A, B, C) \to (B, A^-, C).$$

Composition gives the desired slope:

$$\pi(\beta) = \pi(\sigma_1^{-1} \sigma_2) = \pi(\sigma_2) \circ \pi(\sigma_1^{-1}).$$

Hence $\pi(\beta) : (A, B, C) \to (C^+, A^-, B)$ and so $\gamma = [l]$ i.e. its slope is $0/1$ (no Dehn twist around the meridian).

![Figure 6. Computing the slope of the curve $\gamma_p$.](image)

Concretely the slope of $\gamma_p$ is $p/q = 0/1$. The slope for the other component $T_2$ is $1/0$. We apply Proposition [5.4] at each connected component of $\partial \Sigma \cap T_j$ as follows.

1. For $T_1$ (coordinate $t$): one has $(a_1, b_1) = (3y, s)$. Thus at each connected component (of the $\gcd(3y, s)$ components) there is a singularity of $\mathcal{F}_0$ of type $3y/\gcd(3, s)$-prong.

2. For $T_2$ (coordinate $t$): one has $(a_2, b_2) = (3s, y)$. Thus at each connected component (of the $\gcd(3s, y)$ components) there is a singularity of $\mathcal{F}_0$ of type $y/\gcd(3, y)$-prong.

6.1.6. Orientability of singular foliation. To compute the homological dilatation we will use the Alexander polynomial $\Delta_M(t, u) = u^3 - u(-t^{-1} + 1 - t) + 1$ (up to a factor). By Theorem [5.7] the homological dilatation is the maximal root of $\Delta_M$ (in absolute value) evaluated in $(s, y)$, namely it is the maximal root (in absolute value) of the polynomial:

$$Q(X) = X^{2y} - (1 - X^y - X^{-y})X^y + 1, \quad y > s$$

Recall that the stretch factor is the maximal root of

$$P(X) = X^{2y} - (1 + X^y + X^{-y})X^y + 1.$$  

Since $Q(-X) = P(X)$ when $s$ is odd and $y$ is even, we draw that the invariant measured foliation is orientable if $s$ is odd and $y$ is even.

In the rest of this section we will focus only on computing the Teichmüller polynomial, for the rest of the computations can be performed using the methods presented for the simplest pseudo-Anosov braid.
6.2. The Teichmüller polynomial of $\sigma_2\sigma_1^{-1}\sigma_2 \in B_4$. The link complement $M = S^3 \setminus L(\beta)$ of the braid $\beta = \sigma_2\sigma_1^{-1}\sigma_2$ is homeomorphic to the magic manifold (see [Kin13a] for more details). This braid fixes one strand and permutes the other two, hence the $H$-covering $D_3$ is a $\mathbb{Z}^2$-covering. Let us denote by $(t_1, t_2)$ the variables of the deck transformation group of $\pi : D_3 \to D_3$ corresponding to the permutated and fixed strands, respectively. From the automaton presented in Figure 3 one sees that the path in the automaton $\mathcal{N}(h, \tau_0)$ representing $f_{\sigma_1\sigma_2^{-1}\sigma_2}$ is the composition of folding maps $T_{ab} \circ T_{ba} \circ T_{ab}$. By taking lifts that fix the zero leaf, we get three train track maps:

\[
\tau_0 \rightarrow \tau_0 \rightarrow \tau_0 \rightarrow \tau_0
\]

whose incidence matrices are:

\[
M(T_{ab}) = \begin{pmatrix} 1 & t_1 \\ 0 & t_1 \end{pmatrix}, \quad M(T_{ba}) = \begin{pmatrix} t_1^{-1} & 0 \\ t_1 & 1 \end{pmatrix} \quad \text{and} \quad M(T_{ab}^\prime) = \begin{pmatrix} 1 & t_2 \\ 0 & t_2 \end{pmatrix}.
\]

We want to stress the fact that it is important to keep track how strands are permuted when lifting the maps $f_{\sigma_2}$ and $f_{\sigma_1^{-1}}$. This is the reason why matrices $M(T_{ab})$ and $M(T_{ab}^\prime)$ are not the same. Therefore the incidence matrix of the train track map $\overline{T}$ representing $f_{\sigma_1\sigma_2^{-1}\sigma_2}$ is

\[
M(\overline{T}) = M(T_{ab}^\prime) \circ T_{ba} \circ T_{ab} = \begin{pmatrix} 1 & t_1 \\ 0 & t_1 \end{pmatrix} \begin{pmatrix} t_1^{-1} & 0 \\ t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} t_1^{-1} + 1 & t_1 t_2 + t_2 + t_1 t_2^{-1} \\ 1 & t_1 t_2 + t_2 \end{pmatrix}
\]

And, since remark 6.3 also applies in the context of this example, the corresponding Teichmüller polynomial is given by the characteristic polynomial of the preceding matrix:

\[
\Theta_F(t_1, t_2, u) = u^2 - (t_1 t_2 + t_2 + 1 + t_1^{-1})u + t_2
\]

7. The Teichmüller polynomial of $\sigma_1^{-1}\sigma_2\sigma_3 \in B_4$

7.1. Invariant train track. The homeomorphism $f_{\sigma_1^{-1}\sigma_2\sigma_3}$ is a pseudo-Anosov homeomorphism: it leaves invariant the train track $\tau_0$ (see Figure 7) and the train track map $T : \tau_0 \to \tau_0$ induced by $f_{\sigma_1^{-1}\sigma_2\sigma_3}(\tau_0) < \tau_0$ is given by

\[
\begin{align*}
 a & \to cbac \\
 b & \to c \\
 c & \to d \\
 d & \to ba
\end{align*}
\]

Its incidence matrix $M(T) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is irreducible.

In Figure 7 we depict the folding automaton $\mathcal{N}(\tau_0, h)$. This automaton has three vertices, say $\tau_i$ for $i = 0, 1, 2$. More precisely the folding $F_{ba}$ induces a train track map $T_1 : \tau_0 \to \tau_0$ that represent $[f_{\sigma_1^{-1}}] \in \text{Mod}(D_4)$. On the other hand the folding $F_{ab}$ induces a train track map $T_2 : \tau_0 \to \tau_1$ that represent $[\text{Id}] \in \text{Mod}(D_4)$. Finally the folding $F_{ab}$ induces a train track map $T_3 : \tau_1 \to \tau_0$ that represent $[f_{\sigma_2\sigma_3}] \in \text{Mod}(D_4)$. Hence the closed path representing $f_{\sigma_1^{-1}\sigma_2\sigma_3}$ is given by the sequence $\tau_0 < \tau_1, f_{\sigma_2\sigma_3}(\tau_1) < \tau_0$ and $f_{\sigma_1^{-1}}(\tau_0) < \tau_0$, in this order. The corresponding sequence of train track maps is:

\[
\tau_0 \rightarrow T_2 \rightarrow \tau_1 \rightarrow T_1 \rightarrow \tau_0 \rightarrow \tau_0
\]

The train track map $T_3$ factors as $\tau_1 \to \tau_1 \\ R_3 \to \tau_0$ where $R_3$ is the relabeling. The incidence matrices are

\[
M(T_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M(T_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M(T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Hence our incidence matrix of the train track map $\tau_0 \to \tau_0$ factors as

\[
M(T) = M(T_1 \circ R_3 \circ T_3 \circ T_2) = M(T_2)M(T_3)M(R_3)M(T_1).
\]
7.2. Teichmüller polynomial. We now calculate the Teichmüller polynomial of the fibered face containing the fibration defined by the suspension of $f_{\sigma_1 \sigma_2 \sigma_1}$. Since the braid permutes all the strands cyclically, $\pi : \tilde{D}_4 \to D_4$ is a $\mathbb{Z}$-covering and we fix a labeling by $t \in \mathbb{Z}$ of the set of leaves forming $\mathbb{Z} \tilde{D}_4$ that is coherent with the action of $\text{Deck}(\pi)$. This labeling induces a labeling of the vertices of $\langle \tau_0, h \rangle$. By Remark 6.3, $\Theta_F(u, t)$ is the characteristic polynomial of the incidence matrix of the train track map $\tilde{T} : \tau_0 \to \tau_0$ representing some lift $\tilde{f}_{\sigma_1 \sigma_2 \sigma_1}$ of $f_{\sigma_1 \sigma_2 \sigma_1}$, where only edges that are not infinitesimal are considered.

As discussed in previous sections, one needs to compute the train track maps $\tilde{T}_1 : \tau_0 \to \tau_0$ and $\tilde{T}_3 : \tau_1 \to \tau_0$ representing lifts of $f_{\sigma_1 \sigma_2}$ and $f_{\sigma_2 \sigma_1}$, respectively. Of course for $\tilde{T}_2 : \tau_0 \to \tau_1$ one chooses the lift Id so that $M(\tilde{T}_2) = M(T_2)$. The two train track map are depicted in Figure 8. The corresponding incidence matrices are:

$$M(\tilde{T}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad M(\tilde{T}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot R_3.$$  

(7.16)

We draw the incidence matrix $M(\tilde{T})$:

$$M(\tilde{T}) = M(\tilde{T}_1 \circ \tilde{T}_3 \circ \tilde{T}_2) = \begin{pmatrix} 1 + t^{-1} & t & t & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 1 & t & 0 & 0 \end{pmatrix}.$$  

and its associated characteristic polynomial, that is the Teichmüller polynomial of $f_{\sigma_1 \sigma_2 \sigma_1}$:

$$\Theta_F(t, u) = u^4 - (1 + t^{-1})u^3 - (t^2 + t^3)u + t^2.$$  

We remark that this calculation can also be performed without the use of elementary operations. A direct calculation shows that $\tau_0$ is $f_{\sigma_1 \sigma_2 \sigma_1}$-invariant and that the corresponding incidence matrix is precisely 7.16 (see Figure 8).

APPENDIX A. AN INFINITE FAMILY OF BRAIDS.

In this section we consider, for each $n \in \mathbb{N}$, the braid $\beta_n \in B_{n+3}$ which is given by $\beta_n = \beta_2 \beta_1 \sigma_1$, where $\beta_1 = \sigma_3 \sigma_2 \sigma_1$ and $\beta_2 = \sigma_n \sigma_{n-1} \cdots \sigma_2 \sigma_1$. In Figure 9 we depict a closed path in the folding automaton $N(\tau_0, h)$ representing the train track map $T : \tau_0 \to \tau_0$ defined by $f_{\beta_n}$. We explain the arrows depicted in Figure 9 in what follows. The elementary operation $F_{12}$ folds the edge $a_1$ in $\tau_0$ over the edge $a_2$, and induces a train track map $T_1 : \tau_0 \to \tau_0$ that represents $[f_{\sigma_1}] \in \text{Mod}(B_{n+3})$. The elementary operation $F$
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folds the edge $a_j$ in $\tau_0$ over the edge $a_1$, for $j = n + 3, n + 2, \ldots, a_4, a_3$ in this precise order. If we apply the homeomorphism $f_{\beta_2, \beta_1}$ to $F(\tau_0)$ we obtain a standardly embedded train track $\tau'_0$ which differs from $\tau_0$ by a relabeling. Hence $F$ induces a train track map $T_2 : \tau_0 \to \tau_0$ which represents $[f_{\beta_2, \beta_1}] \in \text{Mod}(B_{n+3})$.

Remark that this train track map factors as $\tau_0 \to \tau'_2 \to \tau_0$, where $R$ is a relabeling. Moreover, we have that $f_{\beta_2, \beta_1} = f_{\beta_3, \beta_2} \circ f_{\beta_1}$, where $f_{\beta_1}$ takes the leftmost vertex of $F(\tau_0)$ and places it between the fourth and fifth vertex (when counting from left to right) of $F(\tau_0)$, and $f_{\beta_3, \beta_2}$ then takes the the leftmost vertex of $f_{\beta_1}(F(\tau_0))$ and places it on the rightmost spot. We note that $f_{\beta_1}(F(\tau_0))$ is an standardly embedded train track $\tau_1$. A direct calculation shows that the corresponding incidence matrices are given by:

\[
M(T_1) = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & Id_{n+1 \times n+1}
\end{pmatrix}
\]

\[
M(T'_2) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & Id_{n \times n}
\end{pmatrix}
\]
Therefore the incidence matrix of the train track map $\tilde{T}$ is given by $M(\tilde{T}) = M(T_1)M(T_2)R$.

We now calculate the Teichmüller of the fibered face $F$ containing the fibration defined by the suspension of $f_{\beta_2}$. Since the braid permutes all the strands cyclically, $\pi : \tilde{D}_n \to D_n$ is a $\mathbb{Z}$-covering and we fix a labeling by $t \in \mathbb{Z}$ of the set of leaves forming $\tilde{D}_n$ that is coherent with the action of $\text{Deck}(\pi)$. This labeling induces a labeling of the vertices of $(\tilde{f}_{\beta_2})^n$.

By Remark 6.3, $\Theta_{f, (u, t)}$ is the characteristic polynomial of the incidence matrix of the train track map $\tilde{T} : \tau_0 \to \tau_0$ representing some lift $\tilde{f}_{\beta_2}$ of $f_{\beta_2}$, where only edges that are not infinitesimal are considered. In figure 10 we depict lifts of the maps $f_{\tau_{1}}, f_{\beta_2}$ and $f_{\beta_{2n}}$, that we denote by $\tilde{f}_{\tau_{1}}, \tilde{f}_{\beta_2}$ and $\tilde{f}_{\beta_{2n}}$ respectively. We remark that $\tilde{f}_{\beta_2}(\tau_0)$ is carried by $\tau_1$ and $\tilde{f}_{\beta_2}(\tau_1)$ is carried by $\tau_0$.

A direct calculation shows that the incidence matrix of the train track map $\tilde{T}_1 : \tau_0 \to \tau_0$ is given by:

$$M(\tilde{T}_1) = \begin{pmatrix}
1 & t & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & t & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix} \oplus \text{Id}_{n+1\times n+1},$$

whereas $M(\tilde{T}_2)$, the incidence matrix of $\tilde{T}_2 : \tau_0 \to \tau_0$, is given by the product of the three matrices $M_{\beta_1}M_{\beta_{2n}}R$, where $M_{\beta_1}$ is computed from the fact that $\tilde{f}_{\beta_2}(\tau_0) < \tau_1$, $M_{\beta_{2n}}$ is computed from the fact that $\tilde{f}_{\beta_2}(\tau_1) < \tau_0$ and $R$ is just the relabeling matrix defined in (A.17). More precisely $M_{\beta_{2n}} = t^{-1}\text{Id}_{n+1\times n+1}$ and:

$$M_{\beta_{2n}} = \begin{pmatrix}
t^{-1} & 0 & 0 & \ldots & 0 \\
0 & t^{-1} & 0 & \ldots & 0 \\
0 & 0 & t^{-1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
t^{-1} & 0 & 0 & \ldots & 0 \\
\end{pmatrix} \oplus t^{-1}\text{Id}_{n\times n}.$$
Therefore the incidence matrix of $M(\tilde{T}) = M(\tilde{T}_1)M_{b_\beta}M_{b_\alpha}R$, more precisely:

$$
M(\tilde{T}) = \begin{pmatrix}
  t^{-2} & 0 & 0 & 0 & \cdots & \cdots & t^{-1} \\
  0 & 0 & 0 & 0 & \cdots & \cdots & t^{-1} \\
  0 & t^{-2} & 0 & 0 & \cdots & \cdots & 0 \\
  \vdots & \vdots & 0 & t^{-1} & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\
  t^{-2} & 0 & 0 & 0 & \cdots & \cdots & t^{-1}
\end{pmatrix}
$$

and its associated characteristic polynomial, that is the Teichmüller polynomial of tori. If

$$(B.18)$$

Therefore the incidence matrix of $M$. The condition "the lamination of the Thurston norm ball $L$ is equivalent to the orientability of a train track $z$" is equivalent to the following condition: there exist a fibration $\Sigma \rightarrow M$ whose pseudo-Anosov fixed point makes use of the Alexander norm for example [Thu86]). Our calculations will make use of the Alexander polynomial $\Delta_\Lambda$ for a unique face $a$ of $\Sigma$. Then we have:

**Theorem B.1** ([McM02]). Let $M$ be a compact, orientable 3-manifold whose boundary, if any, is a union of tori. If $b_1(M) \geq 2$ then for all $[a] \in H^1(M, \mathbb{Z})$:

$$
\|a\|_A := \sup_{b_\alpha \neq 0, b_\beta} a(g - h)
$$

Moreover, equality holds when $\alpha : \pi_1(M) \rightarrow \mathbb{Z}$ is represented by a fibration $\Sigma \rightarrow M_F \rightarrow S^1$, where $\Sigma$ has non-positive Euler characteristic.

**Theorem B.2** ([McM00]). Let $F$ be a fibered face in $H^1(M, \mathbb{R})$ with $b_1(M) \geq 2$. Then we have:

1. $F \subset A$ for a unique face $A$ of the Alexander unit norm ball, and
2. $\Delta = F$ and $\Delta_M$ divides $\Theta_F$ if the lamination $L$ associated to $F$ is transversally orientable.

In particular, the Thurston and Alexander norms agree on integer classes in the cone over a fibered face of the Thurston norm ball.

**Remark B.3.** Since the Alexander polynomial of a 3-manifold is symmetric the Alexander norm ball is dual to the scale by of factor of 2 of the Newton polytope of the Alexander polynomial. For the sake of completeness we end this section discussing the Teichmüller norm and how it can also be used to calculate the Thurston norm. Fix a fibered face $F \subset H^1(M, \mathbb{R})$ and let $\Theta_F = \sum_{g \in G} a_F \cdot g$ be the corresponding Teichmüller polynomial. The Teichmüller norm (relative to $F$) is defined by:

$$
\|a\|_{\Theta_F} := \sup_{a_F \neq 0, a_0} a(g - h)
$$

Compare with $[B.18]$ The unit ball $B_{\Theta_F}$ of the Teichmüller norm is dual to the Newton polytope $N(\Theta_F)$ of the Teichmüller polynomial [McM00]. Moreover,
Theorem B.4 ([McM00]). For any fibered face $F$ of the Thurston norm ball, there exists a face $D$ of the Teichmüller norm ball,

$$D \subset \{ [\alpha] \mid \|[\alpha]\|_{\Theta} = 1 \}$$

such that $\mathbb{R}^+ \cdot F = \mathbb{R}^+ \cdot D$.

REFERENCES


