Countable Fréchet Boolean groups:
An independence result

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Abstract

It is relatively consistent with ZFC that every countable $FU_{fin}$ space of weight $\aleph_1$ is metrizable. This provides a partial answer to a question of G. Gruenhage and P. Szeptycki [GS1].

Introduction

Classical metrization theorem of Birkhoff and Kakutani states that a $T_1$ topological group is metrizable if and only if it is first countable. The results contained here are motivated by the following question:

Question 1. (V.I. Malykhin)[Ar, MT] Is there a countable Fréchet-Urysohn topological group that is not metrizable?

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Recall that a topological space $X$ is Fréchet-Urysohn (or just Fréchet) if whenever a point $x \in X$ is in the closure of a set $A$, there is a sequence of elements of $A$ converging to $x$. In fact, the question can be reformulated by asking for the existence of a separable Fréchet-Urysohn topological group that is not metrizable. On the other hand there are non-separable, even $\sigma$-compact examples, for instance the direct sum of uncountably many copies of the circle.

It is well known (see [Ny1, Ny2, GN]) that the answer to Malykhin’s question is consistently positive, i.e. under either of the following assumptions: $p > \omega_1$, $b = p$ and the existence of an uncountable $\gamma$-set. In fact, either of the assumptions implies the existence of a simply described non-metrizable Fréchet-Urysohn group topology on the Boolean group $G = ([\omega]^\omega, \Delta)$ of finite subsets of $\omega$ with symmetric difference as the group operation.

Let $F$ be a free filter on $\omega$. Then $F^{<\omega} := \{ A \subseteq [\omega]^\omega : \text{there is } F \in F \text{ such that } [F]^{<\omega} \subseteq A \}$ is a filter on $[\omega]^{<\omega}$. By stipulating that $F^{<\omega}$ is a neighbourhood base at $\emptyset$, we introduce a group topology $\tau_F$ on $G$. Note that, by definition, $F^{<\omega}$ is generated by sets of the form $[F]^{<\omega}$, $F \in F$. Also, any set of the form $[F]^{<\omega}$ is a subgroup of $G$. So the $[F]^{<\omega}$, $F \in F$, are in fact open subgroups which generate the neighbourhood base at $\emptyset$.

Observation 1. The following are equivalent:
1. $(G, \tau_F)$ is first-countable (equivalently, metrizable);
2. $F^{<\omega}$ is countably generated;
3. $F$ is countably generated.

A filter $F$ such that the topological group $(G, \tau_F)$ is Fréchet is called an FU$_{\omega}$-filter, see [GS1, GS2, RS, Si]. This is conveniently expressed in the dual language of ideals. If $I = F^\ast$ is the dual ideal of $F$, $I^{<\omega} := \{ A \subseteq [\omega]^{<\omega} : \text{for some } I \in I, a \cap I \neq \emptyset \text{ for all } a \in A \}$ is the dual ideal of $F^{<\omega}$ and is generated by sets of the form $\{ a \in [\omega]^{<\omega} : a \cap I \neq \emptyset \}, I \in I$.

Let $I$ be an ideal on $\omega$. Define the orthogonal ideal $I^{\perp} := \{ A \subseteq [\omega]^{<\omega} : |I \cap A| < \omega \text{ for all } I \in I \}$. Clearly $I^{\perp \perp} \supseteq I$ and $I^{\perp \perp \perp} = I^{\perp}$. We call $I$ Fréchet if $I^{\perp \perp} = I$. This is equivalent to saying that for all $X \in I^+$ there is $C \in [X]^{\omega}$ such that $C \in I^\perp$. $I$ is said to be tall if for all $X \in [\omega]^{\omega}$ there is $C \in [X]^{\omega}$ such that $C \in I$.

Observation 2.
1. Every countably generated ideal is Fréchet.
2. If $I$ is tall, then $I$ is not Fréchet.
3. $I^{\perp}$ is always Fréchet.

The following is a reformulation of a result of [RS]. We include the simple proof for the sake of completeness.

Lemma 1. The topological group $(G, \tau_F)$ is Fréchet iff the ideal $I^{<\omega}$ is Fréchet where $I = F^\ast$.

Proof. ($\Rightarrow$) Let $X \in (I^{<\omega})^+$. This means that for all $I \in I$ there is $a \in X$ with $I \cap a = \emptyset$. Equivalently, for all $F \in F$ there is $a \in X$
First, some additional notions. However, we need a somewhat stronger result.

Assume a filter \( F \) on \( \omega \) and its dual ideal \( I = F^\perp \) are given. For \( a \in [\omega]^{\omega} \) define \( \text{cone}(a) = \{ b \in [\omega]^{\omega} : a \subseteq b \} \), the cone over \( a \).

Define the following families on \( [\omega]^{\omega} \):

- \( G = G_I = \{ A \subseteq [\omega]^{\omega} : \forall I \exists a (a \cap I = \emptyset \land \text{cone}(a) \subseteq A) \} \)
- \( J = J_I = \{ A \subseteq [\omega]^{\omega} : \forall I \exists a (a \cap I = \emptyset) \} \)
- \( J^+ = \{ A \subseteq [\omega]^{\omega} : \exists I \forall a (a \cap I = \emptyset \rightarrow \text{cone}(a) \subseteq A \neq \emptyset) \} \)

Lemma 2. \( G \) is a filter, \( J = G^* \) is the dual ideal, and \( J^+ \) is the collection of \( J \)-positive sets. Furthermore, \( J \) is a Fréchet ideal.

Proof. Only the last assertion needs proof. Let \( A \in J^+ \) and let \( I \) witness this. Let \( \{ k_n : n \in \omega \} \) enumerate \( \omega \setminus I \), and set \( a_n = \{ k_i : i < n \} \). Then for each \( n \) there is \( b_n \supseteq a_n \) with \( b_n \in A \). Clearly \( B = \{ b_n : n \in \omega \} \in J^+ \).

Now consider \( L(G) \), Laver forcing with the filter \( G \). It is defined as the set of those trees \( T \subseteq ([\omega]^{\omega})^{\omega} \) for which there is an \( s \in T \) (called the stem of \( T \)) such that for all \( t \in T \), \( t \subseteq s \) or \( s \subseteq t \) and such that for all \( t \in T \), \( t \supseteq s \) the set \( \text{succ}_T(t) = \{ a \in [\omega]^{\omega} : t^- a \in T \} \in G \). It is ordered by inclusion.

The forcing \( L(G) \) is \( \sigma \)-centered and adds a dominating real \( \dot{\ell} : \omega \rightarrow [\omega]^{\omega} \). Let \( X = \text{ran}(\dot{\ell}) \) be the generic subset of \( [\omega]^{\omega} \) added by \( L(G) \).

Lemma 3. \( \Vdash_{L(G)} X \in (\mathcal{I}^{<\omega})^+ \)

Proof. This is a straightforward genericity argument.

We next intend to prove that if \( I \) is not countably generated, then \( L(G) \) forces that \( X \in (\mathcal{I}^{<\omega})^{+1} \) (Lemma 4 below). For preservation purposes (see Lemmata 5 and 6), however, we need a somewhat stronger result. First, some additional notions.
Say an ideal $K$ is \textit{countably tall} if given $(A_n : n \in \omega) \subseteq [\omega]^{<\omega}$ there is $I \in K$ such that $I \cap A_n$ is infinite for all $n$. Clearly every countably tall ideal is tall. For an ideal $K$ and $X \subseteq K^+$, define the \textit{restriction ideal} by $K \upharpoonright X = \{ I \cup B : I \in K \text{ and } B \subseteq \omega \setminus X \}$.

\textbf{Observation 3.} Assume $K \upharpoonright X$ is tall. Then $X \subseteq K^{+\perp}$.

\textbf{Lemma 4.} Assume $I$ is not countably generated. Then $\Vdash_{\mathbb{L}(\mathcal{G})} \text{“} I^{\subset \omega} \text{ is countably tall”}$. In particular, if $I$ is tall, then $\Vdash_{\mathbb{L}(\mathcal{G})} \text{“} I^{\subset \omega} \text{ is countably tall”}$.

\textbf{Proof.} Let $(A_n : n \in \omega)$ be a sequence of names for infinite subsets of $[\omega]^{<\omega}$. We may suppose the $A_n$ are forced to be subsets of $X$. Assume, by way of contradiction, that for all $I \in \mathcal{I}$ there are $p_I \in \mathbb{L}(\mathcal{G})$, and natural numbers $n_I, m_I$ such that

$$p_I \Vdash \bigcup n_I \cap I \subseteq m_I. \quad (*)$$

Recall the standard rank analysis for Laver forcing [Br1][Br2]. For $s \in ([\omega]^{<\omega})^{\subset \omega}$, say $s$ \textit{favors} $a \in A_n$ if there is no condition $p \in \mathbb{L}(\mathcal{G})$ with stem $s$ such that $p \Vdash a \notin A_n$. Define the \textit{rank} $\rho_n(s)$ by recursion on the ordinals by

$$\rho_n(s) = 0 \quad \text{iff} \quad \exists B \in \mathcal{G}^+ \forall b \in B (s \smallfrown b \text{ favors } b \in A_n)$$

$$\rho_n(s) \leq \alpha \quad \text{iff} \quad \exists B \in \mathcal{G}^+ \forall b \in B (\rho_n(s \smallfrown b) < \alpha)$$

for $\alpha > 0$.

\textbf{Claim 1.} $\rho_n(s) < \infty$ for all $s$ and $n$.

\textbf{Proof.} Fix $n$. Let $k \in \omega$. Define an \textit{auxiliary rank} $\rho_k(s)$ by recursion such that

$$\rho_k(s) = 0 \quad \text{iff} \quad \exists b \not\in k (s \smallfrown b \text{ favors } b \in A_n)$$

and $\rho_k(s) \leq \alpha$ is defined as for $\rho_n(s)$, for $\alpha > 0$. As $A_n$ is forced to be infinite, it is straightforward to see that $\rho_k(s) < \infty$ for all $k$ and $s$. Also note that since $A_n$ is forced to be a subset of the generic $X$, any $s$ can favor only elements of $\text{ran}(s)$.

If $\rho_k(s) = 1$, then there is a $\mathcal{G}$-positive set of $b$ such that $s \smallfrown b$ favors $c \in A_n$ for some $c = c_0$ with $c \not\in k$. If on a $\mathcal{G}$-positive set, the same $c$ works, we get $\rho_k(s) = 0$, a contradiction. Since $c_0 \in \text{ran}(s) \cup \{b\}$, it follows that on a $\mathcal{G}$-positive set, $c_0 = b$. This, however, means that $\rho_n(s) = 0$.

Now, let $k > \max(\bigcup \text{ran}(s))$. Then $\rho_k(s) \geq 1$. By the preceding paragraph and induction, we see that $\rho_n(s) < \infty$, as required.

We continue with the proof of the lemma. Let $s_I$ be the stem of $p_I$. By strengthening the $p_I$, if necessary, we may assume that $\rho_n(s_I) = 0$ for all $I$.

Since the ideal $I$ is not countably generated, there are $s$ and $n$ such that for no $J \in \mathcal{I}$, we have that for all $I$ with $s_I = s$ and $n_I = n$ do we have $I \subseteq J$. Fix such $s$ and $n$.

Let $B \in \mathcal{G}^+$ witness that $\rho_n(s) = 0$. Let $I_0 \in \mathcal{I}$ witness that $B \in \mathcal{G}^+$. Recall that this means that for all $a \in [\omega]^{<\omega}$ with $a \cap I_0 = \emptyset$, we have $\text{cone}(a) \cap B \neq \emptyset$. 

4
Find $I \in \mathcal{I}$ such that $s_I = s$, $n_I = n$, and $I \setminus I_0$ is infinite. By definition of $\mathcal{G}$, there is a with $a \cap I_0 = \emptyset$ such that $\text{cone}(a) \subseteq \text{succ}_{p_I}(s)$. Since $I \setminus I_0$ is infinite, we may assume that $(a \cap I) \setminus m_I \neq \emptyset$. Find $b \in \text{cone}(a) \cap B$. So $b \in B \cap \text{succ}_{p_I}(s)$, and $s \not\vdash b$ favors $b \in \dot{A}_n$ by definition of $B$. Thus we can construct a condition $q \leq p_I$ whose stem extends $s \not\vdash b$ such that $q \upharpoonright b \in \dot{A}_n$. Since $(b \cap I) \setminus m_I \neq \emptyset$, this is a contradiction to the initial assumption (⋆). Thus, for some $I \in \mathcal{I}$,

$$\models \bigcup \dot{A}_n \cap I \text{ is infinite for all } n.$$  

This immediately implies countable tallness of the restriction ideal in the generic extension.  

We turn to the preservation of countable tallness in iterations.

**Lemma 5.** Assume $\mathcal{K}$ is a Fréchet ideal, $\mathcal{H} = \mathcal{K}^*$ is the dual filter, and $\mathcal{L}$ is a countably tall ideal. Then $\models {}^{\mathcal{L}(\mathcal{K})} \mathcal{L}$ is countably tall.

**Proof.** Let $(\dot{A}_n : n \in \omega)$ be names for countable subsets of $\omega$. Assume that for each $I \in \mathcal{L}$ we can find a condition $p_I$ and natural numbers $n_I, m_I$ such that  

$$p_I \upharpoonright I \cap \dot{A}_{n_I} \subseteq m_I.$$  

Define a new rank function $\text{rank}_n$ (cf. the proof of Lemma 4) by recursion on the ordinals as follows:

\[
\text{rank}_n(s) = 0 \quad \text{iff} \quad \exists Z \in [\omega]^\omega \forall k \in Z (s \text{ favors } k \in \dot{A}_n) \\
\text{or } \exists X \in \mathcal{H}^+, f : X \rightarrow \omega \forall \ell \in A (s \not\vdash \ell \text{ favors } f(\ell) \in \dot{A}_n) \\
\text{and } \forall k \in \omega (f^{-1}(k) \in \mathcal{K})
\]

and $\text{rank}_n(s) \leq \alpha$ is defined as for $\text{rk}_n$, for $\alpha > 0$.

**Claim 2.** $\text{rank}_n(s) < \infty$ for all $s$ and $n$.

**Proof.** Fix $n$. Assume $\text{rank}_n(s) = \infty$. So $Z := \{k : s \text{ favors } k \in \dot{A}_n\}$ is finite. Recursively build $p \in \mathcal{L}$ with stem $s$ such that for all $t \in p$ extending $s$,

- $\text{rank}_n(t) = \infty$, and
- $\{k : t \text{ favors } k \in \dot{A}_n\} \subseteq Z$.

Let such $t$ be given. First, there is $X_0 \in \mathcal{H}$ such that $\text{rank}_n(t \not\vdash \ell) = \infty$ for all $\ell \in X_0$. Let $X_1 = \{\ell \in X_0 : \exists k \notin Z (t \not\vdash \ell \text{ favors } k \in \dot{A}_n)\}$. If $X_1 \in \mathcal{H}^+$, then we can define a function as in the definition of $\text{rank}_n$, and so $\text{rank}_n(t) = 0$, a contradiction. Thus $X_1 \in \mathcal{K}$ and $X_0 \setminus X_1 \in \mathcal{H}$. For $t \not\vdash \ell$ with $\ell \in X_0 \setminus X_1$, both clauses above are satisfied, and the construction proceeds.

Now find $q \leq p$ and $k \notin Z$ such that $q \upharpoonright k \in \dot{A}_n$. Then the stem of $q$ in particular favors $k \in \dot{A}_n$, a contradiction.  

We continue with the proof of the lemma. Let $s_I$ be the stem of $p_I$. By strengthening the $p_I$, if necessary, we may assume that $\text{rank}_{n_I}(s_I) = 0$ for all $I$.  

5
Since $L$ is countably tall, there are $s$ and $n$ such that $\{ I \in L : s = s_I \text{ and } n = n_I \}$ is already countably tall. Fix such $s$ and $n$.

We consider two cases, according to the definition of rank $n$.

Case 1. $\exists Z \in [\omega]^\omega \, \forall k \in Z \, (s \text{ favors } k \in A_n)$. Let $I \in L$ be such that $s_I = s$, $n_I = n$, and $I \cap Z$ is infinite. So there is $k > n$ such that $k \in I \cap Z$. Thus there is $q \leq p_I$ with $q \forces k \in A_n$, a contradiction.

Case 2. $\exists X \in \mathcal{H}^+, f : X \rightarrow \omega \, \forall \ell \in A \, (s \thicksim \ell \text{ favors } f(\ell)) \in A_n$ and $\forall k \in \omega \, (f^{-1}(k) \in K)$. Since $K$ is Fréchet, we may assume by shrinking $X$, if necessary, that $X \in K^\omega$. This means that $f^{-1}(k)$ is finite for all $k$. By countable tallness, there is $I \in L$ with $s_I = s$, $n_I = n$, and $I \cap \text{ran}(f)$ is infinite. Since $X \in K^\omega$, we must have $X \subseteq^* \text{succ}_{p_I}(s)$. So there is $k \in I \cap \text{ran}(f)$, $k > m_I$, such that $f^{-1}(k) \cap \text{succ}_{p_I}(s) \neq \emptyset$. Say $\ell \in f^{-1}(k) \cap \text{succ}_{p_I}(s)$. Thus $s \thicksim \ell$ favors $k \in A_n$. Hence there is $q \leq p_I$ whose stem extends $s \thicksim \ell$ such that $q \forces k \in A_n$, again a contradiction. $\square$

**Lemma 6.** Countable tallness is preserved in limit stages of finite support iterations.

**Proof.** This is a standard argument. We provide the details for the sake of completeness. Obviously, it suffices to consider limit stages of cofinality $\omega$.

Let $(P_k, \dot{Q}_k : k \in \omega)$ be a finite support iteration of ccc forcing such that each $P_k$ preserves countable tallness. Also assume $K$ is a countably tall ideal. We need to prove $\forces_{\dot{P}_\omega} \text{"}K \text{ is countably tall"}.$

Let $(A_n : n \in \omega)$ be a sequence of $P_\omega$-names for infinite subsets of $\omega$. In the intermediate extension $V[G_k]$ find a decreasing sequence of conditions $(p_{n,k} : n \in \omega)$ and infinite subsets $A_{m,k}$ of $\omega$ such that

$$p_{n,k} \forces_{\dot{\tau}(k,\omega)} \text{"the first } n \text{ elements of } A_{n,k} \text{ and } \dot{A}_m \text{ agree for } m \leq n".$$  

The $A_{n,k}$ are approximations to $\dot{A}_n$.

Assume, by way of contradiction, that for all $I \in K$ we can find $p_I \in P_\omega$ and $n_I, m_I \in \omega$ such that

$$p_I \forces_{\dot{P}_\omega} I \cap \dot{A}_{n_I} \subseteq m_I.$$  

Clearly there are $k$ and $n$ such that $\mathcal{L}_0 = \{ I \in K : p_I \in P_k \text{ and } n_I = n \}$ is already countably tall. Next notice there is a condition $q \in P_k$ such that

$$q \forces_{\dot{P}_k} \text{"}\dot{\mathcal{L}} = \{ I \in \mathcal{L}_0 : p_I \in G_k \}\text{ is countably tall in } V[\dot{G}_k]".$$  

For assume this was not the case. Then we could build a maximal antichain $(q_i : i \in \omega)$ in $P_k$ and $P_k$-names $(\dot{B}_{j,i} : j, i \in \omega)$ such that

$$q_i \forces_{\dot{P}_k} \text{"}\dot{B}_{j,i} : j \in \omega\text{ witnesses } \{ I \in \mathcal{L}_0 : p_I \in G_k \}\text{ is not countably tall"}.$$  

However, the trivial condition in $P_k$ would then force that $(\dot{B}_{j,i} : j, i \in \omega)$ witnesses that $\mathcal{L}_0$ is not countably tall in $V[\dot{G}_k]$. This contradicts the assumption that $P_k$ preserves countable tallness.

6
Let $G_k$ be a $\mathbb{P}_k$-generic such that $q \in G_k$. Thus there is $I \in \mathcal{L}$ such that $I \cap A_{n,k}$ is infinite. Find $\ell \geq m_I$ with $\ell \in I \cap A_{n,k}$. For large enough $m$, $p_{m,k} \Vdash [\mathbb{P}_k, \omega] \ell \in \dot{A}_n$.

Since $p_I \in G_k$, this contradicts the initial assumption about $p_I$.

The proof of Theorem 2 is now immediate. By taking care of all $\omega_1$-generated ideals $I$ via book-keeping, we iterate forcing notions of the type $L(G)$ for $\omega_2$ steps with finite support. By Lemmata 3 and 4, we add $X \in (I^{<\omega})^+$ such that $I^{<\omega} | X$ is countably tall (and so $I^{<\omega}$ is not Fréchet). By Lemmata 2, 5 and 6, the countable tallness of $I^{<\omega} | X$ is preserved along the iteration, and we are done.

2 Final remarks and questions

Obviously, the question of Gruenhage and Szeptycki remains open. Even though, we do not know whether in the model of $\text{ZFC}$ just constructed there are any $FU_{\text{fin}}$-filters of uncountable character (necessarily of character $\aleph_2$). It should also be noted that there are (consistently) topologies on $(\omega^{<\omega}, \Delta)$ which are not of the form $\tau_f$, yet make $(\omega^{<\omega}, \Delta)$ a non-metrizable Fréchet-Urysohn group. An easy example can be described as follows:

Let $X \subseteq \mathcal{P}(\omega)$ be such that $X$ separates points of $[\omega]^{<\omega}$, i.e. for every $a \in [\omega]^{<\omega} \setminus \{\emptyset\}$ there is an $x \in X$ such that $|a \setminus x|$ is odd. Let

$$F_X = \{A \subseteq [\omega]^{<\omega} : (\exists F \in [X]^{<\omega})(\forall a \in A)(\forall x \in F) \ |a \setminus x| \text{ is even} \}.$$ 

By declaring $F_X$ the neighbourhood base at $\emptyset$, we introduce a Hausdorff group topology $\tau_X$ on $G$. To see this, consider the function $\varphi : [\omega]^{<\omega} \rightarrow 2^X$ defined by $\varphi(a)(x) = 0$ if and only if $|a \setminus x|$ is even. Then $\varphi$ is a group homomorphism and as $X$ separates points of $[\omega]^{<\omega}$, it is an embedding. It is easily seen that the topology $\tau_X$ is just the subspace topology induced by $\varphi$ (viewing $[\omega]^{<\omega}$ as a subgroup of $2^X$).

Now, it is easy to verify that if $X$ is a $\gamma$-set then the topology $\tau_X$ on $([\omega]^{<\omega}, \Delta)$ is Fréchet-Urysohn. Indeed, let for $a \in [\omega]^{<\omega}$

$$U_a = \{x \in \mathcal{P}(\omega) : |a \setminus x| \text{ is even} \}$$

and for $A \subseteq [\omega]^{<\omega}$ let $U_A = \{U_a : a \in A\}$. Note that $U_a$ is a clopen subset of $\mathcal{P}(\omega)$ for every $a \in [\omega]^{<\omega}$. It is now immediate from the definition of $F_X$ that the topology $\tau_X$ is Fréchet-Urysohn at 0 (and hence Fréchet-Urysohn) if and only if for every infinite $A \subseteq [\omega]^{<\omega}$ if $U_A$ is an $\omega$-cover of $X$ then there is an infinite $B \subseteq A$ such that $U_B$ is a $\gamma$-cover of $X$ (see either of [GN, GS1, Ny2] for the definitions of $\gamma$-sets and corresponding covers).

References


