BASE TREE PROPERTY

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Abstract. Building on previous work of Balcar, Pelant and Simon we investigate σ-closed partial orders of size continuum. We provide both an internal and external characterization of such partial orders by showing that (1) every σ-closed partial order of size continuum has a base tree and that (2) σ-closed forcing notions of density $\mathfrak{c}$ correspond exactly to regular suborders of the collapsing algebra $Coll(\omega_1, 2^\omega)$.

We further study some naturally ocurring examples of such partial orders.

Introduction

A partially ordered set $(P, \leq)$ is *σ-closed* if every countable decreasing sequence of elements of p has a lower bound. In this note we study σ-closed partial orders of size continuum. Orders of this type naturally arise in combinatorial and descriptive set-theory, topology and analysis.

An essential example is the *collapsing algebra* $Coll(\omega_1, 2^\omega)$, i.e. the completion, in the sense of Boolean algebra, of the complete binary tree of height ω_1. This forcing notion has several presentations:

- $(\mathrm{Fn}(\omega_1, \{0,1\}, \omega_1), \supseteq)$ - ordering for adding a new subset of ω_1,
- $(\mathrm{Fn}(\omega_1, \mathbb{R}, \omega_1), \supseteq)$ - ordering for the consistency of the continuum hypothesis,
- $(\mathrm{Fn}(2^\omega, \{0,1\}, \omega_1), \supseteq)$ - ordering for adding $\mathfrak{c}$-many subsets of ω_1,
- the natural ordering for adding a $\diamond$-sequence,
- Jech's forcing for adding a Suslin tree by countable conditions.

All these orderings are forcing equivalent, in fact, they have isomorphic base trees (see Theorem 2.1 for the term base tree).

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Consider now the set $[\omega]^\omega$ of all infinite sets of natural numbers ordered by inclusion. This order is not σ-closed, but it is also not *separative.*[1] The separative quotient of $([\omega]^\omega, \subseteq)$ are the positive elements in the Boolean algebra $\mathcal{P}(\omega)/fin$. In [1] the surprising fact that also $\mathcal{P}(\omega)/fin$ has a base tree was established. It was then studied in [8], [10], [12].

Since then many other naturally occuring examples were studied ([2],[3]) and in each case the methods of [1] were used used to prove the corresponding Base Tree Theorem.

In this note we prove this general fact for all partial orders with a dense σ-closed subset of size continuum. We also identify the σ-closed forcings of size continuum as the regular subalgebras of the collapsing algebra $Coll(\omega_1, 2^\omega)$.

We then present some of the standard examples and review the relevant published results.

1. Main results

The *height* of a partial order $(P, \leq)$, $\mathfrak{h}(P)$ shortly, is the minimal cardinality of a system of open dense subsets of P such that the intersection of the system is not dense; i.e. $\mathfrak{h}(P) = \min\{|H| : \forall D \in H(D \text{ is open dense }) \wedge \bigcap H \text{ is not dense}\}$. For a Boolean algebra B we define $\mathfrak{h}(B)$ as the height of the ordering $(B \setminus \{0\}, \leq)$, where $\leq$ is the canonical ordering on B. If B is complete, it coincides with its *distributivity number*. We will deal mostly with non-atomic orderings but for completeness we allow atomic orderings in the definition too. Thus, if $(P, \leq)$ is *atomic*, i.e. there is a set of minimal elements such that every other element is above one of them, then we set $\mathfrak{h}(P) = \infty$.

The height is a forcing invariant, that means every dense subset of an ordering has the same height. In particular, $\mathfrak{h}(P) = \mathfrak{h}(\mathrm{RO}(P))$.

Fact 1.1. *For an ordering P, $\mathfrak{h}(P)$ is the minimal cardinal κ such that forcing with P adds a new subset of κ. In particular, forcing with P preserves all cardinals less than κ.*

An ordering P is *homogeneous in* $\mathfrak{h}$ *(homogeneous in height)* if for every $p \in P$ $\mathfrak{h}(\downarrow p) = \mathfrak{h}(P)$. The following proposition shows that every partial order can be decomposed into factors homogeneous in density. For complete Boolean algebras there is a canonical such decomposition.

[1]Recall that a partial order P is separative if whenever p, q are elements of P such that $p \not\leq q$, there is an $r \in P$ such that $r \leq p$ and $r \perp q$.

Proposition 1.2. *Let B be a complete Boolean algebra. Then $B \cong \prod_{b\in I} B \restriction b$, where I is a partition of unity and $B \restriction b$ is homogeneous in the height for every $b \in I$.*

Moreover, $\mathfrak{h}(B \restriction a) \neq \mathfrak{h}(B \restriction b)$ if $a \neq b$ for $a, b \in I$.

Proof. Let A be the set of all atoms, $\uparrow A = \{b \in B : \exists a \in A\ (a \leq b)\}$, then $B \restriction \bigvee \uparrow A$ is the first factor homogeneous in the height ∞.

Next, we work with an atomless complete algebra $B_0 = B \restriction (- \bigvee \uparrow A)$ $(B \cong B_0 \times B \restriction \bigvee \uparrow A)$. Let $(D_\alpha)_{\alpha<\mathfrak{h}(B_0)}$ be the system of open dense subsets of B_0 such that $\bigcap_{\alpha<\mathfrak{h}(B_0)} D_\alpha$ is not dense. Let A_1 be the subset of elements of B_0 witnessing the non-density, i.e. $\downarrow a \cap \bigcap_{\alpha<\mathfrak{h}(B_0)} D_\alpha = \emptyset$ for every $a \in A_1$. We claim that for every $a \in A_1$ $B \restriction a$ is homogeneous in the height (with height $\mathfrak{h}(B_0)$). Assume not, then there is some $a \in A_1$ and $b < a$ such that $\mathfrak{h}(B \restriction b) < \mathfrak{h}(B \restriction a)$. Thus, there is a system $(S_\alpha)_{\alpha<\mathfrak{h}(B\restriction b)}$ of open dense subsets of $B \restriction b$ with a non-dense intersection below b. But if we set $D_\alpha = S_\alpha \cup B_0 \setminus \downarrow b$ then we get a system of open dense subsets in B_0 without a dense intersection less than $\mathfrak{h}(B_0)$, that is a contradiction.

We take a join $\bigvee A_1$ of all elements from A_1 and the factor $B \restriction \bigvee A_1$ is homogeneous in the height. We continue with the remainder $B_1 = B_0 \restriction (- \bigvee A_1)$ and by the same way get a set A_2 of elements witnessing the non-density of the intersection of a system of open dense subsets of size $\mathfrak{h}(B_1)$. It is possible that $\mathfrak{h}(B_1) = \mathfrak{h}(B_0)$. In this case, we join elements of A_2 with elements of A_1. In the opposite case, $B_1 \restriction \bigvee A_2$ is a new factor homogeneous in the height.

We continue similarly until we treat all elements of B. We end up with the desired decomposition. □

Definition 2 (Base tree property)**.** An ordering $(P, \leq)$ has the base tree property (we shall shortly say it has the *BT-property*) if it contains a dense subset $D \subseteq P$ with the following three properties:

- it is atomless; i.e. for every $d \in D$ there are elements $d_1, d_2 \in D$ below d such that $d_1 \perp d_2$
- it is σ-closed
- $|D| \leq \mathfrak{c}$

It can be easily seen that assuming the Continuum Hypothesis, all partial orders with the BT-property are forcing equivalent with $Coll(\omega_1, 2^\omega)$ and, consequently have a tree base. In fact, the following is true in ZFC.

Theorem 2.1 (The base tree theorem)**.** *Let $(P, \leq)$ be an ordering homogeneous in the height with the BT-property. Then there are $\mathfrak{h}(P)$ maximal antichains $(T_\alpha)_{\alpha<\mathfrak{h}(P)} \subseteq P$ such that:*

(i) $(T = \bigcup_{\alpha<\mathfrak{h}} T_\alpha, \geq)$ *is a tree of height* $\mathfrak{h}(P)$*, where* T_α *is the* α*-th level of the tree,*
(ii) each $t \in T$ *has* $\mathfrak{c}$ *immediate successors,*
(iii) T *is dense in* P.

T *is called the base tree of* P.

Proof. We need to work with some dense subset guaranteed by the definition of the BT-property rather than with P itself. To avoid introducing next new symbols and sets, we assume P itself has the properties.

We use the definition of the height. So we have a system $(A_\alpha)_{\alpha<\mathfrak{h}(P)}$ of open dense subsets with a non-dense intersection. We need to ensure the intersection to be empty. Suppose $\bigcap_{\alpha<\mathfrak{h}(P)} A_\alpha$ is not empty. Since for each $a \in \bigcap_{\alpha<\mathfrak{h}(P)} A_\alpha$ $\mathfrak{h}(\downarrow a) = \mathfrak{h}(P)$ we have a system $(\bar{A}_\alpha)_{\alpha<\mathfrak{h}(P)}$ of open dense sets below a of the same size such that their intersection is non-dense below a. For each such a we replace $\downarrow a \cap A_\alpha$ by $\bar{A}_\alpha$ (i.e. $(A_\alpha \setminus \downarrow a) \cup \bar{A}_\alpha$). We get a new system of open dense subsets with a non-dense intersection. If this intersection is again non-empty we repeat the same procedure for each element from the intersection. We repeat this procedure as long as necessary to get the system $(B_\alpha)_{\alpha<\mathfrak{h}(P)}$ of open dense subsets with an empty intersection.

Next, we extract from each open dense set B_α a maximal antichain C_α. We claim that for every $p \in P$ there is at least one maximal antichain C_α and elements $a, b \in C_\alpha$ such that p is compatible with both of them. Suppose that for some $p \in P$ and for every $\alpha < \mathfrak{h}(P)$ there is only one element c_α from C_α that is compatible with p. But then p is in fact below c_α (since if $p \nleq c_\alpha$ then there is a $p_0 \leq p$ that is disjoint with c_α but necessarily compatible with another element of C_α). But this means that $p \in \bigcap_{\alpha<\mathfrak{h}(P)} \downarrow C_\alpha \subseteq \bigcap_{\alpha<\mathfrak{h}(P)} B_\alpha$ and that is a contradiction with the fact the intersection is empty.

Before constructing the levels of T we modify the antichains into more suitable form, more accurately we modify them into the system $(D_\alpha)_{\alpha<\mathfrak{h}(P)}$ where D_β refines D_α if $\alpha < \beta$. This can be easily done if we set D_α to be the common refinement of $(C_\gamma)_{\gamma\leq\alpha}$ and $(D_\gamma)_{\gamma<\alpha}$.

The levels of the tree T will be the maximal antichains. What we need to care is to ensure that T is dense and that every element of T has $\mathfrak{c}$ immediate successors. We begin by showing that for each element $p \in P$ there is an antichain D_α with $\mathfrak{c}$-many elements compatible with p. There is some D_{α_0} and elements $d_0, d_1 \in D_{\alpha_0}$ compatible with p, i.e. there are elements $p_0 \leq d_0, p_1 \leq d_1$ below p. Then again there is some D_{α_1} and elements $d_{00}, d_{01}, d_{10}, d_{11} \in D_{\alpha_1}$, the first two compatible with p_0, the last two with p_1 (note that this is the place where we need the antichains to be refining; since in general there would be some D_{β_1} with compatible elements with p_0 and some D_{β_2} with compatible elements with p_1 but in our case we can take α_1 to be $\sup\{\beta_1, \beta_2\}$). We again get $p_\zeta \leq p$ for each $\zeta \in {}^2\{0,1\}$. We continue until we get an appropriate $p_\zeta \leq p$ for each $\zeta \in {}^{<\omega}\{0,1\}$. For every $\xi \in {}^\omega\{0,1\}$ we have a descending chain $p \geq p_{\xi\restriction\{0\}} \geq \dots p_{\xi\restriction n} \geq \dots$ with a lower bound (due to σ-closedness) p_ξ. $p_{\xi_1} \perp p_{\xi_2}$ for $\xi_1 \neq \xi_2$. Thus we see that there is a maximal antichain of size $\mathfrak{c}$ below p; we denote it $\mathcal{A}(p)$. Each such p_ξ is compatible with some element d_ξ of D_α where $\alpha = \sup\{\alpha_n : n \in \omega\}$. And again $\xi_1 \neq \xi_2$ implies $d_{\xi_1} \neq d_{\xi_2}$.

Let $P_\alpha = \{p \in P : p \text{ is compatible with } \mathfrak{c}\text{-many elements of } D_\alpha\}$. We see that $P = \bigcup_{\alpha<\mathfrak{h}(P)} P_\alpha$. Since $|P_\alpha| \leq \mathfrak{c}$ for each α there is an injective mapping $f_\alpha : P_\alpha \to {}^\omega 2$ such that $p_{f_\alpha(p)} \leq p$ for every $p \in P_\alpha$, where $p_{f_\alpha(p)}$ is from the construction above.

Now we are ready to start the construction. We set $T_0 = D_0$ and for each $\alpha+1$ we set $T_{\alpha+1}$ to be the common refinement of $D_{\alpha+1}$, $\mathcal{A}(p)$ for all $p \in T_\alpha$ and $\{p_{f_\alpha(p)} : p \in P_\alpha\}$. For α limit, T_α is just the common refinement of $(T_\gamma)_{\gamma<\alpha}$.

Note that by refining $\mathcal{A}(p)$ for all $p \in T_\alpha$ we ensure that each element of the tree has $\mathfrak{c}$-many immediate successors and by refining $\{p_{f_\alpha(p)} : p \in P_\alpha\}$ that T is dense. This finishes the proof. □

Corollary 2.2. *The following statements for an ordering* $(P, \leq)$ *are equivalent:*

(i) P *has the BT-property,*
(ii) P *has a dense subset with the BT-property,*
(iii) Every dense subset of P *has the BT-property,*
(iv) $\mathrm{RO}(P)$ *has the BT-property.*

Proof. Note that (i)⇒(ii) and (iii)⇒(iv) follow from the definition. It suffices to prove (ii)⇒(iii), (iv)⇒(i) is then a consequence.

We need to find a dense subset of a given dense subset that is moreover σ-closed. Atomlessness is determined by the whole ordering, the restriction on size will be clear.

Assuming (ii), we have a base tree T, we are given a dense subset D and we show there is a σ-closed dense subset $S \subseteq D$.

We make S from maximal antichains. For every $t \in T_0$ we find a maximal antichain $A_t \subseteq D$ below. $\bigcup_{t\in T_0} A_t$ is a first maximal antichain S_0.

Then for every $s \in S_0$ we find a maximal antichain $M_s \subseteq T$ below s. Let $P_1 \subseteq T$ be a maximal antichain from T refining $\bigcup_{s\in S_0} A_s$ and T_1. We again for every $p \in P_1$ find a maximal antichain $A_p \subseteq D$ from D, the union $\bigcup_{p\in P_1} A_p$ is S_1.

Next isolated steps are treated similarly. We need not to omit P_α to be refining the tree level T_α. Then we refine it to $S_\alpha \subseteq D$.

For a limit α we take a refinement P_α of all P_β's for $\beta < \alpha$ (which is also a refinement of S_β's) and of T_α. Then we again refine it to $S_\alpha \subseteq D$.

The resulting set $S = \bigcup_{\alpha < \mathfrak{h}(P)} S_\alpha$ is dense and σ-closed. We ensured denseness by refining all levels of T. For σ-closedness observe that for every countable descending chain $s_0 \geq s_1 \geq \ldots$ from S, where $s_n \in S_{\alpha_n}$, there is an inserted descending chain $p_0 \geq p_1 \geq \ldots$ such that $p_0 \geq s_0 \geq p_1 \geq s_1 \geq \ldots$, where $p_n \in P_{\alpha_n}$. This inserted chain has a lower bound p in P_α, where $\alpha = \sup\{\alpha_n : n \in \omega\}$, and p has some successor $s \in S$. □

□

In other words, having a σ-closed dense set is preserved by forcing equivalence among separative partial orders of size continuum. On the other hand, Zapletal in [13] has constructed a model in which the Continuum Hypothesis holds and there are two forcing equivalent separative partial orders of size $\aleph_2$ one σ-closed and the other without a σ-closed dense set. One has to wonder whether such a pair exists in ZFC.

Question 2.3. *Are there, in ZFC, two forcing equivalent separative partial orders, such that one is σ-closed and the other does not have a σ-closed dense set?*

Finally, using this internal characterization of the partial orders with the BT-property one can easily deduce the following external characterization.

Theorem 2.4. *Let $(P, \leq)$ be an arbitrary ordering with the BT-property. Then $\mathrm{RO}(P)$ is a regular subalgebra of $Coll(\omega_1, \mathfrak{c})$.*

Proof. Let $D \subseteq P$ be its dense subset witnessing the BT-property. Then $D \times \mathrm{Fn}(\omega_1, \{0,1\}, \omega_1)$ with induced Cartesian ordering clearly has the BT-property. And the height is ω_1, thus it determines the complete Boolean algebra $\mathrm{Col}(\omega_1, \mathfrak{c})$. Note that there is a regular embedding $e : D \to D \times \mathrm{Fn}(\omega_1, \{0,1\}, \omega_1)$ defined as $e(d) = (d, 1)$ where 1 is the biggest element in $\mathrm{Fn}(\omega_1, \{0,1\}, \omega_1)$, i.e. the empty set. e is extended to $\bar{e} : \mathrm{RO}(P) \to \mathrm{Col}(\omega_1, \mathfrak{c})$ mapping $\mathrm{RO}(P)$ on a regular subalgebra of $\mathrm{Col}(\omega_1, \mathfrak{c})$. □

3. Classical examples

The Boolean algebra $\mathcal{P}(\omega)/fin$ is a prototype of an ordering with the BT-property. Recall the definitions of the cardinal invariants $\mathfrak{p}, \mathfrak{t}, \mathfrak{h}$ [6].

The second fundamental example is $(\mathrm{Dense}(\mathbb{Q}), \subseteq)$, where $\mathrm{Dense}(\mathbb{Q})$ is a set of all dense subsets in rationals. The situation here is similar with the previous example, it is not separative and the ordering $(\mathrm{Dense}(\mathbb{Q}), \subseteq)$ itself does not satisfy the BT-property. We move to the separative modification. The separative modification is $(\mathrm{Dense}(\mathbb{Q}), \subseteq_{\mathrm{nwd}})$, where $A \subseteq_{\mathrm{nwd}} B$ if $A \setminus B$ is nowhere dense in $\mathbb{Q}$, has the BT-property. This ordering is studied in [2].

Let $\mathfrak{p}_{\mathbb{Q}}, \mathfrak{t}_{\mathbb{Q}}, \mathfrak{h}_{\mathbb{Q}}$ be the cardinal invariants of $(\mathrm{Dense}(\mathbb{Q}), \subseteq_{\mathrm{nwd}})$ defined in the same way as their counterparts in $([\omega]^\omega, \subseteq^*)$. It was proved in [2] that $\mathfrak{p}_{\mathbb{Q}} = \mathfrak{p}$ and $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}$ whereas $\mathfrak{h}_{\mathbb{Q}}$ and $\mathfrak{h}$ are incomparable in ZFC, $\mathfrak{h}_{\mathbb{Q}} < \mathfrak{h}$ and $\mathfrak{h}_{\mathbb{Q}} > \mathfrak{h}$ are both consistent (see [2] and [7]); and $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h}$ too of course.

For the third example, let $\mathbb{A}$ be the *Cantor algebra*, i.e. the algebra of all clopen subset of 2^ω, and consider the countable product $\mathbb{A}^\omega$ modulo the ideal $\mathrm{Fin} \subseteq \mathbb{A}^\omega$, where $\mathrm{Fin} = \{f \in \mathbb{A}^\omega : |\{n : f(n) \neq 0\}| < \omega\}$. It satisfies the BT-property, moreover, $\mathbb{A}^\omega/\mathrm{Fin}$ is homogeneous.

$\mathfrak{t}(\mathbb{A}^\omega/\mathrm{Fin}) = \mathfrak{t}$ and $\mathfrak{h}(\mathbb{A}^\omega/\mathrm{Fin}) \leq \min\{\mathfrak{h}, \mathrm{add}(\mathcal{M})\}$ ([3]) and it is consistent that $\mathfrak{h}(\mathbb{A}^\omega/\mathrm{Fin}) < \mathfrak{h}$ ([3],[9]).

For any Boolean algebra B let us consider an infinite product B^ω. Let J be an ideal on ω. By $\mathcal{I}_J \subseteq B^\omega$ we denote the ideal $\{f \in B^\omega : \{n \in \omega : f(n) \neq 0\} \in J\}$. The quotient algebra $B^\omega/\mathcal{I}_J$ consists of equivalence classes where $f, g \in B^\omega$ are equivalent if $\{n : f(n) \neq g(n)\} \in J$ ($f \bigtriangleup g \in \mathcal{I}_J$ equivalently). We state and prove a simple criterion for when such a product has the BT-property.

Theorem 3.1. *Let B be a Boolean algebra and J an ideal on ω. Then the reduced product $B^\omega/\mathcal{I}_J$ has the BT-property if and only if B contains a dense subset of size $\mathfrak{c}$ and either $\mathcal{P}(\omega)/J$ is σ-closed or J is a maximal ideal and B is atomless.*

Proof. Since $B^\omega/\mathcal{I}_J$ contains a dense subset of size $\mathfrak{c}$ if and only if B contains a dense subset of size less or equal to $\mathfrak{c}$ the requirement on the cardinality is satisfied.

Suppose that $\mathcal{P}(\omega)/J$ is not σ-closed. Let $(X_n)_{n\in\omega}$ be a descending chain of infinite subsets of ω such that the chain $([X_n])_{n\in\omega}$ does not have a lower bound in $\mathcal{P}(\omega)/J$, where $[X_n]$ is the equivalence class containing X_n. We define the descending chain $([f_n])_{n\in\omega} \subseteq B^\omega/\mathcal{I}_J$ as follows: $f_n(i) = 1$ if $i \in X_n$ and $f_n(i) = 0$ otherwise (it is the image of the chain $([X_n])_{n\in\omega}$ via the regular embedding of $\mathcal{P}(\omega)/J$ into $B^\omega/\mathcal{I}_J$). Suppose that it has a lower bound $[f]$. Then the support of f, i.e. the set $\{i : f(i) \neq 0\}$, would determine a lower bound for $([X_n])_{n\in\omega}$.

Next we use the fact mentioned in [3] that $B^\omega/\mathcal{I}_J$ can be written as an iteration $\mathcal{P}(\omega)/J \star B^\omega/\dot{\mathcal{U}}$, where $\dot{\mathcal{U}}$ is a name for an ultrafilter added by $\mathcal{P}(\omega)/J$. For $[f] \in B^\omega/\mathcal{I}_J$ We define $\Phi([f]) = (\{i : f(i) \neq 0\}, \dot{[f]})$, where $\dot{[f]}$ is a name for an equivalence class containing f in $B^\omega/\dot{\mathcal{U}}$. Φ is easily verified to be a dense embedding which proves the fact.

Now observe that an ultrapower of any Boolean algebra is σ-closed. For a countable descending chain we can choose representants of equivalence classes $(f_n)_{n\in\omega}$ so that support $f_0 = \omega$, support $f_1 \supseteq$ support $f_2 \supseteq$ support $f_3 \supseteq \ldots$ and $\bigcap_{n\in\omega}$ support $f_n = \emptyset$ since the ultrafilter is non-principal. Then we set $f(i) = f_n(i)$ if n is the smallest number such that $i \in$ support $f_n \setminus$ support f_{n+1}. f clearly determines the lower bound for the chain. Hence, we conclude that $B^\omega/\mathcal{I}_J$ is σ-closed since an iteration of two σ-closed forcings is.

To check atomlessness, if J is not maximal then for any $f \in B^\omega$, where the support of f is not in J, we can always split the support of f into two disjoint infinite sets both outside of J, restrict f on these sets and make two disjoint elements of $B^\omega/\mathcal{I}_J$ below $[f]$. This is no longer possible in case J is a maximal ideal. For such a case we required B to be atomless and we find two disjoint successors coordinatewise. □

4. Orderings made of ideals

We shall deal with orderings that consist of ideals on ω of some concrete type ordered by reverse inclusion.

For an illustration let us consider the following simple example.

4.1. **Non-tall ideals.** An ideal $\mathcal{I}$ on ω is tall if for every $X \in [\omega]^\omega$ there is infinite $Y \subseteq X$ that belongs to $\mathcal{I}$. Consider the set $\mathfrak{T}$ of all non-tall ideals on ω ordered by reverse inclusion.

At first, this ordering is not separative. However, for every $A \in [\omega]^\omega$ consider the ideal I_A of all subsets of ω that have a finite intersection with A. I_A is a non-tall ideal and $B \subseteq^* A$ implies $I_B \supseteq I_A$. Moreover, for every non-tall ideal $\mathcal{I}$ and some infinite set A almost disjoint with every element of $\mathcal{I}$, $I_A \supseteq \mathcal{I}$. Thus we see that $([\omega]^\omega, \subseteq^*)$ is isomorphic with a dense subset of $(\mathfrak{T}, \supseteq)$ and of its seprative modification showing that the separative modification of $(\mathfrak{T}, \supseteq)$ has the BT-property, however it is forcing equivalent to $([\omega]^\omega, \subseteq^*)$.

4.2. **Summable ideals.** The study of summable ideals is in fact mainly the study of sequences because we approach summable ideals via sequences in most cases. We shall focus on an ordering $(c_0^+ \setminus \ell^1, \leq^*)$ where c_0^+ is the set of all sequences of positive reals that tend to zero and ℓ^1

the set of all sequences of reals whose sum converges. The order relation $\leq^*$ is almost domination, i.e. $\bar{f} \leq^* \bar{g}$ if $\{n : g_n > f_n\}$ is finite. The investigation of this ordering was initiated by P. Vojtáš in [11]. $(c_0^+ \setminus \ell^1, \leq^*)$ is not separative but we will show the separative quotient is isomorphic to the set $\mathcal{I}_\Sigma$ of all summable ideals ordered by inverse inclusion.

We check it has the BT-property. Let us verify atomlessness. Let I be a summable ideal determined by a sequence $(a_n)_{n=0}^\infty$, and let $A \in I$. Then $\sum_{i\in\omega\setminus A} a_i$ diverges; we divide $\omega \setminus A$ into two infinite subsets B_1 and B_2 such that the appropriate sums both diverge. We make new sequences $(b_n)_{n=0}^\infty$ and $(c_n)_{n=0}^\infty$ so that $b_i = a_i$ for $i \in A \cup B_1$ and $b_i = z_i$ for $i \in B_2$, where $(z_n)_{n=0}^\infty$ is an arbitrary converging sequence of positive reals. $(c_n)_{n=0}^\infty$ is defined similarly, just B_1 and B_2 change their roles. Both $(b_n)_{n=0}^\infty$ and $(c_n)_{n=0}^\infty$ diverge. We denote the appropriate summable ideals I_b and I_c. It is clear that $I_b, I_c \supseteq I$ and that they are disjoint.

Let $(I_j)_{j\in\omega}$ be an increasing (in inclusion) sequence of summable ideals. Let $(a_n^j)_{n=0}^\infty$ be the sequence of positive reals that determines the ideal I_j. We may assume that $(a_n^0)_{n=0}^\infty \geq (a_n^1)_{n=0}^\infty \geq \dots$. Let n_0 be such that $\sum_{j\leq n_0} a_j^0 > 1$. We set $a_n = a_n^0$ for $n \leq n_0$. Then we find a $n_1 > n_0$ such that $\sum_{j=n_0+1}^{n_1} a_j^1 > 1$ and set $a_n = a_n^1$ for $n_0 < n \leq n_1$. And so on to obtain the whole sequence $(a_n)_{n\in\omega}$ so that $(a_n)_{n\in\omega} \leq^* (a_n^j)_{n\in\omega}$ for all $j \in \omega$.

To verify separativness, consider ideals I_a and I_b, corresponding sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$, such that $I_a \not\supseteq I_b$, i.e. there is a set $B \in I_b$ which does not belong to I_a. That means $\sum_{k\in B} b_k < \infty$ but $\sum_{k\in B} a_k = \infty$. If $\omega\setminus B$ belongs to I_a then I_a and I_b are already disjoint, if this is not that case then we make a new sequence $(c_n)_{n=0}^\infty$ such that $c_n = a_n$ for $n \in B$ and $\sum_{k\in\omega\setminus B} c_k < \infty$. The corresponding ideal I_c is below I_a and disjoint with I_b.

It is easy to check that if $(a_n)_{n=0}^\infty \approx_{\rm sep} (b_n)_{n=0}^\infty$, i.e. $\forall (c_n)_{n=0}^\infty \in (c_0^+ \setminus \ell^1, \leq^*)((c_n)_{n=0}^\infty \perp (a_n)_{n=0}^\infty \Leftrightarrow (c_n)_{n=0}^\infty \perp (b_n)_{n=0}^\infty)$, then $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ determine the same summable ideal and the mapping $\Phi : (c_0^+ \setminus \ell^1, \leq^*) \to (\mathcal{I}_\Sigma, \supseteq)$, defined as $\Phi((c_n)_{n=0}^\infty) = \{A \subseteq \omega : \sum_{n\in A} c_n < \infty\}$, is an onto homomorphism of orderings preserving the disjointness relation. And the preimage of each summable ideal is precisely an equivalence class of sequences in $\approx_{\rm sep}$.

Proposition 4.1. $\mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) = \mathfrak{t}$.

Proof. Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be a descending chain of sequences from $(c_0^+ \setminus \ell^1, \leq^*)$ of length $\kappa < \mathfrak{t}$. We use the methods from [5] to show it has a lower bound.

For each $\alpha < \kappa$ let $h_\alpha : \omega \to \omega$ be a function such that $\forall n \in \omega(\frac{1}{h_\alpha(n)} \leq \bar{a}_{\alpha,n})$. Since $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there is a function $h \in \omega^\omega$ that almost dominates all h_α's, i.e. $h \geq^* h_\alpha$ for all $\alpha < \kappa$.

Similarly, for each $\alpha < \kappa$ let $f_\alpha : \omega \to \omega$ be a function such that $\forall n \in \omega(\sum_{f_\alpha(n) \leq i < f_\alpha(n+1)} \bar{a}_{\alpha,i} > 1)$. Since $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there is a function $f \in \omega^\omega$ that almost dominates all f_α's, i.e. $f \geq^* f_\alpha$ for all $\alpha < \kappa$. Define $g \in \omega^\omega$ recursively so that $g(0) = f(0)$ and $g(n+1) = f(g(n)+1)$. Note that for every $\alpha < \kappa$ and all but finitely many n's $\sum_{g(n) \leq i < g(n+1)} \bar{a}_{\alpha,i} > 1$ since $g(n) < f_\alpha(g(n)) < f_\alpha(g(n)+1) \leq g(n+1)$. We denote I_n the interval $[g(n), g(n+1))$.

For every n, we denote $\mathcal{F}_n$ the set $\{F : \mathrm{dom}(F) \to \mathbb{Q}^+ : \mathrm{dom}(F) \subseteq I_n \wedge \mathrm{rng}(F) \subseteq \{\frac{1}{2|I_n|}, \frac{2}{2|I_n|}, \ldots, 1\} \wedge \sum_{i \in \mathrm{dom}(F)} F(i) > \frac{1}{2}\}$. Let $\mathfrak{F} = \bigcup_{n \in \omega} \mathcal{F}_n$. In the following we shall treat $\mathfrak{F}$ as ω.

For every $\bar{a}_\alpha$, let X_α be the set $\{F : \exists n(F \in \mathcal{F}_n \wedge \forall i \in \mathrm{dom}(F)(F(i) \leq \bar{a}_i))\}$. An easy pigeon-hole type argument shows it is infinite for every $\alpha < \kappa$. It is also clear that $X_\beta \setminus X_\alpha$ is finite for $\alpha < \beta$. Since $\kappa < \mathfrak{t}$, there is a lower bound $X \subseteq \mathfrak{F}$. By reducing if neccessary, we can assume that $|X \cap \mathcal{F}_n| \leq 1$ for every n. Finally, we define a sequence $\bar{a}$ as follows:

For every $m \in \omega$, if there exists $F \in X$ such that $m \in \mathrm{dom}(F)$ then we set $\bar{a}_m = F(m)$. Otherwise, we set $\bar{a}_m = \frac{1}{h(m)}$. It is now easy to check that $\bar{a}$ is the desired lower bound.

To prove the converse, let us at first prove that $\mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) \leq \mathfrak{b}$. Suppose the contrary. Let $(b_\alpha)_{\alpha<\mathfrak{b}}$ be a system of almost increasing functions from ω^ω without an upper bound, $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ a bijection and $(l_n)_{n=0}^\infty$ a strictly decreasing sequence from ℓ^1 such that $l_n < \frac{1}{n}$ for every n. We define a descending chain of sequences from $(c_0^+ \setminus \ell^1, \leq^*)$ $(\bar{a}_\alpha)_{\alpha<\mathfrak{b}}$ as follows: $\bar{a}_{0,\pi(1,k)} = l_k$ for $k \leq b_0(0)$, for $l > b_0(0)$ we set $\bar{a}_{0,\pi(1,l)} = \frac{1}{l}$; generally, $\bar{a}_{0,\pi(n,k)} = l_k$ for $k \leq b_0(n-1)$, for $l > b_0(n-1)$ we set $\bar{a}_{n,\pi(n,l)} = \frac{1}{l}$. $\bar{a}_\alpha$ for other α is defined in the same way.

Let $\bar{a}$ be a lower bound for this chain. Define a function f by $f(n) = \min\{k : \bar{a}_{\pi(n,k)} > l_k\}$. It is easy to check that f almost dominates $(b_\alpha)_{\alpha<\mathfrak{b}}$, a contradiction.

Now assume that $\mathfrak{t} < \mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*))$. Let $(X_\alpha)_{\alpha<\mathfrak{t}} \subseteq [\omega]^\omega$ be a descending chain without a lower bound. We define $f_\alpha \in \omega^\omega$ for every $\alpha < \mathfrak{t}$ so that $f_\alpha(n) = k$ such that $|X_\alpha \cap [f_\alpha(n-1), f_\alpha(n))| \geq n+1$. Since $\mathfrak{t} < \mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) \leq \mathfrak{b}$, by the already used method we find $g \in \omega^\omega$ such that for every $\alpha < \mathfrak{t}$ and for almost all n's $|X_\alpha \cap [g(n-1), g(n))| \geq n+1$.

Define a chain $(\bar{a}_\alpha)_{\alpha<\mathfrak{t}}$ of sequences as follows: $\bar{a}_{\alpha,n} = \frac{1}{k}$ if $n \in X_\alpha \cap [g(k-1), g(k))$; if no such k exists then let $\bar{a}_{\alpha,n} = l_n$.

Finally, let $\bar{a}$ be a lower bound for this descending chain and define a lower bound $X = \{n : \bar{a}_n > l_n\}$ for the chain $(X_\alpha)_{\alpha<\mathfrak{t}}$. $\square$

4.3. **Meager and null ideals.** Next we consider the set of all meager ideals $\mathfrak{M}$ and the set of all ideals $\mathfrak{N}$ of measure zero; i.e. those ideals that are meager sets and null sets respectively in the Cantor space topology. Simultaneously, we study the set of all hereditary meager and null ideals, where an ideal I is hereditary meager (null) if for every $X \in I^+$ the restriction $I \restriction X = \{A \in I : A \subseteq X\}$ is meager (null) in the Cantor space 2^X.

It is obvious they are both σ-closed. We show they are atomless, what their separative quotient is and that there is no dense subset of these orderings that has cardinality $\mathfrak{c}$. In fact, there is $2^{\mathfrak{c}}$ mutually disjoint elements in both orderings.

We will use the following characterization of meager ideals.

Proposition 4.2 (Talagrand; see for example Theorem 4.1.2 [4])**.** *An ideal I is meager if and only if there a partition $(P_i)_{i\in\omega}$ of ω into finite sets such that $\bigcup_{i\in A} P_i \in I$ iff A is finite.*

Proposition 4.3. *There are mappings $\Phi : (\mathfrak{M}, \supseteq) \to (\mathfrak{M}, \supseteq)$ and $\Psi : (\mathfrak{N}, \supseteq) \to (\mathfrak{N}, \supseteq)$ such that $\forall X \in \mathfrak{M} \forall Y \in \mathfrak{N} (\Phi(X) \supseteq X \wedge \Phi(X) \approx_{\text{sep}} X \wedge \Psi(Y) \supseteq Y \wedge \Psi(Y) \approx_{\text{sep}} Y)$.*

Proof. For a meager ideal I consider the set $\tilde{I} = \{A \subseteq \omega : I \restriction A \text{ is not meager}\}$. Let $(P_n)_{n\in\omega}$ be the partition of ω witnessing it is meager. $\tilde{I}$ is a hereditary meager ideal containing I. To see that it is meager check that $(P_n)_{n\in\omega}$ still works. Let $A \in \tilde{I}^+$ be arbitrary. Since $A \notin \tilde{I}$ we have $I \restriction A$ is meager, so there is a partition $(Q_n)_{n\in\omega}$ of A into finite sets such that $\bigcup_{i\in C} Q_i \in I$ iff C is finite. If $\tilde{I} \restriction A$ were not meager then there would be an infinite set $C \subseteq \omega$ such that $B = \bigcup_{i\in C} Q_i \in \tilde{I} \restriction A$. $I \restriction B$ would have to be nonmeager but then there would be an infinite set $D \subseteq C$ such that $\bigcup_{i\in D} Q_i \in I \restriction A$, a contradiction.

For a null ideal I consider the set $\tilde{I} = \{A \subseteq \omega : I \restriction A \text{ is not null}\}$. To show it has measure zero consider the following subset $P = \{(A, B) : A \subseteq \omega \wedge B \in I \restriction A\}$ of $2^\omega \times 2^\omega$. P has measure zero and it follows from Fubini theorem that $\tilde{I} = \{A : P_A \text{ is not null}\}$ has measure zero. It is easy to check that $\tilde{I}$ is downward close in inclusion and use Fubini theorem again to check it is closed under finite unions proving it is a null ideal.

Set $I_0 = I$, $I_1 = \tilde{I_0}$, $I_{n+1} = \tilde{I_n}$ and $\bar{I} = \bigcup_{n\in\omega} I_n$. It is a null ideal and moreover it is hereditary null since for every $A \in \bar{I}^+$ $\bar{I} \restriction A = \bigcup_{n\in\omega} I_n \restriction A$ is a countable union of null sets. $\square$

Corollary 4.4. *$(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$ are atomless, not separative, their separative quotient is isomorphic to the ordering $(\mathfrak{M}_{\mathfrak{H}}, \supseteq)$ of all hereditary meager ideals via the mapping Φ and $(\mathfrak{N}_{\mathfrak{H}}, \supseteq)$ the ordering of all hereditary null ideals via Ψ, respectively.*

Proof. To prove they are atomless, let X be an arbitrary meager ideal, let A and B be two infinite subsets of ω such that $A \cup B = \omega$ and neither A nor B is in $\Phi(X)$ ($\Phi(X)$ is meager, thus not maximal). Extend X by A and by B to obtain two disjoint ideals X_A and X_B that are easily verified to be meager. The proof for null ideals is the same.

We claim they are not separative. Consider some maximal ideal M on odd natural numbers and the ideal F of finite sets on even numbers. Then let $I = \{A \cup B : A \in M \wedge B \in F\}$ and $J = \{A \cup B : A \text{ is an arbitrary subset of odd natural numbers } \wedge B \in F\}$ be two ideals, both easily verified to be meager and null. But $\Phi(I) = \Phi(J)$ and $\Psi(I) = \Psi(J)$ which proves the claim.
On the other hand, if X and Y are two hereditary meager (null) ideals such that $X \not\supseteq Y$, then there is an infinite set $A \in Y$ that is not in X. Let Z be an ideal generated by $X \cup \{\omega \setminus A\}$, it is clearly meager and $\Phi(Z) \supseteq X$ is disjoint with Y.

It remains to show that Φ and Ψ define isomorphisms between separative quotients of $(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$ and orderings $(\mathfrak{M}_{\mathfrak{H}}, \supseteq)$ and $(\mathfrak{N}_{\mathfrak{H}}, \supseteq)$.

Φ and Ψ obviously preserve the inclusion relation and the disjointness between ideals, it suffices to prove that $\Phi(X) \approx_{\text{sep}} X$ and $\Psi(Y) \approx_{\text{sep}} Y$ for any $X \in \mathfrak{M}$ and $Y \in \mathfrak{N}$. But obviously if a meager ideal I is compatible with a meager ideal J, then $\Phi(I)$ is compatible with $\Phi(J)$ and so also with J. The same holds for null ideals. Each equivalence class of meager (null) ideals has its minimal element, the corresponding hereditary meager (null) ideal. $\square$

Finally, we show there is no dense subset of these orderings with size $\mathfrak{c}$, thus preventing them to have the BT-property.

Theorem 4.5. *There are $2^{\mathfrak{c}}$ ideals that are both meager and null and they are mutually disjoint (in both $(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$).*

In particular, neither $(\mathfrak{M}, \supseteq)$ nor $(\mathfrak{N}, \supseteq)$ has the BT-property.

Proof. Let $(I_n)_{n\in\omega}$ be a partition of ω into intervals such that $|I_n| = n+1$. For any $X \subseteq \omega$ let X^I be the set $\bigcup_{m\in X}\{k \in \omega : \exists n(k \text{ is the } m\text{-th element of } I_n\}$.

It is clear that if J is an ideal on ω then $\{X^I : X \in J\}$ is a base of an ideal; we shall denote this ideal as $\mathcal{I}_J$.

Now let $\mathcal{M}$ be the set of all maximal ideals on ω, its size is $2^{\mathfrak{c}}$. For any $J \in \mathcal{M}$, we make an ideal $\mathcal{I}_J$ and obtain a system $\mathfrak{I}$ of $2^{\mathfrak{c}}$ ideals. The disjointness of two ideals $J_1, J_2 \in \mathcal{M}$ is easily seen to be preserved for $\mathcal{I}_{J_1}$ and $\mathcal{I}_{J_2}$.

Claim 1 $\mathfrak{I}$ is a system of meager ideals.

The interval partition $(I_n)_{n\in\omega}$ works for all ideals from $\mathfrak{I}$. Assume that some set set $X \in \mathcal{I}_J$, where $\mathcal{I}_J \in \mathfrak{I}$, contains a union of infinitely many intervals. It is easy to check that once it contains the whole interval I_n then it contains all previous intervals. Thus, we conclude that $X = \omega$ which is a contradiction.

Claim 2 $\mathfrak{I}$ is a system of null ideals.

We use characterization of null ideals (resp. filters) from [4] Theorem 4.1.3. For $m \leq n$, let i_n^m be the m-th element of I_n. Let $A_n = \{a_n = \{i_m^n : n \leq m \leq 2n-1\}\}$. These sets satisfy the first three conditions from the theorem. Let $X \in \mathcal{I}_J$, where $\mathcal{I}_J \in \mathfrak{I}$, be a given set. $X \subseteq Y^I$ for some $Y \in J$. Then it is easy to check that $A_n \cap X = \emptyset$ for $n \in \omega \setminus Y$ and $|\omega \setminus Y| = \omega$; thus we also verified the last fourth condition and proved that every $\mathcal{I}_J$ is null. $\square$

5. Iterations and products

If P and Q are two orderings with the BT-property then their product $P \times Q$ has again the BT-property and the height is less or equal to the minimum of heights of the original orderings. To see this, just realize that if B is a regular subalgebra of C then $\mathfrak{h}(B) \geq \mathfrak{h}(C)$, and that $\mathrm{RO}(P)$ is a regular subalgebra of $\mathrm{RO}(P \times Q)$. The same holds for countable products and iterations. For iterations (or products) of length greater than ω we restrict on countable support and we have to ensure that $\prod_{\alpha<\kappa}^{\mathrm{cs}} P_\alpha$ has size $\mathfrak{c}$, where P_α's are the orderings or their dense subsets witnessing the BT-property.

However, if $\kappa \geq \omega_1$ then the height of such an iteration is ω_1.

Fact 5.1. *Let P_κ be an iteration of length $\kappa \geq \omega_1$ where P_1 has the BT-property and for every isolated $\alpha > 1 \Vdash_{P_\beta} \dot{Q}_\alpha$ has the BT-property, where $\alpha = \beta + 1$. Then assuming the condition on the size is satisfied the iteration has the BT-property and has the height ω_1.*

Proof. Atomlessness is clear, σ-closedness folklor. Let $A_1 \subseteq P_1$ be some maximal antichain. For $\alpha = \beta+1$ and $\alpha < \omega_1$, let $\Vdash_{P_\beta} \dot{A}_\alpha$ is a maximal antichain in $\dot{Q}_\alpha$.

We let $D_1 = \{p \in P_\kappa : \exists a \in A_1(p(0) \leq a)\}$ and for other $\alpha = \beta + 1$ $D_\alpha = \{p \in P_\kappa : p \upharpoonright \beta \Vdash \exists \dot{a} \in \dot{A}_\alpha(p(\beta) \leq \dot{a}\}$. D_α's are dense open sets and any element from the intersection would have to have a support of size at least ω_1. □

Proposition 5.2. *For the ordering $([\omega]^\omega, \subseteq^*)^\alpha$, where $\alpha \leq \omega$, there is a dense subset D such that $\forall d \in D \forall i, j \in \mathrm{dom}(d)(i \neq j \Rightarrow d(i) \cap d(j) = \emptyset)$.*

Proof. It is easy when $\alpha < \omega$. For $\alpha = \omega$, let $f \in ([\omega]^\omega, \subseteq^*)^\omega$ be given. We look for some infinite subset $d(0) \subseteq f(0)$ such that

$$\forall n > 0(|f(n) \setminus d(0)| = \omega) \tag{1}$$

Let $\{a_\gamma : \gamma < \omega_1\}$ be some AD system of size ω_1 on $f(0)$. For every $n > 0$ there is at most one γ such that $|f(n) \setminus a_\gamma| < \omega$. Thus there is some a_δ that satisfies the condition (1). We set $d(0) = a_\delta$ and subtract $d(0)$ from $f(n)$ for every $n > 0$. We continue similarly and finally obtain $d = \langle d(n) : n \in \omega\rangle \in D$ below f. □

References

[1] B. Balcar, J. Pelant, P. Simon, *The space of ultrafilters on* $\mathbb{N}$ *covered by nowhere dense sets*, Fund. Math. 110 (1980), 11-24

[2] B. Balcar, F. Hernandéz-Hernandéz, M. Hrušák, *Combinatorics of dense subsets of rationals*, Fund. Math. 183(2004), 59-79

[3] B. Balcar, M. Hrušák, *Distributivity of the algebra of regular open subsets of* $\beta\mathbb{R} \setminus \mathbb{R}^*$, Topol. Appl. 149 (2005), 1-7

[4] T. Bartoszyński, H. Judah, *Set Theory: On the Structure of the Real Line*, A. K. Peters, Wellesley, 1995

[5] T. Bartoszyński, M. Scheepers, *Remarks on small sets related to trigonometric series*, Topol. Appl. 64 (1995), 133-140

[6] A. Blass, *Combinatorial Cardinal Characteristics of the Continuum*, In: Handbook of Set Theory, Springer, Dordrecht, 2010

[7] J. Brendle, *Van Douwen's diagram for dense sets of rationals*, Ann. Pure Appl. Logic 143, Issues 13(2006), 54-69

[8] P. Dordal, *A model in which the base matrix tree has no cofinal branches*, J. Symb. Logic 52 (1987), Issue 3, 651-664

[9] A. Dow, *The regular open algebra of* $\beta\mathbb{R} \setminus \mathbb{R}$ *is not equal to the completion of* $\mathcal{P}(\omega)/\mathrm{fin}$, Fund. Math. 157 (1998), 33-41

[10] A. Dow, *Tree* π*-Bases for* $\beta\mathbb{N} - \mathbb{N}$ *in Various Models*, Topol. Appl. 33 (1989), 3-19

[11] P. Vojtáš, *Boolean isomorphism between partial orderings of convergent and divergent series and infinite subsets of* $\mathbb{N}$, Proc. Amer. Math. Soc. 117, (1993), 235-242

[12] S.W. Williams, *Trees, Gleason spaces, and Coabsolutes of* $\beta\mathbb{N}-\mathbb{N}$, Trans. Amer. Math. Soc. 271 (1982), Number 1, 83-100

[13] J. Zapletal, *On the existence of a σ-closed dense subset*, Commentat. Math. Univ. Carol. 51 (2010), issue 3, 513-517

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